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# On quasi-monoidal comonads and their corepresentations

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**Abstract:** In this paper, we define and study quasi-monoidal comonads on a monoidal category. It generalize the (Hom type) coquasi-bialgebras to a non-braided setting. We investigate their corepresentations and their coquasitriangular structures. We also discuss their gauge equivalence relations.

**Keywords:** quasi-monoidal comonads; coquasitriangular structures; gauge transformations; monoidal categories

## 1. Introduction

The theory of (co)monads can be used as a tool in various fields of mathematics such as algebra, logic or operational semantics, and theoretical computer science. Note that in algebra theory, there are two different "bimonads". On the one hand, bimonads and Hopf monads without monoidal structures were introduced in [1], and developed in [2–4]. On the other hand, bimonads on monoidal categories were introduced in [5]. In 2002, Moerdijk used an opmonoidal monad to define a bimonad. This bimonad $F$ is both a monad and an opmonoidal functor satisfying the multiplication and the unit of $F$ are all monoidal natural transformations (see [5] for details). Although Moerdijk called his bimonad "Hopf monad", the antipode was not involved in his definition. In 2007, A. Bruguières and A. Virelizier introduced the notion of Hopf monad with antipode in the rigid categories in [6], and then put it in the non-dual monoidal categories [7]. We refer to [7–11] for the recent research on A. Bruguières and A. Virelizier's bimonads.

Quasi-bialgebras were introduced by V. G. Drinfel'd in [12]. The dual definition, a $k$-coquasi-bialgebra $H$ (or a Majid algebra), was introduced by S. Majid in [13]. The associativity of the multiplication are replaced by a weaker property, called coquasi-associativity. The multiplication is associative up to conjugation by a convolution invertible linear form $\omega \in (H \otimes H \otimes H)^*$, called the coassociator. Note that the definition of a coquasi-bialgebra is not selfdual, and the category of (left or right) comodules over a coquasi-bialgebra is a monoidal category with nontrivial associativity constraint and nontrivial unit constraints. Coquasi-bialgebras in a braided monoidal category also have been studied in [14].

Taking into account the results proved A. Bruguières and A. Virelizier in [6], it is now very natural to ask how to extend coquasi-bialgebras to the non-braided setting. This is the main motivation of the present paper.

In this paper, we present a dual version of the second author's results about quasi-bimonads which appeared in [15]. We mainly provide a generalization of coquasi-bialgebras by introducing the notion of quasi-monoidal comonad. Actually, a quasi-monoidal comonad $F$ is both a comonad and a quasi-monoidal functor such that its corepresentations is a non-strict monoidal category. The notion of quasi-monoidal comonad is very general. For example, the tensor functor of a (Hom-type) coquasi-bialgebras and bicomonads are all special cases of quasi-monoidal comonads.

The paper is organized as follows. In Section 2 we recall some notions of comonads, quasi-monoidal functors, $\pi$-categories and so on. In Section 3, we introduce the definition of quasi-monoidal comonads and discuss their corepresentations. In Section 4, we mainly investigate the coquasitriangular structures of a quasi-monoidal comonad. At last, we introduce the gauge equivalent relation on quasi-monoidal comonads.

## 2. Preliminaries

Throughout the paper, we let $k$ be a fixed field and $char(k) = 0$ and $Vec_k$ be the category of finite dimensional $k$-spaces. All the algebras and coalgebras, modules and comodules are supposed to be in $Vec_k$. For the comultiplication $\Delta$ of a $k$-space $C$, we use the Sweedler-Heyneman's notation: $\Delta(c) = \sum c_1 \otimes c_2$ for any $c \in C$.

### 2.1. *Quasi-monoidal functor*

Let $(C, \otimes, I, a, l, r)$ and $(C', \otimes', I', a', l', r')$ be two monoidal categories. Recall that a *quasi-monoidal functor* from $C$ to $C'$ is a triple $(F, F_2, F_0)$, where $F : C \to C'$ is a functor, $F_2 : F \otimes' F \to F\otimes$ is a natural transformation, and $F_0 : I' \to FI$ is a morphism in $C'$.

Furthermore, if the following equations hold for any $X, Y, Z \in C$:

$$\begin{aligned} &F_2(X, Y \otimes Z) \circ (id_{FX} \otimes' F_2(Y, Z)) \circ a'_{FX,FY,FZ} \\ &\qquad = F(a_{X,Y,Z}) \circ F_2(X \otimes Y, Z) \circ (F_2(X, Y) \otimes' id_{FZ}), \end{aligned} \tag{2.1}$$

$$F(l_X) \circ F_2(I, X) \circ (F_0 \otimes' id_{FX}) = l'_{FX}, \tag{2.2}$$

$$F(r_X) \circ F_2(X, I) \circ (id_{FX} \otimes' F_0) = r'_{FX}, \tag{2.3}$$

then $F = (F, F_2, F_0)$ is called a *monoidal functor*.

### 2.2. *Monoidal comonad*

Let $C$ be a category, $F$: $C \to C$ be a functor. Recall from [16] or [17] that if there exist natural transformations $\delta$: $F \to FF$ and $\varepsilon$: $F \to id_C$, such that the following identities hold

$$F\delta \circ \delta = \delta F \circ \delta, \quad \text{and} \quad id_F = F\varepsilon \circ \delta = \varepsilon F \circ \delta,$$

then we call the triple $(F, \delta, \varepsilon)$ a *comonad* on $C$.

Let $X \in C$, and $(F, \delta, \varepsilon)$ a comonad on $C$. If there exists a morphism $\rho^X$: $X \to FX$, satisfying

$$F\rho^X \circ \rho^X = \delta_X \circ \rho^X, \text{ and } \varepsilon_X \circ \rho^X = id_X,$$

then we call the couple $(X, \rho^X)$ an *F-comodule*.

A morphism between $F$-comodules $g$: $X \to X'$ is called $F$-colinear, if $g$ satisfies: $Fg \circ \rho^X = \rho^{X'} \circ g$. The category of $F$-comodules is denoted by $C^F$.

Let $(C, \otimes, I, a, l, r)$ be a monoidal category, $(F, \delta, \varepsilon)$ be a comonad on $C$, and $(F, F_2, F_0) : C \to C$ be a monoidal functor. Then recall from [18] or [19] that $F$ is called a *monoidal comonad* (or a *bicomonad*) on $C$ if $\delta$ and $\varepsilon$ are both monoidal natural transformations, i.e. the following *compatibility conditions* hold for any $X, Y \in C$:

$$\begin{cases} (C1)\ F(F_2(X,Y)) \circ F_2(FX,FY) \circ (\delta_X \otimes \delta_Y) = \delta_{X\otimes Y} \circ F_2(X,Y), \\ (C2)\ \varepsilon_{X\otimes Y} \circ F_2(X,Y) = \varepsilon_X \otimes \varepsilon_Y, \\ (C3)\ F(F_0) \circ F_0 = \delta_I \circ F_0, \\ (C4)\ \varepsilon_I \circ F_0 = id_I. \end{cases}$$

*2.3. Convolution product*

Given a category $C$ and a positive integer $n$, we denote $C^n = C \times C \times \cdots \times C$ the $n$-tuple cartesian product of $C$. If $F$ is a comonad on $C$, then $F^{\times n}$ (the $n$-tuple cartesian product of $F$) is a comonad on $C^n$, and we have $C^{nF^{\times n}} = (C^F)^n$.

Assume that $U : C^F \to C$ is the forgetful functor and $P, Q : C^n \to \mathcal{D}$ are functors. Then from [ [9], Proposition 4.1], we have the following results.

**Lemma 2.1.** *There is a canonical bijection:*

$$Nat(PU^{\times n}, QU^{\times n}) \cong Nat(PF^{\times n}, Q).$$

*Proof.* Define $?^\flat : Nat(PU^{\times n}, QU^{\times n}) \to Nat(PF^{\times n}, Q)$, $f \mapsto f^\flat$, by

$$f^\flat_{(X_1,\cdots,X_n)} :\ P(FX_1 \times \cdots \times FX_n) \xrightarrow{f_{(FX_1,\cdots,FX_n)}} Q(FX_1 \times \cdots \times FX_n) \xrightarrow{Q(\varepsilon_{X_1},\cdots,\varepsilon_{X_n})} Q(X_1 \times \cdots \times X_n),$$

and $?^\sharp : Nat(PF^{\times n}, Q) \to Nat(PU^{\times n}, QU^{\times n})$, $\alpha \mapsto \alpha^\sharp$, by

$$\alpha^\sharp_{(M_1,\cdots,M_n)} :\ P(M_1 \times \cdots \times M_n) \xrightarrow{P(\rho^{M_1},\cdots,\rho^{M_n})} P(FM_1 \times \cdots \times FM_n) \xrightarrow{\alpha_{(M_1,\cdots,M_n)}} Q(M_1 \times \cdots \times M_n),$$

for any $f \in Nat(PU^{\times n}, QU^{\times n})$, $\alpha \in Nat(PF^{\times n}, Q)$ and $X_i \in C$, $(M_i, \rho^{M_i}) \in C^F$. It is easy to check that $?^\flat$ and $?^\sharp$ are well defined and are inverse with each other. □

Let $P, Q, R : C^n \to \mathcal{D}$ be functors. For any $\alpha \in Nat(PF^{\times n}, Q)$ and $\beta \in Nat(QF^{\times n}, R)$, define their *convolution product* $\beta * \alpha \in Nat(PF^{\times n}, R)$ by setting, for any objects $X_1, \cdots, X_n$ in $C$,

$$\beta * \alpha_{X_1,\cdots,X_n} = \beta_{X_1,\cdots,X_n} \circ \alpha_{FX_1,\cdots,FX_n} \circ P(\delta_{X_1}, \cdots, \delta_{X_n}).$$

We say that $\alpha \in Nat(PF^{\times n}, Q)$ is $*$*-invertible* if there exists $\beta \in Nat(QF^{\times n}, P)$ such that $\beta * \alpha = P(\varepsilon^{\times n}) \in Nat(PF^{\times n}, P)$ and $\alpha * \beta = Q(\varepsilon^{\times n}) \in Nat(QF^{\times n}, Q)$. We denote $\beta$ by $\alpha^{*-1}$.

**Proposition 2.2.** *The $*$-invertible elements in $Nat(PF^{\times n}, Q)$ are in corresponding with the natural isomorphisms in $Nat(PU^{\times n}, QU^{\times n})$.*

*Proof.* Suppose that $f \in Nat(PU^{\times n}, QU^{\times n})$ is a natural isomorphism. Then we immediately get that $(f^{\flat})^{*-1} = (f^{-1})^{\flat}$.

Conversely, if $\alpha \in Nat(PF^{\times n}, Q)$ is $*$-invertible, then $\alpha^{\sharp^{-1}} = (\alpha^{*-1})^{\sharp}$. □

## 3. Quasi-monoidal comonads

Suppose that $(C, \otimes, I, a, l, r)$ is a monoidal category, $F : C \to C$ is a functor, $(F, \delta, \varepsilon)$ is a comonad and $(F, F_2, F_0)$ is a quasi-monoidal functor.

**Lemma 3.1.** *If we define the $F$-coaction on $I$ by $F_0$, and define the $F$-coaction on $M \otimes N$ (as the tensor product in $C$) for any $(M, \rho^M), (N, \rho^N) \in C^F$ by:*

$$\rho^{M\otimes N} : \ M \otimes N \xrightarrow{\rho^M \otimes \rho^N} FM \otimes FN \xrightarrow{F_2(M,N)} F(M \otimes N),$$

*then $(I, F_0)$ and $(M \otimes N, \rho^{M\otimes N})$ are all objects in $C^F$ if and only if the compatibility conditions Eqs (C1)–(C4) hold.*

*Proof.* It is straightforward to check that Eqs (C1) and (C2) hold if and only if $(M \otimes N, \rho^{M\otimes N}) \in C^F$, Eqs (C3) and (C4) hold if and only if $(I, F_0) \in C^F$. □

From now on, we always assume that the compatibility conditions Eqs (C1)–(C4) hold.

We suppose that there are natural transformations $\vartheta : (\_ \otimes \_) \otimes \_ \circ F^{\times 3} \Rightarrow \_ \otimes (\_ \otimes \_) : C^{\times 3} \to C$, and $\iota : I \otimes F\_ \Rightarrow \_ : C \to C$, $\kappa : F\_ \otimes I \Rightarrow \_ : C \to C$. From Lemma 2.1, for any objects $(M, \rho^M), (N, \rho^N), (P, \rho^P) \in C^F$, $\vartheta, \iota, \kappa$ can induce the following natural transformations

$$A_{M,N,P} = \vartheta^{\sharp}_{M,N,P}, \quad L_M = \iota^{\sharp}_M, \quad R_M = \kappa^{\sharp}_M.$$

Conversely, if there are natural transformations $A : (\_ \otimes \_) \otimes \_ \Rightarrow \_ \otimes (\_ \otimes \_) : C^{\times 3} \to C$ and $L : I \otimes \_ \Rightarrow id : C \to C$, $R : \_ \otimes I \Rightarrow id : C \to C$, then from Lemma 2.1, for any $X, Y, Z \in C$, they can induce natural transformations

$$\vartheta_{X,Y,Z} = A^{\flat}_{X,Y,Z}, \quad \iota_X = L^{\flat}_X, \quad \kappa_X = R^{\flat}_X.$$

Next, we will discuss when $A$ is the associativity constraint and $L, R$ are the unit constraints in $C^F$.

**Lemma 3.2.** *$A$, $L$ and $R$ are isomorphisms if and only if $\vartheta$, $\iota$ and $\kappa$ are $*$-invertible.*

*Proof.* Straightforward from Proposition 2.2. □

**Lemma 3.3.** *$A$ is $F$-colinear if and only if $\vartheta$ satisfies*

$$\begin{array}{ccccc}
(FX \otimes FY) \otimes FZ & \xrightarrow{\delta_X \otimes \delta_Y \otimes \delta_X} & (FFX \otimes FFY) \otimes FFZ & \xrightarrow{F_2 \otimes id} & F(FX \otimes FY) \otimes FFZ \\
\downarrow{\scriptstyle \delta_X \otimes \delta_Y \otimes \delta_X} & & & & \downarrow{\scriptstyle F_2} \\
(FFX \otimes FFY) \otimes FFZ & & & & F((FX \otimes FY) \otimes FZ) \\
\downarrow{\scriptstyle \vartheta_{FX,FY,FZ}} & & & & \downarrow{\scriptstyle F\vartheta_{X,Y,Z}} \\
FX \otimes (FY \otimes FZ) & \xrightarrow[id \otimes F_2]{} & FX \otimes F(Y \otimes Z) & \xrightarrow[F_2]{} & F(X \otimes (Y \otimes Z))
\end{array} \tag{3.1}$$

*for any $X, Y, Z \in C$.*

*Proof.* $\Leftarrow$): Since the following diagram

$$\begin{array}{ccccc}
(M\otimes N)\otimes P & \xrightarrow{\rho^M\otimes\rho^N\otimes\rho^P} & (FM\otimes FN)\otimes FP & \xrightarrow{\vartheta} & M\otimes(N\otimes P) \\
\downarrow{\scriptstyle \rho^M\otimes\rho^N\otimes\rho^P} & & \downarrow{\scriptstyle \delta_M\otimes\delta_N\otimes\delta_P} & & \downarrow{\scriptstyle \rho^M\otimes\rho^N\otimes\rho^P} \\
(FM\otimes FN)\otimes FP & & (FFM\otimes FFN)\otimes FFP & \xrightarrow[\vartheta]{} & FM\otimes(FN\otimes FP) \\
\downarrow{\scriptstyle F_2\otimes id} & & \downarrow{\scriptstyle F_2\otimes id} & & \downarrow{\scriptstyle id\otimes F_2} \\
F(M\otimes N)\otimes FP & \xrightarrow{F\rho^M\otimes F\rho^N\otimes F\rho^P} & F(FM\otimes FN)\otimes FFP & & FM\otimes F(N\otimes P) \\
\downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle F_2} \\
F(M\otimes(N\otimes P)) & \xrightarrow[F(\rho^M\otimes\rho^N\otimes\rho^P)]{} & F(FM\otimes(FN\otimes FP)) & \xrightarrow[F\vartheta]{} & F(M\otimes(N\otimes P))
\end{array}$$

is commutative for any $M, N, P \in C^F$, $A_{M,N,P}$ is $F$-colinear.

$\Rightarrow$): Notice that $A_{FX,FY,FZ}$ is $F$-colinear for any $X, Y, Z \in C$, then it follows

$$\begin{aligned}
&F(\varepsilon_X\otimes\varepsilon_Y\otimes\varepsilon_Z)\circ FA_{FX,FY,FZ}\circ\rho^{(FX\otimes FY)\otimes FZ} \\
&= F(\varepsilon_X\otimes\varepsilon_Y\otimes\varepsilon_Z)\circ\rho^{FX\otimes(FY\otimes FZ)}\circ A_{FX,FY,FZ}.
\end{aligned}$$

After a direct computation, we obtain (3.1). □

**Lemma 3.4.** *A satisfies the Pentagon Axiom in $C^F$ if and only if $\vartheta$ satisfies*

$$\begin{aligned}
&(id\otimes\vartheta_{X,Y,Z})\circ\vartheta_{W,FX\otimes FY,FZ}\circ(id\otimes F_2\otimes id)\circ(\vartheta_{FW,FFX,FFY}\otimes id) \\
&\quad\circ(\delta_W\otimes\delta_X^2\otimes\delta_Y^2\otimes\delta_Z) \\
&= \vartheta_{W,X,Y\otimes Z}\circ(id\otimes id\otimes F_2)\circ\vartheta_{FW\otimes FX,FY,FZ}\circ(F_2\otimes id\otimes id)\circ(\delta_W\otimes\delta_X\otimes\delta_Y\otimes\delta_Z)
\end{aligned}\tag{3.2}$$

*for any $W, X, Y, Z \in C$.*

*Proof.* $\Leftarrow$): Since we have

$$\begin{aligned}
&(id\otimes\vartheta_{N,P,Q})\circ(id\otimes\rho^N\otimes\rho^P\otimes\rho^Q)\circ\vartheta_{M,N\otimes P,Q}\circ(id\otimes F_2\otimes id)\circ(\rho^M\otimes\rho^N\otimes\rho^P\otimes\rho^Q) \\
&\quad\circ(\vartheta_{M,N,P}\otimes id)\circ(\rho^M\otimes\rho^N\otimes\rho^P\otimes id) \\
&= (id\otimes\vartheta_{N,P,Q})\circ\vartheta_{M,FN\otimes FP,FQ}\circ(id\otimes F(\rho^N\otimes\rho^P)\otimes\rho^Q)\circ(id\otimes F_2\otimes id)\circ(\vartheta_{FM,FN,FP}\otimes id) \\
&\quad\circ(F\rho^M\otimes F\rho^N\otimes F\rho^P\otimes\rho^Q)\circ(\rho^M\otimes\rho^N\otimes\rho^P\otimes id) \\
&= (id\otimes\vartheta_{N,P,Q})\circ\vartheta_{M,FN\otimes FP,FQ}\circ(id\otimes F_2\otimes id)\circ(\vartheta_{FM,FFN,FFP}\otimes id)\circ(\delta_M\otimes\delta_N^2\otimes\delta_P^2\otimes\delta_Q) \\
&\quad\circ(\rho^M\otimes\rho^N\otimes\rho^P\otimes\rho^Q) \\
&= \vartheta_{M,N,P\otimes Q}\circ(id\otimes id\otimes F_2)\circ\vartheta_{FM\otimes FN,FP,FQ}\circ(F_2\otimes id\otimes id)\circ(\delta_M\otimes\delta_N\otimes\delta_P\otimes\delta_Q) \\
&\quad\circ(\rho^M\otimes\rho^N\otimes\rho^P\otimes\rho^Q) \\
&= \vartheta_{M,N,P\otimes Q}\circ(id\otimes id\otimes F_2)\circ(\rho^M\otimes\rho^N\otimes\rho^P\otimes\rho^Q)\circ\vartheta_{M\otimes N,P,Q}\circ(F_2\otimes id\otimes id) \\
&\quad\circ(\rho^M\otimes\rho^N\otimes\rho^P\otimes\rho^Q)
\end{aligned}$$

for any $M, N, P, Q \in C^F$, $A$ satisfies the Pentagon Axiom.

⇒): For any $W, X, Y, Z \in \mathcal{C}$, we have cofree $F$-comodules $FW, FX, FY, FZ$. Consider the following Pentagon Axiom:

$$\begin{aligned} & A_{FW,FX,FY\otimes FZ} \circ A_{FW\otimes FX,FY,FZ} \\ = & (id \otimes A_{FX,FY,FZ}) \circ A_{FW,FX\otimes FY,FZ} \circ (A_{FW,FX,FY} \otimes id). \end{aligned}$$

Applying $\varepsilon_W \otimes \varepsilon_X \otimes \varepsilon_Y \otimes \varepsilon_Z$ to both sides of the above identity, we get Diagram (3.2). □

**Lemma 3.5.** *For any $X \in \mathcal{C}$,*

*(1) $L$ is $F$-colinear if and only if $\iota$ satisfies*

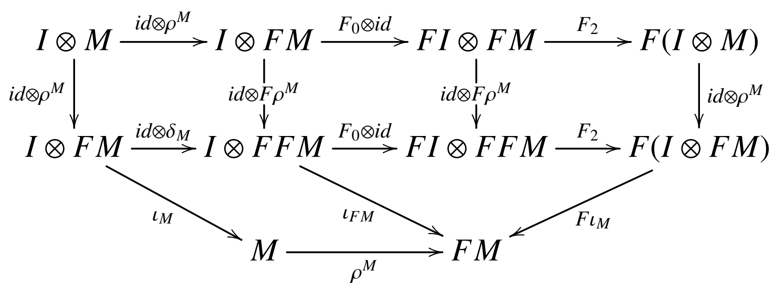
$$\begin{array}{ccccc} I\otimes FX & & \xrightarrow{id\otimes\delta_X} & & I\otimes FFX \\ \downarrow{\scriptstyle F_0\otimes\delta_X} & & & & \downarrow{\scriptstyle \iota_{FX}} \\ FI\otimes FFX & \xrightarrow[F_2]{} & F(I\otimes FX) & \xrightarrow[F\iota_X]{} & FX. \end{array} \tag{3.3}$$

*(2) $R$ is $F$-colinear if and only if $\kappa$ satisfies*

$$\begin{array}{ccccc} FX\otimes I & & \xrightarrow{\delta_X\otimes id} & & FFX\otimes I \\ \downarrow{\scriptstyle \delta_X\otimes F_0} & & & & \downarrow{\scriptstyle \kappa_{FX}} \\ FFX\otimes FI & \xrightarrow[F_2]{} & F(FX\otimes I) & \xrightarrow[F\kappa_X]{} & FX. \end{array} \tag{3.4}$$

*Proof.* We only prove (1).

⇐): From the following commutative diagram



for any $M \in \mathcal{C}^F$, $L_M$ is $F$-colinear.

⇒): Conversely, since $FX$ is an $F$-comodule and $L_{FX}$ is $F$-colinear for any $X \in \mathcal{C}$, it is directly to get Diagram (3.3). □

**Lemma 3.6.** *$A$, $L$ and $R$ satisfy the Triangle Axiom in $\mathcal{C}^F$ if and only if $\vartheta$, $\iota$ and $\kappa$ satisfy*

$$\begin{array}{ccccc} (FX\otimes I)\otimes FY & \xrightarrow{\kappa_X\otimes id} & X\otimes FY & \xrightarrow{id\otimes\varepsilon_Y} & X\otimes Y \\ \downarrow{\scriptstyle id\otimes F_0\otimes\delta_Y} & & & & \uparrow{\scriptstyle id\otimes\iota_Y} \\ (FX\otimes FI)\otimes FFY & & \xrightarrow[\vartheta_{X,I,FY}]{} & & X\otimes(I\otimes FY) \end{array} \tag{3.5}$$

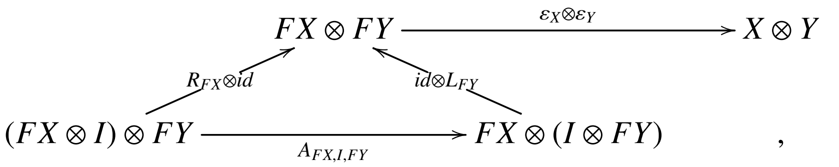
*for any $X, Y, Z \in \mathcal{C}$.*

*Proof.* ⇐): For any $M, N \in C^F$, we compute

$$\begin{aligned}&(id_M \otimes \iota_N) \circ (id_M \otimes id_I \otimes \rho^N) \circ \vartheta_{M,I,N} \circ (\rho^M \otimes F_0 \otimes \rho^N)\\ =\ &(id_M \otimes \iota_N) \circ \vartheta_{M,I,FN} \circ (id_{FM} \otimes F_0 \otimes \delta_N) \circ (\rho^M \otimes id_I \otimes \rho^N)\\ =\ &(id_M \otimes \varepsilon_N) \circ (\kappa_M \otimes id_{FN}) \circ (\rho^M \otimes id_I \otimes \rho^N)\\ =\ &(\kappa_M \otimes id_N) \circ (\rho^M \otimes id_I \otimes id_N)\end{aligned}$$

thus the Triangle Axiom in $C^F$ holds.

⇒): Conversely, for any $X, Y \in C$, since we have

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it is a direct computation to get Diagram (3.5). □

**Definition 3.7.** Let $(C, \otimes, I, a, l, r)$ be a monoidal category on which $(F, \delta, \varepsilon)$ is a monad and $(F, F_2, F_0)$ is a quasi-monoidal functor such that the compatible conditions Eqs (C1)–(C4) are satisfied. If there are $*$-invertible natural transformations $\vartheta$, $\iota$ and $\kappa$ satisfying (3.1)–(3.5), then we call $(F, \delta, \varepsilon, F_2, F_0, \vartheta, \iota, \kappa)$ a *quasi-monoidal comonad* on $C$,

Then by Lemma 3.1–3.6, one gets the following result.

**Theorem 3.8.** *Let $(C, \otimes, I, a, l, r)$ be a monoidal category on which $(F, \delta, \varepsilon)$ is a monad and $(F, F_2, F_0)$ is a quasi-monoidal functor such that the compatible conditions Eqs (C1)–(C4) is satisfied. Then there exist natural transformations $\vartheta$, $\iota$ and $\kappa$ such that $(F, \delta, \varepsilon, F_2, F_0, \vartheta, \iota, \kappa)$ is a quasi-monoidal comonad if and only if there are natural transformations A, L and R such that $(C^F, \otimes, I, A, L, R)$ is a monoidal category.*

**Example 3.9.** Let $(C, \otimes, I, a, l, r)$ be a monoidal category on which $(F, \delta, \varepsilon)$ is a monad and $(F, F_2, F_0)$ is a quasi-monoidal functor such that the compatible conditions Eqs (C1)–(C4) are satisfied. If we define

$$\vartheta_{X,Y,Z} = a^{\flat}_{X,Y,Z}, \ \ \iota_X = l^{\flat}_X, \ \ \kappa_{X,Y,Z} = r^{\flat}_{X,Y,Z}$$

for any $X, Y, Z \in C$, then Eq (3.2) holds because of the Pentagon Axiom of $a$; Eq (3.5) holds because of the Triangle Axiom of $a, l, r$; Eqs (3.1), (3.3) and (3.4) hold if and only if $(F, F_2, F_0)$ is a monoidal functor. That means, the quasi-monoidal comonad $(F, \delta, \varepsilon, F_2, F_0, \vartheta, \iota, \kappa)$ is exactly a monoidal comonad.

**Example 3.10.** Recall from [9] or [10], we consider the following monoidal category $\overline{\mathcal{H}}^{i,j}(Vec_k)$ for any $i, j \in \mathbb{Z}$:

- the objects of $\overline{\mathcal{H}}^{i,j}(Vec_k)$ are pairs $(X, \alpha_X)$, where $X \in Vec_k$ and $\alpha_X \in Aut_k(X)$;
- the morphism $f : (X, \alpha_X) \to (Y, \alpha_Y)$ in $\overline{\mathcal{H}}^{i,j}(Vec_k)$ is a $k$-linear map from $X$ to $Y$ such that $\alpha_Y \circ f = f \circ \alpha_X$;
- the monoidal structure is given by

$$(X, \alpha_X) \otimes (Y, \alpha_Y) = (X \otimes Y, \alpha_X \otimes \alpha_Y),$$

and the unit is $(k, id_k)$;

• the associativity constraint $a$, the unit constraints $l$ and $r$ are given by

$$a_{X,Y,Z} : (x \otimes y) \otimes z \mapsto \alpha_X^{i+1}(x) \otimes (y \otimes \alpha_Z^{-j-1}(z));$$
$$l_X(1_k \otimes x) = \alpha_X^{j+1}(x), \quad r_X(x \otimes 1_k) = \alpha_X^{i+1}(x), \quad \forall X \in Vec_k.$$

Now assume that $(H, \alpha_H)$ is an object in $\overline{\mathcal{H}}^{i,j}(Vec_k)$, $m_H : H \otimes H \to H$ (with notation $m_H(a \otimes b) = ab$), $\eta_H : k \to H$ (with notation $\eta_H(1_k) = 1_H$), and $\Delta_H : H \to H \otimes H$ (with notation $\Delta_H(h) = h_1 \otimes h_2$), and $\varepsilon_H : H \to k$ are all morphisms in $\overline{\mathcal{H}}^{i,j}(Vec_k)$. Further, we write

$$\ddot{H} = \_ \otimes H : \overline{\mathcal{H}}^{i,j}(Vec_k) \to \overline{\mathcal{H}}^{i,j}(Vec_k), \quad (X, \alpha_X) \mapsto (X \otimes H, \alpha_X \otimes \alpha_H)$$

for the right tensor functor of $H$.

If we define the following structures on $\ddot{H}$:

• $\delta : \ddot{H} \to \ddot{H}\ddot{H}$ and $\epsilon : \ddot{H} \to id_{\overline{\mathcal{H}}^{i,j}(Vec_k)}$ are defined by

$$\delta_X : x \otimes h \mapsto (\alpha_X(x) \otimes h_1) \otimes \alpha_H^{-1}(h_2),$$
$$\epsilon_X : x \otimes h \mapsto \varepsilon_H(h)\alpha_X^{-1}(x);$$

• $\ddot{H}_2 : \ddot{H} \otimes \ddot{H} \to \ddot{H} \otimes$ and $\ddot{H}_0 : k \to \ddot{H}(k)$ are given by

$$\ddot{H}_2(X, Y) : (x \otimes a) \otimes (y \otimes b) \mapsto (x \otimes y) \otimes \alpha_H^i(a)\alpha_H^j(b),$$
$$\ddot{H}_0(1_k) = 1_k \otimes 1_H,$$

for any $X, Y \in \overline{\mathcal{H}}^{i,j}(Vec_k)$. Then obviously $\ddot{H} = (\ddot{H}, \delta, \epsilon)$ forms a comonad on $\overline{\mathcal{H}}^{i,j}(Vec_k)$ if and only if $(H, \alpha_H, \Delta_H, \varepsilon_H)$ is a Hom-coalgebra over $k$, Eqs (C1)–(C4) hold if and only if $m_H$ and $\eta_H$ are all morphisms of Hom-coalgebras.

Suppose that there are $\alpha_H$-invariant convolution invertible linear forms $\omega \in (H \otimes H \otimes H)^*$ and $p, q \in H^*$, then we can define the following $*$-invertible natural transformations

$$\vartheta_{X,Y,Z} : ((x \otimes a) \otimes (y \otimes b)) \otimes (z \otimes c) \mapsto \omega(\alpha_H^{2i}(a), \alpha_H^{i+j}(b), \alpha_H^{j-1}(c))(\alpha_X^i(x) \otimes (\alpha_Y^{-1}(y) \otimes \alpha_Z^{-j-2}(z))),$$
$$\iota_X : 1_k \otimes (x \otimes a) \mapsto p(a)\alpha_X^j(x), \quad \kappa_X : (x \otimes a) \otimes 1_k \mapsto q(a)\alpha_X^i(x),$$

where $a, b, c \in H$, $x \in X$, $y \in Y$, $z \in Z$ and $X, Y, Z \in Vec_k$. Thus we immediately get that $\vartheta$ satisfies Eq (3.1) if and only if $\omega$ satisfies

$$\sum \alpha_H(a_1)(b_1c_1)\omega(a_2, b_2, c_2) = \sum \omega(a_1, b_1, c_1)(a_2b_2)\alpha_H(c_2); \tag{3.6}$$

$\vartheta$ satisfies Eq (3.2) if and only if $\omega$ satisfies

$$\sum \omega(\alpha_H(a_1), \alpha_H(b_1), c_1d_1)\omega(a_2b_2, \alpha_H(c_2), \alpha_H(d_2))$$
$$= \sum \omega(b_1, c_1, \alpha_H(d_1))\omega(\alpha_H(a_1), \alpha_H^{-1}(b_{21})\alpha_H^{-1}(c_{21}), \alpha_H(d_2))\omega(\alpha_H(a_2), b_{22}, c_{22}); \tag{3.7}$$

$\iota$ satisfies Eq (3.3) and $\kappa$ satisfies Eq (3.4) if and only if $p, q$ satisfy

$$\sum p(a_1)1_Ha_2 = \alpha_H(a_1)p(a_2), \quad \sum q(a_1)a_21_H = \alpha_H(a_1)q(a_2); \tag{3.8}$$

$\vartheta, \iota$ and $\kappa$ satisfy Eq (3.5) if and only if $\omega$, $p$ and $q$ satisfy

$$\omega(a, 1_H, b) = q(a)p^{*-1}(b). \tag{3.9}$$

This means, $\ddot{H} = (\ddot{H}, \delta, \epsilon, \ddot{H}_2, \ddot{H}_0, \vartheta, \iota, \kappa)$ forms a quasi-monoidal comonad on $\overline{\mathcal{H}}^{i,j}(Vec_k)$ if and only if $H = (H, \alpha_H, m_H, \eta_H, \Delta_H, \varepsilon_H, \omega, p, q)$ forms a *Hom-coquasi-bialgebra* over $k$ (see [20] for the dual definition). Further, from Theorem 3.10, one get that $Corep(H) = (\overline{\mathcal{H}}^{i,j}(Vec_k))^{\ddot{H}}$, the category of right $H$-Hom-comodules, is a monoidal category and its associativity constraint, unit constraints are given as follows:

$$A_{M,N,P}((m \otimes n) \otimes p) = \sum \omega(\alpha^{2i}(m_1), \alpha^{i+j}(n_1), \alpha^{j-1}(p_1))\alpha^i_M(m_0) \otimes (\alpha^{-1}_N(n_0) \otimes \alpha^{-j-2}_P(p_0)),$$
$$L_M(1_k \otimes m) = \sum p(m_1)\alpha^j_M(m_0), \quad R_M(m \otimes 1_k) = \sum q(m_1)\alpha^i_M(m_0),$$

where $m \in M$, $n \in N$, $p \in P$, $M, N, P \in Corep(H)$.

**Example 3.11.** Under the consideration of Example 3.10, if all the Hom-structure maps $\alpha$ are identity maps, then the Hom-coquasi-bialgebra is exactly the *coquasi-bialgebra* (also called a *Majid algebra*, see [13] for details) over $k$.

**Example 3.12.** Let $B = (B, \mu, 1_B, \Delta, \varepsilon)$ be a bialgebra over $k$, $\alpha_B : B \to B$ be an endo-isomrophism. Recall that a $k$-linear form $g \in B^*$ is called

(1) *dual central* if $g(x_1)x_2 = x_1 g(x_2)$ for any $x \in B$;

(2) *dual group-like* if it is convolution invertible and satisfies $g(xy) = g(x)g(y)$ for any $x, y \in B$;

(3) $\alpha_B$-*invariant* if $g(\alpha_B(x)) = g(x)$.

Now suppose that $p, q \in B^*$ are all dual central dual group-like and $\alpha_B$-invariant linear forms. Define a $k$-linear form $\omega : B \otimes B \otimes B \to k$ by

$$\omega(x, y, z) = p(x)\varepsilon(y)q^{*-1}(z), \quad \text{for any } x, y, z \in B,$$

define the new multiplication $\mu^{\alpha_B}$ and comultiplication $\Delta^{\alpha_B}$ by

$$\mu^{\alpha_B} = \alpha_B \circ \mu, \quad \Delta^{\alpha_B} = \Delta \circ \alpha_B.$$

Then it is a direct calculation to check that $\alpha_B, \omega, p, q$ satisfy Eqs.(3.6) - (3.9) (under $\mu^{\alpha_B}$ and $\Delta^{\alpha_B}$), hence $B^{p,q}_{\alpha_B} = (B, \alpha_B, \mu^{\alpha_B}, 1_B, \Delta^{\alpha_B}, \varepsilon, \omega, p, q)$ forms a nontrivial Hom-coquasi-bialgebra.

## 4. Coquasitriangular structures

Recall that a *braiding* in a monoidal category $(C, \otimes, I, a, l, r)$ is a natural isomorphism $\tau: \otimes \Rightarrow \otimes^{op} : C \times C \to C$ such that the following identities hold

$$a_{Y,Z,X} \circ \tau_{X,Y\otimes Z} \circ a_{X,Y,Z} = (id_Y \otimes \tau_{X,Z}) \circ a_{Y,X,Z} \circ (\tau_{X,Y} \otimes id_Z), \tag{B1}$$

$$a^{-1}_{Z,X,Y} \circ \tau_{X\otimes Y,Z} \circ a^{-1}_{X,Y,Z} = (\tau_{X,Z} \otimes id_Y) \circ a^{-1}_{X,Z,Y} \circ (id_X \otimes \tau_{Y,Z}) \tag{B2}$$

for any $X, Y, Z \in C$.

Now let $F$ be a quasi-monoidal comonad on $C$. Suppose that there is a natural transformation $\sigma$: $\otimes \circ (F \times F) \Rightarrow \otimes^{op} : C^{\times 2} \to C$. From Lemma 2.1, for any objects $M, N$ in $C^F$, $\sigma$ can induce a natural transformation

$$\tau_{M,N} = \sigma^{\sharp}_{M,N} : \ M \otimes N \xrightarrow{\rho^M \otimes \rho^N} FM \otimes FN \xrightarrow{\sigma_{M,N}} N \otimes M.$$

Conversely, if there exists $\tau : \otimes \Rightarrow \otimes^{op} : C \times C \to C$, then from Lemma 2.1, for any $X, Y \in C$, $\tau$ can induce the following

$$\sigma_{X,Y} = \tau^{\flat}_{X,Y} : \ FX \otimes FY \xrightarrow{\tau_{FX,FY}} FY \otimes FX \xrightarrow{\varepsilon_Y \otimes \varepsilon_X} Y \otimes X.$$

Next we will discuss when $\tau$ is a braiding in $C^F$.

**Lemma 4.1.** *$\tau$ is an isomorphism if and only if $\sigma$ is $*$-invertible.*

*Proof.* Straightforward from Proposition 2.2. □

**Lemma 4.2.** *$\tau$ is $F$-colinear if and only if $\sigma$ satisfies*

$$\begin{array}{ccccc} FX \otimes FY & \xrightarrow{\delta_X \otimes \delta_Y} & FFX \otimes FFY & \xrightarrow{\sigma_{FX,FY}} & FY \otimes FX \\ \downarrow{\scriptstyle \delta_X \otimes \delta_Y} & & & & \downarrow{\scriptstyle F_2} \\ FFX \otimes FFY & \xrightarrow[F_2]{} & F(FX \otimes FY) & \xrightarrow[F\sigma_{X,Y}]{} & F(Y \otimes X) \end{array} \tag{4.1}$$

*for any $X, Y \in C$.*

*Proof.* $\Leftarrow$): We compute

$$\begin{array}{ccccc} M \otimes N & \xrightarrow{\rho^M \otimes \rho^N} & FM \otimes FN & \xrightarrow{\sigma_{M,N}} & N \otimes M \\ \downarrow{\scriptstyle \rho^M \otimes \rho^N} & & \downarrow{\scriptstyle F\rho^M \otimes F\rho^N} & & \downarrow{\scriptstyle \rho^N \otimes \rho^M} \\ FM \otimes FN & \xrightarrow[=F\rho^M \otimes F\rho^N]{\delta_M \otimes \delta_N} & FFM \otimes FFN & \xrightarrow{\sigma_{FM,FN}} & FN \otimes FM \\ \downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle F_2} \\ F(M \otimes N) & \xrightarrow[F(\rho^M \otimes \rho^N)]{} & F(FM \otimes FN) & \xrightarrow[F\sigma_{M,N}]{} & F(N \otimes M) \end{array}$$

for any $M, N \in C^F$. Hence $\tau_{M,N}$ is $F$-colinear.

$\Rightarrow$): Conversely, notice that $\tau_{FX,FY}$ is $F$-colinear for any $X, Y \in C$, we have

$$F(\varepsilon_Y \otimes \varepsilon_X) \circ F\tau_{FX,FY} \circ \rho^{FX \otimes FY} = F(\varepsilon_Y \otimes \varepsilon_X) \circ \rho^{FY \otimes FX} \circ \tau_{FX,FY},$$

which implies Diagram (4.1) holds. □

**Lemma 4.3.** *Diagram (B1) holds in $C^F$ if and only if $\sigma$ satisfies*

$$\begin{aligned} &\vartheta_{Y,Z,X} \circ \sigma_{FX,FY \otimes FZ} \circ (id \otimes F_2) \circ \vartheta_{FFX,FFY,FFZ} \circ (\delta^2_X \otimes \delta^2_Y \otimes \delta^2_Z) \\ &= (id \otimes \sigma_{X,Z}) \circ \vartheta_{Y,FX,FZ} \circ (\sigma_{FFX,FY} \otimes id) \circ (\delta^2_X \otimes \delta_Y \otimes \delta_Z) \end{aligned} \tag{4.2}$$

*for any $X, Y, Z \in C$.*

*Proof.* ⇐): Take $X = M$, $Y = N$, $Z = P$ for any $F$-comodules $M, N, P$. Multiplied by $\rho^M \otimes \rho^N \otimes \rho^P$ right on both sides of Eq (4.2), we immediately get Diagram (B1).

⇒): Since Diagram (B1) is commutative for any $FX, FY, FZ \in C$, multiplied by $\varepsilon \otimes \varepsilon \otimes \varepsilon$ left on both sides of the above equation, we get Eq (4.2). □

**Lemma 4.4.** *For any $X, Y, Z \in C$, Diagram (B2) holds in $C^F$ if and only if $\sigma$ satisfies*

$$\begin{aligned}&\vartheta^{*-1}_{Z,X,Y} \circ \sigma_{FX\otimes FY,FZ} \circ (F_2 \otimes id)) \circ \vartheta^{*-1}_{FFX,FFY,FFZ} \circ (\delta^2_X \otimes \delta^2_Y \otimes \delta^2_Z)\\ &= (\sigma_{X,Z} \otimes id) \circ \vartheta^{*-1}_{FX,FZ,Y} \circ (id \otimes \sigma_{FY,FFZ}) \circ (\delta_X \otimes \delta_Y \otimes \delta^2_Z),\end{aligned} \tag{4.3}$$

*where $\vartheta^{*-1}$ means the $*$-inverse of $\vartheta$.*

*Proof.* The proof is similar to Lemma 4.3. □

**Definition 4.5.** Let $(F, \delta, \varepsilon, F_2, F_0, \vartheta, \iota, \kappa)$ be a quasi-monoidal comonad on a monoidal category $C$. If there is a $*$-invertible natural transformation $\sigma \in Nat(F \otimes F, \otimes^{op})$, satisfying Eqs (4.1)–(4.3) for any $X, Y, Z \in C$, then $\sigma$ is called a *coquasitriangular structure* of $F$, and $(F, \sigma)$ is called a *coquasitriangular quasi-monoidal comonad*.

Combining Lemma 4.1–Definition 4.5, we obtain the following result.

**Theorem 4.6.** *Let $(F, \delta, \varepsilon, F_2, F_0, \vartheta, \iota, \kappa)$ be a quasi-monoidal comonad on a monoidal category $C$. Then $C^F$ is a braided monoidal category if and only if there exists a natural transformation $\sigma : F\otimes F \to \otimes^{op}$ such that $(F, \sigma)$ is a coquasitriangular quasi-monoidal comonad. Further, the braiding in $C^F$ is given by $\tau = \sigma^\sharp$.*

**Corollary 4.7.** *Let $(F, \sigma)$ be a coquasitriangular quasi-monoidal comonad on a monoidal category $C$. Then for any $X, Y, Z \in C$, $\sigma$ satisfies the following generalized Yang-Baxter equation:*

$$\begin{aligned}&(id \otimes \sigma_{X,Y}) \circ \vartheta_{Z,FX,FY} \circ (\sigma_{FFX,FZ} \otimes id) \circ \vartheta^{*-1}_{F^3X,FFZ,FFY} \circ (id \otimes \sigma_{F^3Y,F^3Z})\\ &\quad \circ \vartheta_{F^4X,F^4Y,F^4Z} \circ (\delta^4_X \otimes \delta^4_Y \otimes \delta^4_Z)\\ &= \vartheta_{Z,Y,X} \circ (\sigma_{FY,FZ} \otimes id) \circ \vartheta^{*-1}_{FFY,FFZ,FX} \circ (id \otimes \sigma_{FFX,F^3Z}) \circ \vartheta_{F^3Y,F^3X,F^4Z}\\ &\quad \circ (\sigma_{F^4X,F^4Y} \otimes id) \circ (\delta^4_X \otimes \delta^4_Y \otimes \delta^4_Z).\end{aligned}$$

*Proof.* Straightforward. □

**Example 4.8.** If $F$ is a monoidal comonad on $C$, and $\sigma : \otimes \circ F^{\times 2} \Rightarrow \otimes^{op}$ is a $*$-invertible natural transformation satisfying Eqs (4.1)–(4.3), then $(F, \sigma)$ is exactly a coquasitriangular monoidal comonad (see [9], Definition 4.12).

**Example 4.9.** With the notations in Example 3.10, if $Q \in (H \otimes H)^*$ is $\alpha_H$-invariant and convolution invertible, then we have the following $*$-invertible natural transformation

$$\sigma_{X,Y} : \ddot{H}X \otimes \ddot{H}Y \to Y \otimes X, \quad (x \otimes a) \otimes (y \otimes b) \mapsto Q(\alpha^i_H(a), \alpha^j_H(b))\alpha^{j-i-1}_Y(y) \otimes \alpha^{i-j-1}_X(x),$$

where $x \in X$, $y \in Y$ and $X, Y \in \overline{\mathcal{H}}^{i,j}(Vec_k)$. Thus we immediately get that $\sigma$ satisfies Eq (4.1) if and only if $Q$ satisfies

$$\sum Q(a_1, b_1)a_2b_2 = \sum b_1a_1Q(a_2, b_2),$$

$\sigma$ satisfies Eqs (4.2) and (4.3) if and only if $Q$ satisfies

$$\begin{aligned}
&\sum \omega(b_1, c_1, a_1) Q(\alpha_H(a_{21}), b_{21}c_{21}) \omega(a_{22}, b_{22}, c_{22}) \\
&\quad = \sum Q(a_1, c_1) \omega(b_1, \alpha_H^{-1}(a_{21}), c_2) Q(\alpha_H^{-1}(a_{22}), b_2), \\
&\sum \omega^{*-1}(c_1, a_1, b_1) Q(a_{21}b_{21}, \alpha_H(c_{21})) \omega^{*-1}(a_{22}, b_{22}, c_{22}) \\
&\quad = \sum Q(a_1, c_1) \omega^{*-1}(a_2, \alpha_H^{-1}(c_{21}), b_1) Q(b_2, \alpha_H^{-1}(c_{22})),
\end{aligned}$$

where $a, b, c \in H$. That is, $(\ddot{H}, \sigma)$ forms a coquasitriangular quasi-monoidal comonad if and only if $(H, Q)$ is a *coquasitriangular Hom-coquasi-bialgebra*. Further, from Theorem 4.6, one get that $Corep(H) = (\overline{\mathcal{H}}^{i,j}(Vec_k))^{\ddot{H}}$ is a braided monoidal category.

**Example 4.10.** With the notations in Example 3.12, if $p \in B^*$ is a dual central dual group-like $\alpha_B$-invariant $k$-linear form on a bialgebra $B$, then we get a coquasi-bialgebra $B^{p,p}_{\alpha_B}$. Now suppose that $Q \in (B \otimes B)^*$ is the coquasitriangular structure over $B$. If $Q \circ (\alpha_B \otimes \alpha_B) = Q$, then after a straightforward compute we get that $Q$ is also a coquasitriangular structure over the Hom-coquasi-bialgebra $B^{p,p}_{\alpha_B}$.

## 5. Gauge transformations

Let $F = (F, \delta, \varepsilon, F_2, F_0)$ be a quasi-monoidal comonad on a monoidal category $(C, \otimes, I, a, l, r)$.

**Definition 5.1.** A *gauge transformation* on $F$ is a $*$-invertible natural transformation $\xi : F \otimes F \Rightarrow \otimes$.

Using a gauge transformation $\xi$ on $F$, we can build a new quasi-monoidal comonad $F^\xi$ as follows.
Firstly, as a functor, $F^\xi = F : C \to C$.
Secondly, the comonad structure of $F^\xi$ is $F^\xi = F = (F, \delta, \varepsilon)$.
Thirdly, the quasi-monoidal functor structure of $F^\xi$ is given by:
• for any $X, Y \in C$, $F_2^\xi : F \otimes F \Rightarrow F\otimes$ is defined as follows

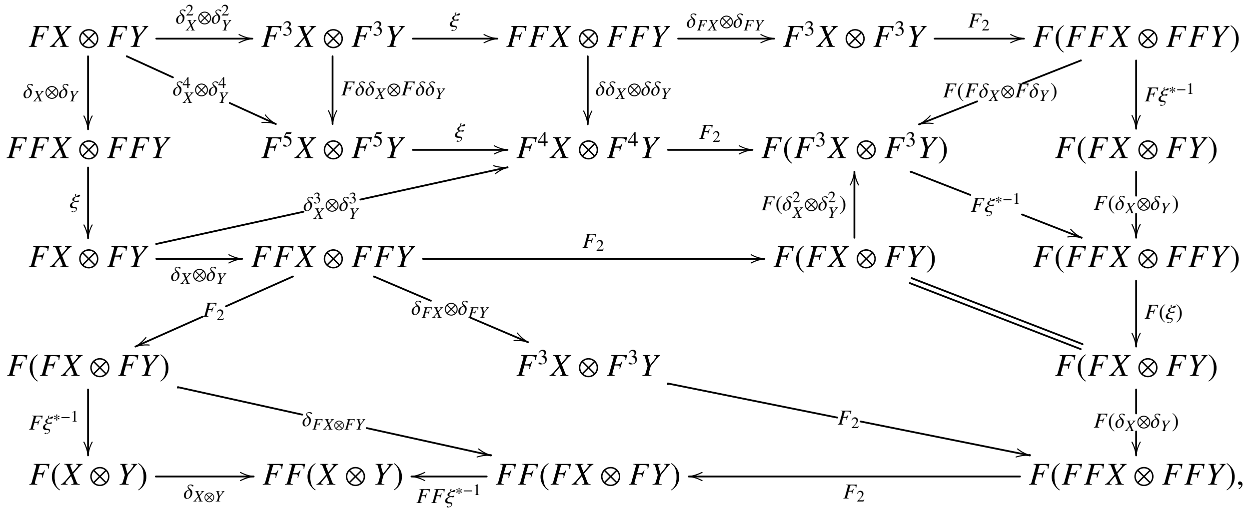
$$F_2^\xi(X, Y) : \ FX \otimes FY \xrightarrow{\delta_X^2 \otimes \delta_Y^2} F^3X \otimes F^3Y \xrightarrow{\xi} FFX \otimes FFY \xrightarrow{F_2} F(FX \otimes FY) \xrightarrow{F(\xi^{*-1}_{X,Y})} F(X \otimes Y) \tag{5.1}$$

where $\xi^{*-1}$ means the $*$-inverse of $\xi$;
• $F_0^\xi = F_0 : FI \to I$.

**Proposition 5.2.** *With the above notations, $\delta$ and $\varepsilon$ are both monoidal natural transformations*

*Proof.* We only need to show the compatible conditions Eqs (C1)–(C4) hold.

To prove Eq (C1), we compute




for any $X, Y \in C$. The rest are straightforward. $\square$

For any $X, Y \in C$, define the natural transformation $\vartheta^\xi : (F \otimes F) \otimes F \Rightarrow \_ \otimes (\_ \otimes \_)$ by

$$\begin{aligned} \vartheta^\xi_{X,Y,Z} = & (id \otimes \xi^{*-1}_{Y,Z}) \circ \xi^{*-1}_{X,FY\otimes FZ} \circ (id \otimes F_2) \circ \vartheta_{FX,FFY,FFZ} \circ (\xi_{FFX\otimes F^3Y,F^3Z}) \\ & \circ (F_2 \otimes id) \circ (\xi_{F^3X,F^4Y} \otimes id) \circ (\delta^3_X \otimes \delta^4_Y \otimes \delta^3_Z), \end{aligned} \tag{5.2}$$

and define the followings natural transformations:

$$\iota^\xi_X : \ I \otimes FX \xrightarrow{F_0\otimes\delta_X} FI \otimes FFX \xrightarrow{\xi} I \otimes FX \xrightarrow{\iota_X} X, \tag{5.3}$$

and

$$\kappa^\xi_X : \ FX \otimes I \xrightarrow{\delta_X\otimes F_0} FFX \otimes FI \xrightarrow{\xi} FX \otimes I \xrightarrow{\kappa_X} X. \tag{5.4}$$

It is easy to get that $\vartheta^\xi$, $\iota^\xi$ and $\kappa^\xi$ are all $*$-invertible. Further, we have the following properties.

**Lemma 5.3.** *With the above notations, $\vartheta^\xi$ satisfies Eqs (3.1) and (3.2).*

*Proof.* We only prove Eq (3.1). For any $X, Y, Z \in C$, we compute

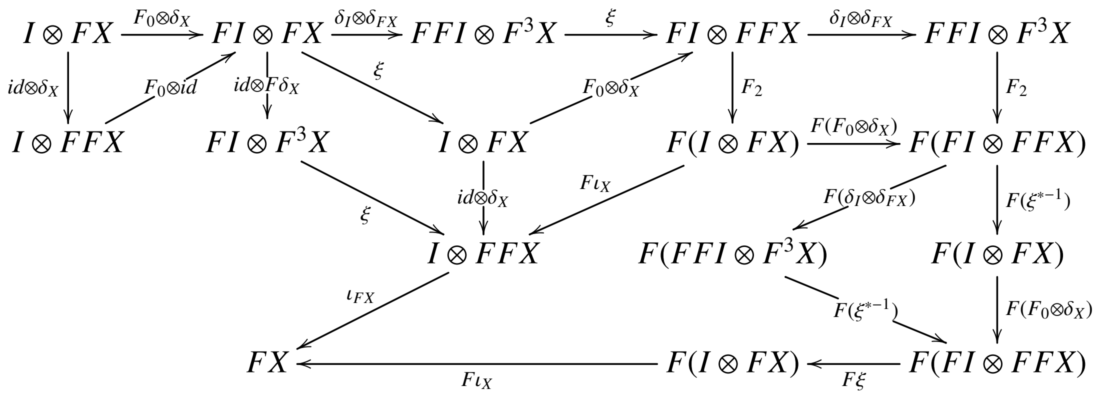
$$\begin{aligned} & F(\vartheta^\xi_{X,Y,Z}) \circ F^\xi_2 \circ (F^\xi_2 \otimes id) \circ (\delta_X \otimes \delta_Y \otimes \delta_Z) \\ = & F(id \otimes \xi^{*-1}_{Y,Z}) \circ F(\xi^{*-1}_{X,FY\otimes FZ}) \circ F(id \otimes F_2) \circ F(\vartheta_{FX,FFY,FFZ}) \circ F(\xi_{FFX\otimes F^3Y,F^3Z}) \\ & \circ F(F_2 \otimes id) \circ F(\xi_{F^4X,F^3Y} \otimes id) \circ F(\delta^3_X \otimes \delta^4_Y \otimes \delta^3_Z) \circ F(\xi^{*-1}_{FX\otimes FY,FZ}) \circ F_2 \\ & \circ(\delta_{FX\otimes FY} \otimes \delta_{FZ}) \circ \xi_{F(FX\otimes FY),FFZ} \circ (\delta_{FX\otimes FY} \otimes \delta_{FZ}) \circ (F(\xi^{*-1}_{FX,FY}) \otimes id) \\ & \circ(F_2 \otimes id) \circ (\delta_{FX} \otimes \delta_{FY} \otimes id) \circ (\xi_{FFX,FFY} \otimes id) \circ (\delta_{FX} \otimes \delta_{FY} \otimes id) \circ (\delta_X \otimes \delta_Y \otimes \delta_Z) \\ = & F(id \otimes \xi^{*-1}_{Y,Z}) \circ F(\xi^{*-1}_{X,FY\otimes FZ}) \circ F(id \otimes F_2) \circ F(\vartheta_{FX,FFY,FFZ}) \circ F(\xi_{FFX\otimes F^3Y,F^3Z}) \\ & \circ F(F_2 \otimes id) \circ F(\delta_{FX} \otimes \delta^2_{FY} \otimes \delta^3_Z) \circ F(\xi^{*-1}_{FFX\otimes FFY,FZ}) \circ F_2 \circ (\delta_{FFX\otimes FFY} \otimes \delta_{FZ}) \\ & \circ\xi_{F(FFX\otimes FFY),FFZ} \circ (\delta_{FFX\otimes FFY} \otimes \delta_{FZ}) \circ (F_2 \otimes id) \circ (\delta_{FX} \otimes \delta_{FY} \otimes id) \circ (\xi_{FFX,FFY} \otimes id) \end{aligned}$$

$$
\begin{aligned}
&\circ(\delta_X^2 \otimes \delta_Y^2 \otimes \delta_Z) \\
&= F(id \otimes \xi^{*-1}_{Y,Z}) \circ F(\xi^{*-1}_{X,FY\otimes FZ}) \circ F(id \otimes F_2) \circ F(\vartheta_{FX,FFY,FFZ}) \circ F_2 \circ (F_2 \otimes id) \circ (\delta^2_{FX} \otimes \delta^2_{FY} \\
&\quad \otimes \delta^2_{FZ}) \circ \xi_{F^3X\otimes F^3Y,FFZ} \circ (F_2 \otimes id) \circ (\delta_{FX} \otimes \delta^2_{FY} \otimes \delta_{FZ}) \circ (\xi_{FFX,FFY} \otimes id) \circ (\delta_X^2 \otimes \delta_Y^2 \otimes \delta_Z) \\
&= F(id \otimes \xi^{*-1}_{Y,Z}) \circ F(\xi^{*-1}_{X,FY\otimes FZ}) \circ F(id \otimes F_2) \circ F_2 \circ (id \otimes F_2) \circ (\delta_X \otimes \delta^2_Y \otimes \delta^2_Z) \circ \vartheta_{FX,FY,FZ} \\
&\quad \circ \xi_{FFX\otimes FFY,FFZ} \circ (F_2 \otimes id) \circ (\xi_{F^3X,F^3Y} \otimes id) \circ (\delta^3_X \otimes \delta^3_Y \otimes \delta^2_Z) \\
&= F(\xi^{*-1}_{X,Y\otimes Z}) \circ F(id \otimes F(\xi^{*-1}_{Y,Z})) \circ F_2 \circ (\delta_X \otimes \delta_{FY\otimes FZ}) \circ (id \otimes F_2) \circ (id \otimes \delta_Y \otimes \delta_Z) \\
&\quad \circ (\varepsilon_{FX} \otimes \varepsilon_{FY} \otimes \varepsilon_{FZ}) \circ \vartheta_{FFX,FFY,FFZ} \circ \xi_{F^3X\otimes F^3Y,F^3Z} \circ (F_2 \otimes id) \circ (\xi_{F^4X,F^4Y} \otimes id) \\
&\quad \circ (\delta^4_X \otimes \delta^4_Y \otimes \delta^3_Z) \\
&= F(\xi^{*-1}_{X,Y\otimes Z}) \circ F_2 \circ (\delta_X \otimes \delta_{Y\otimes Z}) \circ \xi_{FX,F(Y\otimes Z)} \circ \xi^{*-1}_{FFX,FF(Y\otimes Z)} \circ (F\delta_X \otimes F\delta_{Y\otimes Z}) \circ (id \\
&\quad \otimes FF(\xi^{*-1}_{Y,Z})) \circ (id \otimes F(F_2)) \circ (id \otimes F(\delta_Y \otimes \delta_Z)) \circ (id \otimes F_2) \circ \vartheta_{FFX,FFY,FFZ} \circ \xi_{F^3X,F^3Y\otimes F^3Z} \\
&\quad \circ (F(id \otimes FF\varepsilon_{FY}) \otimes FFF\varepsilon_{FZ}) \circ (F_2 \otimes id) \circ (\xi_{F^4X,F^5Y} \otimes id) \circ (\delta^4_X \otimes \delta^5_Y \otimes \delta^4_Z) \\
&= F(\xi^{*-1}_{X,Y\otimes Z}) \circ F_2 \circ (\delta_X \otimes \delta_{Y\otimes Z}) \circ \xi_{FX,F(Y\otimes Z)} \circ (\delta_X \otimes \delta_{Y\otimes Z}) \circ (F(id \otimes \xi^{*-1}_{Y,Z})) \circ (id \otimes F_2(FY, FZ)) \\
&\quad \circ (id \otimes \delta_Y \otimes \delta_Z) \circ (id \otimes \xi_{FX,FY}) \circ (id \otimes \xi^{*-1}_{FFY,FFZ}) \circ (id \otimes \delta_{FY} \otimes \delta_{FZ}) \circ \xi^{*-1}_{FX,FFY\otimes FFZ} \\
&\quad \circ (id \otimes F_2(FFY \otimes FFZ)) \circ \vartheta_{FFX,F^3Y,F^3Z} \circ \xi_{F^3X,F^4Y\otimes F^4Z} \circ (F_2 \otimes id) \circ (\xi_{F^4X,F^5Y} \otimes id) \\
&\quad \circ (\delta^4_X \otimes \delta^5_Y \otimes \delta^4_Z) \\
&= F_2^{\xi}(FX, FY \otimes FZ) \circ (id \otimes F_2^{\xi}(FY, FZ)) \circ \vartheta^{\xi}_{FX,FY,FZ} \circ (\delta_X \otimes \delta_Y \otimes \delta_Z).
\end{aligned}
$$

Thus the conclusion holds. □

**Lemma 5.4.** *With the above notations, $\iota^{\xi}$ satisfies Eq (3.3) and $\kappa^{\xi}$ satisfies Eq (3.4).*

*Proof.* We only prove Eq (3.3). For any $X \in C$, we have




which implies Eq (3.3). □

**Lemma 5.5.** *With the above notations, $\vartheta^{\xi}$ and $\iota^{\xi}$, $\kappa^{\xi}$ satisfy Eq (3.5).*

*Proof.* For any $X, Y \in C$, we obtain

$$
\begin{aligned}
&(id \otimes \iota^{\xi}_Y) \circ (\vartheta^{\xi}_{X,I,FY}) \circ (id \otimes F_0 \otimes \delta_Y) \\
&= (id \otimes \iota_Y) \circ (id \otimes \xi_{I,FFY}) \circ (id \otimes F_0 \otimes \delta_Y) \otimes (id \otimes \xi^{*-1}_{I,FY}) \circ \xi^{*-1}_{X,FI,FFY} \circ (id \otimes F_2) \circ \vartheta_{FX,FFI,F^3Y} \\
&\quad \circ \xi_{FFX\otimes F^3I,F^4Y} \circ (F_2 \otimes id) \circ (\xi_{F^3X,F^4I} \otimes id) \circ (\delta^3_X \otimes \delta^4_I \otimes \delta^3_{FY}) \otimes (\delta_X \otimes F_0 \otimes \delta_Y)
\end{aligned}
$$

$$= \xi^{*-1}_{X,Y} \circ (id \otimes F\iota_Y) \circ (id \otimes F_2) \circ \vartheta_{FX,FI,FFY} \circ \xi_{FFX\otimes FFI,F^3Y} \circ (F_2 \otimes id) \circ (\xi_{F^3X,F^3I} \otimes id)$$
$$\circ(\delta^3_X \otimes \delta^3_I \otimes \delta^2_{FY}) \circ (\delta_X \otimes F_0 \otimes \delta_Y)$$
$$= \xi^{*-1}_{X,Y} \circ (id \otimes \iota_{FY}) \circ \vartheta_{FX,I,FFY} \circ (id \otimes F_0 \otimes \delta_Y) \circ \xi_{FFX\otimes I,FFY} \circ (F_2 \otimes id) \circ (\xi_{F^3X,FI} \otimes id)$$
$$\circ(\delta^3_X \otimes \delta_I \otimes \delta^2_Y) \circ (\delta_X \otimes F_0 \otimes id)$$
$$= (id \otimes \varepsilon_Y) \circ \xi^{*-1}_{X,FY} \circ (\kappa_{FX} \otimes id) \circ \xi_{FFX\otimes I,FFY} \circ (F_2 \otimes id) \circ (\delta_{FX} \otimes F_0 \otimes id) \circ (\xi_{FFX,I} \otimes id)$$
$$\circ(\delta^2_X \otimes F_0 \otimes \delta^2_Y)$$
$$= (id \otimes \varepsilon_Y) \circ \xi^{*-1}_{X,FY} \circ \xi_{FX,FFY} \circ (\kappa_{FFX} \otimes id) \circ (\delta_{FX} \otimes id \otimes id) \circ (\xi_{FFX,I} \otimes id) \circ (\delta^2_X \otimes F_0 \otimes \delta^2_Y)$$
$$= (id \otimes \varepsilon_Y) \circ (\kappa_X \otimes id) \circ (\xi_{FX,I} \otimes id) \circ (\delta_X \otimes F_0 \otimes id) = (id \otimes \varepsilon_Y) \circ (\kappa^{\xi}_X \otimes id)$$

hence Eq (3.5) holds. □

**Theorem 5.6.** *$F^{\xi} = (F, \delta, \varepsilon, F^{\xi}_2, F_0, \vartheta^{\xi}, \iota^{\xi}, \kappa^{\xi})$ is a quasi-monoidal comonad.*

*Remark* 5.7. $(C^{F^{\xi}}, \otimes, I, A^{\xi}, L^{\xi}, R^{\xi})$ is a monoidal category, where $A^{\xi} = (\vartheta^{\xi})^{\sharp}$, $L^{\xi} = (\iota^{\xi})^{\sharp}$, $R^{\xi} = (\kappa^{\xi})^{\sharp}$.

Now consider a coquasitriangular quasi-monoidal comonad $(F, \sigma)$. For any gauge transformation $\xi$ on $F$, for any $X, Y \in C$, define

$$\sigma^{\xi}_{X,Y} : \ FX \otimes FY \xrightarrow{\delta^2\otimes\delta^2} F^3X \otimes F^3Y \xrightarrow{\xi} FFX \otimes FFY \xrightarrow{\sigma} FY \otimes FX \xrightarrow{\xi^{*-1}} Y \otimes X. \tag{5.5}$$

**Proposition 5.8.** *With the above notations, $\sigma^{\xi}$ is a coquasitriangular structure of $F^{\xi}$. Thus $F^{\xi}$ is a coquasitriangular quasi-monoidal comonad. Hence $C^{F^{\xi}}$ is a braided monoidal category with the braiding $\tau^{\xi} = (\sigma^{\xi})^{\sharp}$.*

*Proof.* Firstly, it is straightforward to get that $\sigma^{\xi}$ is $*$-invertible.

Secondly, to prove Eq (4.1), for any $X, Y \in C$, we compute

$$F^{\xi}_2 \circ \sigma^{\xi}_{FX,FY} \circ (\delta_X \otimes \delta_Y)$$
$$= F(\xi^{*-1}_{Y,X}) \circ F_2 \circ (\delta_Y \otimes \delta_X) \circ \xi_{FY,FX} \circ \xi^{*-1}_{FFY,FFX} \circ (\delta^2_Y \otimes \delta^2_X) \circ \sigma_{FX,FY} \circ \xi_{FFX,FFY} \circ (\delta^2_X \otimes \delta^2_Y)$$
$$= F(\xi^{*-1}_{Y,X}) \circ F_2 \circ \sigma_{FFX,FFY} \circ \xi_{F^3X,F^3Y} \circ (\delta^3_X \otimes \delta^3_Y)$$
$$= F(\xi^{*-1}_{Y,X}) \circ F(\sigma_{FX,FY}) \circ F(\xi_{FFX,FFY}) \circ F(\xi^{*-1}_{F^3X,F^3Y}) \circ F(\delta^3_X \otimes \delta^3_Y) \circ F_2 \circ (\delta_X \otimes \delta_Y)$$
$$= F(\xi^{*-1}_{Y,X}) \circ F(\sigma_{FX,FY}) \circ F(\xi_{FFX,FFY}) \circ F(\delta^2_X \otimes \delta^2_Y) \circ F(\xi^{*-1}_{FX,FY}) \circ F_2 \circ (\delta_{FX} \otimes \delta_{FY})$$
$$\circ\xi_{FFX,FFY} \circ (\delta^2_X \otimes \delta^2_Y)$$
$$= F(\sigma^{\xi}_{Y,X}) \circ F^{\xi}_2(FX, FY) \circ (\delta_X \otimes \delta_Y).$$

Thirdly, for Eq (4.2), we have

$$\vartheta^{\xi}_{Y,Z,X} \circ \sigma^{\xi}_{FX,FY\otimes FZ} \circ (id \otimes F^{\xi}_2) \circ \vartheta^{\xi}_{FFX,FFY,FFZ} \circ (\delta^2_X \otimes \delta^2_Y \otimes \delta^2_Z)$$
$$= (id \otimes \xi^{*-1}_{Z,X}) \circ \xi^{*-1}_{Y,FZ\otimes FX} \circ (id \otimes F_2) \circ \vartheta_{FY,FFZ,FFX} \circ \xi_{FFY\otimes F^3Z,F^3X} \circ (F_2 \otimes id)$$
$$\circ(\xi_{F^3Y\otimes F^4Z} \otimes id) \circ (\delta^3_Y \otimes \delta^4_Z \otimes \delta^3_X) \circ \xi^{*-1}_{FY\otimes FZ,FX} \circ \sigma_{FFX,F(FY\otimes FZ)} \circ \xi_{F^3X,FF(FY\otimes FZ)})$$
$$\circ(\delta^2_{FX} \otimes \delta^2_{FY\otimes FZ}) \circ (id \otimes F(\xi^{*-1}_{FY,FZ})) \circ (id \otimes F_2) \circ (id \otimes \xi_{F^3Y,F^3Z}) \circ (id \otimes \delta^2_{FY} \otimes \delta^2_{FZ})$$
$$\circ(id \otimes \xi^{*-1}_{F^2Y,F^2Z}) \circ \xi^{*-1}_{FFX,F^3Y\otimes F^3Z} \circ (id \otimes F_2) \circ \vartheta_{F^3X,F^4Y,F^4Z} \circ \xi_{F^4X\otimes F^5Y,F^5Z}$$
$$\circ(F_2 \otimes id) \circ (\xi_{F^5X,F^6Y} \otimes id) \circ (\delta^5_X \otimes \delta^6_Y \otimes \delta^5_Z)$$

$$= (id\otimes\xi^{*-1}_{Z,X})\circ\xi^{*-1}_{Y,FZ\otimes FX}\circ(id\otimes F_2)\circ\vartheta_{FY,FFZ,FFX}\circ\sigma_{F^3X,FFY\otimes F^3Z}\circ(id\otimes F_2)$$
$$\circ\vartheta_{F^4X,F^3Y,F^4Z}\circ(\delta^2_{FFX}\otimes\delta^2_{FY}\otimes\delta^2_{FFZ})\circ\xi_{F^3X\otimes FFY,F^3Z}\circ(F_2\otimes id)\circ(\xi_{F^4X,F^3Y}\otimes id)$$
$$\circ(\delta^4_X\otimes\delta^3_Y\otimes\delta^3_Z)$$
$$= (id\otimes\xi^{*-1}_{Z,X})\circ(id\otimes\sigma_{FX,FZ})\circ\xi^{*-1}_{Y,FFX\otimes FFZ}\circ(id\otimes F_2)\circ\vartheta_{FY,F^3X,F^3Z}\circ(\sigma_{F^4X,FFY}\otimes id)$$
$$\circ(\delta^2_{FFX}\otimes\delta_{FY}\otimes\delta_{FFZ})\circ\xi_{F^3X\otimes FFY,F^3Z}\circ(F_2\otimes id)\circ(\xi_{F^4X,F^3Y}\otimes id)\circ(\delta^4_X\otimes\delta^3_Y\otimes\delta^3_Z)$$
$$= (id\otimes\xi^{*-1}_{Z,X})\circ(id\otimes\sigma_{FX,FZ})\circ\xi^{*-1}_{Y,FFX\otimes FFZ}\circ(id\otimes F_2)\circ\vartheta_{FY,F^3X,F^3Z}\circ\xi_{F^2Y\otimes F^4X,F^4Z}$$
$$\circ(F(\sigma_{F^4X,FFY})\otimes id)\circ(F_2\otimes id)\circ(\delta^4_{FX}\otimes\delta^3_{FY}\otimes\delta_{FY})\circ(\xi_{FFX,FFY}\otimes id)\circ(id\otimes\delta^2_Z\otimes\delta^2_Z)$$
$$= (id\otimes\sigma^{\xi}_{X,Z})\circ\vartheta^{\xi}_{Y,FX,FZ}\circ(\sigma^{\xi}_{FFX,FF}\otimes id)\circ(\delta^2_X\otimes\delta_Y\otimes\delta_Z).$$

At last, we can prove Eq (4.3) in a similar way. Thus the conclusion holds. □

Now consider the corepresentations of $F$ and $F^{\xi}$.

**Theorem 5.9.** *$C^F$ and $C^{F^{\xi}}$ are isomorphic as monoidal categories. Further, if $F$ is a coquasitriangular quasi-monoidal comonad, then $C^F$ and $C^{F^{\xi}}$ are braided isomorphic.*

*Proof.* For any morphism $f$ and objects $M, N$ in $C$, the monoidal functor is defined as follows

$$\mathbb{E} = (\mathbb{E}, \mathbb{E}^{\xi}_2, \mathbb{E}_0) : (C^F, \otimes, I, A, L, R) \to (C^{F^{\xi}}, \otimes, I, A^{\xi}, L^{\xi}, R^{\xi}),$$

where

$$\mathbb{E}(M) := M \text{ as an } F\text{-comodule},\ \ \mathbb{E}(f) := f,\ \ \mathbb{E}_0 = id_I,$$

and $\mathbb{E}^{\xi}_2(M,N) : \mathbb{E}(M)\otimes\mathbb{E}(N)\to\mathbb{E}(M\otimes N)$ is given by

$$\mathbb{E}^{\xi}_2(M,N) = \xi^{\sharp}:\ M\otimes N \xrightarrow{\rho^M\otimes\rho^N} FM\otimes FN \xrightarrow{\xi_{M,N}} M\otimes N.$$

Obviously $\mathbb{E}$ is well-defined.

Now we will check relation (2.1). Indeed, we have

$$\mathbb{E}^{\xi}_2(M,N\otimes P)\circ(id\otimes\mathbb{E}^{\xi}_2(N,P))\circ A^{\xi}_{M,N,P}$$
$$= \xi_{M,N\otimes P}\circ(id\otimes F_2)\circ(\rho^M\otimes\rho^N\otimes\rho^P)\circ(id\otimes\xi_{N,P})\circ(id\otimes\rho^N\otimes\rho^P)\circ(id\otimes\xi^{*-1}_{N,P})$$
$$\circ\xi^{*-1}_{M,FN\otimes FP}\circ(id\otimes F_2)\circ\vartheta_{FM,FFN,FFP}\circ\xi_{FFM\otimes F^3N,F^3P}\circ(F_2\otimes id)\circ(\xi_{F^3M,F^4N}\otimes id)$$
$$\circ(\delta^3_M\otimes\delta^4_N\otimes\delta^3_P)\circ(\rho^M\otimes\rho^N\otimes\rho^P)$$
$$= \xi_{M,N\otimes P}\circ(\rho^M\otimes F_2)\circ\xi^{*-1}_{M,FN\otimes FP}\circ(id\otimes F_2)\circ\vartheta_{FM,FFN,FFP}\circ\xi_{FFM\otimes F^3N,F^3P}\circ(F_2\otimes id)$$
$$\circ(\xi_{F^3M,F^4N}\otimes id)\circ(\delta^3_M\otimes\delta^4_N\otimes\delta^3_P)\circ(\rho^M\otimes\rho^N\otimes\rho^P)$$
$$= \xi_{M,N\otimes P}\circ\xi^{*-1}_{FM,F(N\otimes P)}\circ(\delta_M\otimes\delta_{N\otimes P})\circ(id\otimes F_2)\circ\vartheta_{FM,FN,FP}\circ\xi_{FFM\otimes FFN,FFP}\circ(F_2\otimes id)$$
$$\circ(\xi_{F^3M,F^3N}\otimes id)\circ(\delta^3_M\otimes\delta^3_N\otimes\delta^2_P)\circ(\rho^M\otimes\rho^N\otimes\rho^P)$$
$$= \vartheta_{M,N,P}\circ\xi_{FM\otimes FN,FP}\circ(F_2\otimes id)\circ(\xi_{F^2M,F^2N}\otimes id)\circ(\delta^2_M\otimes\delta^2_N\otimes\delta_P)\circ(\rho^M\otimes\rho^N\otimes\rho^P)$$
$$= \mathbb{E}(A_{M,N,P})\circ\mathbb{E}^{\xi}_2(M\otimes N,P)\circ(\mathbb{E}^{\xi}_2(M,N)\otimes id),$$

which implies Eq (2.1).

Further, we can obtain (2.2) and (2.3) by straightforward computation. Hence the conclusion holds.

Moreover, if $\sigma$ is a coquasitriangular structure of $F$, then from Theorem 5.6, $(F^{\xi}, \sigma^{\xi})$ is also a coquasitriangular quasi-monoidal comonad. Then we have

$$\begin{aligned}
&\mathbb{E}_2^{\xi}(N, M) \circ \tau^{\xi}_{M,N} \\
&= \xi_{N,M} \circ (\rho^N \otimes \rho^M) \circ \xi^{*-1}_{N,M} \circ \sigma_{FM,FN} \circ \xi_{FFM,FFN} \circ (\delta^2_M \otimes \delta^2_N) \circ (\rho^M \otimes \rho^N) \\
&= (\varepsilon_N \otimes \varepsilon_M) \circ \sigma_{FM,FN} \circ \xi_{FFM,FFN} \circ (\delta^2_M \otimes \delta^2_N) \circ (\rho^M \otimes \rho^N) \\
&= \sigma_{FM,FN} \circ (\rho^N \otimes \rho^M) \circ \xi_{M,N} \circ (\rho^M \otimes \rho^N) = \mathbb{E}(\tau_{M,N}) \circ \mathbb{E}_2^{\xi}(M, N),
\end{aligned}$$

which implies $(\mathbb{E}, \mathbb{E}_2^{\xi}, \mathbb{E}_0)$ is a braided monoidal functor. □

**Example 5.10.** With the notations in Example 3.10, if there is a convolution invertible linear form $\chi \in (H \otimes H)^*$ satisfying $\chi \circ (\alpha_H \otimes \alpha_H) = \chi$, then we have the following $*$-invertible natural transformation in $\overline{\mathcal{H}}^{i,j}(Vec_k)$

$$\xi_{X,Y} : \ddot{H}X \otimes \ddot{H}Y \to X \otimes Y, \quad (x \otimes a) \otimes (y \otimes b) \mapsto \chi(\alpha^i_H(a), \alpha^j_H(b))\alpha^{-1}_X(x) \otimes \alpha^{-1}_Y(y),$$

where $a, b \in H$, $x \in X$, $y \in Y$ and $X, Y \in \overline{\mathcal{H}}^{i,j}(Vec_k)$. It is not hard to check that $\ddot{H}^{\xi}_2$, $\vartheta^{\xi}$, $\iota^{\xi}$ and $\kappa^{\xi}$ in Eqs (5.1)–(5.4) are deduced from the following

$$m^{\chi}(a \otimes b) = \sum \chi^{*-1}(a_1, b_1)\alpha^{-2}_H(a_{21})\alpha^{-2}_H(b_{21})\chi(a_{22}, b_{22}),$$

where $\chi^{*-1}$ means the convolution inverse of $\chi$, and

$$\begin{aligned}
\omega^{\chi}(a, b, c) = \sum \chi^{*-1}(b_{11}, c_{11})\chi^{*-1}(\alpha_H(a_{11}), \alpha^{-1}_H(b_{121})c_{12}) \\
\omega(a_{12}, \alpha^{-1}_H(b_{122}), c_{21})\chi(a_{21}b_{21}, \alpha_H(c_{22}))\chi(a_{22}, b_{22}), \\
p^{\chi}(a) = \sum p(a_1)\chi(1_H, a_2), \quad q^{\chi}(a) = \sum q(a_1)\chi(a_2, 1_H),
\end{aligned}$$

respectively. Thus from Example 3.10 and Theorem 5.6, $H^{\chi} = (H, \alpha_H, m^{\chi}, 1_H, \Delta, \varepsilon, \omega^{\chi}, p^{\chi}, q^{\chi})$ is also a Hom-coquasi-bialgebra.

**Example 5.11.** With the notations in Example 3.12, note that the $B_{\alpha_B} = (B, \alpha_B, \alpha_B \circ \mu, 1_B, \Delta \circ \alpha_B, \varepsilon)$ is a Hom-bialgebra, and it can be seen as a Hom-coquasi-bialgebra $B_{\alpha_B} = (B, \alpha_B, \alpha_B \circ \mu, 1_H, \Delta \circ \alpha_B, \varepsilon, \varepsilon \otimes \varepsilon \otimes \varepsilon, \varepsilon, \varepsilon)$. If there are $\alpha_B$-invariant and dual central dual group-like $k$-linear forms $p, q \in B^*$, then we have the following gauge transformation $\chi \in (B \otimes B)^*$ by

$$\chi(a, b) = q^{*-1}(a)p(b), \quad \text{where } a, b \in B.$$

Obviously $B^{\chi}_{\alpha_B} = B^{p,q}_{\alpha_B}$.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## Conflict of interest

The authors declare there is no conflict of interest.

## References


1. B. Mesablishvili, R. Wisbauer, Bimonads and Hopf monads on categories, *J. K-Theory*, **7** (2011), 349–388. https://doi.org/10.1017/is010001014jkt105
2. M. Livernet, B. Mesablishvili, R. Wisbauer, Generalised bialgebras and entwined monads and comonads, *J. Pure Appl. Algebra,* **219** (2015), 3263–3278. https://doi.org/10.1016/j.jpaa.2014.10.013
3. B. Mesablishvili, R. Wisbauer, Galois functors and generalised Hopf modules, *J. Homotopy Relat. Struct.* **9** (2014), 199–222. https://doi.org/10.1007/s40062-013-0072-1
4. B. Mesablishvili, R. Wisbauer, The fundamental theorem for weak braided bimonads, *J. Algebra*, **490** (2017), 55–103. https://doi.org/10.1016/j.jalgebra.2017.06.023
5. I. Moerdijk, Monads on tensor categories, *J. Pure Appl. Algebra*, **168** (2002), 189–208. https://doi.org/10.1016/S0022-4049(01)00096-2
6. A. Bruguières, A. Virelizier, Hopf monads, *Adv. Math.*, **215** (2007), 679–733. https://doi.org/10.1016/j.aim.2007.04.011
7. A. Bruguières, S. Lack, A. Virelizier, Hopf monads on monoidal categories, *Adv. Math.*, **227** (2011), 745–800. https://doi.org/10.1016/j.aim.2011.02.008
8. G. Böhm, S. Lack, R. Street, Weak bimonads and weak Hopf monads, *J. Algebra*, **328** (2011), 1–30. https://doi.org/10.1016/j.jalgebra.2010.07.032
9. X. H. Zhang, L. H. Dong, Braided mixed datums and their applications on Hom-quantum groups, *Glasgow Math. J.*, **60** (2018), 231–251. https://doi.org/10.1017/S0017089517000088
10. X. H. Zhang, W. Wang, X. F. Zhao, Smash coproducts of monoidal comonads and Hom-entwining structures, *Rocky Mountain J. Math.*, **49** (2019), 2063–2105. https://doi.org/10.1216/RMJ-2019-49-6-2063
11. H. X. Zhu, The crossed structure of Hopf bimodules, *J. Algebra Appl.*, **17** (2018), 1850172. https://doi.org/10.1142/S0219498818501724
12. V. G. Drinfel'd, Quasi-Hopf algebras, *Leningrad Math. J.*, **1** (1990), 1419–1457.
13. M. Gerstenhaber, J. Stasheff, Deformation theory and quantum groups with applications to mathematical physics, *Contemp. Math.*, **134** (1992), 219–232. https://doi.org/10.1090/conm/134/1187289
14. I. Angiono, A. Ardizzoni, C. Menini, Cohomology and coquasi-bialgebras in the category of Yetter-Drinfeld modules, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **17** (2017), 609–653. https://doi.org/10.2422/2036-2145.201509$_0$18
15. X. H. Zhang, X. F. Zhao, W. Wang, Quasi-bimonads and their representations. *J. Pure Appl. Algebra*, **225** (2021), 106459. https://doi.org/10.1016/j.jpaa.2020.106459

16. J. Beck, Distributive laws, *Lect. Notes Math.*, **80** (1969), 119–140. https://doi.org/10.1007/BFb0083084
17. R. Street, The formal theory of monads, *J. Pure Appl. Algebra*, **2** (1972), 149–168. https://doi.org/10.1016/0022-4049(72)90019-9
18. D. Chikhladze, S. Lack, R. Street, Hopf monoidal comonads, *Theory Appl. Categ.*, **24** (2010), 554–563.
19. X. H. Zhang, D. G. Wang, Cotwists of Bicomonads and BiHom-bialgebras, *Algebra Repr. Theory*, **23** (2020), 1355–1385. https://doi.org/10.1007/s10468-019-09888-2
20. M. Elhamdadi, A. Makhlouf,Hopf algebras and tensor categories, *Contemp. Math.*, **585** (2013), 227–245. https://doi.org/10.1090/conm/585/11617
21. A. Bruguières, A. Virelizier, Quantum double of Hopf monads and categorical centers, *Trans. Amer. Math. Soc.*, **364** (2012), 1225–1279. https://doi.org/10.1090/S0002-9947-2011-05342-0
22. G. Böhm, T. Brzeziński, R. Wisbauer, Monads and comonads on module categories, *J. Algebra*, **322** (2009), 1719–1747. https://doi.org/10.1016/j.jalgebra.2009.06.003
23. X. L. Fang, Gauge transformations for quasitriangular quasi-Turaev group coalgebras, *J. Algebra Appl.*, **17** (2018), 1850080. https://doi.org/10.1142/S0219498818500809
24. S. H. Wang, A. Van Daele, Y. H. Zhang, Constructing quasitriangular multiplier Hopf algebras by twisted tensor coproducts, *Comm. Algebra*, **37** (2009), 3171–3199. https://doi.org/10.1080/00927870902747894