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Brauer configuration algebras defined by snake graphs and Kronecker modules

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Abstract: Recently, Çanakçi and Schroll proved that associated with a string module $M(w)$ there is an appropriated snake graph $\mathscr{G}$. They established a bijection between the corresponding perfect matching lattice $\mathscr{L}(\mathscr{G})$ of $\mathscr{G}$ and the canonical submodule lattice $\mathscr{L}(M(w))$ of $M(w)$. We introduce Brauer configurations whose polygons are defined by snake graphs in line with these results. The developed techniques allow defining snake graphs, which after suitable procedures, build Kronecker modules. We compute the dimension of the Brauer configuration algebras and their centers arising from the different processes. As an application, we estimate the trace norm of the canonical non-regular Kronecker modules and some families of trees associated with some snake graphs classes.

Keywords: Auslander-Reiten quiver; Brauer configuration algebra; Kronecker algebra; Snake graph; trace norm

1. Introduction

In the last few years, it has been proved that Brauer configuration algebras (BCAs) are a helpful tool in different fields of applied mathematics [1–3]. In particular, they were used in cryptography and the theory of graph energy. Green and Schroll introduced such algebras as a generalization of Brauer graph algebras. Bearing in mind that any Brauer graph algebra is a Brauer configuration algebra. Perhaps, the main characteristic of BCAs is that their theory of representation is based on combinatorial data [4, 5].

Snake graphs is another helpful combinatorial tool for a better understanding of the theory of representation of some algebras with applications in number theory. In particular, in the theory of continued fractions. Such graphs were studied by Propp [6] in the context of the investigation of the Laurent phenomenon associated with cluster algebras.

Çanakçi and Schiffler developed a complete theory for snake graphs, which were used to compute the Laurent expansions of the cluster variables in cluster algebras of surface type [7–11]. Perfect matchings of suitable snake graphs parametrize terms in the Laurent polynomial of such variables. Çanakçi and Schiffler proved that each snake graph $\mathcal{G}$ defines a unique continued fraction whose numerator is given by the number of perfect matchings of a suitable snake graph.

Regarding applications of the snake graph theory, we recall that recently Çanakçi and Schroll [12] defined abstract string modules associating to each of such modules a suitable snake graph, whose lattice of perfect matchings is in bijective correspondence with the lattice of submodules of such abstract module. Conversely, they proved that each snake graph defines a string module given by an orientation of Dynkin type $\mathbb{A}$, where every vertex is replaced by a copy of an algebraically closed field $\mathbb{F}$, and the arrows correspond to the identity map.

We establish interesting interactions between snake graph theory, matrix problems and the algebra representation theory, in line with Çanakçi and Schroll's work. We prove in this paper that some suitable snake graphs can be used to build non-regular Kronecker modules, recall that if $\mathbb{F}$ is a field, and $\Lambda = \begin{pmatrix} \mathbb{F} & \mathbb{F}^2 \\ 0 & \mathbb{F} \end{pmatrix}$ is the Kronecker algebra then the finite dimensional right Λ-modules are said to be *Kronecker modules*. It is worth noticing that the category of Kronecker modules is equivalent to the category of pairs (A, B) of matrices A, B over $\mathbb{F}$ of the same size. We also define Brauer configuration algebras, for which it is possible associating a string snake graph or a subset of the set of vertices of the preprojective (or preinjective) component of the Auslander-Reiten quiver of the Kronecker algebra Λ. Dimensions of these Brauer configuration algebras are also given in this work.

Figure 1 shows how Brauer configuration algebras, snake graphs, and solutions of the Kronecker problem are related to the main results presented in this paper.

This paper is distributed as follows; in Section 2, we recall definitions and notation used throughout the document. In particular, we recall notions of Brauer configuration algebra (2.1), the Kronecker problem (2.3) and snake graph (2.4). In Section 3, we make an overview of Çanakçi and Schroll's work regarding interactions between snake graphs and string modules. In Section 4, we give our main results. We prove that some non-regular Kronecker modules can be built with some suitable snake graphs. We introduce Brauer configuration algebras associated with snake graphs and estimate the trace norm of some Kronecker trees and non-regular Kronecker modules. Concluding remarks are given in Section 5.

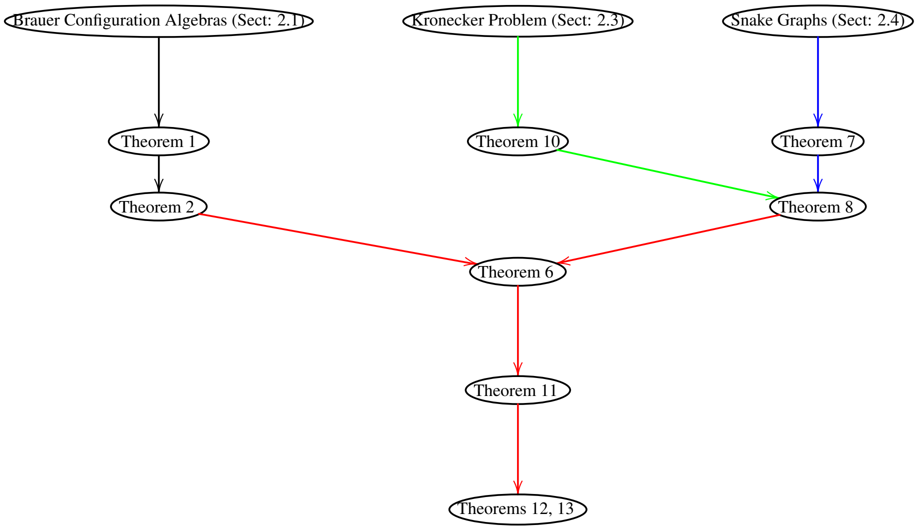


Figure 1. Brauer configuration algebras, the Kronecker problem, and the snake graph theory are related via red arrows. Theorem 6 gives a way of building snake graphs with some suitable Brauer configuration algebras. Theorems 7 and 8 prove that preprojective Kronecker modules (of type II), and preinjective Kronecker modules (of type III), arise from some suitable snake graphs. Theorem 11 gives the structure of a Brauer configuration algebra induced by preprojective Kronecker modules. Theorem 12 provides the trace norm of some Kronecker trees used to enumerate helices, also named Kronecker snake graphs (see Theorem 10). Theorem 13 gives the trace norm of some non-regular Kronecker modules.

2. Background and related work

In this section, we introduce some definitions and notations to be used throughout the paper. In particular, it is given a brief overview regarding Brauer configuration algebras, and snake graph theory.

2.1. Brauer configuration algebras

Green and Schroll introduced Brauer configuration algebras as a generalization of Brauer graph algebras [4, 5]. Its definition goes as follows:

A Brauer configuration algebra Λ_Γ (or simply Λ if no confusion arises) is a bound quiver algebra induced by a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, O)$, where:

- Γ_0 is a finite set of vertices.
- Γ_1 is a collection of polygons, which are labeled multisets consisting of vertices (vertices repetition allowed). Each polygon contains more than one vertex.
- μ is a map from the set of vertices Γ_0 to the set of positive integers $\mathbb{N}\backslash\{0\} = \mathbb{N}^{+}$, $\mu : \Gamma_0 \to \mathbb{N}^{+}$.
- O is a choice for each vertex $\alpha \in \Gamma_0$, of a cyclic ordering of the polygons in which α occurs as a vertex including repetitions (see [4] for more details). For instance, if a vertex $\alpha \in \Gamma_0$ occurs in

polygons $U_{i_i}, U_{i_2}, \ldots, U_{i_m}$, for suitable indices $i_1, i_2, \ldots, i_m \in \{1, 2, 3, \ldots, n\}$, then the cyclic order is obtained by linearly ordering the list, say

$$U_{i_1}^{\alpha_1} < U_{i_2}^{\alpha_2} < \cdots < U_{i_m}^{\alpha_m}, \quad \alpha_{i_s} > 0. \tag{2.1}$$

Where, $U_{i_s}^{\alpha_s} = U_{i_s}^{(1)} < U_{i_s}^{(2)} < \cdots < U_{i_s}^{(\alpha_s)}$ means that vertex α occurs α_s times in polygon U_{i_s}, denoted $\alpha_s = occ(\alpha, U_{i_s})$. The cyclic order is completed by adding the relation $U_{i_m} < U_{i_1}$. Note that, if $U_{i_1} < \cdots < U_{i_t}$ is the chosen ordering at vertex α. Then the same ordering can be represented by any cyclic permutation.
The sequence (2.1) is said to be the successor sequence at vertex α denoted S_α, which is unique up to permutations. Note that, Green and Schroll [4] mentioned that different orientations choice are typically associated to non-isomorphic Brauer configuration algebras.
Henceforth, in this paper, if a vertex $\alpha' \neq \alpha$ belongs to some polygons $U_{j_1}, U_{j_2}, \ldots U_{j_k}$ ordered according to the already defined cyclic ordering associated with the vertex α. Then, we will assume that up to permutations, the cyclic ordering associated with the vertex α' is built, taking into account that polygons $U_{j_1}, U_{j_2}, \ldots U_{j_k}$ inherit the order given by the successor sequence S_α.
- If $\alpha \in \Gamma_0$ then there is at least one polygon U_i such that $\alpha \in U_i$.

If $\alpha \in \Gamma_0$ then the valency $val(\alpha)$ of α is given by the identity

$$val(\alpha) = \sum_{U \in \Gamma_1} occ(\alpha, U). \tag{2.2}$$

If $\alpha \in \Gamma_0$ is such that $\mu(\alpha)val(\alpha) = 1$ then α is said to be *truncated* (it occurs once in just one polygon). Otherwise α is a non-truncated vertex. It is worth pointing out that each polygon in a Brauer configuration has at least one non-truncated vertex. A Brauer configuration without truncated vertices is said to be *reduced*.

Latter on, we will assume that successor sequences associated with non-truncated vertices are of the form (2.1). As Green and Schroll mentioned in [4], if α is a non-truncated vertex and $val(\alpha) = 1$. Then, there is only one choice for the associated cyclic ordering.

In [3], Cañadas et al. introduced Algorithm 1 to build the Brauer quiver Q_Γ and the Brauer configuration algebra $\Lambda_\Gamma = \mathbb{F}Q_\Gamma/I_\Gamma$ induced by a Brauer configuration Γ, where I_Γ is an admissible ideal generated by suitable relations associated with the vertices occurrences.

From now on, if no confusion arises, we will assume notations Q, I and Λ instead of Q_Γ, I_Γ and Λ_Γ, for a quiver, an admissible ideal, and the Brauer configuration algebra induced by a fixed Brauer configuration Γ.

Since polygons in Brauer configurations are multisets, we will often assume that such polygons are given by words of the form

$$w = x_1^{s_1} x_2^{s_2} \ldots x_{t-1}^{s_{t-1}} x_t^{s_t} \tag{2.3}$$

Where for each i, $1 \leq i \leq t$, x_i is an element of the polygon called vertex and s_i is the number of times that the vertex x_i occurs in the polygon [13]. In particular, if vertices x_i in a polygon V of a Brauer configuration are integer numbers then the corresponding word w will be interpreted as a partition of

Algorithm 1: Construction of a Brauer configuration algebra

1. **Input** A reduced Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, O)$.
2. **Output** The Brauer configuration algebra $\Lambda_\Gamma = \mathbb{F}Q_\Gamma/I_\Gamma$
3. Construct the quiver $Q_\Gamma = ((Q_\Gamma)_0, (Q_\Gamma)_1, s : (Q_\Gamma)_1 \to (Q_\Gamma)_0, t : (Q_\Gamma)_1 \to (Q_\Gamma)_0)$
   (a) $(Q_\Gamma)_0 = \Gamma_1$,
   (b) For each cover $U_i < U_{i+1} \in \Gamma_1$ define an arrow $a \in (Q_\Gamma)_1$, such that $s(a) = U_i$ and $t(a) = U_{i+1}$,
   (c) Each relation $U_i < U_i$ defines a loop in Q_Γ,
   (d) Each ordered set S_α defines a cycle $C_\alpha = S_\alpha \cup \{U_{i_m} < U_{i_1}\}$ in Q_Γ called *special cycle*. Special cycles are obtained from successor sequences by defining a suitable circular relation, without loss of generality, we assume that a relation of the form $U_{i_m} < U_{i_1}$ holds in special cycles.
4. Define the path algebra $\mathbb{F}Q_\Gamma$,
5. Construct the admissible ideal I_Γ, which is generated by the following relations:
   (a) If $\alpha_i, \alpha_j \in U$, $U \in \Gamma_1$ and $C_{\alpha_i}, C_{\alpha_j}$ are corresponding special cycles then $C_{\alpha_i}^{\mu(\alpha_i)} - C_{\alpha_j}^{\mu(\alpha_j)} = 0$,
   (b) If C_{α_i} is a special cycle associated to the vertex α_i then $C^{\mu(\alpha_i)}a = 0$, if a is the first arrow of C_{α_i},
   (c) If $\alpha, \alpha' \in \Gamma_0$, $\alpha \neq \alpha'$, $a, b \in (Q_\Gamma)_1$, $a \neq b$, $ab \notin C_\alpha$ for any $\alpha \in \Gamma_0$ then $ab = 0$, if $a \in C_\alpha$, $b \in C_{\alpha'}$ and $ab \in \mathbb{F}Q_\Gamma$,
   (d) If a is a loop associated to a vertex α with $val(\alpha) = 1$ and $\mu(\alpha) > 1$ then $a^{\mu(\alpha)+1} = 0$.
6. $\Lambda_\Gamma = \mathbb{F}Q_\Gamma/I_\Gamma$ is the Brauer configuration algebra.
7. For the construction of a basis of Λ_Γ follow the next steps:
   (a) For each $U \in \Gamma_1$ choose a non-truncated vertex α_U and exactly one special α-cycle C_{α_U} at U,
   (b) Define:

$$A = \{\overline{p} \mid p \textit{ is a proper prefix of some } C_\alpha^{\mu(\alpha)}\},$$
$$B = \{\overline{C_{\alpha_U}^{\mu(\alpha)}} \mid U \in \Gamma_1\}.$$

   (c) $A \cup B$ is a $\mathbb{F}$-basis of Λ_Γ.

an integer number n_V associated with the polygon V where it is assumed that each vertex x_i is a part of the partition and s_i is the number of times that the part x_i occurs in the partition and $n_V = \sum_{i=1}^{t} s_i x_i$. If the order of the letters (or parts) x_i matters in the construction of a word (partition) $w\,(n_V)$ then we will say that $w\,(n_V)$ is a *composition*. Note that, in this case, each ordering of the letters define a polygon. We let $\mathcal{N}(w)$ denote the set of compositions associated with a word w. $\mathcal{N}(w)$ is endowed with an injective numbering $f_w : \mathcal{N}(w) \longrightarrow \mathcal{N}(\mathbf{w})$. If c is a composition. Then, $f_w(c) = j \in \mathcal{N}(\mathbf{w}) = \{1, 2, \ldots, |\mathcal{N}(w)|\}$. $\mathcal{N}(\mathbf{w})$ is endowed with the usual order of natural numbers. In such a case, if $w', w'' \in \mathcal{N}(w)$ and $f_w(w') < f_w(w'')$ is a covering, then we will assume that as polygons $w' < w''$ is a covering in the corresponding successor sequences.

The following results describe the structure of Brauer configuration algebras [4, 14].

Theorem 1 (Theorem B, Proposition 2.7, Theorem 3.10, Corollary 3.12, [4])**.** *Let Λ be a Brauer configuration algebra with Brauer configuration Γ.*

1) *There is a bijective correspondence between the set of indecomposable projective Λ-modules and the polygons in Γ.*
2) *If P is an indecomposable projective Λ-module corresponding to a polygon V in Γ. Then* rad *P is a sum of r indecomposable uniserial modules, where r is the number of (non-truncated) vertices of V and where the intersection of any two of the uniserial modules is a simple Λ-module.*
3) *A Brauer configuration algebra is a multiserial algebra.*
4) *The number of summands in the heart $ht(P) = \operatorname{rad} P/\operatorname{soc} P$ of an indecomposable projective Λ-module P such that $\operatorname{rad}^2 P \neq 0$ equals the number of non-truncated vertices of the polygons in Γ corresponding to P counting repetitions.*
5) *If Λ' is a Brauer configuration algebra obtained from Λ by removing a truncated vertex of a polygon in Γ_1 with $d \geq 3$ vertices then Λ is isomorphic to Λ'.*

Proposition 1 and Theorem 2 give formulas for the dimensions $\dim_{\mathbb{F}} \Lambda$, and $\dim_{\mathbb{F}} Z(\Lambda)$ of a Brauer configuration algebra Λ and its center $Z(\Lambda)$ [4, 14].

Proposition 1 (Proposition 3.13, [4])**.** *Let Λ be a Brauer configuration algebra associated with the Brauer configuration Γ and let $C = \{C_1, \ldots, C_t\}$ be a full set of equivalence class representatives of special cycles. Assume that for $i = 1, \ldots, t$, C_i is a special α_i-cycle where α_i is a non-truncated vertex in Γ. Then*

$$\dim_{\mathbb{F}} \Lambda = 2|Q_0| + \sum_{C_i \in C} |C_i|(n_i|C_i| - 1),$$

where $|Q_0|$ denotes the number of vertices of Q, $|C_i|$ denotes the number of arrows in the α_i-cycle C_i and $n_i = \mu(\alpha_i)$.

Theorem 2 (Theorem 4.9, [14])**.** *Let $\Lambda = \mathbb{F}Q/I$ be the Brauer configuration algebra associated to the connected and reduced Brauer configuration Γ. Then*

$$\dim_{\mathbb{F}} Z(\Lambda) = 1 + \sum_{\alpha \in \Gamma_0} \mu(\alpha) + |\Gamma_1| - |\Gamma_0| + \#(Loops\ Q) - |\mathscr{C}_\Gamma|,$$

where $\mathscr{C}_\Gamma = \{\alpha \in \Gamma_0 \mid val(\alpha) = 1,\ and\ \mu(\alpha) > 1\}$.

In this case, rad M denotes the radical of a module M, rad M is the intersection of all the maximal submodules of M.

2.2. The message of a Brauer configuration

The notion of the message of a Brauer configuration and labeled Brauer configurations were introduced by Espinosa et al. [1, 15] to define suitable specializations of some Brauer configurations. According to them, since polygons in a Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ are multisets, it is possible to assume that any polygon $U \in \Gamma_1$ is given by a word $w(U)$ of the form

$$w(U) = \alpha_1^{s_1}\alpha_2^{s_2}\dots\alpha_{t-1}^{s_{t-1}}\alpha_t^{s_t} \tag{2.4}$$

Where for each i, $1 \le i \le t$, $s_i = \mathrm{occ}(\alpha_i, U)$.

The message is in fact an algebra of words element $\mathscr{W}_\Gamma$ associated with a fixed Brauer configuration such that for a given field $\mathbb{F}$ the word algebra $\mathscr{W}_\Gamma$ consists of formal sums of words with the form $\sum_{\substack{\alpha_i\in\mathbb{F}\\ U\in\Gamma_1}} \alpha_i w(U)$, $0w(U) = \varepsilon$ is the empty word, and $1w(U) = w(U)$ for any $U \in \Gamma_1$. The product in this case is given by the usual word concatenation. The formal product (or word product)

$$M(\Gamma) = \prod_{U\in\Gamma_1} w(U) \tag{2.5}$$

is said to be the *message of the Brauer configuration* Γ.

The notion of labeled Brauer configurations is helpful to define suitable specializations of some Brauer configuration algebras [15].

An *integer specialization* of a reduced Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1, \mu, \mathcal{O})$ is a Brauer configuration $\Gamma^e = (\Gamma_0^e, \Gamma_1^e, \mu^e, \mathcal{O}^e)$ endowed with a suitable map $e : \Gamma_0 \to \mathbb{N}$, such that

$$\begin{aligned}
\Gamma_0^e &= \mathrm{Img}\, e \subset \mathbb{N},\\
\Gamma_1^e &= e(\Gamma_1) = \{e(H) \mid H \in \Gamma_1\}, \quad \text{if } H \in \Gamma_1 \text{ then } e(H) = \{e(\alpha_i) \mid \alpha_i \in H\} \in e(\Gamma_1),\\
w^e(U) &= ((e(\alpha_1))^{f_1}(e(\alpha_2))^{f_2}\dots(e(\alpha_n))^{f_n}), \text{is the specialization under } e \text{ of a word}\\
w(U) &= \alpha_1^{f_1}\alpha_2^{f_2}\dots\alpha_n^{f_n} \text{ associated with a polygon } U \in \Gamma_1,\\
\mu^e(e(\alpha)) &= \mu(\alpha), \text{ for any vertex } \alpha \in \Gamma_0.
\end{aligned} \tag{2.6}$$

The orientation $\mathcal{O}^e$ is defined by the orientation $\mathcal{O}$, in such a way that if

$$S_\alpha = U_{i_1}^{\alpha_1} < U_{i_2}^{\alpha_2} < \dots < U_{i_m}^{\alpha_m}$$

is a successor sequence associated with a vertex $\alpha \in \Gamma_0$ (see, (2.1)). Then,

$$S'_{e(\alpha)} = (e(U_{i_1}))^{\alpha_1} < (e(U_{i_2}))^{\alpha_2} < \dots < (e(U_{i_m}))^{\alpha_m}$$

is contained in the successor sequence $S_{e(\alpha)}$ associated with $e(\alpha) \in \Gamma_0^e$.

$M(\Gamma^e) = \sum_{U\in\Gamma_1^e} w^e(U)$ is the *specialized message* of the Brauer configuration Γ.

A Brauer configuration $\Gamma = (\Gamma_0, \Gamma_1 = \{U_{i_1}, \dots, U_{i_m}\}, \mu, \mathcal{O})$ is said to be *S-labeled* (or simply labeled, if no confusion arises) by an integer sequence $S = \{n_1, n_2, \dots, n_{|\Gamma_1|}\}$ if each polygon U_{i_j} is labeled by an integer number n_j, $1 \le j \le |\Gamma_1|$. In such a case we often write

$\Gamma_1 = \{(U_{i_1}, n_1), (U_{i_2}, n_2), \ldots, (U_{i_m}, n_m)\}$,

For each vertex $\alpha \in \Gamma_0$, it is defined a corresponding cyclic ordering of labeled polygons where α occurs. One advantage of labeling Brauer configurations is that the set S can be used to systematically define the orientation associated with each vertex or obtain the polygons recursively [15].

It is worth noticing that the set S used to label a Brauer configuration can be any finite set. In this paper, we also use finite well-ordered sets of matrices to label Brauer configurations.

As an example, we recall that Espinosa [15] defined the following labeled Brauer configuration $\mathscr{K} = (\mathscr{K}_0, \mathscr{K}_1, \mu, O)$, where:

$$\begin{aligned} \mathscr{K}_0 &= \{\alpha_1^h, \alpha_2^h, \ldots, \alpha_k^h \mid 1 \leq h \leq k\}, \\ \mathscr{K}_1 &= \{(U_1, n_1), (U_2, n_2), \ldots, (U_k, n_k)) \mid \alpha_w^i \in (U_i, n_i), n_i \geq 2\}. \end{aligned} \tag{2.7}$$

Vertices $\alpha_w^i \in (U_i, n_i) \in \mathscr{K}_1$ are given by the following formula

$$\alpha_w^i = n_i - g(w_{i-1}, i) - g(w_i, i) - 2, \tag{2.8}$$

where for a given vector $w = (w_1, w_2, \ldots, w_k) \in \{0, 1\}^k$, $w_i \in \{0, 1\}$, g is a map $g := \{0, 1\} \times \mathbb{Z}^+ \to \{1, 2\}$ defined by

$$g(0, i) = \begin{cases} 2, & \text{if } i \text{ is even;} \\ 1, & \text{if } i \text{ is odd;} \end{cases} \quad \text{and} \quad g(1, i) = \begin{cases} 1, & \text{if } i \text{ is even;} \\ 2, & \text{if } i \text{ is odd;} \end{cases}.$$

In particular, $g(w_0, 1) = g(w_k, k) = 0$. The definition of g can be reformulated by the rule $g(x, n) = 2 - (x + n \pmod 2)$.

$\mu(\alpha) = 2$, for any vertex $\alpha \in \Gamma_0$.

Successor sequences are defined by relations of the form $(U_i, n_i) < (U_s, n_s)$, if $i < s$ (see (2.1)).

Suitable specializations of $\mathscr{K}$ are used in Theorem 5 to describe the number of perfect matchings of some snake graphs.

2.3. The Kronecker problem

The Kronecker matrix problem consists of finding canonical Jordan form of pairs (A, B) of matrices (with the same size) with respect to the following elementary transformations:

(i) All elementary transformations on rows of the block matrix (A, B).
(ii) All elementary transformations made simultaneously on columns of A and B having the same index number.

Weierstrass solved this problem in 1867 for some particular cases, whereas Kronecker in 1890 solved the complex number field case. We recall that solutions of the Kronecker problem allow solving systems of linear differential equations of the form $Ax'(t) + Bx(t) = f(t)$.

Nowadays, it is known that if $\mathbb{F}$ is an algebraically closed field, then up to isomorphism every indecomposable Kronecker pair (A, B) belongs to one of the following four classes shown in Figures 2–4 [16].

$$0:\quad \begin{array}{|c|c|}\hline \mathrm{I}_n & C(p) \\ \hline \end{array}\,,$$

Figure 2. Regular Kronecker modules of type 0. $C(p)$ is a Frobenius matrix or companion matrix of a minimal polynomial $p^s(t)$, with $n = s\partial p(t)$, $\partial p(t)$ denotes the degree of the polynomial $p(t)$.

$$\mathrm{I} = \mathrm{I}^*:\quad (a)\ \begin{array}{|c|c|}\hline \mathrm{I}_n & \mathrm{J}_n(0) \\ \hline \end{array}\,,$$

$$(b)\ \begin{array}{|c|c|}\hline \mathrm{J}_n(0) & \mathrm{I}_n \\ \hline \end{array}\,,$$

Figure 3. Regular Kronecker modules of type I. $J_n(0) \in \{J_n^+(0), J_n^-(0)\}$ and $J_n^{\pm}(0)$ denotes a corresponding upper or lower Jordan block. Whereas, I^* denotes the dual case defined by the classification problem.

$$\mathrm{II} = \mathrm{III}^*:\quad \begin{array}{|c|c|}\hline \overrightarrow{\mathrm{I}}_n & \overleftarrow{\mathrm{I}}_n \\ \hline \end{array}\,,$$

$$\mathrm{III} = \mathrm{II}^*:\quad \begin{array}{|c|c|}\hline \mathrm{I}_n^{\uparrow} & \mathrm{I}_n^{\downarrow} \\ \hline \end{array}\,.$$

Figure 4. Non-regular Kronecker modules. $\overrightarrow{\mathrm{I}}_n$ ($\overleftarrow{\mathrm{I}}_n$, respectively) denotes an $n \times (n+1)$ matrix obtained from the identity I_n by adding a column of zeroes in fact the last column (the first column, respectively) in these matrices consists only of zeroes. $\mathrm{I}_n^{\uparrow}$ ($\mathrm{I}_n^{\downarrow}$, respectively) denotes an $(n+1) \times n$ matrix obtained from the identity I_n by adding a row of zeroes.

Solving the Kronecker problem is equivalent to give a complete classification of the $\mathbb{F}Q$-modules over the path algebra $\mathbb{F}Q$, where Q is the quiver shown in Figure 5.

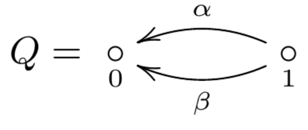


Figure 5. 2-Kronecker quiver.

Figure 6 shows the preprojective component of the Auslander-Reiten quiver of the 2-Kronecker quiver Q, which has as vertices indecomposable representations of type III. The preinjective component has indecomposable representations of type III^* as vertices.

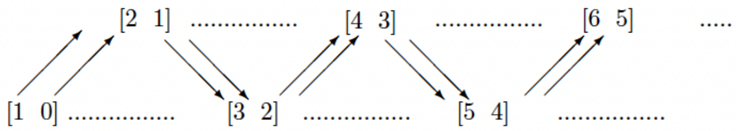


Figure 6. Preprojective component of the Auslander-Reiten quiver associated with the 2-Kronecker quiver.

Henceforth, we let $(n+1, n)$ ($(n, n+1)$) denote a representative of the indecomposable preprojective (preinjective) Kronecker modules class obtained from a representation of type III (II) via elementary transformations of type (ii). Several invariants can be defined for Kronecker modules, for instance, the Frobenius cells F_n are invariants for indecomposable Kronecker modules of type 0, whereas Jordan blocks $J_n(0)$ are invariants for indecomposable Kronecker modules of type I.

This paper proves that non-regular Kronecker modules can be defined by using some suitable snake graphs. So, it is possible to argue that each non-regular Kronecker module has associated a set of snake graphs whose cardinality is invariant under admissible transformations.

2.4. Snake graphs

A *tile* G is a square in the plane whose sides are parallel or orthogonal to the elements in the standard orthonormal basis of the plane (as in [10] in this work a tile G is considered as a graph with four vertices and four edges in the obvious way).

A *snake graph* $\mathcal{G}$ is a connected planar graph consisting of a finite sequence of tiles $G_1, G_2, \ldots, G_d$, such that G_i and G_{i+1} share exactly one edge e_i and this edge is either the north edge of G_i and the south edge of G_{i+1} or the east edge of G_i and the west edge of G_{i+1}, (cf. [7–11]). Denote by $\text{Int}(\mathcal{G}) = \{e_1, e_2, \ldots, e_{d-1}\}$ the set of interior edges of the snake graph $\mathcal{G}$. We will use the natural ordering of the set of interior edges of $\mathcal{G}$, so that e_i is the edge shared by tiles G_i and G_{i+1}. A snake graph is called *straight* if all its tiles lie in one column or one row, and a snake graph is called *zigzag* if no three consecutive tiles are straight. Two snake graphs are *isomorphic* if they are isomorphic as graphs.

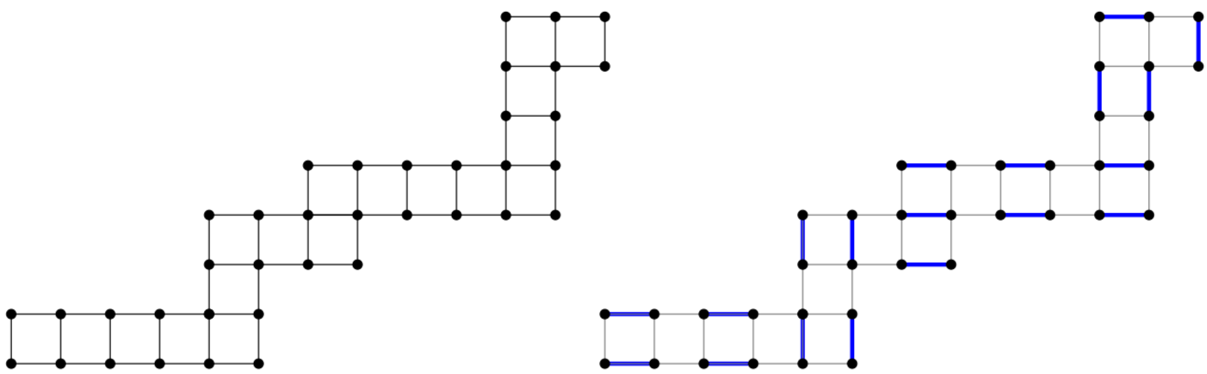


Figure 7. Example of a snake graph and one of its perfect matchings.

For positive integers $n_1, n_2, \ldots, n_k$, we let $\mathscr{G}_f(n_1, n_2, \ldots, n_k)$ denote a snake graph, with $n_1 \geq 2$ tiles in the first row, $n_2 \geq 2$ in the first column, $n_3 \geq 2$ tiles in the second row and so on up to $n_k \geq 2$. In this case the last tile in a given row is the first tile in the next column (if it exists) vice versa the last tile in a given column coincides with the first tile in the next row. As an example, in Figure 7, it is shown the snake graph $\mathscr{G}_f(5, 3, 3, 2, 5, 4, 2)$.

A *perfect matching* P of a graph G is a subset of the edges of G such that every vertex of G is incident to exactly one edge in P. We denote by Match(G) the set of perfect matchings of G.

A *sign function* f of a snake graph $\mathscr{G}$ is a map f from the set of edges of $\mathscr{G}$ to the set of signs $\{+, -\}$, such that on every tile in $\mathscr{G}$ the north and the west edge have the same sign, the south and the east have the same sign, and the sign on the north edge is opposite to the sign on the south edge. For example, in Figure 8, we show a labeling of the snake graph $\mathscr{G}_f(5, 3, 3, 2, 5, 4, 2)$.

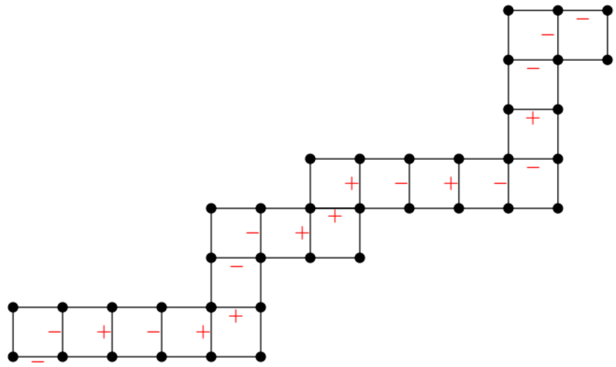


Figure 8. Snake graph $\mathscr{G}_f(5, 3, 3, 2, 5, 4, 2) = \mathscr{G}[2, 1, 1, 2, 2, 3, 1, 1, 2, 1, 3]$. The first notation (second notation) is adopted according to the number of tiles in each contained straight snake graph (signs associated with internal tiles) of the graph $\mathscr{G}$.

Note that, on every snake graph there are exactly two sign functions. A snake graph is determined up to symmetry by its sequence of tiles together with a sign function on its interior edges $\{e_1, e_2, \ldots, e_{d-1}\}$. Henceforth, it will be assumed the notation $e_0 = sw(\mathscr{G})$ (the edge at the southwest of the first tile).

If $e_d \in ne(\mathscr{G})$ (the edge at the northeast of the last tile) then sign function can be extended in a unique way to all edges in $\mathscr{G}$ and it is obtained a sign sequence

$$sgn(\mathscr{G}) = \{f(e_0), f(e_1), f(e_2), \ldots, f(e_{d-1}), f(e_d)\}$$

Actually this sequence uniquely determines the snake graph and a choice of a north east edge $e_d \in ne(\mathscr{G})$.

A positive finite *continued fraction* is a function

$$[a_1, a_2, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}} \tag{2.9}$$

on n variables $a_1, a_2, \ldots, a_n$, $a_i \in \mathbb{Z}_{\geq 1}$. Now let $[a_1, a_2, \ldots, a_n]$ be a positive continued fraction and let $d = a_1 + a_2 + \cdots + a_n - 1$ and consider the sign sequence:

$$\underbrace{(-\varepsilon,\dots,-\varepsilon}_{a_1},\underbrace{\varepsilon,\dots,\varepsilon}_{a_2},\dots,\underbrace{\pm\varepsilon,\dots,\pm\varepsilon}_{a_n}), \tag{2.10}$$

where $\varepsilon \in \{+,-\}$,

$$-\varepsilon = \begin{cases} +, & \text{if } \varepsilon = -; \\ -, & \text{if } \varepsilon = +; \end{cases}$$

$$sgn(a_i) = \begin{cases} -\varepsilon, & \text{if } i \text{ is odd;} \\ \varepsilon, & \text{if } i \text{ is even;} \end{cases}$$

Thus each integer a_i corresponds to a maximal subsequence of constant sign $sgn(a_i)$ in the sequence (2.10).

The snake graph $\mathscr{G}[a_1, a_2, \dots, a_n]$ of the positive continued fraction $[a_1, a_2, \dots, a_n]$ is the snake graph with d tiles determined by the sign sequence (2.10). In particular, $\mathscr{G}[1]$ is a single edge and the continued fraction of the graph in Figure 8 is $[2, 1, 1, 2, 2, 3, 1, 1, 2, 1, 3]$.

Çanakçi and Schiffler proved Theorems 3 and 4 regarding snake graphs and their relationships with continued fractions [7–11].

Theorem 3. *The number of snake graphs with exactly N perfect matchings is $\phi(N)$, where ϕ is the totient Euler function.*

Theorem 4. *1) The number of perfect matchings of $\mathscr{G}[a_1, a_2, \dots, a_n]$ is equal to the numerator of the continued fraction $[a_1, a_2, \dots, a_n]$.*

2) The number of perfect matchings of $\mathscr{G}[a_2, a_3, \dots, a_n]$ is equal to the denominator of the continued fraction $[a_1, a_2, \dots, a_n]$.

3) If Match$(\mathscr{G})$ *denotes the number of perfect matchings of the snake graph $\mathscr{G}$ then* $[a_1, a_2, \dots, a_n] = \frac{\mathrm{Match}(\mathscr{G})[a_1,a_2,\dots,a_n]}{\mathrm{Match}(\mathscr{G})[a_2,a_3,\dots,a_n]}$.

For instance, the snake graph $\mathscr{G}[2, 1, 1, 2, 2, 3, 1, 1, 2, 1, 3]$ shown in Figure 8 has 3221 perfect matchings.

The following theorem proves that the number of perfect matchings of a snake graph is given by a message specialization of the Brauer configuration $\mathscr{K}$ [15] (see, formulas (2.7)).

Theorem 5 (Theorem 14, [15])**.** *For all $n_1, n_2 \dots, n_k \geq 2$, we have*

$$\mathrm{Match}(\mathscr{G}_f(n_1, n_2, \dots, n_k)) = M(\mathscr{K}^e),$$

where $e : \mathscr{K}_0 \to \mathbb{N}$ is a specialization of the Brauer configuration given by identities (2.7) *and* (2.8) *such that $e(\alpha^i_{w_i}) = F_{\alpha^i_{w_i}}$ with F_j being the jth Fibonacci number.*

3. String modules and snake graphs

An abstract string is a word of the form $w = a_1a_2 \dots a_n$, with $a_j \in \{\to, \leftarrow\}$, $1 \leq j \leq n$. $\varnothing$ is also considered an abstract string. If $a_j = \to (\leftarrow)$, for any j then w is said to be a *direct string* (*inverse string*).

According to Çanakçi and Schroll [12], the following procedure allows building a snake graph with $n + 1$ tiles from an abstract string $w = a_1a_2 \dots a_n$:

1) If $w = \varnothing$ then the corresponding abstract snake graph is given by a single tile.
2) If there is at least one letter, then $a_1, a_2, \ldots, a_n$ is a concatenation of a collection of alternating maximal direct and inverse strings w_i such that $w = w_1 w_2 \ldots w_k$. Each w_i might be of length 1.
3) For each w_i, it is constructed a zigzag snake graph $\mathscr{G}_i$ with $l(w_i)+1$ tiles. where $l(w_i)$ is the number of direct or inverse arrows in w_i. Let $\mathscr{G}_i$ be the zigzag snake graph with tiles $T^i_1, \ldots, T^i_{l(w_i)+1}$, such that T^i_2 is glued to the right (resp. on top) of T^i_1 if w_i is direct (resp. inverse).
4) We now glue $\mathscr{G}_{i+1}$ to $\mathscr{G}_i$, for all i, by identifying the last tile $T^i_{l(w_i)+1}$ of $\mathscr{G}_i$ and the first tile T^{i+1}_1 of $\mathscr{G}_{i+1}$, such that, $T^i_{l(w_i)}$, $T^i_{l(w_i)+1}$, T^{i+1}_2 is a straight piece.

$$w = \; 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longleftarrow 5$$

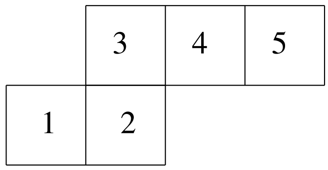


Figure 9. Example of the snake graph associated with a 4-arrow string. The string module $M(w)$ over the corresponding Dynkin algebra of type $\mathbb{A}$ is obtained replacing every vertex by a copy of a field $\mathbb{F}$ and the arrows correspond to the identity.

Çanakçi and Schroll proved that if $A = \mathbb{F}Q/I$ is a bound quiver algebra and $M(w)$ is a string module over A with string w and with associated snake graph $\mathscr{G}$. Then the perfect matching lattice $\mathscr{L}(\mathscr{G})$ of $\mathscr{G}$ is in bijection with the canonical submodule lattice $\mathscr{L}(M(w))$ of $M(w)$.

4. Main results

Results in this section allow establishing interactions between Brauer configuration algebras, Kronecker modules and snake graphs. We start by defining a labeled Brauer configuration whose polygons have associated a unique snake graph as Çanakçi and Schroll describe in [12].

4.1. Interactions between Brauer configuration algebras and snake graphs

We let $\mathcal{N}(w)$ denote the set of all possible compositions associated with a word $w = x_1^{s_1} x_2^{s_2} \ldots x_{t-1}^{s_{t-1}} x_t^{s_t}$ (see (2.4)).

For $n \geq 3$ fixed. We denote by $\mathfrak{S}^n$, the labeled Brauer configuration $(\mathfrak{S}^n_0, \mathfrak{S}^n_1, \mu, O)$, such that

$$\begin{aligned} \mathfrak{S}^n_0 &= \{0, 1\}, \\ \mathfrak{S}^n_1 &= \{(U_i, j) \mid w(U_i) \in \mathcal{N}(w), w = 0^{(i)} 1^{(n-i)}, 0 < i \leq n, j = f_w(w(U_i))\}, \\ \mu(0) &= \mu(1) = 1. \end{aligned} \tag{4.1}$$

Note that a labeled polygon (U_i, j) contains i zeroes and $n - i$, 1's. Successor sequences associated with vertices 0 and 1 are defined by the following relations:

$$
\begin{aligned}
&(U_i, j) < (U_i, k), \ \text{ if } \ j < k,\\
&(U_i, j) < (U_{i+1}, k), \ \text{ for any } \ j, k,\\
&(U_i, M_i) < (U_{i+1}, m_{i+1}) \ \text{ is a covering if } \ M_i = \mathrm{Max}\{f_w(w(U_i)) \mid w = 0^{(i)}1^{(n-i)}\}, \ \text{and}\\
&m_{i+1} = \mathrm{Min}\{f_w(w(U_{i+1})) \mid w = 0^{(i+1)}1^{(n-i-1)}\},\\
&S_0 = (U_1, 1)^{(1)} < (U_1, 2)^{(1)} < \cdots < (U_1, \binom{n}{1})^{(1)} < (U_2, 1)^{(2)} < \cdots < (U_2, \binom{n}{2})^{(2)} < \cdots < (U_n, 1)^{(n)},\\
&S_1 = (U_1, 1)^{(n-1)} < (U_1, 2)^{(n-1)} < \cdots < (U_1, \binom{n}{1})^{(n-1)} < (U_2, 1)^{(n-2)} < \cdots < (U_2, \binom{n}{2})^{(n-2)} <\\
&\quad \ldots (U_{n-1}, \binom{n}{n-1} - 1)^{(1)} < (U_{n-1}, \binom{n}{n-1})^{(1)} \text{ (see (2.1)).}
\end{aligned}
\tag{4.2}
$$

Theorem 6. *For $n \geq 3$ fixed, the following properties hold for the Brauer configuration algebra $\Lambda_{\mathfrak{S}^n}$*

$$
\begin{aligned}
\dim_{\mathbb{F}} \Lambda_{\mathfrak{S}^n} &= 2(2^n + t_{e_k^n - 1} + t_{f_k^n - 1} - 1),\\
\dim_{\mathbb{F}} Z(\Lambda_{\mathfrak{S}^n}) &= (n-1)2^n - n + 3.
\end{aligned}
\tag{4.3}
$$

Where

$$
\begin{aligned}
e_k^n &= \sum_{k=1}^{n} k\binom{n}{k},\\
f_k^n &= \sum_{k=1}^{n-1} (n-k)\binom{n}{k},\\
t_i &= \frac{i(i+1)}{2} \ \text{is the ith triangular number}
\end{aligned}
\tag{4.4}
$$

Furthermore, each labeled polygon $(U_i, j) \in \mathfrak{S}_1^n$ is defined by a $n+1$-tile snake graph.

Proof. For $n \geq 3$ fixed, we note that, the number of compositions associated with a word-polygon $w = 0^{(k)}1^{(n-j)} \in \mathcal{N}(w)$ is $\binom{n}{k}$. Thus,

$$
|\mathfrak{S}_1^n| = 2^n - 1,
\tag{4.5}
$$

Then, the occurrences of 0 in w is $k\binom{n}{k}$. Therefore, $val(0) = \sum_{k=1}^{n} k\binom{n}{k} = e_k^n$ and $val(1) = \sum_{k=1}^{n-1} (n-k)\binom{n}{k} = f_k^n$. The dimension of the algebra is obtained by definition.

The number of loops associated with a word of the form $0^{(n)} = n-1$, and the number of loops associated with a word of the form $w(U_k) = 0^{(k)}1^{(n-k)}$ is $n-2$. Then, $\#(Loops\ (\mathcal{N}(w)) = (n-2)\binom{n}{k}$, $1 \leq k \leq n-1$. Thus, $\#(Loops\ Q_{\mathfrak{S}_n}) = \sum_{k=1}^{n-1} (n-2)\binom{n}{k} + (n-1)$, $1 \leq k \leq n-1$. Then, the dimension $\dim_{\mathbb{F}} Z(\Lambda_{\mathfrak{S}^n})$ of the center is given by the identities $\dim_{\mathbb{F}} Z(\Lambda_{\mathfrak{S}^n}) = 1 + |\mathfrak{S}_1^n| + (n-2)(2^n - 2) + (n-1) = (n-1)2^n - n + 3$.

The map $\sigma : \mathfrak{S}_0^n \longrightarrow \{\rightarrow, \leftarrow\}$, such that $\sigma(0) = \rightarrow$, $\sigma(1) = \leftarrow$, with $\sigma(w') = \sigma(0^{(i)}1^{(n-i)}) = (\sigma(0))^{(i)}(\sigma(1))^{(n-i)}$, for any composition-polygon $w' \in \mathfrak{S}_1^n$. Thus, each polygon has associated an abstract string, for which it is possible defining a snake graph $\mathcal{G}$ with $n+1$ tiles. We are done. □

4.2. Interactions between Brauer configuration algebras, Kronecker modules and Snake graphs

In this section, we recall the notion of helix introduced by Cañadas et al. [17] These helices are nothing but snake graphs, if we consider entries of the considered matrices as tiles.

As in the case of string modules (see Section 3), the following theorems prove that both preprojective and preprojective Kronecker modules can be built from suitable snake graphs.

Theorem 7. *For $n \geq 2$ fixed, the $(2n+1)-$terms snake graph*

$$\mathcal{G}_{p_nk} = \mathcal{G}(n+1, n+1, 2, 2, n+1, 2, n+3, 2, n+1, 2, \dots, n+3, 2, n+1, 2, \dots, 2, n+3)$$

induces the indecomposable preprojective Kronecker module $(n+1, n)$. Furthermore the corresponding continued fraction of $\mathcal{G}_{p_nk}$ has the following form

$$[2, \underbrace{1, 1, \dots, 1}_{n-2}, 2, \underbrace{1, 1, \dots, 1}_{n-2}, 4, \Delta, 2],$$

where

$$\Delta = \underbrace{1, 1, \dots, 1}_{n-2}, 3, \underbrace{1, 1, \dots, 1}_{n}, 3, \underbrace{1, 1, \dots, 1}_{n-2}, 3\ \underbrace{1, 1, \dots, 1}_{n}, \dots$$

and the length $l(\Delta)$ of Δ is given by the following identities:

$$l(\Delta) = \begin{cases} n(n-1) - 1, & \text{if } n \text{ is odd;} \\ n(n-1) - 2, & \text{if } n \text{ is even;} \end{cases}$$

Proof. Let us consider that an indecomposable matrix block has n columns and $n+1$ rows, which are defined by the straight snake graphs in alternative fashion. First subsnake graph corresponds to the first row of the matrix block, second subsnake graph corresponds to the first column of the matrix block and so on.

Odd rows and the first column are labeled taking into account that if the last tile of a given row r_i is not labeled, then the first tile of the next column c_i is not labeled. Besides, the labels determine the way that rows and columns of a matrix block must be constructed. The procedure goes as follows:

Labeled horizontal straight snake graphs indicates that each tile corresponds to an entry of a row of the matrix block developed from the right to the left. Whereas, a labeled vertical straight snake graph indicates that the tiles correspond to the entries of a column of the matrix block developed from the top to the bottom.

An indecomposable preprojective Kronecker module $(n+1, n)$ is obtained from $\mathcal{G}_{p_nk}$ by assigning alternatively either 0 or a 1 to the ends of the straight snake graphs constituting $\mathcal{G}_{p_nk}$. In this case, a 0 is assigned to the first tile in the first row, then a 1 is assigned to the corresponding last tile, which is the first tile of the next straight snake graph, which has assigned a 0 in its last tile, and the procedure goes on. Numbers 1's are entries of the identities in the matrix block, which can be completed by definition. □

The following example illustrates step by step the arguments posed in the proof of Theorem 7. Considering the labeled snake graph $\mathcal{G}_{p_3k}$ presented in Figure 10. In Figure 11, 0 and 1 are assigned alternately to the ends of each straight snake subgraph, in Figure 12, $\mathcal{G}_{p_3k}$ is rolled up into the matrix block and finally using the definition of a preprojective Kronecker module we complete each matrix block (see Figure 4).

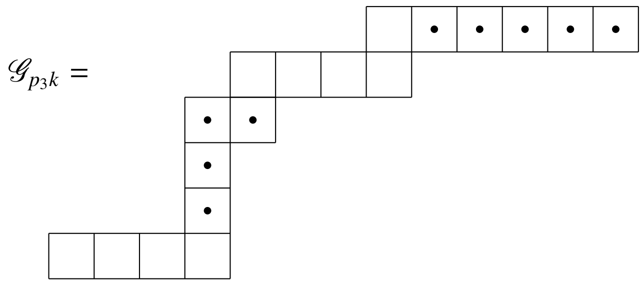


Figure 10. Labeled snake graph $\mathcal{G}_{p_3k}$.

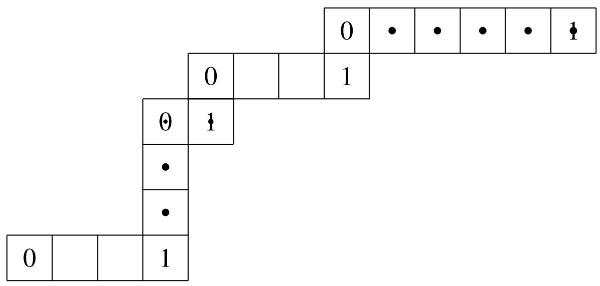


Figure 11. Assigned a 0 or 1 to the ends of each straight snake subgraph.

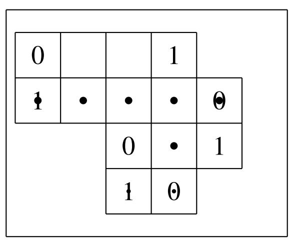


Figure 12. The defined snake graph is rolled up into the matrix block.

0	•	•	1	•	•
1	•	•	•	•	0
•	•	0	•	•	1
•	•	1	0	•	•

Figure 13. The preprojective Kronecker module is completed by definition.

0	0	0	1	0	0
1	0	0	0	1	0
0	1	0	0	0	1
0	0	1	0	0	0

The following theorem is the preinjective version of Theorem 7.

Theorem 8. *For $n \geq 3$ fixed. The $(2n+1)-$terms snake graph*

$$\mathcal{G}_{i_n k} = \mathcal{G}(3, n, 4, 2, n+2, 2, n+4, 2, n+2, \dots, 2, h),$$

where $h \in \{n+2, n+4\}$ induces the indecomposable preinjective Kronecker module $(n+1, n)$. Furthermore the corresponding continued fraction of $\mathcal{G}_{i_n k}$ has the following form

$$[2, 2, \underbrace{1, 1, \dots, 1}_{n-3}, 2, 1, 3, \underbrace{1, 1, \dots, 1}_{n-1}, \Delta, 2],$$

where $\Delta = 3, \underbrace{1, 1, \dots, 1}_{n+1}, 3, \underbrace{1, 1, \dots, 1}_{n-1}, 3, \underbrace{1, 1, \dots, 1}_{n+1}, 3\ \underbrace{1, 1, \dots, 1}_{n-1}, \dots$ *and the length of* Δ, $l(\Delta)$ *is*

$$l(\Delta) = \begin{cases} n(n-2) - 3, & \textit{if } n \textit{ is odd;} \\ n(n-2) - 2, & \textit{if } n \textit{ is even;} \end{cases}$$

The following result regarding snake graphs $\mathcal{G}_{p_n k}$ and $\mathcal{G}_{i_n k}$ is easy to see.

Proposition 2. *If $l(\mathcal{G})$ denotes the length of a snake graph $\mathcal{G}$, then*

$$l(\mathcal{G}_{p_n k}) = \begin{cases} n(n+2) - 2, & \textit{if } n \textit{ is odd;} \\ n(n+2) - 1, & \textit{if } n \textit{ is even;} \end{cases}$$

and

$$l(\mathcal{G}_{i_n k}) = \begin{cases} n(n+2) - 3, & \textit{if } n \textit{ is odd;} \\ n(n+2) - 2, & \textit{if } n \textit{ is even;} \end{cases}$$

4.2.1. Helices

Helices were introduced by the first autor et al. in [17]. In this paper, we observe that such helices are nothing but snake graphs associated with non-regular Kronecker modules. For $n \geq 1$, let P_n be an $(n+1) \times 2n$, k-matrix then P_n can be partitioned into two $(n+1) \times n$ matrix blocks A and B. In such a case we write $P_n = (P_n, A, B, n)$, where $A = (a_{i,j}) = [C^A_{i_1}, \ldots, C^A_{i_n}]$, $B = (b_{i,j}) = [C^B_{j_1}, \ldots, C^B_{j_n}]$, with $C^A_{i_r}$ ($C^B_{j_s}$) columns of P_n, if I_A (I_B) is the set of indices $I_A = \{i_r \mid 1 \leq r \leq n\}$ ($I_B = \{j_s \mid 1 \leq s \leq n\}$) then $I_A \cap I_B = \emptyset$, and $|I_A| = |I_B| = n$. In this case, each column of the matrix P_n belongs either to the matrix A or to the matrix B and a word $W_{P_n} = l_{m_1} \ldots l_{m_n} \ldots l_{m_{2n}}$, $\quad l_{m_h} \in \{A, B\}$, $1 \leq h \leq 2n$ is used to denote matrix P_n by specifying the way that columns of P_n have been assigned to the matrices A and B.

A row r_{P_n} of P_n has the form (r_A, r_B) with r_A (r_B) being a row of the matrix block A (B). We let R_A (R_B) denote the set of rows of the matrix block A (B), whereas $\mathscr{H}_n$ denotes the set of all matrices P_n with the aforementioned properties.

An *helix* associated with a matrix P_n of type $\mathscr{H}_n$ is a connected directed graph h whose construction goes as follows:

1) (*Vertices*) Vertices of h are entries of blocks A and B. We let h_0 denote the set of vertices of h.

   (a) Fix two different rows $i_{P_n} = (i_A, i_B)$ and $j_{P_n} = (j_A, j_B)$ of P_n.

   (b) Choose sets P_A and P_B of *pivoting entries* also called *pivoting vertices*, $P_A \subset A$, $P_B \subset B$ such that $|P_A| = |P_B| = n$. Entries in $A \backslash P_A$ and $B \backslash P_B$ are said to be *exterior entries* or *exterior vertices*. In this case, if $x \in P_A$ ($x \in P_B$) then $x \notin i_A$ ($x \notin j_B$).

   P_A and P_B are sets of the form:

$$\begin{aligned} P_A &= \{a_{i_1,j_1}, a_{i_2,j_2}, \ldots, a_{i_s,j_s}\}, \quad j_x \neq j_y \text{ if and only if } i_x \neq i_y, \\ P_B &= \{b_{t_1,h_1}, b_{t_2,h_2}, \ldots, b_{t_s,h_s}\}, \quad h_x \neq h_y \text{ if and only if } t_x \neq t_y. \end{aligned} \tag{4.7}$$

   Where, $a_{i_r,j_r} \in R_A \backslash i_A$, $\quad b_{t_m,h_m} \in R_B \backslash j_B$, $1 \leq r, m \leq s$. It is chosen just only one entry a_{i_r,j_r} (b_{t_m,h_m}) for each row in $R_A \backslash i_A$ ($R_B \backslash j_B$) and for each column C_A (C_B) of A (B).

2) (*Arrows*) arrows in h are defined in the following fashion:

   (a) Arrows in h are either horizontal or vertical. We let h_1 denote the set of arrows of h.

   (b) Horizontal arrows connect a vertex of the matrix block A (B) with a vertex of the matrix block B (A). Vertical arrows only connect vertices in the same matrix block. Starting and ending vertices of horizontal (vertical) arrows are entries of the same row (column) of P_n.

   (c) The starting vertex of a horizontal (vertical) arrow is an exterior (pivoting) vertex. The ending point of a horizontal (vertical) arrow is a pivoting (exterior) vertex.

   (d) A pivoting vertex occurs as ending (starting) vertex just once. Thus, h does not cross itself.

   (e) The first and last arrow of h are horizontal and its starting vertex belongs to i_A.

   (f) Each vertical arrow is preceded by a unique horizontal arrow, and unless the first arrow, any horizontal arrow is preceded by a vertical arrow.

   (g) All the rows of P_n are visited by h, and no row or column of P_n is visited by arrows of h more than once.

   (h) There are not horizontal arrows connecting exterior vertices of j_A with vertices of j_B.

Figure 14 is an example of an helix of type

$$(4_{P_3}, 2_{P_3}, P_A = \{p_{3,2}, p_{1,3}, p_{2,5}\}, P_B = \{p_{4,1}, p_{1,4}, p_{3,6}\}),$$

associated with a matrix P_3 of type $\mathscr{H}_3$ and defined by the word $W_{P_3} = BAABAB$.

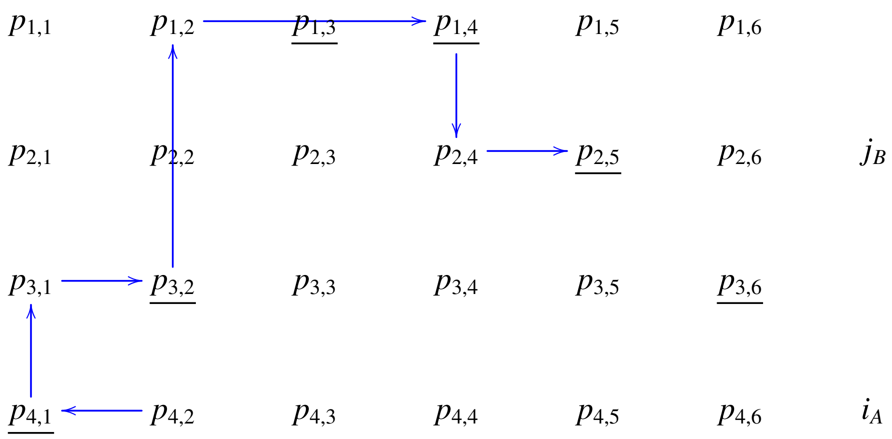


Figure 14. Example of a helix or snake graph (considering vertices as tiles of a snake graph glued according to the orientation of the arrows. Bearing in mind that the cases ← and ↓ must be reversed for the gluing process).

The following Theorems 9 and 10 were proved by the first author et al. in [17].

Theorem 9 (Corollary 9, [17])**.** *If for $n \geq 1$, P_n and P'_n are equivalent preprojective Kronecker modules with dimension vector of the form $[n+1 \;\; n]$ and corresponding sets of helices H_{P_n} and $H_{P'_n}$ then $|H_{P_n}| = |H_{P'_n}|$.*

Helices associated with preprojective (preinjective) Kronecker modules are said to be *preprojective (preinjective) Kronecker snake graphs* or simply Kronecker snake graphs, if no confusion arises (note that, preprojective Kronecker snake graphs $\mathscr{G}_{p_n k}$ are helices).

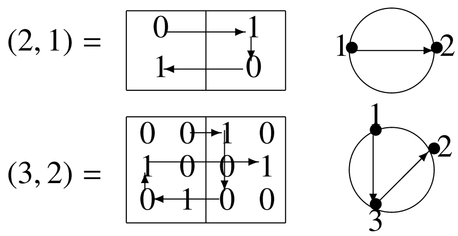


Figure 15. Examples of Kronecker snake graphs. The number of such graphs equals the number $a(n)$ of ways of connecting $n+1$ equally spaced points on a circle with a path of n line segments ignoring reflections.

Theorem 10 (Theorem 10, [17]). *If for $n \geq 1$, P_n denotes a preprojective Kronecker module then the number of helices associated with P_n is $h_n^{P_n} = n!\lceil \frac{n}{2} \rceil$ where $\lceil x \rceil$ denotes the smallest integer greatest than x.*

The proof of Theorem 10 is based on a bijection between maximal paths (connecting the vertex-root with vertices at the last level) of trees (Kronecker trees) T_n shown in Figure 16 and Kronecker snake graphs, for $n \geq 1$ fixed.

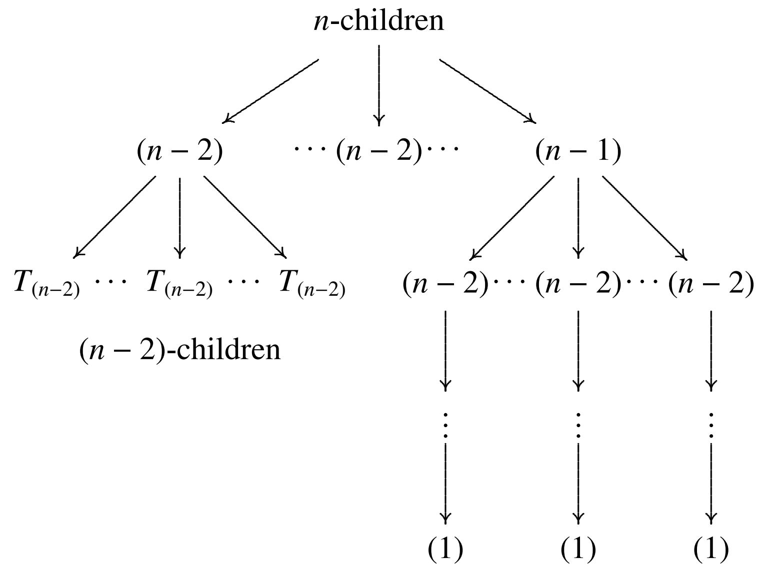


Figure 16. Kronecker tree of type T_n associated with the preprojective Kronecker module $(n+1, n)$.

4.2.2. Brauer configuration algebras induced by preprojective Kronecker modules

This section introduces Brauer configuration algebras induced by Kronecker modules, which according to Theorem 7 can be constructed from a snake graph.

For $n \geq 1$, let Ψ^n be a labeled Brauer configuration, such that $\Psi^n = (\Psi^n_0, \Psi^n_1, \mu, O)$. The labeling is given by $(n+1) \times n$ matrix blocks A, and B related to canonical preprojective Kronecker modules (see Figure 4). Ψ^n satisfies the following conditions:

$$\begin{aligned}
\Psi^n_0 &= \{0, 1\},\\
\Psi^n_1 &= \{(U^n_1, A), (U^n_2, B)\},\\
w(U^n_1) &= w(U^n_2) = 0^{(n^2)}1^{(n)}, \text{ are the words associated with the polygons.}\\
\mu(0) &= \mu(1) = 1. \quad \text{The following are the corresponding successor sequences}\\
S_0 &= (U^n_1, A)^{(1)} < (U^n_1, A)^{(2)} < \cdots < (U^n_1, A)^{(n^2)} < (U^n_2, B)^{(1)} < (U^n_2, B)^{(2)} < \cdots < (U^n_2, B)^{(n^2)},\\
S_1 &= (U^n_1, A)^{(1)} < (U^n_1, A)^{(2)} < \cdots < (U^n_1, A)^{(n)} < (U^n_2, B)^{(1)} < (U^n_2, B)^{(2)} < \cdots < (U^n_2, B)^{(n)}.
\end{aligned} \tag{4.8}$$

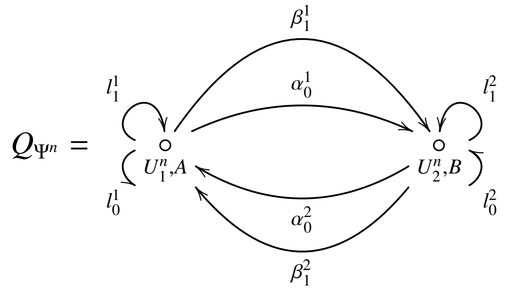


Figure 17. Example of the Brauer quiver defined by the Brauer configuration Ψ^n, l_1^1 (l_2^1) denotes the family of n loops associated with the vertex 1 in (U_1, A) $((U_2, B))$. In the same way, l_1^0 (l_2^0) denotes the family of $2n^2 - 1$ loops associated with the vertex 0 at the corresponding polygon.

Admissible ideal I such that the Brauer configuration algebra $\Lambda_{\Psi^n} = \mathbb{F}Q_{\Psi^n}/I$ is generated by the following relations, in this case, we use symbols $\alpha^h_{0,i}$, and $\beta^{h'}_{1,j}$ to denote arrows in sets α^h_0, and β^h_1, respectively, $h = 1, 2$.

1) $l^h_{0,j}\beta^h_1$, for all $l^h_{0,j} \in l^h_0$, $h = 1, 2$.
2) $l^h_{1,i}\alpha^h_0$, for all $l^h_{1,i} \in l^h_1$, $\alpha^h_0 \in \beta^h_1$.
3) $(l^h_{m,n})^2 = l^h_{m,n}$, for all possible values of n, $m = 0, 1$.
1. $\alpha^h_0\beta^{h'}_1$, $\beta^h_1\alpha^h_0$.

Figure 17 shows the shape of a Brauer quiver Q_{Ψ^n} associated with a Brauer configuration Ψ^n, for $n \geq 1$ fixed.

Theorem 11. *For $n \geq 1$ fixed, the following properties hold for the Brauer configuration algebra Λ_{Ψ^n}*

$$\begin{aligned} \dim_{\mathbb{F}} \Lambda_{\Psi^n} &= 2(2 + t_{2n^2-1} + t_{2n-1}), \\ \dim_{\mathbb{F}} Z(\Lambda_{\Psi^n}) &= 2n^2 + 2n - 1. \end{aligned} \tag{4.9}$$

Where t_i denotes the ith triangular number.

Proof. It suffices to observe that $val(0) = 2n^2$, and $val(1) = 2n$. □

4.2.3. The Kronecker energy

If $M(Q)$ is the $m \times n$-adjacency matrix of a quiver Q. And $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ is the set of singular values of $M(Q)$ [3]. Then, the *trace norm* $||M(Q)||_*$ of $M(Q)$ is given by the sum

$$||M(Q)||_* = \sum_{i=1}^{\min\{m,n\}} \sigma_i. \tag{4.10}$$

The trace norm of a quiver is a generalization of the energy $\mathcal{E}(G)$ of a graph G, which is the sum of the absolute values of its adjacency matrix eigenvalues.

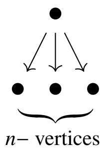
The main problem in the theory of graph energy consists of estimating the energy value (trace norm value) of significant classes of graphs (matrices). Bearing in mind that the trace norm of a given matrix is the sum of its singular values as defined in identity (4.10).

In this paper, we estimate the trace norm value of a Kronecker tree of type T_n (see Figure 16) defined by preprojective Kronecker snake graphs.

Theorem 12. *For $n \geq 1$ fixed, the trace norm $||M(T_n)||_*$ of the Kronecker tree T_n satisfies the following inequalities:*

$$\frac{(n-1)(n)(4n+1)}{6} + d\lceil \sqrt{n} \rceil \leq ||M(T_n)||_* \leq \frac{2n_2(n-2)^{3/2} + 6\sqrt{n}}{3}. \tag{4.11}$$

Where n_2 is the number of bifurcations of T_n.

Proof. By construction, we note that, there is a bijection $f : B \longrightarrow S$, where B is the set of ramifications of the form $b_n =$ 

and the set of singular values of $M(T_n)$ given by the rule $f(b_n) = \sqrt{n}$. In fact, the characteristic polynomial $\mathcal{P}_n(\lambda)$ has the form:

$$\mathcal{P}_n(\lambda) = \lambda^{n_0}(\lambda - 1)^{n_1}(\lambda - \sqrt{2})^{n_2} \ldots (\lambda - \sqrt{n})^{n_k}, \tag{4.12}$$

$n_k = 1$. In this case, n_2 is the largest multiplicity. Thus,

$\sum\limits_{j=0}^{n} \sqrt{j} = ||M(T_n)||_* \leq n_2(\sum\limits_{i=1}^{n-2} \sqrt{i}) + \sqrt{n} \leq n_2 \frac{2(n-2)^{3/2}}{3} + 2\sqrt{n} = \frac{2n_2(n-2)^{3/2}+6\sqrt{n}}{3}$.

On the other hand,

$||M(T_n)||_* \geq \sum\limits_{j=0}^{n} \sqrt{n} = \sum\limits_{i=1}^{n} i(2i+1) + d\lceil \sqrt{n} \rceil = \frac{(n-1)(n)(4n+1)}{6} + d\lceil \sqrt{n} \rceil$. We are done. □

The following result regards the trace norm of non-regular Kronecker modules. Note that, these modules are matrix interpretations of polygons Ψ_1^n in Brauer configuration (4.8).

Theorem 13. *For $n \geq 1$ fixed, $||(n+1, n)||_* = ||(n, n+1)||_* = 2n$*

Proof. Considered as matrices, it holds that $(n+1, n)^t(n+1, n) = diag(1, 2, 2, \ldots, 2, 1)_{n+1}$. The corresponding singular values are $\sigma_1 = \sigma_2 = \cdots = \sigma_{n-1} = 2$, $\sigma_n = \sigma_{n+1} = 1$. On the other hand, $(n, n+1)(n, n+1)^t = diag(2, 2, \ldots, 2, 2)$, therefore the associated singular values are $\sigma_1 = \sigma_2 = \cdots = \sigma_n = 2$. □

5. Concluding remarks

Brauer configuration algebras defined by Kronecker modules can be associated with string snake graphs. In particular, non-regular Kronecker modules can be built via suitable snake graphs. Since non-regular Kronecker modules have a matrix presentation, giving their trace norm values is also possible.

To obtain results as those presented in this paper for other matrix problems as the four subspace problem is an interesting task for the future.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. A. M. Cañadas, I. D. M. Gaviria, J. D. C. Vega, Relationships between the Chicken McNugget problem, Mutations of Brauer configuration algebras and the advanced encryption standard, *Mathematics*, **9** (2021), 1937. https://doi.org/10.3390/math9161937
2. A. M. Cañadas, M. A. O. Angarita, Brauer configuration algebras for multimedia based cryptography and security applications, *Multimed. Tools Appl.*, **80** (2021), 23485–23510. https://doi.org/10.1007/s11042-020-10239-3
3. N. Agudelo, A. M. Cañadas, I. D. M. Gaviria, P. F. F. Espinosa, $\{0,1\}$-Brauer configuration algebras and their applications in the graph energy theory, *Mathematics*, **9** (2021), 3042. https://doi.org/10.3390/math9233042
4. E. L. Green, S. Schroll, Brauer configuration algebras: A generalization of Brauer graph algebras, *Bull. Sci. Math.*, **121** (2017), 539–572. https://doi.org/10.1016/j.bulsci.2017.06.001
5. S. Schroll, Brauer graph algebras, in *Homological Methods, Representation Theory, and Cluster Algebras*, Springer, (2018), 177–223. https://doi.org/10.1007/978-3-319-74585-5
6. J. Propp, The combinatorics of frieze patterns and Markoff numbers, *Integers*, **20** (2020), 1–38. https://doi.org/10.48550/arXiv.math/0511633
7. I. Çanakçi, R. Schiffler, Cluster algebras and continued fractions, *Compos. Math.*, **54** (2018), 565–593. https://doi.org/10.1112/S0010437X17007631
8. I. Çanakçi, R. Schiffler, Snake graphs and continued fractions, *Eur. J. Combin.*, **86** (2020), 1–19. https://doi.org/10.1016/j.ejc.2020.103081
9. I. Çanakçi, R. Schiffler, Snake graphs calculus and cluster algebras from surfaces, *J. Algebra*, **382** (2013), 240–281. https://doi.org/10.1016/j.jalgebra.2013.02.018
10. I. Çanakçi, R. Schiffler, Snake graphs calculus and cluster algebras from surfaces II: Self-crossings snake graphs, *Math. Z.*, **281** (2015), 55–102. https://doi.org/10.1007/s00209-015-1475-y
11. I. Çanakçi, R. Schiffler, Snake graphs calculus and cluster algebras from surfaces III: Band graphs and snake rings, *Int. Math. Res. Not. IMRN*, (2019), 1145–1226.
12. I. Çanakçi, S. Schroll, Lattice bijections for string modules snake graphs and the weak Bruhat order, *Adv. Appl. Math.*, **126**, (2021), 102094. https://doi.org/10.1016/j.aam.2020.102094
13. G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, Cambridge UK, 2010.
14. A. Sierra, The dimension of the center of a Brauer configuration algebra. *J. Algebra*, **510** (2018), 289–318. https://doi.org/10.1016/j.jalgebra.2018.06.002

15. P. F. F. Espinosa, *Categorification of some integer sequences and its applications*, Ph.D thesis, Universidad Nacional de Colombia, BTA, Colombia, 2021.

16. D. Simson, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Gordon and Breach, London UK, 1992.

17. A. M. Cañadas, I. D. M. Gaviria, P. F. F. Espinosa, Brauer configuration algebras and Kronecker modules to categorify integer sequences, *ERA*, **30** (2022), 661–682. https://doi.org/10.3934/era.2022035

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