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Research article

Reconstruction of initial heat distribution via Green function method

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Abstract: In this paper, layer potential techniques are investigated for solving the thermal diffusion problem. We construct the Green function to get the analytic solution. Moreover, by combining Fourier transform some attractive relation between initial heat distribution and the final observation is obtained. Finally iteration scheme is developed to solve the inverse heat conduction problem and convergence results are presented.

Keywords: linear heat equation; analytic solution; fundamental solution; Green function; iteration

1. Introduction

The thermal diffusion problem has long been studied. Applications can be found in various physical and engineering settings, in particular in hydrology [1], material sciences, heat transfer [2] and transport problems, etc. The inverse problems for determination of initial temperature, thermal conductivity or heat source from final over determination or other additional measurements have been widely considered by lots of researchers (see, e.g., [3–11]). In [11], the authors explored the Tikhonov regularization for simultaneous reconstruction of the initial temperature and heat radiative coefficient in a heat conductive system. They used the final observation and temperature in a small region as the additional data. In [9], an exact and analytical representation of the initial heat distribution was given by using only the measurements of temperature and heat flux at one point. In [12], the authors proved the uniqueness of the identification of unknown source locations in two-dimensional heat equations from scattered measurements and presented some numerical methods to identify the locations. Hon and Wei introduced the fundamental solution methods in solving inverse heat conduction problems [6,7]. In recent years a special kind of inverse heat conduction problem called sideways parabolic equations have also been widely studied [13–17]. Various regularization methods are presented to solve the severely ill-posed problem and optimal convergence results are obtained in different stopping rules.

Reconstruction of thermophysical properties such as thermal conductivity and heat capacity is another interesting and meaningful research area. Many theoretical and experimental methods have been developed in the literature, they include, among others, the steady-state method, the probe method [18], the periodic heating method [19], the least-squares method [20] and the pulse heating method. For a steady case, the layer potential technique is widely used for analysis of the solution and reconstruction of the thermal conductivity [21, 22]. In [21], the authors construct the Generalized Polarization Tensors based on layer potential method in reconstruction of the shape of the homogeneous conductive body. Besides, layer potential techniques are widely used in wave prorogation problems, including Helmholtz problems, Maxwell problems, Elastic problems and so on (see [23–29] and there references there in).

In this paper, we shall consider the reconstruction of the initial heat distribution from final observation in the upper half plane. The Dirichlet boundary condition is given. We firstly use the layer potential method to represent the solution of the forward heat conduction problem in an integral form. Then by exploring the Green function, the Dirichlet boundary problem can be solved analytically without knowing the Neumann boundary condition. By using appropriate extension and Fourier transform, we derive a very simple relation between the initial heat distribution and the final observation. The severely ill-posed problem for reconstruction of initial temperature then is done by introducing a new iteration scheme. Convergence results are investigated under both the *a priori* and *a posteriori* stopping rules. The main contribution of this paper is to extend the layer potential techniques to the inverse heat diffusion problem and designing of a new iterative scheme in solving the related inverse heat distribution problem, theoretically. We shall consider the numerical implementation in a forthcoming work.

The organization of this paper is as follows. In the second section, we shall introduce the layer potential techniques together with Green's function in representing the integral solution to the heat diffusion problem, while proofs of the main theorems can be found in Appendix A. In Section 3, the attractive relation between initial heat distribution and the final observation shall be derived by making use of the properties of layer potentials. Base on the relation, a new iterative methods is designed in solving the inverse heat distribution problem. We then analyze some properties of the Fourier transform and derived the convergence results by using the aforementioned iterative methods, in Section 4 and 5, respectively. Some conclusions are made in Section 6.

2. Representation of solution

We shall first consider the forward heat conduction problem in the half plane

$$\begin{cases} u_t = k\Delta u & (\mathbf{x}, t) \in \mathbb{R} \times \mathbb{R}_+ \times (0, T), \\ u(\mathbf{x}, 0) = \mu_0(\mathbf{x}) & \mathbf{x} \in \mathbb{R} \times \mathbb{R}_+, \\ u(x_1, 0, t) = f(x_1, t) & (x_1, t) \in \mathbb{R} \times (0, T) \end{cases} \tag{2.1}$$

where $\mathbb{R}_+ := (0, +\infty)$ and $\mathbf{x} := (x_1, x_2)$. The thermal diffusion coefficient k is supposed to be a positive constant. The initial temperature and boundary temperature are given and denoted by $\mu_0(x_1, x_2)$ and $f(x_1, t)$, respectively. In this paper, we suppose that $\mu_0(\mathbf{x}) \in L^2(\mathbb{R} \times \mathbb{R}_+)$ and $f(x, t) \in L^2(\mathbb{R} \times (0, T))$.

We shall consider the analytic solution to the forward problem (2.1) by using the Green function method. Let $\Gamma(\mathbf{x}, t)$ be the fundamental solution to the heat equation, i.e.,

$$\Gamma_k(\mathbf{x}, t) = \frac{1}{4\pi kt} e^{-\frac{|\mathbf{x}|^2}{4kt}} \tag{2.2}$$

which satisfies

$$\begin{cases} (\Gamma_k)_t = k\Delta\Gamma_k & (\mathbf{x}, t) \in \mathbb{R}^2 \times \mathbb{R}_+ \\ \Gamma_k(\mathbf{x}, 0) = \delta_0 & \mathbf{x} \in \mathbb{R}^2 \end{cases}$$

where δ_0 is the Dirac function at the origin [31]. To derive the solution to (2.1), we firstly introduce the two well known quantities, single layer potential and double layer potential for heat equation. They are extensions of the classical layer potentials in electrostatic problem [32]. For a domain D, we denote by $\mathcal{S}_D^k$ (if $k = 1$ then we simply denote by $\mathcal{S}_D$) the single layer potential

$$\mathcal{S}_D^k[\varphi](\mathbf{x}, t) := \int_0^t \int_{\partial D} \Gamma_k(\mathbf{x} - \mathbf{y}, t - s)\varphi(\mathbf{y}, s) d\sigma_y ds, \quad \mathbf{x} \in \mathbb{R}^2 \setminus \partial D \tag{2.3}$$

for a density function $\varphi(\mathbf{x}, t) \in H^{-1/2,1}(\partial D, \mathbb{R}_+)$ and denote by $\mathcal{D}_D^k$ (if $k = 1$ then denote by $\mathcal{D}_D$) the double layer potential

$$\mathcal{D}_D^k[\psi](\mathbf{x}, t) := \int_0^t \int_{\partial D} \frac{\partial}{\partial \nu_y} \Gamma_k(\mathbf{x} - \mathbf{y}, t - s)\psi(\mathbf{y}, s) d\sigma_y ds, \quad \mathbf{x} \in \mathbb{R}^2 \setminus \partial D \tag{2.4}$$

for a density funtion $\psi(\mathbf{x}, t) \in H^{1/2,1}(\partial D, \mathbb{R}_+)$. We shall further define a volume potential by

$$\mathcal{V}_D^k[\mu](\mathbf{x}, t) := \int_D \Gamma_k(\mathbf{x} - \mathbf{y}, t)\mu(\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^2 \tag{2.5}$$

for a function $\mu(\mathbf{x}) \in L^2(D)$. Here $\mathbf{y} = (y_1, y_2)$. By using Green's formula, the solution to (2.1) can be written in the form of layer potentials. We state the result in the following theorem, for the sake of convenience to the reader, where proofs can be found in the appendix.

Theorem 2.1. The solution to the heat conduction problem

$$\begin{cases} u_t = k\Delta u & (\mathbf{x}, t) \in D \times (0, T), \\ u(\mathbf{x}, 0) = \mu_0(\mathbf{x}) & \mathbf{x} \in D, \\ u(\mathbf{x}, t) = f(\mathbf{x}, t) & (x, t) \in \partial D \times (0, T) \end{cases}$$

has the following form

$$u(\mathbf{x}, t) = \mathcal{V}_D^k[\mu_0](\mathbf{x}, t) + \mathcal{S}_D^k[k\partial u/\partial \nu](\mathbf{x}, t) - k\mathcal{D}_D^k[f](\mathbf{x}, t). \tag{2.6}$$

We see from the solution form (2.6) that it contains both Dirichlet and Neumann boundary conditions. Thus, in order to get the solution of (2.1) by using the integral form (2.6), additional Neumann boundary condition is required. In this paper, we shall not use the Neumann boundary measurement thus we need to avoid the appearance of it in the solution form.

If for a bounded smooth domain D, we can introduce the jump formula for layer potentials. For the single layer potential we have on ∂D

$$\frac{\partial}{\partial \boldsymbol{\nu}}\mathcal{S}_D[\varphi]\Big|_{\pm}(\mathbf{x},t) = \mp\frac{1}{2}\varphi(\mathbf{x},t) + \mathcal{K}_D^*[\varphi](\mathbf{x},t), \tag{2.7}$$

where the $\pm$ signs mean the limits are taken from outside the domain D and inside the domain D, respectively. The integral operator $\mathcal{K}_D^*$ is the adjoint operator of $\mathcal{K}_D$, which is defined by

$$\mathcal{K}_D[\psi](\mathbf{x},t) = \text{p.v.}\int_0^t\int_{\partial D}\frac{\partial}{\partial \boldsymbol{\nu}_y}\Gamma(\mathbf{x}-\mathbf{y},t-s)\psi(\mathbf{y},s)d\sigma_y ds, \tag{2.8}$$

where p.v. means the Cauchy principle value of the integral. For the jump formula of double layer potential we have

$$\mathcal{D}_D[\psi]\Big|_{\pm}(\mathbf{x},t) = \pm\frac{1}{2}\psi(\mathbf{x},t) + \mathcal{K}_D[\psi](\mathbf{x},t). \tag{2.9}$$

By taking the trace of the solution $u(\mathbf{x},t)$ and using the jump formulas there holds

$$\mathcal{S}_D[g](\mathbf{x},t) = \mathcal{V}_D^k[\mu_0](\mathbf{x},t) - (\frac{I}{2} + \mathcal{K}_D)[f(\cdot, s/k)](\mathbf{x},kt)$$

for $\mathbf{x} \in \partial D$, where $g := k\partial u/\partial \boldsymbol{\nu}$. The above integral equation can be used to solve the Neumann boundary condition g. The equation is a Volterra integral equation of the first kind with kernel $\Gamma_k(\mathbf{x},t)$ and uniqueness of solution has been proved in [30].

However, for a unbounded domain D the jump formula for the layer potentials may fail. Besides, it is also not easy and stable to solve the Volterra integral equation of the first kind. In what follows, we seek for the Green function to solve (2.1) without getting involved with the Neumann boundary condition. Denote by $G(\mathbf{x},t)$ the Green function which satisfies

$$\begin{cases} G_t = \Delta G & (\mathbf{x},t) \in D\times\mathbb{R}_+ \\ G(\mathbf{x},0) = \delta_0 & \mathbf{x}\in D \\ G(\mathbf{x},t) = 0. & (\mathbf{x},t)\in\partial D\times\mathbb{R}_+. \end{cases} \tag{2.10}$$

With the help of Green's function, solution to heat equation can be written by integral formulation.

Theorem 2.2. Let $\mathbf{x} \in D = \mathbb{R}\times\mathbb{R}_+$ then the solution $u(\mathbf{x},t)$ to (2.1) has the form

$$u(\mathbf{x},t) = \int_D G(\mathbf{x}-\mathbf{y},kt)\mu_0(\mathbf{y})d\mathbf{y} - \int_0^t\int_{\partial D} k\frac{\partial}{\partial \nu_y}G(\mathbf{x}-\mathbf{y},k(t-s))f(y,s)d\sigma_y ds.$$

The solution form by using Green function method gives us a direct way to compute the solution of (2.1). However, to compute the solution we need first compute the Green's function (2.10), which does not have an explicit form in most cases. However for $D = \mathbb{R}\times\mathbb{R}_+$ we can find the Green function by using the reflection method. We have the following theorem

Theorem 2.3. The solution to (2.1) can be represented by

$$u(x_1,x_2,t)=\int_{-\infty}^{\infty}\frac{1}{4k\pi t}e^{-\frac{(x_1-y_1)^2}{4kt}}\int_0^{\infty}\Big(e^{-\frac{(x_2-y_2)^2}{4kt}}-e^{-\frac{(x_2+y_2)^2}{4kt}}\Big)\mu_0(y_1,y_2)dy_2dy_1$$
$$-y\int_0^t\frac{1}{4k\pi(t-s)^2}e^{-\frac{x_2^2}{4k(t-s)}}\int_{-\infty}^{\infty}e^{-\frac{(x_1-y_1)^2}{4k(t-s)}}f(y_1,s)dy_1ds$$

The proofs of Theorem 2.2 and 2.3 can be found in Appendix. We have shown the solution to the forward problem (2.1) with nonhomogeneous boundary condition. The solution is an integral form associated with the Green function. We shall use this solution form to reconstruct the initial temperature distribution from final overdetermination.

3. The inverse problem

The inverse problem we consider in this paper is the reconstruction of the initial temperature distribution $\mu_0(\mathbf{x})$ and generally the solution $u(\mathbf{x},t)$ from the following problem

$$\begin{cases} u_t=k\Delta u & (\mathbf{x},t)\in\mathbb{R}\times\mathbb{R}_+\times(0,T),\\ u(\mathbf{x},T)=\mu_T(\mathbf{x}) & \mathbf{x}\in\mathbb{R}\times\mathbb{R}_+,\\ u(x_1,0,t)=f(x_1,t) & (x_1,t)\in\mathbb{R}\times(0,T).\end{cases}\tag{3.1}$$

We suppose $f(x_1,t)\in L^2(\mathbb{R}\times(0,T))$ and $\mu_T(\mathbf{x})\in L^2(\mathbb{R}\times\mathbb{R}_+)$. For convenience to use the Fourier transform we also suppose those functions are in L^1 for any fixed time t. Define $\|\cdot\|$ as the L^2 norm with respect to $\mathbf{x}$ in $\mathbb{R}\times\mathbb{R}_+$. Denote by $\mathcal{G}_D$ the operator on $L^2(\mathbb{R}\times\mathbb{R}_+)$

$$\mathcal{G}_D[\mu](\mathbf{x}):=\int_D G(\mathbf{x},\mathbf{y},kT)\mu(\mathbf{y})d\mathbf{y}\tag{3.2}$$

where

$$G(\mathbf{x},\mathbf{y},kT)=\frac{1}{4k\pi T}e^{-\frac{(x_1-y_1)^2}{4kT}}\Big(e^{-\frac{(x_2-y_2)^2}{4kT}}-e^{-\frac{(x_2+y_2)^2}{4kT}}.\Big)$$

By setting $t=T$ in the solution form in Theorem 2.3 we get

$$\mathcal{G}_D[\mu_0](\mathbf{x})=h(\mathbf{x}),\quad \mathbf{x}\in\mathbb{R}\times\mathbb{R}_+\tag{3.3}$$

where

$$h(\mathbf{x})=\mu_T(\mathbf{x})+y\int_0^T\frac{1}{4k\pi(T-s)^2}e^{-\frac{x_2^2}{4k(T-s)}}\int_{-\infty}^{\infty}e^{-\frac{(x_1-y_1)^2}{4k(T-s)}}f(y_1,s)dy_1ds.\tag{3.4}$$

The inverse problem here is to solve the integral equation (3.3) to get the initial temperature distribution $\mu_0(\mathbf{x})$. We see that $\mathcal{G}_D$ is an integral operator with the kernel $G(\mathbf{x},kT)$. The following properties on the operator $\mathcal{G}_D$ is straight forward

Lemma 3.1. The operator $\mathcal{G}_D$ in (3.3) is a self-adjoint and bounded operator on $L^2(\mathbb{R}\times\mathbb{R}_+)$.

Proof. Since the kernel of the operator $\mathcal{G}_D$ is symmetric we immediately get that $\mathcal{G}_D$ is self-adjoint on $L^2(\mathbb{R}\times\mathbb{R}_+)$. We shall show the bound of $\mathcal{G}_D$ on $L^2(\mathbb{R}\times\mathbb{R}_+)$. Define

$$\mu(x_1,x_2)=\begin{cases}\mu_0(x_1,x_2) & y>0\\ -\mu_0(x_1,-x_2) & y<0.\end{cases}$$

Then there holds

$$\mathcal{G}_D[\mu_0](\mathbf{x}) = \frac{1}{4k\pi T}\int_{\mathbb{R}^2} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4kT}}\mu(\mathbf{y})d\mathbf{y}$$

where $\mathbf{x} = (x_1, x_2)$. It is easy to see that the right hand side above is a convolution of functions $\frac{1}{4k\pi T}e^{-\frac{|\mathbf{x}|^2}{4kT}}$ and $\mu(\mathbf{x})$. Thus by Young's inequality we have

$$\|\mathcal{G}_D[\mu_0]\| \le \|\frac{1}{4k\pi T}e^{-\frac{|\mathbf{x}|^2}{4kT}}\|_{L^1(\mathbb{R}^2)}\|\mu\|_{L^2(\mathbb{R}^2)} = 2\|\mu_0\|$$

which completes the proof. □

We shall explore a method to solve (3.3). To proceed, we present some preliminary results

Lemma 3.2. There holds the following identities

$$\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} y^{2n}e^{-\frac{y^2}{4kt}}dy = \begin{cases} 1 & n = 0 \\ (kt)^n\frac{(2n)!}{n!} & n = 1, 2, \dots \end{cases} \tag{3.5}$$

and

$$\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty}\cos(y\eta)e^{-\frac{y^2}{4kt}}dy = e^{-\eta^2 kt}. \tag{3.6}$$

Proof. For $n = 0$ the result is obvious. We assume $n \ge 1$. Define

$$I_n := \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} y^{2n}e^{-\frac{y^2}{4kt}}dy$$

then by integral by parts we have

$$I_n = \frac{1}{2}\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} y^{2n-1}e^{-\frac{y^2}{4kt}}dy^2 = 2kt(2n-1)I_{n-1}$$

and thus by recursion we obtain

$$I_n = (2kt)^n\prod_{j=1}^{n}(2j-1)I_0 = (kt)^n\frac{(2n)!}{n!}.$$

Next by Taylor expansion of $\cos(m\pi y)$ we have

$$\begin{aligned}\int_{-\infty}^{\infty}\cos(y\eta)\frac{1}{\sqrt{4k\pi t}}e^{-\frac{y^2}{4kt}}dy &= \int_{-\infty}^{\infty}\sum_{n=0}^{\infty}(-1)^n\frac{(y\eta)^{2n}}{(2n)!}\frac{1}{\sqrt{4k\pi t}}e^{-\frac{y^2}{4kt}}dy \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n\eta^{2n}}{(2n)!}\int_{-\infty}^{\infty}y^{2n}\frac{1}{\sqrt{4k\pi t}}e^{-\frac{y^2}{4kt}}dy \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n\eta^{2n}}{(2n)!}(kt)^n\frac{(2n)!}{n!} = e^{-\eta^2 kt}\end{aligned}$$

which completes the proof. □

Lemma 3.3. There holds the following identity

$$\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{iy\eta}e^{-\frac{(x-y)^2}{4kt}}dy = e^{-\eta^2 kt}e^{ix\eta} \tag{3.7}$$

where i is the imaginary unit.

Proof. By using (3.6) and some elementary calculations we have

$$\begin{aligned}
&\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{iy\eta}e^{-\frac{(x-y)^2}{4kt}}dy = \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} e^{i(x+y)\eta}e^{-\frac{y^2}{4kt}}dy\\
=\ & \frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty} \left[\cos((x+y)\eta) + i\sin((x+y)\eta)\right]e^{-\frac{y^2}{4kt}}dy\\
=\ & \cos(x\eta)\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty}\cos(y\eta)e^{-\frac{y^2}{4kt}}dy + i\sin(x\eta)\frac{1}{\sqrt{4k\pi t}}\int_{-\infty}^{\infty}\cos(y\eta)e^{-\frac{y^2}{4kt}}dy\\
=\ & (\cos(x\eta) + i\sin(x\eta))e^{-\eta^2\pi^2 kt} = e^{ix\eta}e^{-\eta^2 kt}.
\end{aligned}$$

□

With (3.7) on hand, we can get the following result

Theorem 3.1. There holds the following relation

$$\mathcal{G}_D[e^{ix_0\xi}\sin(y_0\eta)](x,y) = e^{-(\xi^2+\eta^2)kT}e^{ix\xi}\sin(y\eta). \tag{3.8}$$

Proof. By using the definition of $\mathcal{G}_D$ in (3.2) we have

$$\begin{aligned}
&\mathcal{G}_D[e^{ix_0\xi}\sin(y_0\eta)](x,y)\\
=&\frac{1}{4k\pi T}\int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{4kT}}e^{ix_0\xi}dx_0\int_0^{\infty}\left(e^{-\frac{(y-y_0)^2}{4kT}} - e^{-\frac{(y+y_0)^2}{4kT}}\right)\sin(y_0\eta)dy_0.
\end{aligned}$$

By using change of variables we have

$$\int_0^{\infty}\left(e^{-\frac{(y-y_0)^2}{4kT}} - e^{-\frac{(y+y_0)^2}{4kT}}\right)\sin(y_0\eta)dy_0 = \int_{-\infty}^{\infty} e^{-\frac{(y-y_0)^2}{4kT}}\sin(y_0\eta)dy_0$$

thus by using (3.7) we finally get

$$\mathcal{G}_D[e^{ix_0\xi}\sin(y_0\eta)](x,y) = e^{-\xi^2 kT}e^{ix\xi}e^{-\eta^2 kT}\sin(y\eta)$$

which completes the proof. □

The relation (3.8) inspires us that we can actually use the Fourier transform to solve (3.3). To do this, we shall actually extend the functions $\mu_0(\mathbf{x})$ and $h(\mathbf{x})$ to $\mathbb{R}^2$ by setting

$$\mu_0(x_1,x_2) = -\mu_0(x_1,-x_2) \quad h(x_1,x_2) = -h(x_1,-x_2) \quad x_2 < 0.$$

Under this kind of extension we have

$$\mathcal{G}_D[\mu_0](\mathbf{x}) = \frac{1}{4k\pi T}\int_{\mathbb{R}^2} e^{-\frac{(x_1-y_1)^2+(x_2-y_2)^2}{4kT}}\mu_0(y_1,y_2)dy_1dy_2. \tag{3.9}$$

and

$$\mathcal{G}_D[\mu_0](\mathbf{x}) = h(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2. \tag{3.10}$$

Define the Fourier transform operator $\mathcal{F}$ by

$$\hat{\mu}(\boldsymbol{\xi}) = \mathcal{F}\{\mu\}(\boldsymbol{\xi}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mu(\mathbf{x}) e^{-i\mathbf{x}\cdot\boldsymbol{\xi}} d\mathbf{x} \tag{3.11}$$

and the corresponding inverse Fourier transform $\mathcal{F}^{-1}$ by

$$\mu(\mathbf{x}) = \mathcal{F}^{-1}\{\hat{\mu}\}(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{\mu}(\boldsymbol{\xi}) e^{i\mathbf{x}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi} \tag{3.12}$$

where $\boldsymbol{\xi} = (\xi, \eta)$. Then by taking the Fourier transform on both sides of (3.10) and using (3.7) we obtain

$$\mathcal{F}\{h\}(\boldsymbol{\xi}) = \mathcal{F}\{\mathcal{G}_D[\mu_0]\}(\boldsymbol{\xi}) = e^{-kT|\boldsymbol{\xi}|^2}\hat{\mu}_0(\boldsymbol{\xi}).$$

Thus we have

$$\hat{\mu}_0(\boldsymbol{\xi}) = e^{kT|\boldsymbol{\xi}|^2}\hat{h}(\boldsymbol{\xi}) \tag{3.13}$$

and so

$$\mu_0(\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{kT|\boldsymbol{\xi}|^2}\hat{h}(\boldsymbol{\xi}) e^{i\mathbf{x}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi}.$$

We mention that to get $\hat{\mu}_0(\boldsymbol{\xi})$ directly from (3.13) is quite unstable especially when the data $h(\mathbf{x})$ is not exactly given. We shall use the iteration schemes to solve this kind of problem when the final over determination $\mu_T(\mathbf{x})$ and the boundary measurement $f(x,t)$ are not exactly given. For the sake of simplicity we denote by $h^\epsilon(\mathbf{x})$ the perturbation of the function $h(\mathbf{x})$ in (3.3) and satisfies

$$||h^\epsilon - h|| \le \epsilon, \quad \epsilon > 0 \tag{3.14}$$

where ϵ is a small number which can be treated as the noise level of the measurement. Similar to [13, 14], we introduce the following iteration scheme to solve (3.13)

$$\hat{\mu}_j^\epsilon(\boldsymbol{\xi}) = (1 - \sqrt[n]{v(\boldsymbol{\xi})})\hat{\mu}_{j-1}^\epsilon(\boldsymbol{\xi}) + \frac{\sqrt[n]{v(\boldsymbol{\xi})}}{v(\boldsymbol{\xi})}\chi_\varsigma \hat{h}^\epsilon(\boldsymbol{\xi}), \tag{3.15}$$

where

$$v(\boldsymbol{\xi}) = e^{-kT|\boldsymbol{\xi}|^2}$$

and χ_ς is the characteristic function of $\varsigma = \{\boldsymbol{\xi} |\, k|\boldsymbol{\xi}|^2 \le \rho\}$, that is

$$\chi_\varsigma = \begin{cases} 1 & k|\boldsymbol{\xi}|^2 \le \rho \\ 0 & k|\boldsymbol{\xi}|^2 > \rho \end{cases}.$$

The parameter ρ here is a selective positive number which can be chosen in a flexible way depending on the stopping rule that is used for terminating the iteration scheme. The parameter n is a selective nature number which is used to control the iteration speed. Generally speaking, the lager n is the less iteration steps are required to get an approximate solution. The iterative steps actually decrease exponentially with the parameter n increase normally. However, the error estimation for the solution behaviors badly as n increases as well.

4. Fourier transform of h

The iteration scheme (3.15) requires us to get the Fourier transform of the perturbation $h^\epsilon(\mathbf{x})$. In fact, the perturbation comes from the noise of the final observation μ_T and the boundary condition f in (3.4). In this section we consider the Fourier transform of h. To extend h to $\mathbb{R}^2$ we only need to extend the final observation $\mu_T(\mathbf{x})$ by

$$\mu_T(x_1, x_2) = -\mu_T(x_1, -x_2), \quad x_2 < 0.$$

We observe that (3.4) can be written as follows

$$h(\mathbf{x}) = \mu_T(\mathbf{x}) - 2k \int_0^T \frac{1}{\sqrt{4k\pi(T-s)}} \frac{\partial}{\partial y} e^{-\frac{x_2^2}{4k(T-s)}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{4k\pi(T-s)}} e^{-\frac{(x_1-y_1)^2}{4k(T-s)}} f(y_1, s) dy_1 ds. \tag{4.1}$$

Taking the Fourier transform on both sides of (4.1) and using (3.7) we get

$$\hat{h}(\boldsymbol{\xi}) = \hat{\mu}_T(\boldsymbol{\xi}) - \frac{2ik\eta}{\sqrt{2\pi}} \int_0^T e^{-k|\boldsymbol{\xi}|^2(T-s)} \hat{f}(\xi, s) ds \tag{4.2}$$

where by $\hat{f}(\xi, s)$ we mean the one dimensional Fourier transform of $f(x_1, s)$ on x_1, i.e.,

$$\hat{f}(\xi, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x_1, s) e^{-i\xi x_1} dx_1$$

and so the inverse one dimensional Fourier transform is

$$f(x_1, s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi, s) e^{i\xi x_1} d\xi.$$

5. Error estimates

In this section, we consider the convergence results of iteration scheme (3.15) by using both the *a priori* and the *a posteriori* stopping rules. Before presenting the error estimate of the initial temperature distribution $\mu_0(\mathbf{x})$, we suppose the *a priori* bound on $\mu_0(\mathbf{x})$

$$||\mu_0||_p \leq M, \quad p \geq 0 \tag{5.1}$$

where

$$||\mu_0||_p := \left(\int_{\mathbb{R}^2} (1 + |\boldsymbol{\xi}|^2)^p |\hat{\mu}_0(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \right)^{\frac{1}{2}}.$$

We see that if $p = 0$ then the norm $||\cdot||_p$ turns to L^2 norm.

5.1. *The* a priori *stopping rule*

By using the prior information (5.1) we can design the following stopping rule

$$j \sim \left\lfloor (M/\epsilon)^{1/n} \right\rfloor \tag{5.2}$$

where $\lfloor a \rfloor$ denotes the largest integer that is not greater than a. We note that in the stopping rule (5.2), if n increases then the iteration steps j are greatly reduced.

Theorem 5.1. Let $\mu_0(\mathbf{x})$ be the initial temperature distribution in (3.1). Suppose $\mu_T^\epsilon(\mathbf{x})$ is the measured temperature at $t = T$ and $f^\epsilon(x_1, t)$ is the measured boundary condition. Suppose the corresponding function $h^\epsilon(\mathbf{x})$ by (3.4) satisfies (3.14). Let $\mu_j^\epsilon(\mathbf{x})$ be the iterates defined by (3.15) with initial iterate $\mu_0^\epsilon(\mathbf{x}) = 0$. If (5.1) is satisfied for $M > 0$, $p > 0$ and we select $j \sim \left\lfloor (M/\epsilon)^{1/n} \right\rfloor$ as the prior stopping rule and

$$\rho = \frac{1}{T}\ln(\frac{M}{\delta}(\ln\frac{M}{\delta})^{-\frac{p}{2}})$$

then we have the following error estimate

$$\left\|\mu_j^\epsilon - \mu_0\right\|^2 \quad\leq\quad 2M^2\Big(\ln\frac{M}{\delta}\Big)^{-p}\left(1 + Cn^{2n} + 2k^pT^p\left(\frac{\ln\frac{M}{\delta}}{\ln\frac{M}{\delta} + \ln\,(\ln\frac{M}{\delta})^{-\frac{p}{2}}}\right)^p\right) \tag{5.3}$$

where C is a constant depending on j and n.

Proof. Set $\alpha = \sqrt[n]{\nu(\boldsymbol{\xi})}$ then $0 < \alpha \leq 1$. By (3.15) we obtain

$$\begin{aligned}\hat{\mu}_j^\epsilon(\boldsymbol{\xi}) &= (1-\alpha)\hat{\mu}_{j-1}^\epsilon(\boldsymbol{\xi}) + \alpha^{1-n}\chi_\varsigma\hat{h}^\epsilon(\boldsymbol{\xi}) \\ &= \sum_{i=0}^{j-1}(1-\alpha)^i\alpha^{1-n}\chi_\varsigma\hat{h}^\epsilon(\boldsymbol{\xi}).\end{aligned}$$

Take $p_j(\alpha) = \sum\limits_{i=0}^{j-1}(1-\alpha)^i$, $r_j(\alpha) = 1 - \alpha p_j(\alpha) = (1-\alpha)^j$, we have the following elementary results (cf. [33]):

$$p_j(\lambda)\lambda^m \leq j^{1-m}, \text{for all } 0 \leq m \leq 1, \tag{5.4}$$

$$r_j(\lambda)\lambda^q \leq \theta_q(j+1)^{-q}, \tag{5.5}$$

where

$$\theta_q = \begin{cases} 1 & 0 \leq q \leq 1 \\ q^q & q > 1. \end{cases}$$

From (3.13) we have

$$\hat{\mu}_0(\boldsymbol{\xi}) = \alpha^{-n}\hat{h}(\boldsymbol{\xi})$$

then there holds the estimates

$$\begin{aligned}\left\|\hat{\mu}_j^\epsilon - \hat{\mu}_0\right\|^2 &= \left\|p_j(\alpha)\alpha^{1-n}\chi_\varsigma\hat{h}^\epsilon - \hat{\mu}_0\right\|^2 \\ &= \left\|p_j(\alpha)\alpha^{1-n}(\chi_\varsigma\hat{h}^\epsilon - \hat{h}) - r_j(\alpha)\hat{\mu}_0\right\|^2 \\ &\leq 2\left\|p_j(\alpha)\alpha^{1-n}(\chi_\varsigma\hat{h}^\epsilon - \hat{h})\right\|^2 + 2\left\|r_j(\alpha)\hat{\mu}_0\right\|^2 \\ &:= 2I_1 + 2I_2.\end{aligned}$$

Next we derive the estimates for I_1 and I_2, respectively.

$$I_1 \quad=\quad \left\|p_j(\alpha)\alpha^{1-n}(\chi_\varsigma\hat{h}^\epsilon - \hat{h})\right\|^2$$

$$
\begin{aligned}
&= \int_{\varsigma}\left|p_j(\alpha)\alpha^{1-n}(\chi_{\varsigma}\hat{h}^{\epsilon}(\boldsymbol{\xi})-\hat{h}(\boldsymbol{\xi}))\right|^2 d\boldsymbol{\xi} + \int_{\mathbb{R}^2\setminus\varsigma}\left|p_j(\alpha)\alpha^{1-n}(\chi_{\varsigma}\hat{h}^{\epsilon}(\boldsymbol{\xi})-\hat{h}(\boldsymbol{\xi}))\right|^2 d\boldsymbol{\xi} \\
&\le \int_{\varsigma}\left|\alpha^{-n}(\hat{h}^{\epsilon}(\boldsymbol{\xi})-\hat{h}(\boldsymbol{\xi}))\right|^2 d\boldsymbol{\xi} + \int_{\mathbb{R}^2\setminus\varsigma}(1+|\boldsymbol{\xi}|^2)^{-p}(1+|\boldsymbol{\xi}|^2)^{p}|\hat{\mu}_0(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}.
\end{aligned}
$$

By using the definition of ς and the choice of ρ we thus have

$$
\begin{aligned}
I_1 &\le e^{2\rho T}\epsilon^2 + \sigma^p\rho^{-p}M^2 \\
&\le (\frac{M}{\epsilon})^2(\ln\frac{M}{\epsilon})^{-p}\epsilon^2 + k^pT^p\left(\ln(\frac{M}{\epsilon}(\ln\frac{M}{\epsilon})^{-\frac{p}{2}})\right)^{-p}M^2 \\
&\le M^2(\ln\frac{M}{\epsilon})^{-p}(1+k^pT^p\left(\frac{\ln\frac{M}{\epsilon}}{\ln\frac{M}{\epsilon}+\ln(\ln\frac{M}{\epsilon})^{-\frac{p}{2}}}\right)^p).
\end{aligned}
$$

With similar strategy and (5.2) we estimate I_2

$$
\begin{aligned}
I_2 &= \left\|r_j(\alpha)\hat{\mu}_0\right\|^2 = \int_{\varsigma}\left|r_j(\alpha)\hat{\mu}_0(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} + \int_{\mathbb{R}^2\setminus\varsigma}\left|r_j(\alpha)\hat{\mu}_0(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \\
&= \int_{\varsigma}\left|r_k(\alpha)\alpha^n\frac{v(\boldsymbol{\xi},t)}{v(\boldsymbol{\xi},T)}\hat{\varphi}_1(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} + \int_{\mathbb{R}^2\setminus\varsigma}(1+|\boldsymbol{\xi}|^2)^{-p}(1+|\boldsymbol{\xi}|^2)^{p}\left|r_j(\alpha)\hat{\mu}_0(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \\
&\le n^{2n}(j+1)^{-2n}e^{2\rho T}M^2 + k^p\rho^{-p}M^2 \\
&\le Cn^{2n}(\frac{M}{\epsilon})^2(\ln\frac{M}{\epsilon})^{-p}\epsilon^2 + k^pT^p\left(\ln(\frac{M}{\epsilon}(\ln\frac{M}{\epsilon})^{-\frac{p}{2}})\right)^{-p}M^2 \\
&\le M^2(\ln\frac{M}{\epsilon})^{-p}(Cn^{2n}+k^pT^p\left(\frac{\ln\frac{M}{\epsilon}}{\ln\frac{M}{\epsilon}+\ln(\ln\frac{M}{\epsilon})^{-\frac{p}{2}}}\right)^p),
\end{aligned}
$$

where (5.5) and the choice of ρ are taken into account and the selection of constant C is only dependent on j and n. We come to the conclusion by combining the estimates of I_1 and I_2 and using Parseval equality. □

5.2. The a posteriori *stopping rule*

The prior stopping rule needs the *a priori* bound of μ_0 to stop the iteration process. However if such bound can not be obtained accurately, the iteration may not stop at a "good time" and thus the result may not be satisfactory. Thus an *a posteriori* stopping rule is required. We introduce the widely-used "discrepancy principle" due to Morozov [34]. We set the discrepancy principle in the following form

$$\left\|\hat{h}^{\epsilon}-v\hat{\mu}^{\epsilon}_{j_*}\right\| \le \tau\epsilon < \left\|\hat{h}^{\epsilon}-v\hat{\mu}^{\epsilon}_{j}\right\|, \quad 0<j<j_* \tag{5.6}$$

where k_* is the first iteration step which satisfies the left inequality of (5.6).

Theorem 5.2. Let $\mu_0(\mathbf{x})$ be the initial temperature distribution in (3.1). Suppose $\mu^{\epsilon}_T(\mathbf{x})$ is the measured temperature at $t = T$ and $f^{\epsilon}(x,t)$ is the measured boundary condition. Suppose the corresponding function $h^{\epsilon}(\mathbf{x})$ by (3.4) satisfies (3.14). Let $\mu^{\epsilon}_j(\mathbf{x})$ be the iterates defined by (3.15) with initial iterate

$\mu_0^\epsilon(\mathbf{x}) = 0$. If (5.1) is satisfied for $M > 0$, $p > 0$ and we select the discrepancy principle (5.6) as the stopping rule and

$$\rho = \frac{1}{T}\ln\Big(\frac{M}{\delta}\big(\ln\frac{M}{\delta}\big)^{-\frac{p}{2}}\Big)$$

then we have the following error estimate

$$\left\|\mu_j^\epsilon - \mu_0\right\|^2 \leq 2M^2\left(\ln\frac{M}{\delta}\right)^{-p}\left(1 + Cn^{2n} + 2k^pT^p\left(\frac{\ln\frac{M}{\delta}}{\ln\frac{M}{\delta} + \ln\left(\ln\frac{M}{\delta}\right)^{-\frac{p}{2}}}\right)^p\right) \tag{5.7}$$

where C is a constant depending on j and n.

Proof. It sufficient to prove $j_* \sim (M/\epsilon)^{1/n}$ by Theorem 5.1. We have

$$\begin{aligned}\left\|\chi_\varsigma\hat{h}^\epsilon - \nu\hat{\mu}_j^\epsilon\right\| &= \left\|(1-\alpha)(\chi_\varsigma\hat{h}^\epsilon - \nu\hat{\mu}_{j-1}^\epsilon)\right\| = \left\|(1-\alpha)^j\chi_\varsigma\hat{h}^\epsilon\right\| \\ &\leq \left\|(1-\alpha)^j(\hat{h}^\epsilon - \hat{h})\right\| + \left\|(1-\alpha)^j\hat{h}\right\| \\ &\leq \epsilon + \left\|r_j(\alpha)\alpha^n\hat{\mu}_0\right\| \leq \epsilon + n^n j^{-n}M,\end{aligned}$$

furthermore

$$\begin{aligned}\left\|(1-\chi_\varsigma)\hat{h}^\epsilon\right\| &\leq \left\|(1-\chi_\varsigma)(\hat{h}^\epsilon - \hat{h})\right\| + \left\|(1-\chi_\varsigma)\hat{h}\right\| \\ &\leq \epsilon + \left(\int_{\mathbb{R}^2\backslash\varsigma}|\nu(\boldsymbol{\xi})\hat{\mu}_0(\boldsymbol{\xi})|^2 d\boldsymbol{\xi}\right)^{1/2} \\ &\leq \epsilon + e^{-\rho T}M \leq 2\epsilon\end{aligned}$$

thus we obtain

$$\left\|\hat{h}^\epsilon(\cdot) - \nu\hat{\mu}_j^\epsilon(\cdot,t)\right\| \leq 3\epsilon + n^n j^{-n}M,$$

Set $\tau > 3$ and according to (5.6) we have $(\tau - 3)\epsilon < n^n j^{-n}M$ and thus $j \sim \left\lfloor\left(\frac{M}{\epsilon}\right)^{1/n}\right\rfloor$. □

We remark that although the parameter ρ appears in (3.15) requires the *a priori* bound on μ_0, we do not care much on the choice. In fact the choice of ρ is just for technique setup. We can actually set a very large number M such that it is lager than the sharp upper bound on μ_0. By using the discrepancy principle (5.6) as the stopping rule, the convergence rate and reconstruction performance are still ensured.

6. Conclusions

We have introduced the layer potential technique to get the analytic solution of the heat diffusion problem in the half plane. The Green function has been constructed to avoid the usage of the Neumann boundary condition. We have combined the Fourier transform and the Green function method to get the relation between the initial temperature distribution and the given final observation. We designed an iteration scheme to reconstruct the initial temperature distribution. Although the whole analysis is on two dimensional space heat conduction problem, it can be similar used to analyze higher dimensional problem. The method can also be used to reconstruct the initial heat distribution on any kind of domain and numerical realization of two and three dimensional inverse heat conduction problem will be a forthcoming work.

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A. Proofs of Theorem 2.1 to 2.3

In this section, we show the main proofs to some Theorems for the sake of convenience to the reader.

Proof of Theorem 2.1 By using the Green's theorem, we have

$$\begin{aligned}
&\int_0^t \int_{\partial D} k\frac{\partial}{\partial \nu_y}\Gamma(\mathbf{x}-\mathbf{y}, k(t-s))u(\mathbf{y}, s) - \Gamma(\mathbf{x}-\mathbf{y}, k(t-s))k\frac{\partial}{\partial \nu_y}u(\mathbf{y}, s)d\sigma_y ds \\
=& \int_0^t \int_D k\Delta_y\Gamma(\mathbf{x}-\mathbf{y}, k(t-s))u(\mathbf{y}, s) - \Gamma(\mathbf{x}-\mathbf{y}, k(t-s))k\Delta_y u(\mathbf{y}, s)d\mathbf{y}ds \\
=& \int_0^t \int_D \Gamma_t(\mathbf{x}-\mathbf{y}, k(t-s))u(\mathbf{y}, s) - \Gamma(\mathbf{x}-\mathbf{y}, k(t-s))u_s(\mathbf{y}, s)d\mathbf{y}ds \\
=& \int_0^t \int_D -\Gamma_s(\mathbf{x}-\mathbf{y}, k(t-s))u(\mathbf{y}, s) - \Gamma(\mathbf{x}-\mathbf{y}, k(t-s))u_s(\mathbf{y}, s)d\mathbf{y}ds \\
=& \int_D -\Gamma(\mathbf{x}-\mathbf{y}, k(t-s))u(\mathbf{y}, s)\Big|_{s=0}^{t} d\mathbf{y} = -u(\mathbf{x}, t) + \int_D \Gamma(\mathbf{x}-\mathbf{y}, kt)u(\mathbf{y}, 0)d\mathbf{y}
\end{aligned}$$

where we have changed the order of integration and used the integral by parts. Thus there holds

$$u(\mathbf{x}, t) = \mathcal{V}_D^k[\mu_0](\mathbf{x}, t) + \mathcal{S}_D^k[\partial u/\partial \nu](\mathbf{x}, t) - k\mathcal{D}_D[f](\mathbf{x}, t)$$

which completes the proof. □

Proof of Theorem 2.2 We first show the relation between $G(\mathbf{x}, t)$ and $\Gamma(\mathbf{x}, t)$. If fact, let $v(\mathbf{x}, t)$ be the solution to

$$\begin{cases} v_t = \Delta v & \text{in } D \times \mathbb{R}_+ \\ v(\mathbf{x}, 0) = 0 & \text{in } D \\ v(\mathbf{x}, t) = \Gamma(\mathbf{x}, t) & \text{on } \partial D \times \mathbb{R}_+ \end{cases}$$

then it is easy to see that

$$G(\mathbf{x}, t) = \Gamma(\mathbf{x}, t) - v(\mathbf{x}, t).$$

By using the Green's formula there holds for $u(\mathbf{x}, t)$ and $v(\mathbf{x}, t)$

$$\begin{aligned}
&\int_D v(\mathbf{x}-\mathbf{y}, kt)\mu_0(\mathbf{y})d\mathbf{y} + \int_0^t \int_{\partial D} v(\mathbf{x}-\mathbf{y}, k(t-s))k\frac{\partial}{\partial \nu_y}u(\mathbf{y}, s)d\sigma_y ds \\
&- \int_0^t \int_{\partial D} k\frac{\partial}{\partial \nu_y}v(\mathbf{x}-\mathbf{y}, k(t-s))u(\mathbf{y}, s)d\sigma_y ds = 0
\end{aligned}$$

By subtracting the above equation from (2.6) and using the fact that

$$G(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \partial D \times \mathbb{R}_+$$

we immediately get the conclusion. □

Proof of Theorem 2.3 Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$. By using the reflection method, it is easy to find the Green function of (2.10) for $D = \mathbb{R} \times \mathbb{R}_+$

$$G(\mathbf{x} - \mathbf{y}, t) = \frac{1}{4k\pi t}\left(e^{-\frac{(x_1-x_2)^2+(y_1-y_2)^2}{4kt}} - e^{-\frac{(x_1-x_2)^2+(y_1+y_2)^2}{4kt}}\right)$$

To use the result in Theorem 2.2, we compute

$$\begin{aligned}\frac{\partial}{\partial \nu_y} G(\mathbf{x} - \mathbf{y}, k(t-s))\Big|_{\partial D} &= -\frac{\partial}{\partial y_0} G(\mathbf{x} - \mathbf{y}, k(t-s))\Big|_{y_0=0} \\ &= \frac{y}{4k^2\pi(t-s)^2} e^{-\frac{(x_1-y_1)^2+x_2^2}{4k(t-s)}}.\end{aligned}$$

By substituting the Green function and the above formula into the solution form in Theorem 2.2 we come to the conclusion. □

Conflict of interest

The authors declare there is no conflict of interest.

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