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http://www.aimspress.com/journal/era

ERA, 30(9): 3238–3248.
DOI: 10.3934/era.2022164
Received: 26 November 2021
Revised: 17 April 2022
Accepted: 20 April 2022
Published: 20 June 2022

Research article

Regularity criteria for 3D MHD flows in terms of spectral components

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Abstract: We extend the spectral regularity criteria of the Prodi-Serrin kind for the Navier-Stokes equations in a torus to the MHD equations. More precisely, the following is established: for any $N > 0$, let $\boldsymbol{x}_N$ and $\boldsymbol{y}_N$ be the sum of all spectral components of the velocity and magnetic field whose wave numbers possess absolute value greater that N; then, it is possible to show that for any N the finiteness of the Prodi-Serrin norm of $\boldsymbol{x}_N$ implies the regularity of the weak solution $(\boldsymbol{u}, \boldsymbol{h})$; thus, no restriction on the magnetic field is needed.

Keywords: MHD equations; Prodi-Serrin criteria; spectral Galerkin approximations; regularity

1. Introduction

In many situations, the motion of an incompressible electricity conductor fluid can be modeled by the *magneto-hydrodynamic* equations (MHD). They form a Navier-Stokes-like system coupled to the Maxwell equations. In the case of free motion of heavy ions not directly due to the electrical field (see Schlüter [1] and Pikelner [2]), the related initial-value problem for the equations in the torus can be reduced to

$$\begin{aligned}
\boldsymbol{u}_t - \frac{\eta}{\rho}\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \frac{\mu}{\rho}(\boldsymbol{h}\cdot\nabla)\boldsymbol{h} &= \boldsymbol{f} - \frac{1}{\rho}\nabla\left(p^* + \frac{\mu}{2}|\boldsymbol{h}|^2\right)\\
\boldsymbol{h}_t - \frac{1}{\mu\sigma}\Delta\boldsymbol{h} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{h} - (\boldsymbol{h}\cdot\nabla)\boldsymbol{u} &= -\nabla w\\
\text{div}\ \boldsymbol{u} = \text{div}\ \boldsymbol{h} &= 0\\
\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), \qquad \boldsymbol{h}(x,0) &= \boldsymbol{h}_0(x).
\end{aligned} \tag{1.1}$$

Here, $\boldsymbol{u}$ and $\boldsymbol{h}$ are respectively the unknown velocity and magnetic fields; p^* is the unknown hydrostatic pressure; w is an unknown function related to the motion of heavy ions (the density of electric current j_0 generated by this motion satisfies the identity curl $j_0 = -\sigma\nabla w$); ρ, μ, σ and η are positive constants (respectively, the mass density of the fluid, the magnetic permeability, the electric conductivity and the dynamic viscosity); finally, $\boldsymbol{f}$ is a given external force field. We complete (1.1) with periodic in space boundary conditions for $\boldsymbol{u}$ and $\boldsymbol{h}$.

In the context of the three-dimensional Navier-Stokes equations, uniqueness of weak solution is one of the most dificult and misterious questions (see for instance [3]). For a sufficiently smooth solution, uniqueness is proved quite easily; however, in the class of weak solutions, the problem remains open (see however [4] for a recent result concerning non-uniqueness in a class of "very weak" solutions).

Among other works devoted to clarify and/or provide partial answers to this question, we can mention [5] and [6], where one finds the so-called *Prodi-Serrin condition* (see [3]). By this we mean an additional convenient hypothesis for the velocity making it possible to demonstrate that the weak solution is global, regular and unique.

Let us set

$$\mathcal{V} := \{\varphi \in C^\infty(\mathbb{T}^3)^3 : \operatorname{div}\varphi = 0\ ,\ \int \varphi\, dx = 0\}$$

and let us introduce the spaces $\boldsymbol{L}^2(\mathbb{T}^3) := L^2(\mathbb{T}^3)^3$, $\boldsymbol{H}^r(\mathbb{T}^3) = H^r(\mathbb{T}^3)^3$,

$$\boldsymbol{H} := \text{ the closure of } \mathcal{V} \text{ in } \boldsymbol{L}^2(\mathbb{T}^3) \tag{1.2}$$

and

$$\boldsymbol{V} := \text{ the closure of } \mathcal{V} \text{ in } \boldsymbol{H}^1(\mathbb{T}^3). \tag{1.3}$$

Recall that, for the Navier-Stokes equations, it is well known that, if $\boldsymbol{u}_0 \in \boldsymbol{H}$ and $\boldsymbol{f} \in L^2(0,T;\boldsymbol{H})$, there exists at least one weak solution

$$\boldsymbol{u} \in L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H}),$$

which is usually known as the *Leray-Hopf solution*. It is also known that, if $\boldsymbol{u}_0 \in \boldsymbol{V}$, there exists $T_* \in (0,+\infty]$ such that the solution is strong and unique and, in particular,

$$\boldsymbol{u} \in L^2(0,T';\boldsymbol{H}^2(\Omega)) \cap L^\infty(0,T';\boldsymbol{V})$$

for all $T' < T_*$. Therefore, it is completely natural to ask under which conditions one has a global in time strong solution for any initial data $\boldsymbol{u}_0 \in \boldsymbol{V}$, without any restriction on the size of its norm.

The results in [5] and [6] assert that any weak solution to the Navier-Stokes equations is actually strong up to $t = T$ if

$$\boldsymbol{u} \in L^r(0,T;\boldsymbol{L}^q(\Omega)), \tag{1.4}$$

where (r,q) is a *Prodi-Serrin pair*, that is,

$$\frac{2}{r} + \frac{3}{q} = 1 \ \text{ with } r \in [2,\infty) \text{ and } q \in (3,\infty].$$

See also [7] for other properties satisfied by the weak solutions under the assumption (1.4) and [8, 9] for the case $q = 3$.

In the case of the torus, using the expansion of the solutions in terms of the spectral components, the following was shown in [10]: let the $\widehat{\boldsymbol{u}}^k = \widehat{\boldsymbol{u}}^k(t)$ be the Fourier coefficients of $\boldsymbol{u}(\cdot\,,t)$ associated to the eigenfunctions $e^{ik\cdot x}$ for $k = (k_1, k_2, k_3) \in \mathbb{Z}^3$ and let $\boldsymbol{x}_N$ be for each $N \geq 1$ the partial sum corresponding to all k with $|k_1|, |k_2|, |k_3| > N$; then, if for some Prodi-Serrin pair (r, q) with $q > 3$ one has

$$\liminf_{N\to+\infty} \|\boldsymbol{x}_N\|_{L^r(0,T;\boldsymbol{L}^q(\mathbb{T}^3))} < +\infty, \tag{1.5}$$

the regularity of the solution $\boldsymbol{u}$ is guaranteed up to $t = T$. The same holds if one has

$$\liminf_{N\to+\infty} \|\nabla \boldsymbol{x}_N\|_{L^r(0,T;\boldsymbol{L}^q(\mathbb{T}^3))} < +\infty \tag{1.6}$$

for some (r, q) such that $(2r, 2q)$ is a Prodi-Serrin pair with $q > 3/2$.

In this paper, our main purpose is to extend this regularity criteria to the framework of (1.1). Note that, for this system, there exist many results in the literature concerning the existence of weak and strong solutions and their regularity; see for instance [11–19]. In particular, some regularity results of the Prodi-Serrin kind can be found. More precisely, if (r, q) is a Prodi-Serrin pair with $q > 3$ and a weak solution to (1.1) satisfies

$$\boldsymbol{u} \in L^r(0, T; \boldsymbol{L}^q(\Omega)), \quad \boldsymbol{h} \in L^r(0, T; \boldsymbol{L}^q(\Omega)), \tag{1.7}$$

then the regularity of the solution is ensured (see for instance [13, 16]). A "better" result will be established below (see Theorem 2.6).

The plan of the paper is as follows. In Section 2, some notations and function spaces are introduced and the statement of the main result is given. Specifically we see that, if a Prodi-Serrin condition on the velocity field is satisfied, the solution $(\boldsymbol{u}, p^*, \boldsymbol{h}, w)$ to (1.1) is smooth. Section 3 is devoted to prove this result. Finally, we present some additional comments in Section 4.

2. Preliminaries and main result

Let us start by recalling some definitions and elementary results.

The $L^2(\mathbb{T}^3)$-inner product and norm will be respectively denoted by $(\cdot\,,\cdot)$ and $\|\cdot\|$; for any p with $1 \leq p \leq \infty$, $\|\cdot\|_{L^p}$, $1 \leq p \leq \infty$ will stand for the norm in $L^p(\mathbb{T}^3)$; also, the norms in the usual Sobolev spaces $H^m(\mathbb{T}^3)$ and $W^{k,p}(\mathbb{T}^3)$ will be respectively denoted by $\|\cdot\|_{H^m}$ and $\|\cdot\|_{W^{k,p}}$. The same notations will be used for the corresponding vector-valued spaces.

In the sequel, the symbol C will stand for a generic positive constant. Sometimes, we will write C_q, C_ϵ, etc. in order to indicate explicitly the data on which it depends.

For any Banach space B and any $q \in [1, +\infty)$, we will denote by $L^q(0, T; B)$ the space formed by the B-valued (classes of) functions in $(0, T)$ that are L^q-integrable in the sense of Bochner. On the other hand, $L^\infty(0, T; B)$ will stand for the space of measurable and essentially bounded (classes of) functions $v : (0, T) \mapsto B$.

The $L^q(0, T; B)$ are Banach spaces for the norms $\|\cdot\|_{L^q(0,T;B)}$, where

$$\|v\|_{L^q(0,T;B)} := \begin{cases} \left(\displaystyle\int_0^T \|v(t)\|_B^q \, dt \right)^{1/q} & \text{if } q < +\infty \\ \operatorname{ess\,sup}_{[0,T]} \|v(t)\|_B & \text{if } q = +\infty. \end{cases}$$

Note that, in terms of Fourier expansions, the spaces $\boldsymbol{H}$ and $\boldsymbol{V}$, respectively given by (1.2) and (1.3), can also be described as follows (see [20, 21]):

$$\boldsymbol{H} = \left\{ \boldsymbol{h} \in \boldsymbol{L}^2(\mathbb{T}^3) : \widehat{\boldsymbol{h}}^0 = 0, \ \ k \cdot \widehat{\boldsymbol{h}}^k = 0 \ \ \forall k \in \mathbb{Z}^3 \right\}$$

and

$$\boldsymbol{V} = \left\{ \boldsymbol{h} \in \boldsymbol{H}^1(\mathbb{T}^3) : \widehat{\boldsymbol{h}}^0 = 0, \ \ k \cdot \widehat{\boldsymbol{h}}^k = 0 \ \ \forall k \in \mathbb{Z}^3 \right\},$$

where, for any $\boldsymbol{h}$, we denote by $\widehat{\boldsymbol{h}}^k$ the associated Fourier coefficients.

Throughout this paper, $P : \boldsymbol{L}^2\left(\mathbb{T}^3\right) \mapsto \boldsymbol{H}$ will stand for the usual orthogonal projector. We will also use the Stokes operator $A : D(A) \subset \boldsymbol{H} \mapsto \boldsymbol{H}$, given by

$$A\boldsymbol{v} = -\Delta \boldsymbol{v} \quad \forall \boldsymbol{v} \in D(A) := \boldsymbol{V} \cap \boldsymbol{H}^2(\mathbb{T}^3)$$

and the trilinear form $b\,(\cdot\,,\cdot\,,\cdot)$, with

$$b\,(\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\zeta}) = \sum_{i,j=1}^{3} \int_{\mathbb{T}^3} \varphi_i\, \partial_i \psi_j\, \zeta_j\, dx.$$

We will need in the sequel the Fourier expansions of $\boldsymbol{u}$ and $\boldsymbol{h}$, respectively given by

$$\boldsymbol{u} = \sum_{k \in \mathbb{Z}^3} \widehat{\boldsymbol{u}}^k e^{2\pi i k \cdot x} \ \text{ and } \ \boldsymbol{h} = \sum_{k \in \mathbb{Z}^3} \widehat{\boldsymbol{h}}^k e^{2\pi i k \cdot x}, \tag{2.1}$$

where $\widehat{\boldsymbol{u}}^k = \widehat{\boldsymbol{u}}^k(t)$ and $\widehat{\boldsymbol{h}}^k = \widehat{\boldsymbol{h}}^k(t)$ are the Fourier coefficients. As in [13], for any $N \geq 1$ we introduce

$$\boldsymbol{x}_N = Q_N(\boldsymbol{u}) := \sum_{|k_1|,|k_2|,|k_3| > N} \widehat{\boldsymbol{u}}^k e^{2\pi i k \cdot x}. \tag{2.2}$$

In our main result, one of the following assumptions will be imposed:

- **Hyp 1:** (r, q) is a Prodi-Serrin pair with $q > 3$ and

$$\liminf_{N \to \infty} \|Q_N(\boldsymbol{u})\|_{L^r\left(0,T;\boldsymbol{L}^q(\mathbb{T}^3)\right)} < +\infty.$$

- **Hyp 2:** $(2r, 2q)$ is a Prodi-Serrin pair and one has $q > 3/2$ and

$$\liminf_{N \to \infty} \|\nabla Q_N(\boldsymbol{u})\|_{L^r\left(0,T;\boldsymbol{L}^q(\mathbb{T}^3)\right)} < +\infty.$$

For convenience, let us introduce the positive constants $\alpha := \rho/\mu$, $\nu := \eta/\mu$ and $\gamma := 1/(\mu\sigma)$. Then, the equations in (1.1) can be rewritten as follows:

$$\begin{cases} \alpha \boldsymbol{u}_t - \nu \Delta \boldsymbol{u} + \alpha(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{h} \cdot \nabla)\boldsymbol{h} &= \alpha \boldsymbol{f} - \nabla p \\ \boldsymbol{h}_t - \gamma \Delta \boldsymbol{h} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{h} - (\boldsymbol{h} \cdot \nabla)\boldsymbol{u} &= -\nabla w \\ \text{div } \boldsymbol{u} = \text{div } \boldsymbol{h} = 0 & \end{cases}$$

for some *effective pressure* p.

Definition 2.1. Let us assume that $\boldsymbol{u}_0, \boldsymbol{h}_0 \in \boldsymbol{H}$ and $\boldsymbol{f} \in \boldsymbol{L}^2(0, T; \boldsymbol{H})$ are given. A weak solution to the MHD equations (1.1) is a couple $(\boldsymbol{u}, \boldsymbol{h})$ with $\boldsymbol{u}, \boldsymbol{h} \in L^2(0, T; \boldsymbol{V}) \cap L^\infty(0, T; \boldsymbol{H})$ such that

$$\begin{cases} \alpha(\boldsymbol{u}_t, \boldsymbol{\varphi}) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + \alpha b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{\varphi}) - b(\boldsymbol{h}, \boldsymbol{h}, \boldsymbol{\varphi}) = \alpha(\boldsymbol{f}, \boldsymbol{\varphi}) \;\; \forall \boldsymbol{\varphi} \in \boldsymbol{V} \\ (\boldsymbol{h}_t, \boldsymbol{\phi}) + \gamma(\nabla \boldsymbol{h}, \nabla \boldsymbol{\phi}) + b(\boldsymbol{u}, \boldsymbol{h}, \boldsymbol{\phi}) - b(\boldsymbol{h}, \boldsymbol{u}, \boldsymbol{\phi}) = 0 \;\; \forall \boldsymbol{\phi} \in \boldsymbol{V} \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad \boldsymbol{h}|_{t=0} = \boldsymbol{h}_0. \end{cases} \tag{2.3}$$

It is known that the MHD equations are solvable, see for instance Theorem 2.1 in [17]. More precisely, the following holds:

Theorem 2.2. *For any $\boldsymbol{u}_0, \boldsymbol{h}_0 \in \boldsymbol{H}$ and $\boldsymbol{f} \in \boldsymbol{L}^2(0, T; \boldsymbol{H})$, there exists at least one weak solution $(\boldsymbol{u}, \boldsymbol{h})$ to* (1.1).

In the sequel, for any $\boldsymbol{w} \in \boldsymbol{L}^2(\mathbb{T}^3)$ with Fourier coefficients $\widehat{\boldsymbol{w}}^k$ and any $s \in \mathbb{R}$, whenever it makes sense, we will set

$$(-\Delta)^s \boldsymbol{w} := \sum_{k \in \mathbb{Z}^3} \widehat{\boldsymbol{\omega}}^k \lambda_k^s e^{2\pi i k \cdot x},$$

with the λ_k given by $\lambda_k := 4\pi |k|^2$. On the other hand, for any weak solution $(\boldsymbol{u}, \boldsymbol{h})$ to (1.1) and any $N \geq 1$, we will introduce the corresponding $\boldsymbol{v}_N$, with

$$\boldsymbol{v}_N = \boldsymbol{u} - \boldsymbol{x}_N = \sum_{|k_1| \leq N} \widehat{\boldsymbol{u}}^k e^{2\pi i k \cdot x} + \sum_{|k_1| > N, \; |k_2| \leq N} \widehat{\boldsymbol{u}}^k e^{2\pi i k \cdot x} + \sum_{|k_1|, \; |k_2| > N, \; |k_3| \leq N} \widehat{\boldsymbol{u}}^k e^{2\pi i k \cdot x}.$$

For the $\boldsymbol{x}_N$, we have the following Sobolev embedding result, whose proof can be found for instance in [10] (see Theorem 2.0.3):

Theorem 2.3. *Let q and s satisfy*

$$q \in (2, +\infty), \quad s \in (0, 1) \quad \text{and} \quad \frac{1}{q} = \frac{1}{2} - \frac{2s}{3}.$$

Then, for any $\boldsymbol{w} \in \boldsymbol{H}^{2s}(\mathbb{T}^3)$, we have

$$\|\boldsymbol{w}\|_{L^q} \leq C_{q,s} \left(\|(-\Delta)^s \boldsymbol{w}\| + \|\boldsymbol{w}\| \right). \tag{2.4}$$

Although they may possess an infinite amount of spectral components and "live" in a space of functions defined in a three-dimensional domain, the $\boldsymbol{v}_N$ satisfy appropriate two-dimensional Sobolev inequalities with constants depending on N. This is detailed in the following result (see [10], Lemma 2.0.4):

Lemma 2.4. *Let p and s be given with*

$$p \in (2, +\infty), \quad s \in [0, 1/2) \quad \text{and} \quad \frac{1}{p} = \frac{1}{2} - s.$$

Let the v_i be the components of $\boldsymbol{v}_N$. Then, if $\boldsymbol{v}_N \in \boldsymbol{H}^{2s}(\mathbb{T}^3)$, one has

$$\|v_i\|_{L^p} \leq C_{q,s} N^{1/2} \|(-\Delta)^s v_i\|, \quad i = 1, 2, 3. \tag{2.5}$$

If $\boldsymbol{v}_N \in \boldsymbol{V}$, one also has

$$\|\boldsymbol{v}_N\|_{L^p} \leq C_p N^{1/2} \|\boldsymbol{v}_N\|^{2/p} \|\nabla \boldsymbol{v}_N\|^{1-2/p}. \tag{2.6}$$

We will also need the following *Interpolation Lemma:*

Lemma 2.5. *Let q and a satisfy*

$$q \in [2,6], \quad a \in [0,1] \quad \text{and} \quad \frac{1}{q} = \frac{a}{2} + \frac{1-a}{6}. \tag{2.7}$$

Then, for any function $\varphi \in H^1(\mathbb{T}^3)$ with zero mean, one has

$$\|\varphi\|_{L^q} \le C_q \, \|\varphi\|^a \, \|\nabla \varphi\|^{1-a}. \tag{2.8}$$

The main result in this paper is the following:

Theorem 2.6. *Let $(\boldsymbol{u}, \boldsymbol{h})$ be a weak solution to* (1.1) *and let us assume that* **Hyp 1** *or* **Hyp 2** *holds. Then,* $\boldsymbol{u}$ *and* $\boldsymbol{h}$ *belong to $L^2(0,T;D(A)) \cap L^\infty(0,T;\boldsymbol{V})$ and, consequently, $(\boldsymbol{u}, \boldsymbol{h})$ is a strong solution to* (1.1) *up to $t = T$.*

3. Proof of Theorem 2.6

A crucial fact used in the proof is that, due to periodicity, the Stokes and Laplace operators coincide on $D(A)$; in other words, $-\Delta$ sends $D(A)$ into $\boldsymbol{H}$.

In the sequel, we assume for simplicity that $\boldsymbol{f} = 0$. A nonzero right hand side in the motion equation in (1.1) does not change the proof significantly. Moreover, the usual convention of repeated indices will be adopted and the index N will be omitted.

We will first establish the result under the assumption **Hyp 2**. Thus, let us take $\boldsymbol{\varphi} = A\boldsymbol{u}$ and $\boldsymbol{\phi} = A\boldsymbol{h}$ in (2.3). We get:

$$\begin{cases} \dfrac{\alpha}{2}\dfrac{d}{dt}\|\nabla \boldsymbol{u}\|^2 + \nu\|A\boldsymbol{u}\|^2 & = -\alpha b(\boldsymbol{u},\boldsymbol{u},A\boldsymbol{u}) + b(\boldsymbol{h},\boldsymbol{h},A\boldsymbol{u}) \\ \dfrac{1}{2}\dfrac{d}{dt}\|\nabla \boldsymbol{h}\|^2 + \gamma\|A\boldsymbol{h}\|^2 & = -b(\boldsymbol{u},\boldsymbol{h},A\boldsymbol{h}) + b(\boldsymbol{h},\boldsymbol{u},A\boldsymbol{h}). \end{cases}$$

After integration by parts the right hand sides of these identities, we have:

$$\begin{cases} \dfrac{\alpha}{2}\dfrac{d}{dt}\|\nabla \boldsymbol{u}\|^2 + \nu\|A\boldsymbol{u}\|^2 & = & -\alpha b(\partial_i\boldsymbol{u},\boldsymbol{u},\partial_i\boldsymbol{u}) + b(\partial_i\boldsymbol{h},\boldsymbol{h},\partial_i\boldsymbol{u}) \\ & & -b(\boldsymbol{h},\partial_i\boldsymbol{u},\partial_i\boldsymbol{h}) \\ \dfrac{1}{2}\dfrac{d}{dt}\|\nabla \boldsymbol{h}\|^2 + \gamma\|A\boldsymbol{h}\|^2 & = & -b(\partial_i\boldsymbol{u},\boldsymbol{h},\partial_i\boldsymbol{h}) + b(\partial_i\boldsymbol{h},\boldsymbol{u},\partial_i\boldsymbol{h}) \\ & & +b(\boldsymbol{h},\partial_i\boldsymbol{u},\partial_i\boldsymbol{h}). \end{cases}$$

Therefore, after addition, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\alpha\|\nabla \boldsymbol{u}\|^2 + \|\nabla \boldsymbol{h}\|^2\right) + \nu\|A\boldsymbol{u}\|^2 + \gamma\|A\boldsymbol{h}\|^2 \\ &\quad = -\alpha b(\partial_i\boldsymbol{u},\boldsymbol{u},\partial_i\boldsymbol{u}) + b(\partial_i\boldsymbol{h},\boldsymbol{h},\partial_i\boldsymbol{u}) - b(\partial_i\boldsymbol{u},\boldsymbol{h},\partial_i\boldsymbol{h}) + b(\partial_i\boldsymbol{h},\boldsymbol{u},\partial_i\boldsymbol{h}). \end{aligned} \tag{3.1}$$

The terms in the right hand side of (3.1) must be bounded appropriately. In the sequel, we denote by ϵ a small positive constant (to be chosen below) and we will use Hölder's and Young's inequalities, Lemma 2.4 and Lemma 2.5.

First, $b(\partial_i \boldsymbol{u}, \boldsymbol{u}, \partial_i \boldsymbol{u})$ can be bounded arguing as in [10], pp. 79–80. More precisely,

$$|b(\partial_i \boldsymbol{u}, \boldsymbol{u}, \partial_i \boldsymbol{u})| \leq |b(\partial_i \boldsymbol{v}, \boldsymbol{u}, \partial_i \boldsymbol{u})| + |b(\partial_i \boldsymbol{x}, \boldsymbol{u}, \partial_i \boldsymbol{u})|$$

and we get

$$\begin{aligned} |b(\partial_i \boldsymbol{v}, \boldsymbol{u}, \partial_i \boldsymbol{u})| &\leq |b(\partial_i \boldsymbol{v}, \boldsymbol{u}, \partial_i \boldsymbol{v})| + |b(\partial_i \boldsymbol{v}, \boldsymbol{u}, \partial_i \boldsymbol{x})| \\ &\leq C\|\nabla \boldsymbol{v}\|_{L^4}^2 \|\nabla \boldsymbol{u}\| + C\|\nabla \boldsymbol{x}\|_{L^q} \|\nabla \boldsymbol{v}\|_{L^p} \|\nabla \boldsymbol{u}\|_{L^p} \\ &\leq CN\|\nabla \boldsymbol{v}\| \|A\boldsymbol{v}\| \|\nabla \boldsymbol{u}\| \\ &\quad + C\|\nabla \boldsymbol{x}\|_{L^q} \|\nabla \boldsymbol{v}\|^a \|A\boldsymbol{v}\|^{1-a} \|\nabla \boldsymbol{u}\|^a \|A\boldsymbol{u}\|^{1-a} \\ &\leq C_\epsilon \left(\|\nabla \boldsymbol{x}\|_{L^q}^r + N^2 \|\nabla \boldsymbol{v}\|^2 \right) \|\nabla \boldsymbol{u}\|^2 + \epsilon \|A\boldsymbol{u}\|^2 \end{aligned}$$

and

$$\begin{aligned} |b(\partial_i \boldsymbol{x}, \boldsymbol{u}, \partial_i \boldsymbol{u})| &\leq C\|\nabla \boldsymbol{x}\|_{L^q} \|\nabla \boldsymbol{u}\|_{L^p}^2 \\ &\leq C\|\nabla \boldsymbol{x}\|_{L^q} \|\nabla \boldsymbol{u}\|^{2a} \|A\boldsymbol{u}\|^{2-2a} \\ &\leq C_\epsilon \|\nabla \boldsymbol{x}\|_{L^q}^r \|\nabla \boldsymbol{u}\|^2 + \epsilon \|A\boldsymbol{u}\|^2, \end{aligned}$$

where $p = 6r/(r+2)$ and $a = 1/r$.

Hence, for any small $\epsilon > 0$, one has

$$|b(\partial_i \boldsymbol{u}, \boldsymbol{u}, \partial_i \boldsymbol{u})| \leq C_\epsilon \left(\|\nabla \mathbf{x}\|_{L^q}^r + N^2 \|\nabla \boldsymbol{v}\|^2 \right) \|\nabla \boldsymbol{u}\|^2 + 2\epsilon \|A\boldsymbol{u}\|^2. \tag{3.2}$$

Secondly, we observe that

$$|b(\partial_i \boldsymbol{h}, \boldsymbol{h}, \partial_i \boldsymbol{u})| \leq |b(\partial_i \boldsymbol{h}, \boldsymbol{h}, \partial_i \boldsymbol{v})| + |b(\partial_i \boldsymbol{h}, \boldsymbol{h}, \partial_i \boldsymbol{x})|$$

and

$$\begin{aligned} |b(\partial_i \boldsymbol{h}, \boldsymbol{h}, \partial_i \boldsymbol{v})| &\leq |b(A\boldsymbol{h}, \boldsymbol{h}, \boldsymbol{v})| + |b(\partial_i \boldsymbol{h}, \partial_i \boldsymbol{h}, \boldsymbol{v})| \\ &\leq 2\|\boldsymbol{v}\|_{L^4} \|\nabla \boldsymbol{h}\|_{L^4} \|A\boldsymbol{h}\| \\ &\leq CN\|\boldsymbol{v}\|^{1/2} \|\nabla \boldsymbol{v}\|^{1/2} \|\nabla \boldsymbol{h}\|^{1/2} \|A\boldsymbol{h}\|^{3/2} \\ &\leq C_\epsilon N^4 \|\boldsymbol{v}\|^2 \|\nabla \boldsymbol{v}\|^2 \|\nabla \boldsymbol{h}\|^2 + \epsilon \|A\boldsymbol{h}\|^2, \end{aligned} \tag{3.3}$$

while

$$\begin{aligned} |b(\partial_i \boldsymbol{h}, \boldsymbol{h}, \partial_i \boldsymbol{x})| &\leq C\|\nabla \boldsymbol{x}\|_{L^q} \|\nabla \boldsymbol{h}\|_{L^p}^2 \\ &\leq C\|\nabla \boldsymbol{x}\|_{L^q} \|\nabla \boldsymbol{h}\|^{2a} \|A\boldsymbol{h}\|^{2-2a} \\ &\leq C_\epsilon \|\nabla \boldsymbol{x}\|_{L^q}^r \|\nabla \boldsymbol{h}\|^2 + \epsilon \|A\boldsymbol{h}\|^2. \end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned} |b(\partial_i \boldsymbol{h}, \boldsymbol{h}, \partial_i \boldsymbol{u})| \leq\; & C_\epsilon \left(\|\nabla \boldsymbol{x}\|_{L^q}^r + N^4 \|\boldsymbol{v}\|^2 \|\nabla \boldsymbol{v}\|^2 \right) \|\nabla \boldsymbol{h}\|^2 \\ & + 2\epsilon \left(\|A\boldsymbol{u}\|^2 + \|A\boldsymbol{h}\|^2 \right). \end{aligned} \tag{3.5}$$

Observe that the remaining terms in (3.1) can be bounded in the same way. Thus,

$$\begin{aligned} & |b(\partial_i \boldsymbol{u}, \boldsymbol{h}, \partial_i \boldsymbol{h})| + |b(\partial_i \boldsymbol{h}, \boldsymbol{u}, \partial_i \boldsymbol{h})| \\ & \quad \leq C_\epsilon \left(\|\nabla \boldsymbol{x}\|_{L^q}^r + N^4 \|\boldsymbol{v}\|^2 \|\nabla \boldsymbol{v}\|^2 \right) \|\nabla \boldsymbol{h}\|^2 + 4\epsilon \left(\|A\boldsymbol{u}\|^2 + \|A\boldsymbol{h}\|^2 \right). \end{aligned} \tag{3.6}$$

From (3.1), (3.2) and (3.6), the following differential inequality is obtained:

$$\frac{1}{2}\frac{d}{dt}\left(\alpha\|\nabla \boldsymbol{u}\|^2 + \|\nabla \boldsymbol{h}\|^2\right) + \nu\|A\boldsymbol{u}\|^2 + \gamma\|A\boldsymbol{h}\|^2$$
$$\leq C_\epsilon N^4\left(\|\nabla \boldsymbol{x}\|_{L^q}^r + \|\boldsymbol{v}\|^2\|\nabla \boldsymbol{v}\|^2\right)(\|\nabla \boldsymbol{u}\|^2 + \|\nabla \boldsymbol{h}\|^2)$$
$$+ 8\epsilon\left(\|A\boldsymbol{u}\|^2 + \|A\boldsymbol{h}\|^2\right). \tag{3.7}$$

If we take ϵ small enough, then (3.7) yields:

$$\frac{d}{dt}\left(\alpha\|\nabla \boldsymbol{u}\|^2+\|\nabla \boldsymbol{h}\|^2\right)+\nu\|A\boldsymbol{u}\|^2+\gamma\|A\boldsymbol{h}\|^2 \leq C\beta(t)\left(\alpha\|\nabla \boldsymbol{u}\|^2+\|\nabla \boldsymbol{h}\|^2\right), \tag{3.8}$$

where we have introduced $\beta := \|\nabla \boldsymbol{x}\|_{L^q}^r + \|\boldsymbol{v}\|^2\|\nabla \boldsymbol{v}\|^2$. Note that $\beta \in L^1(0,T)$ in view of **Hyp 2** and the fact that $\boldsymbol{u}$ belong to the energy space $L^2(0,T;\boldsymbol{V}) \cap L^\infty(0,T;\boldsymbol{H})$.

Let us set $Y(t) := \left(\alpha\|\nabla \boldsymbol{u}\|^2 + \|\nabla \boldsymbol{h}\|^2\right)(t)$ and $\nu_0 := \min\{\nu,\gamma\}$. Then

$$\frac{dY}{dt} + \nu_0\left(\|A\boldsymbol{u}\|^2 + \|A\boldsymbol{h}\|^2\right) \leq C\beta(t)Y \ \text{ in } \ (0,T). \tag{3.9}$$

Hence, from *Gronwall's Lemma,* we obtain an estimate of Y in $L^\infty(0,T)$ and, additionally, estimates of $A\boldsymbol{u}$ and $A\boldsymbol{h}$ in $L^2(0,T;\boldsymbol{L}^2(\mathbb{T}^3))$.

More precisely, with $B := \int_0^T \beta(\tau)\,d\tau$, we find that

$$Y(t) \leq Y(0)e^{CB} \quad \forall\, t \in [0,T] \tag{3.10}$$

and

$$\nu_0\int_0^T \left(\|A\boldsymbol{u}\|^2 + \|A\boldsymbol{h}\|^2\right)(\tau)\,d\tau \leq Y(0)e^{KB}. \tag{3.11}$$

This ends the proof when (2) is satisfied for some (r,q) with $q > 3/2$.

Now, let us assume that **Hyp 1** is satisfied with $q > 3$. Let p and a be such that

$$\frac{1}{q} + \frac{1}{p} = \frac{1}{2} \quad \text{and} \quad \frac{1}{p} = \frac{1}{6} + \frac{a}{3}. \tag{3.12}$$

Then, the terms of the right hand side of (3.1) can be estimated in a way similar to (3.2)–(3.6). Indeed, we have

$$\begin{aligned} |b(\partial_i\boldsymbol{u},\boldsymbol{u},\partial_i\boldsymbol{u})| &\leq |b(\partial_i\boldsymbol{v},\boldsymbol{u},\partial_i\boldsymbol{v})| + |b(\partial_i\boldsymbol{v},\boldsymbol{u},\partial_i\boldsymbol{x})| + |b(\partial_i\boldsymbol{x},\boldsymbol{u},\partial_i\boldsymbol{u})| \\ &\leq |b(\partial_i\boldsymbol{v},\boldsymbol{u},\partial_i\boldsymbol{v})| + |b(A\boldsymbol{v},\boldsymbol{u},\boldsymbol{x})| + |b(\partial_i\boldsymbol{v},\partial_i\boldsymbol{u},\boldsymbol{x})| \\ &\quad +|b(\boldsymbol{x},\boldsymbol{u},A\boldsymbol{u})|. \end{aligned} \tag{3.13}$$

In the right hand side, the first term can be bounded as follows:

$$\begin{aligned} |b(\partial_i\boldsymbol{v},\boldsymbol{u},\partial_i\boldsymbol{v})| &\leq C\|\nabla \boldsymbol{v}\|_{L^4}^2\|\nabla \boldsymbol{u}\| \\ &\leq CN\|\nabla \boldsymbol{v}\|\|A\boldsymbol{v}\|\|\nabla \boldsymbol{u}\| \\ &\leq C_\epsilon N^2\|\nabla \boldsymbol{v}\|^2\|\nabla \boldsymbol{u}\|^2 + \epsilon\|A\boldsymbol{u}\|^2. \end{aligned}$$

The second term satisfies

$$\begin{aligned}|b(A\boldsymbol{v},\boldsymbol{u},\boldsymbol{x})| &\le C\|\boldsymbol{x}\|_{L^q}\|\nabla\boldsymbol{u}\|_{L^p}\|A\boldsymbol{v}\|\\ &\le C\|\boldsymbol{x}\|_{L^q}\|\nabla\boldsymbol{u}\|^{a}\|A\boldsymbol{u}\|^{2-a}\\ &\le C_\epsilon\|\boldsymbol{x}\|_{L^q}^{r}\|\nabla\boldsymbol{u}\|^{2}+\epsilon\|A\boldsymbol{u}\|^{2},\end{aligned}$$

with $a = 2/r$. Moreover, similar estimates can be found for the third and fourth term:

$$\begin{aligned}|b(\partial_i\boldsymbol{v},\partial_i\boldsymbol{u},\boldsymbol{x})|+|b(\boldsymbol{x},\boldsymbol{u},A\boldsymbol{u})| &\le C\|\boldsymbol{x}\|_{L^q}\|\nabla\boldsymbol{u}\|_{L^p}\|A\boldsymbol{u}\|\\ &\le C_\epsilon\|\boldsymbol{x}\|_{L^q}^{r}\|\nabla\boldsymbol{u}\|^{2}+\epsilon\|A\boldsymbol{u}\|^{2}.\end{aligned}$$

Consequently,

$$|b(\partial_i\boldsymbol{u},\boldsymbol{u},\partial_i\boldsymbol{u})| \le C_\epsilon\left(\|\boldsymbol{x}\|_{L^q}^{r}+N^2\|\nabla\boldsymbol{v}\|^2\right)\|\nabla\boldsymbol{u}\|^2+4\epsilon\|A\boldsymbol{u}\|^2. \tag{3.14}$$

On the other hand, taking into account (3.3), with similar arguments and estimates, we find that

$$\begin{aligned}|b(\partial_i\boldsymbol{h},\boldsymbol{h},\partial_i\boldsymbol{u})| \;\;\le\;\; & C_\epsilon\left(\|\boldsymbol{x}\|_{L^q}^{r}+N^4\|\boldsymbol{v}\|^2\|\nabla\boldsymbol{v}\|^2\right)\|\nabla\boldsymbol{h}\|^2\\ &+3\epsilon\left(\|A\boldsymbol{u}\|^2+\|A\boldsymbol{h}\|^2\right).\end{aligned} \tag{3.15}$$

The remaining terms can be bounded in the same way:

$$\begin{aligned}&|b(\partial_i\boldsymbol{u},\boldsymbol{h},\partial_i\boldsymbol{h})|+|b(\partial_i\boldsymbol{h},\boldsymbol{u},\partial_i\boldsymbol{h})|\\ &\quad\le C_\epsilon\left(\|\boldsymbol{x}\|_{L^q}^{r}+N^2\|\nabla\boldsymbol{v}\|^2\right)\|\nabla\boldsymbol{h}\|^2+\epsilon\|A\boldsymbol{h}\|^2.\end{aligned} \tag{3.16}$$

Therefore, from (3.1), (3.14), (3.15) and (3.16), we easily find as before that $\boldsymbol{u}$ and $\boldsymbol{h}$ are bounded in $L^\infty(0,T;\boldsymbol{V})$, $A\boldsymbol{u}$ and $A\boldsymbol{h}$ are bounded in $L^2(0,T;\boldsymbol{L}^2(\mathbb{T}^3))$ and, consequently, $\boldsymbol{u}$ and $\boldsymbol{h}$ are smooth. This ends the proof when **Hyp 1** is assumed.

4. Some additional comments and open questions

To our knowledge, it is unknown if the assumption **Hyp 1** with $q = 3$ suffices to get the regularity of $(\boldsymbol{u},\boldsymbol{h})$. On the other hand, it is also unknown if Theorem 2.6 holds for systems similar to (1.1) in a cylinder $\Omega\times(0,T)$, completed with Dirichlet boundary conditions on $\partial\Omega\times(0,T)$ (here, we assume that $\Omega\subset\mathbb{R}^3$ is a bounded connected open set).

Note to this respect that a relevant point that allowed good estimates in our proof in Section 3 is that $A\boldsymbol{v} = -\Delta\boldsymbol{v}$ for any $\boldsymbol{v}\in\boldsymbol{V}$, a property that disappears in the non-periodic framework independently of the regularity of $\boldsymbol{v}$. It would be very interesting to know how generic are the assumptions in Theorem 2.6, that is, to answer the following question: How "large" (in a sense to specify) is the set of initial data $(\boldsymbol{u}_0,\boldsymbol{h}_0)$ such that at least one of the assumptions **Hyp 1** or **Hyp 2** holds? This will be the purpose of a forthcoming paper.

Also, it makes sense to explore extensions and generalizations of Theorem 2.6 in several directions. Thus, can **Hyp 1** be weakened and replaced by a similar property in weak Lebesgue spaces? Or, can **Hyp 2** be relaxed to a condition on only some few derivatives? See [22–26] for some related work in the Navier-Stokes context. Finally, let us indicate that it is completely meaningful to try to prove results similar to those in this paper for variable density Navier-Stokes (and variable density MHD) systems.

Acknowledgment

The authors were partially supported by ANID/PCI/ MEC-Atracción de Capital Humano Avanzado del Extranjero No. 80170081 (Chile). E. Fernández-Cara was partially supported by MICINN (Spain), Grant MTM2016-76990-P, E. Notte-Cuello was partially supported by DIDULS-PTE16151, Universidad de La Serena and M.A. Rojas-Medar was partially supported by MATH-AMSUD project 21-MATH-03 (CTMicrAAPDEs), CAPES-PRINT 88887.311962/2018-00 (Brazil), Project UTA-Mayor, 4753-20, Universidad de Tarapacá (Chile).

Conflict of interest

The authors declare there is no conflict of interest.

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