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http://www.aimspress.com/journal/era

ERA, 30(8): 3042–3057.
DOI: 10.3934/era.2022154
Received: 25 July 2021
Revised: 20 April 2022
Accepted: 18 May 2022
Published: 08 June 2022

Research article

Dynamics of L^p multipliers on harmonic manifolds

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Abstract: Let X be a complete, simply connected harmonic manifold of purely exponential volume growth. This class contains all non-flat harmonic manifolds of nonpositive curvature, and in particular all known examples of non-compact harmonic manifolds except for the flat spaces. We use the Fourier transform from [1] to investigate the dynamics on $L^p(X)$ for $p > 2$ of certain bounded linear operators $T : L^p(X) \to L^p(X)$ which we call "L^p-multipliers" in accordance with standard terminology. Examples of L^p-multipliers are given by the operator of convolution with an L^1 radial function, or more generally convolution with a finite radial measure. In particular elements of the heat semigroup $e^{t\Delta}$ act as multipliers. Given $2 < p < \infty$, we show that for any L^p-multiplier T which is not a scalar multiple of the identity, there is an open set of values of $\nu \in \mathbb{C}$ for which the operator $\frac{1}{\nu}T$ is chaotic on $L^p(X)$ in the sense of Devaney, i.e., topologically transitive and with periodic points dense. Moreover such operators are topologically mixing. We also show that there is a constant $c_p > 0$ such that for any $c \in \mathbb{C}$ with $\operatorname{Re} c > c_p$, the action of the shifted heat semigroup $e^{ct}e^{t\Delta}$ on $L^p(X)$ is chaotic. These results generalize the corresponding results for rank one symmetric spaces of noncompact type and harmonic NA groups (or Damek-Ricci spaces).

Keywords: L^p multipliers; heat semigroup; harmonic manifolds; Fourier transform; Devaney chaotic

1. Introduction

The study of chaos in linear dynamics originated in the work of Godefroy and Shapiro [2]. The dynamics of a linear operator T on a Frechet space X is said to be *chaotic* (in the sense of Devaney) if T is *hypercyclic* (i.e., has a dense orbit, equivalently is topologically transitive), and has a dense set of periodic points. There is now an extensive literature on chaotic and hypercyclic operators, of which a summary may be found in the books [3, 4]. We mention also the references [5–7], on the universality and hyperyclicity of the heat equation solution family of operators and semigroups, in which the study of the dynamics of these operators was initiated and continued.

In a geometric context, linear chaos has been investigated for the heat semigroup $e^{t\Delta}$ acting on the

Lebesgue spaces $L^p(X)$, for certain complete Riemannian manifolds X (where $\Delta = div\ grad$ is the Laplace-Beltrami operator on X). Ji and Weber considered finite volume locally symmetric spaces of rank one in [8], where they showed that for $p \in (1,2)$ there is a constant $c_p \in \mathbb{R}$ such that for $c > c_p$ the shifted semigroup $e^{t(\Delta+c)}$ is *subspace chaotic* on $L^p(X)$, i.e., there is a closed, invariant subspace such that the semigroup restricted to the subspace is chaotic. In [9], Ji and Weber investigated the case of symmetric spaces of noncompact type, and showed that in this setting for $p \in (2,\infty)$ there is a constant $c_p \in \mathbb{R}$ such that for $c > c_p$ the shifted semigroup $e^{t(\Delta+c)}$ is subspace chaotic on $L^p(X)$. In [10], Sarkar improved on the result of Ji and Weber for rank one symmetric spaces, by showing that for the *Damek-Ricci spaces* (these are certain solvable Lie groups equipped with a left-invariant metric, which include as a particular case rank one symmetric spaces of noncompact type [11]), for $p \in (2,\infty)$ there is a constant $c_p \in \mathbb{R}$ such that for $c > c_p$ the shifted semigroup $e^{t(\Delta+c)}$ is chaotic on $L^p(X)$, and not just subspace chaotic. Sarkar and Pramanik later showed that the same result also holds for higher rank symmetric spaces of noncompact type [12]. Ji and Weber also extended their results for locally symmetric spaces to the case of higher rank in [13]. Finally, in [14], Sarkar and Ray generalized the results on chaotic dynamics of the heat semigroup to the case of more general operators on symmetric spaces of noncompact type known as *Fourier multipliers* (these include as a particular case the operators $e^{t\Delta}$), showing that for $p \in (2,\infty)$, for any such operator T on $L^p(X)$ which is not a scalar multiple of the identity, there is a $z \in \mathbb{C}$ such that the operator zT is chaotic.

The aim of the present article is to generalize this last result to the case of a class of Riemannian manifolds known as *harmonic manifolds*. These include the rank one symmetric spaces and Damek-Ricci spaces as particular examples. A Riemannian manifold X is said to be *harmonic* if for any $x \in X$, sufficiently small geodesic spheres centered at x have constant mean curvature depending only on the radius of the sphere. Harmonic manifolds may be characterized in various equivalent ways, one characterization being that harmonic functions on the manifold satisfy the mean-value property with respect to geodesic spheres. The Lichnerowicz conjecture asserts that harmonic manifolds are either flat or locally symmetric of rank one. The conjecture holds in dimension less than or equal to 5 [15–17] and for compact simply connected harmonic manifolds [18], though it is false in general, with the Damek-Ricci spaces giving a family of counterexamples [11]. Heber showed however that the only complete, simply connected, homogeneous harmonic manifolds are the Euclidean spaces, rank one symmetric spaces, and the Damek-Ricci spaces [19]. For a survey of results on general noncompact harmonic manifolds we refer to [20].

In [21], a study of harmonic analysis on noncompact harmonic manifolds in terms of eigenfunctions of the Laplace-Beltrami operator Δ was initiated, where a Fourier transform was defined and a Plancherel theorem and Fourier inversion formula were proved. The paper [21] however only considered harmonic manifolds of strictly negative curvature. While this class contains the rank one symmetric spaces, it does not include the non-symmetric Damek-Ricci spaces, which are nonpositively curved, but not negatively curved. In [1], the results of [21] were generalized to the larger class of harmonic manifolds of *purely exponential volume growth* (see section 2 for the definition of purely exponential volume growth). This last class does include all Damek-Ricci spaces, and in particular contains all known examples of non-compact harmonic manifolds apart from the Euclidean spaces.

When X is a rank one symmetric space of noncompact type, an *L^p-multiplier* is a bounded operator $T : L^p(X) \to L^p(X)$ which is translation invariant. Examples of L^p-multipliers are given by convolution on the right with radial L^1-functions, or more generally convolution on the right with finite radial measures. For a general harmonic manifold as in our case, we define in section 2.4 a notion of L^p-multiplier as a bounded operator on $L^p(X)$ satisfying certain natural properties. Our class of L^p-multipliers includes the operators of convolution with radial L^1-functions, or more generally convolution with radial complex measures of finite total variation (see section 2.4 for the definition of convolution with radial functions and measures in a harmonic manifold).

The terminology "multiplier" is motivated by the following: for $p > 2$, if $T : L^p(X) \to L^p(X)$ is an L^p-multiplier, we show in section 2.4 that there exists a holomorphic function m_T on a certain horizontal strip $S_p \subset \mathbb{C}$ (see section 2.2 for the definition of S_p), called the *symbol* of T, such that for all C_c^∞-functions ϕ radial around a point $o \in X$, the spherical Fourier transform of $T\phi$ is given by

$$\widehat{T\phi}^{o}(\lambda) = m_T(\lambda) \cdot \widehat{\phi}^{o}(\lambda)\ ,\ \lambda \in S_p$$

(see section 2.3 for the definition of the spherical Fourier transform of a radial function). Moreover if T is not a scalar multiple of the identity, then we show that the function m_T is a nonconstant holomorphic function. We can now state our main theorem:

Theorem 1.1. *Let X be a complete, simply connected, harmonic manifold of purely exponential volume growth. Let $2 < p < \infty$ and let $T : L^p(X) \to L^p(X)$ be an L^p-multiplier with symbol m_T such that T is not a scalar multiple of the identity. Then for all $\lambda \in S_p$ such that $m_T(\lambda) \neq 0$, for any $\nu \in \mathbb{C}$ such that $|\nu| = |m_T(\lambda)|$ the dynamics of the operator $\frac{1}{\nu}T$ on $L^p(X)$ is topologically mixing with periodic points dense, in particular the dynamics is chaotic in the sense of Devaney.*

A particular case of multipliers is given by the heat semigroup $e^{t\Delta}$ on X. For a simply connected harmonic manifold, the heat kernel $H_t(x, y)$ is radial, i.e., there exists an L^1 function h_t radial around a basepoint $o \in X$ such that $H_t(x, y) = (\tau_x h_t)(y)$ (see [18]; here $\tau_x h_t$ denotes the x-translate of h_t as defined in section 2.4). The action of $e^{t\Delta}$ is thus given by convolution with the radial L^1 function h_t, so $e^{t\Delta}$ is an L^p-multiplier for all $p \in [1, +\infty]$. We determine the symbol of $e^{t\Delta}$ and then apply the previous theorem to obtain the following corollary:

Corollary 1.2. *Let X be a complete, simply connected, harmonic manifold of purely exponential volume growth, and let $2 < p < \infty, 1 < q < 2$ be such that $1/p + 1/q = 1$. There exists a constant $c_p = \frac{4\rho^2}{pq}$ such that the action of the shifted heat semigroup $(e^{ct}e^{t\Delta})_{t>0}$ on $L^p(X)$ is chaotic in the sense of Devaney for all $c \in \mathbb{C}$ with $\operatorname{Re} c > c_p$. In fact for any $t_0 > 0$, the operator $e^{ct_0}e^{t_0\Delta}$ on $L^p(X)$ is chaotic for all $c \in \mathbb{C}$ with $\operatorname{Re} c > c_p$.*

In section 2 we recall some basic facts about eigenfunctions of the Laplacian, the Fourier transform, and convolution on harmonic manifolds, show that convolution with a radial measure of finite variation is an L^p-multiplier, and prove existence of the symbol of a multiplier. In section 3 we prove the main theorem. We also prove the corollary by determining the symbol of the multiplier $e^{t\Delta}$.

2. Preliminaries

In this section we briefly recall the facts about the Fourier transform on harmonic manifolds which we will require. For details the reader is referred to [1]. Throughout, X will denote a complete, simply connected harmonic n-manifold of purely exponential volume growth. Here by purely exponential volume growth we mean that there are constants $C > 1, h > 0$ such that the volumes of geodesic balls $B(x, r)$ satisfy

$$\frac{1}{C}e^{hr} \leq vol(B(x, r)) \leq Ce^{hr}$$

for all $x \in X, r > 0$. We fix a basepoint $o \in X$.

2.1. Boundary, visibility measures and Busemann functions

In [22], it was shown that the hypotheses on X imply that X is a Gromov hyperbolic space, and so we have a *boundary at infinity* ∂X of the space X, defined as the set of equivalence classes of geodesic rays $\gamma : [0, \infty) \to X$, where two rays are equivalent if they stay at bounded distance from each other. We denote the equivalence class of a geodesic ray γ by $\gamma(\infty) \in \partial X$. There is a natural topology on $\overline{X} := X \cup \partial X$ called the *cone topology* for which $\overline{X}$ becomes a compactification of X (for details on Gromov hyperbolic spaces we refer to Chapter III.H of [23]).

Given a point $x \in X$, let λ_x be normalized Lebesgue measure on the unit tangent sphere $T^1_x X$, i.e., the unique probability measure on $T^1_x X$ invariant under the orthogonal group of the tangent space $T_x X$. For $v \in T^1_x X$, let $\gamma_v : [0, \infty) \to X$ be the unique geodesic ray with initial velocity v. Then we have a homeomorphism $p_x : T^1_x X \to \partial X, v \mapsto \gamma_v(\infty)$. The *visibility measure* on ∂X (with respect to the basepoint x) is defined to be the push-forward $(p_x)_* \lambda_x$ of λ_x under the map p_x; for notational convenience, we will however denote the visibility measure on ∂X by the same symbol λ_x.

Given a point $x \in X$ and a boundary point $\xi \in \partial X$, the *Busemann function at* ξ *based at* x is defined by

$$B_{\xi,x}(y) := \lim_{t \to \infty}(d(y, \gamma(t)) - d(x, \gamma(t)))$$

where $\gamma : [0, \infty) \to X$ is any geodesic ray such that $\gamma(\infty) = \xi$ (it is shown in [1] that the above limit exists and is independent of the choice of the ray γ). The Busemann functions $B_{\xi,x}$ are C^2 convex functions, and their level sets are called *horospheres* based at ξ.

2.2. Radial and horospherical eigenfunctions of the Laplacian

Let Δ denote the Laplace-Beltrami operator of X, or Laplacian. As X is harmonic, X is also *asymptotically harmonic*, i.e., all horospheres have constant mean curvature, so there is a constant h such that $\Delta B_{\xi,x} \equiv h$ for all $\xi \in \partial X, x \in X$. From [1], we know that in fact $h > 0$. We let

$$\rho := \frac{1}{2}h.$$

A function f on X is called radial around a point $x \in X$ if f is constant on geodesic spheres centered at x. For any $x \in X$ and $\lambda \in \mathbb{C}$, there is a unique eigenfunction $\phi_{\lambda,x}$ of Δ for the eigenvalue $-(\lambda^2 + \rho^2)$ which is radial around x and satisfies $\phi_{\lambda,x}(x) = 1$. Moreover for any fixed $y \in Y$, $\lambda \mapsto \phi_{\lambda,x}(y)$ is an entire function of λ. The functions $\phi_{\lambda,x}$ are real-valued for $\lambda \in \mathbb{R} \cup i\mathbb{R}$, and bounded by 1 for $|\operatorname{Im} \lambda| \leq \rho$. Given $p > 2$, for all λ in the strip $S_p := \{|\operatorname{Im} \lambda| < (1 - 2/p)\rho\}$, the function $\phi_{\lambda,x}$ is in $L^p(X)$.

For any $x \in X, \xi \in \partial X$ and $\lambda \in \mathbb{C}$, the function $e^{(i\lambda - \rho)B_{\xi,x}}$ is an eigenfunction of Δ for the eigenvalue $-(\lambda^2 + \rho^2)$. Note that this eigenfunction is constant on horospheres based at ξ.

2.3. The spherical and Helgason Fourier transforms

Let $f \in L^1(X)$. Given a point $x \in X$, the *spherical Fourier transform of f based at x* is the function $\widehat{f}^x$ on $\mathbb{R}$ defined by pairing f with the radial eigenfunctions $\phi_{\lambda,x}$:

$$\widehat{f}^x(\lambda) := \int_X f(y)\phi_{\lambda,x}(y)dvol(y)\ ,\ \lambda \in \mathbb{R}.$$

There exists a function c on $\mathbb{C} - \{0\}$ satisfying, for some constants $C, K > 0$, the estimates

$$\frac{1}{C}|\lambda| \leq |c(\lambda)|^{-1} \leq C|\lambda|, \qquad 0 < |\lambda| \leq K,$$
$$\frac{1}{C}|\lambda|^{(n-1)/2} \leq |c(\lambda)|^{-1} \leq C|\lambda|^{(n-1)/2}, \quad |\lambda| \geq K,$$

such that the following inversion formula for the spherical Fourier transform from [1] holds:

Theorem 2.1. *Let $f \in C_c^\infty(X)$ be radial around x. Then*

$$f(y) = \int_0^\infty \widehat{f}^x(\lambda)\phi_{\lambda,x}(y)|c(\lambda)|^{-2}d\lambda$$

for all $y \in X$.

Given $1 \leq q < 2$, if $p > 2$ is the conjugate exponent such that $1/p + 1/q = 1$, then using the fact that the functions $\phi_{\lambda,o}$ are in $L^p(X)$ for λ in the strip S_p, we have the following proposition from [1]:

Proposition 2.2. *Let $1 \leq q < 2$ and $p > 2$ be such that $1/p + 1/q = 1$. Then for any $x \in X$ and $f \in L^q(X)$, the spherical Fourier transform of f based at x is well-defined and extends to a holomorphic function on the strip S_p.*

Let $f \in C_c^\infty(X)$. Given $x \in X$, the *Helgason Fourier transform of f based at x* is the function $\widetilde{f}^x : \mathbb{C} \times \partial X \to \mathbb{C}$ defined by

$$\widetilde{f}^x(\lambda, \xi) := \int_X f(y)e^{(-i\lambda-\rho)B_{\xi,x}(y)}dvol(y)\ , \lambda \in \mathbb{C}, \xi \in \partial X$$

We have the following relation between the Helgason Fourier transforms based at two different basepoints $o, x \in X$:

$$\widetilde{f}^x(\lambda, \xi) = e^{(i\lambda+\rho)B_{\xi,o}(x)}\widetilde{f}^o(\lambda, \xi) \tag{2.1}$$

If f is radial around the point x then the Helgason Fourier transform reduces to the spherical Fourier transform,

$$\widetilde{f}^x(\lambda, \xi) = \widehat{f}^x(\lambda)\ , \lambda \in \mathbb{C}, \xi \in \partial X$$

From [1] we have the following inversion formula for the Helgason Fourier transform:

Theorem 2.3. *Let $x \in X$ and let $f \in C_c^\infty(X)$. Then*

$$f(y) = \int_0^\infty \int_{\partial X} \widetilde{f}^x(\lambda, \xi)e^{(i\lambda-\rho)B_{\xi,x}(y)}d\lambda_x(\xi)|c(\lambda)|^{-2}d\lambda$$

for all $y \in X$.

2.4. Convolution operators and L^p multipliers

Convolution with radial functions in harmonic manifolds was first considered in [18]. We describe below this notion of convolution.

For a point $x \in X$, let d_x denote the distance function from the point x, defined by $d_x(y) := d(x, y), y \in X$.

Given a function f on X radial around a point x, let u be a function on $[0, \infty)$ such that $f = u \circ d_x$. Given a point y in X, the y-translate of f is the function $\tau_y f$ radial around y defined by $\tau_y f := u \circ d_y$. It follows from the fact that X is harmonic that $\|\tau_y f\|_p = \|f\|_p$ for all $p \in [1, +\infty]$. Moreover if f is also in L^1, then the spherical Fourier transforms satisfy

$$\widehat{\tau_y f}^y(\lambda) = \widehat{f}^x(\lambda)$$

We note also from [1] that there is an even C^∞ function on $\mathbb{R}$ which we denote by ϕ_λ such that $\phi_{\lambda,x} = \phi_\lambda \circ d_x$. Thus the x-translate of the eigenfunction $\phi_{\lambda,o}$ radial around o is the eigenfunction $\phi_{\lambda,x}$ radial around x, $\tau_x \phi_{\lambda,o} = \phi_{\lambda,x}$.

For simplicity, in the sequel, unless otherwise mentioned, by "radial function" we will mean a function which is radial around the basepoint o. Likewise, by "spherical Fourier transform" we will mean the spherical Fourier transform based at o, unless otherwise mentioned.

Given $f, g \in L^1(X)$ with g radial, the convolution of f with g is the function $f * g$ on X defined by

$$(f * g)(x) = \int_X f(y)\tau_x g(y) dvol(y)$$

The integral above converges for a.e. x, and satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

We note that if $f \in L^\infty(X)$ and $g \in L^1(X)$ with g radial, then the integral defining $(f * g)(x)$ converges for all x and satisfies

$$\|f * g\|_\infty \leq \|f\|_\infty \|g\|_1$$

It follows by interpolation that for any $p \in [1, +\infty]$, convolution with a radial L^1 function g defines a bounded linear operator on $L^p(X)$ satisfying

$$\|f * g\|_p \leq \|f\|_p \|g\|_1$$

for all $f \in L^p(X)$.

A standard argument using the above inequality and density of $C_c^\infty(X)$ in $L^p(X)$ gives that if $\{\phi_n\}$ is an approximate identity, i.e., $\phi_n \geq 0$, $\int_X \phi_n dvol = 1$ and $\int_{B(o,r)} \phi_n dvol \to 1$ for any $r > 0$, then for any $f \in L^p(X)$,

$$\|f * \phi_n - f\|_p \to 0$$

as $n \to \infty$.

In [1] it is shown that for $\phi, \psi \in C_c^\infty(X)$ with ψ radial, the Helgason Fourier transform of the convolution $\phi * \psi$ satisfies

$$\widetilde{\phi * \psi}^o(\lambda, \xi) = \widetilde{\phi}^o(\lambda, \xi)\widehat{\psi}^o(\lambda) \,, \lambda \in \mathbb{C}, \xi \in \partial X$$

In particular, if both ϕ, ψ are radial, then

$$\widehat{\phi * \psi}^o(\lambda) = \widehat{\phi}^o(\lambda)\widehat{\psi}^o(\lambda)$$

We also have from [1] that the radial L^1 functions form a commutative Banach algebra under convolution. It follows, using density of radial C_c^∞-functions in radial L^p functions, that for a radial L^1 function g the convolution operator $T_g : f \mapsto f * g$ on $L^p(X)$ preserves the subspace of radial L^p functions and satisfies, for all radial C_c^∞-functions ϕ, ψ,

$$T_g\phi * \psi = \phi * T_g\psi$$

In fact for any $x \in X$ the convolution operator T_g preserves the subspace of L^p functions radial around x. This is a consequence of the following lemma:

Lemma 2.4. *Let ϕ, ψ be radial C_c^∞-functions. Then for any $x \in X$,*

$$\tau_x\phi * \psi = \tau_x(\phi * \psi)$$

Proof: We compute Helgason Fourier transforms:

$$\begin{aligned}
\widetilde{\tau_x\phi * \psi}^o(\lambda, \xi) &= \widetilde{\tau_x\phi}^o(\lambda, \xi)\widehat{\psi}^o(\lambda) \\
&= e^{-(i\lambda+\rho)B_{\xi,o}(x)}\widetilde{\tau_x\phi}^x(\lambda, \xi)\widehat{\psi}^o(\lambda) \\
&= e^{-(i\lambda+\rho)B_{\xi,o}(x)}\widehat{\tau_x\phi}^x(\lambda)\widehat{\psi}^o(\lambda) \\
&= e^{-(i\lambda+\rho)B_{\xi,o}(x)}\widehat{\phi}^o(\lambda)\widehat{\psi}^o(\lambda) \\
&= e^{-(i\lambda+\rho)B_{\xi,o}(x)}\widehat{\phi * \psi}^o(\lambda) \\
&= e^{-(i\lambda+\rho)B_{\xi,o}(x)}\widehat{\tau_x(\phi * \psi)}^x(\lambda) \\
&= e^{-(i\lambda+\rho)B_{\xi,o}(x)}\widetilde{\tau_x(\phi * \psi)}^x(\lambda, \xi) \\
&= \widetilde{\tau_x(\phi * \psi)}^o(\lambda, \xi)
\end{aligned}$$

It follows from the Fourier inversion formula (Theorem 2.3) that $\tau_x\phi * \psi = \tau_x(\phi * \psi)$. $\diamond$

Now given g a radial L^1 function and $\phi \in C_c^\infty(X)$, let $\{\psi_n\}$ be a sequence of radial C_c^∞-functions converging to g in L^1. Given $x \in X$, since ϕ and $\tau_x\phi$ are in L^∞, it follows that $\phi * \psi_n$ and $\tau_x\phi * \psi_n$ converge pointwise to $\phi * g$ and $\tau_x\phi * g$ respectively, so $\tau_x(\phi * \psi_n)$ converges pointwise to $\tau_x(\phi * g)$. Applying the previous Lemma, we obtain $\tau_x\phi * g = \tau_x(\phi * g)$. Thus the convolution operator T_g satisfies

$$T_g\tau_x\phi = \tau_x T_g\phi$$

for all radial C_c^∞ functions ϕ and all $x \in X$.

This leads us to the following definition:

Definition 2.5. *(**L^p-multipliers**) For $p \in [1, +\infty]$, an L^p-multiplier is a bounded operator $T : L^p(X) \to L^p(X)$ such that:*
(1) T preserves the subspace of radial L^p functions.

(2) For all radial C_c^∞-functions ϕ, ψ we have

$$T\phi * \psi = \phi * T\psi$$

(3) For all radial C_c^∞-functions ϕ and all $x \in X$ we have

$$T\tau_x\phi = \tau_x T\phi$$

Thus convolution operators given by radial L^1 functions are L^p multipliers for all $p \in [1, +\infty]$. For more general examples of L^p-multipliers we can consider convolution with radial complex measures μ of finite total variation, which is defined as follows:

We say that a complex measure μ on X is radial around o if there exists a complex measure $\widetilde{\mu}$ on $[0, \infty)$ such that for any continuous bounded function f on X we have

$$\int_X f(x)d\mu(x) = \int_0^\infty \left(\int_{S(o,r)} f(y)d\lambda_{o,r}(y) \right) d\widetilde{\mu}(r)$$

where $S(o, r)$ denotes the geodesic sphere of radius r around o and $\lambda_{o,r}$ denotes the volume measure on $S(o, r)$ induced from the metric on X. For $x \in X$, the x-translate of such a measure μ is the measure $\tau_x\mu$ radial around x defined by requiring that

$$\int_X f(y)d\tau_x\mu(y) = \int_0^\infty \left(\int_{S(x,r)} f(y)d\lambda_{x,r}(y) \right) d\widetilde{\mu}(r)$$

for all continuous bounded functions f on X (where $S(x, r)$ is the geodesic sphere of radius r around x and $\lambda_{x,r}$ is the volume measure on $S(x, r)$).

For an L^1 function f on X and a radial complex measure μ on X of finite variation, the convolution $f * \mu$ is the function on X defined by

$$(f * \mu)(x) := \int_X f(y)d\tau_x\mu(y)$$

We note that any L^1 function g which is radial around o gives a complex measure $\mu = g dvol$ which is radial around o and satisfies $||\mu|| = ||g||_1$ (where $||\mu||$ is the total variation norm of μ), and $f * \mu = f * g$, so convolution with finite variation radial measures generalizes convolution with L^1 radial functions.

Given a finite variation radial measure μ, we can approximate μ in the weak-* topology by measures $g_n dvol$ where g_n's are radial L^1 functions such that $||g_n||_1 \to ||\mu||$, then for any $f \in C_c^\infty(X)$ we have $f * g_n \to f * \mu$ pointwise, and an application of Fatou's Lemma then leads to the inequality

$$||f * \mu||_1 \leq ||f||_1 ||\mu||$$

valid for all $f \in C_c^\infty(X)$ and all finite variation radial measures μ. The inequality then continues to hold for all $f \in L^1(X)$ by density of $C_c^\infty(X)$ in $L^1(X)$.

Moreover for $f \in L^\infty(X)$ and μ a finite variation radial measure, it is straightforward to see that the integral defining $f * \mu$ exists for all x and satisfies

$$||f * \mu||_\infty \leq ||f||_\infty ||\mu||$$

Thus by interpolation for any $p \in [1, +\infty]$, convolution with a finite variation radial measure μ defines a bounded operator on $L^p(X)$ satisfying

$$||f * \mu||_p \leq ||f||_p ||\mu||$$

for all $f \in L^p(X)$.

Proposition 2.6. *Let μ be a radial complex measure of finite total variation. Then for any $p \in [1, +\infty]$, the operator $T_\mu : f \mapsto f * \mu$ is an L^p multiplier.*

Proof: Fix $p \in [1, \infty]$. Let $\{g_n\}$ be a sequence of radial L^1 functions such that $g_n dvol \to \mu$ in the weak-* topology and such that $||g_n||_1 \to ||\mu||$. Then for any radial C_c^∞-function ϕ, the functions $\phi * g_n$ are radial and converge to $\phi * \mu$ pointwise, so $\phi * \mu$ is radial. It follows that T_μ preserves the subspace of radial L^p functions.

Let ϕ, ψ be radial C_c^∞-functions. Then

$$||\phi * g_n||_\infty \leq ||\phi||_\infty ||g_n||_1 \leq C||\phi||_\infty$$

for some constant $C > 0$, so for any $x \in X$ the functions $\phi * g_n$ are uniformly bounded on the support of $\tau_x \psi$, and converge to $\phi * \mu$ pointwise, so it follows from dominated convergence that $(\phi * g_n) * \psi(x) \to (\phi * \mu) * \psi(x)$ for all $x \in X$. A similar argument gives that $\phi * (\psi * g_n)(x) \to \phi * (\psi * \mu)(x)$ for all $x \in X$. Since $(\phi * g_n) * \psi = \phi * (\psi * g_n)$ for all n, it follows that $(\phi * \mu) * \psi = \phi * (\psi * \mu)$.

Let ϕ be a radial C_c^∞-function and let $x \in X$. Then $\phi * g_n$ and $\tau_x \phi * g_n$ converge to $\phi * \mu$ and $\tau_x \phi * \mu$ respectively, so $\tau_x(\phi * g_n)$ converges pointwise to $\tau_x(\phi * \mu)$. Since $\tau_x \phi * g_n = \tau_x(\phi * g_n)$ for all n, it follows that $\tau_x \phi * \mu = \tau_x(\phi * \mu)$. $\diamond$

Let $1 \leq q < 2$ and $p > 2$ such that $1/p + 1/q = 1$. Let f be a radial L^q function, then the spherical Fourier transform $\widehat{f}$ is holomorphic in the strip S_p, and it turns out that for any radial C_c^∞-function ψ, we have

$$\widehat{f * \psi}(\lambda) = \widehat{f}(\lambda)\widehat{\psi}(\lambda) \,, \lambda \in S_p$$

This can be seen as follows: let $\{\phi_n\}$ be a sequence of radial C_c^∞-functions converging to f in $L^q(X)$, then since $\phi_{\lambda,o} \in L^p(X)$ for $\lambda \in S_p$, it follows from Holder's inequality that $\widehat{\phi_n}(\lambda) \to \widehat{f}(\lambda)$ for $\lambda \in S_p$. Moreover, since $\psi \in L^1(X)$, $\phi_n * \psi$ converges to $f * \psi$ in $L^q(X)$, so as before $\widehat{\phi_n * \psi}(\lambda) \to \widehat{f * \psi}(\lambda)$ for $\lambda \in S_p$. The desired equality follows by passing to the limit in the equality $\widehat{\phi_n * \psi}(\lambda) = \widehat{\phi_n}(\lambda)\widehat{\psi}(\lambda)$.

Other examples of L^p-multipliers can be obtained by using the *Kunze-Stein phenomenon* proved in [1]. This asserts that if $1 \leq q < 2$, then there is a constant $C_q > 0$ such that for all C_c^∞-functions f, g with g radial, we have

$$||f * g||_2 \leq C_q ||f||_2 ||g||_q.$$

Combining this with the trivial estimate

$$||f * g||_\infty \leq ||f||_\infty ||g||_1,$$

it follows from interpolation that for any $p > 2$, if $1 \leq r < 2$ is such that $1/r < 1 + 1/p$, then there is a constant $C_p > 0$ such that

$$||f * g||_p \leq C_p ||f||_p ||g||_r.$$

The above inequality then implies that convolution with a radial L^r-function g defines an L^p-multiplier $T_g : L^p(X) \to L^p(X)$.

The following proposition justifies the use of the term ”multiplier”:

Proposition 2.7. *Let $1 \leq q < 2$ and $p > 2$ be such that $1/p + 1/q = 1$. Let $T : L^p(X) \to L^p(X)$ be an L^p multiplier. Then there exists a holomorphic function m_T on the strip S_p such that, for any radial C_c^∞-function ϕ, we have $T\phi \in L^q(X)$, and*

$$\widehat{T\phi}(\lambda) = m_T(\lambda)\widehat{\phi}(\lambda) \; , \lambda \in S_p$$

Proof: We first show that given a radial C_c^∞ function ϕ, $T\phi \in L^q(X)$. For any radial C_c^∞-function ψ, we have

$$\begin{aligned}\left|\int_X T\phi(x)\psi(x)dvol(x)\right| &= |T\phi * \psi(o)| \\ &= |\phi * T\psi(o)| \\ &= \left|\int_X \phi(x)T\psi(x)dvol(x)\right| \\ &\leq ||\phi||_q ||T\psi||_p \\ &\leq (||T|| ||\phi||_q)||\psi||_p\end{aligned}$$

Since $T\phi$ is radial and the above inequality holds for all radial C_c^∞-functions ψ, it follows that $||T\phi||_q \leq ||T|| ||\phi||_q < +\infty$.

Thus for any radial C_c^∞-function ϕ which is not identically zero, $\widehat{T\phi}$ is a holomorphic function in the strip S_p, and we can define a meromorphic function m_ϕ on S_p by

$$m_\phi := \frac{\widehat{T\phi}}{\widehat{\phi}}$$

If ψ is another radial C_c^∞-function which is not identically zero, then the equality $T\phi * \psi = \phi * T\psi$ implies $\widehat{T\phi}\widehat{\psi} = \widehat{\phi}\widehat{T\psi}$ on S_p and hence $m_\phi = m_\psi$. Thus the meromorphic function m_ϕ is independent of the choice of ϕ, and we may denote it by m_T.

It suffices to show that m_T is in fact holomorphic in S_p. For this it is enough to show that given any $\lambda_0 \in S_p$, there is a radial C_c^∞-function ϕ such that $\widehat{\phi}(\lambda_0) \neq 0$, since then $m_T = \widehat{T\phi}/\widehat{\phi}$ will be holomorphic near λ_0. If $\widehat{\phi}(\lambda_0) = 0$ for all radial C_c^∞-functions ϕ, then

$$\int_X \phi(x)\phi_{\lambda_0,o}(x)dvol(x) = 0$$

for all such ϕ, and since $\phi_{\lambda_0,o}$ is radial this implies that $\phi_{\lambda_0,o} \equiv 0$, a contradiction. Thus m_T is holomorphic in S_p and by definition satisfies $\widehat{T\phi} = m_T\widehat{\phi}$ for all radial C_c^∞-functions ϕ. ⋄

Remark. If for $1 \leq q < 2$ we have an L^q-multiplier T, then by definition $T\phi \in L^q$ for ϕ a radial C_c^∞-function, and then the proof of the above proposition applies to show that for any L^q-multiplier T there is a function m_T holomorphic in the strip S_p such that $\widehat{T\phi}(\lambda) = m_T(\lambda)\widehat{\phi}(\lambda)$ for $\lambda \in S_p$ and ϕ a radial C_c^∞-function. Thus the conclusion of the proposition holds in fact for all L^p-multipliers with $p \neq 2$.

We will call the holomorphic function m_T given by the above proposition the *symbol* of the L^p-multiplier T. Note that if T is given by convolution with a radial L^1-function g, then the symbol m_T equals the spherical Fourier transform $\widehat{g}^o$ of g, since $\widehat{\phi * g}^o = \widehat{\phi}^o\widehat{g}^o$ for all radial C_c^∞-functions ϕ.

Proposition 2.8. *Let $1 \leq q < 2$ and $p > 2$ be such that $1/p + 1/q = 1$. Let $T : L^p(X) \to L^p(X)$ be an L^p-multiplier. Then for all $\lambda \in S_p$ and $x \in X$, we have*

$$T\phi_{\lambda,x} = m_T(\lambda)\phi_{\lambda,x}$$

Proof: Let $\lambda \in S_p$ and let $\{\phi_n\}$ be a sequence of radial C_c^∞-functions converging to $\phi_{\lambda,o}$ in $L^p(X)$. Then $T\phi_n$ converges to $T\phi_{\lambda,o}$ in $L^p(X)$. For any radial C_c^∞-function ψ, since $\psi \in L^q(X)$ it follows from Holder's inequality that

$$\int_X T\phi_n(x)\psi(x)dvol(x) \to \int_X T\phi_{\lambda,o}(x)\psi(x)dvol(x)$$

as $n \to \infty$. On the other hand, again using Holder's inequality and the fact that ϕ_n converges to $\phi_{\lambda,o}$ in $L^p(X)$, we have

$$\begin{aligned}
\int_X T\phi_n(x)\psi(x)dvol(x) &= T\phi_n * \psi(o) \\
&= \phi_n * T\psi(o) \\
&= \int_X \phi_n(x)T\psi(x)dvol(x) \\
&\to \int_X \phi_{\lambda,o}T\psi(x)dvol(x) \\
&= \widehat{T\psi}(\lambda) \\
&= m_T(\lambda)\widehat{\psi}(\lambda) \\
&= m_T(\lambda)\int_X \phi_{\lambda,o}(x)\psi(x)dvol(x)
\end{aligned}$$

Thus

$$\int_X T\phi_{\lambda,o}(x)\psi(x)dvol(x) = m_T(\lambda)\int_X \phi_{\lambda,o}(x)\psi(x)dvol(x)$$

for all radial C_c^∞-functions ψ, so it follows that $T\phi_{\lambda,o} = m_T(\lambda)\phi_{\lambda,o}$.

Now given $x \in X$ and $\lambda \in S_p$, the functions $\tau_x\phi_n$ converge to $\phi_{\lambda,x}$ in $L^p(X)$, and so

$$\begin{aligned}
T\phi_{\lambda,x} &= \lim_{n\to\infty} T\tau_x\phi_n \\
&= \lim_{n\to\infty} \tau_x T\phi_n \\
&= \tau_x T\phi_{\lambda,o} \\
&= m_T(\lambda)\tau_x\phi_{\lambda,o} \\
&= m_T(\lambda)\phi_{\lambda,x}
\end{aligned}$$

◇

3. Dynamics of L^p multipliers

3.1. General multipliers

We show in this section that the dynamics of appropriately scaled L^p-multipliers is chaotic in the sense of Devaney if $2 < p < \infty$. The following lemma is the key to the results which follow:

Lemma 3.1. *Let $1 < q < 2$ and $2 < p < \infty$ be such that $1/p + 1/q = 1$. Let $K \subset S_p$ be a subset such that K has a limit point in S_p. Then the subspace*

$$V_K := Span\{\tau_x \phi_{\lambda,o} | x \in X, \lambda \in K\}$$

is dense in $L^p(X)$.

Proof: It suffices to show that if $f \in L^q(X)$ is such that $\int_X f(y)\tau_x\phi_{\lambda,o}(y)dvol(y) = 0$ for all $x \in X, \lambda \in K$, then $f = 0$. Given such an $f \in L^q(X)$, the hypothesis on f means that for any $x \in X$, the spherical Fourier transform of f based at x vanishes on the set K. By Proposition 2.2, $\widehat{f^x}$ is holomorphic in S_p and K has a limit point in S_p, thus $\widehat{f^x}$ vanishes identically in S_p, in particular on $\mathbb{R}$. Thus for all $x \in X$ and $\lambda \in \mathbb{R}$, we have

$$(f * \phi_{\lambda,o})(x) = \int_X f(y)\phi_{\lambda,x}(y)dvol(y) = \widehat{f^x}(\lambda) = 0.$$

Let ϕ be a radial C_c^∞-function, then by the Fourier inversion formula (Theorem 2.1) we have

$$\phi(y) = \int_0^\infty \widehat{\phi}(\lambda)\phi_{\lambda,o}(y)|c(\lambda)|^{-2}d\lambda$$

for all $y \in X$, so it follows from Fubini's theorem that

$$(f * \phi)(x) = \int_0^\infty (f * \phi_{\lambda,o})(x)\widehat{\phi}(\lambda)|c(\lambda)|^{-2}d\lambda = 0$$

for all $x \in X$. Thus $f * \phi = 0$ for all radial C_c^∞-functions ϕ. Now letting $\{\phi_n\}$ be a sequence of radial C_c^∞-functions which forms an approximate identity, we have $f * \phi_n = 0$ for all n, and $f * \phi_n$ converges to f in $L^q(X)$, thus $f = 0$. ⋄

We will also need the following lemma:

Lemma 3.2. *Let $2 < p < \infty$ and let $T : L^p(X) \to L^p(X)$ be an L^p-multiplier. Suppose T is not a scalar multiple of the identity. Then the symbol m_T is a nonconstant holomorphic function in the strip S_p.*

Proof: Suppose to the contrary that $m_T \equiv C$ for some constant $C \in \mathbb{C}$. By Proposition 2.8 we then have $T\phi_{\lambda,x} = C\phi_{\lambda,x}$ for all $\lambda \in S_p$ and $x \in X$. Thus $T = CId$ on the subspace $V = Span\{\phi_{\lambda,x} | \lambda \in S_p, x \in X\}$, which is dense by the previous Lemma, hence $T = CId$ on $L^p(X)$, a contradiction. ⋄

The main tool to prove that the dynamics of L^p multipliers is chaotic is the following criterion of Godefroy-Shapiro (see [4], Theorem 3.1):

Theorem 3.3. *(***Godefroy-Shapiro criterion***) Let X be a separable Banach space and let $T : X \to X$ be a bounded operator. Suppose the subspaces X^+, X^- defined by*

$$X^+ = Span\{v \in X | Tv = \lambda v \text{ for some } \lambda \in \mathbb{C} \text{ such that } |\lambda| < 1\}$$

$$X^- = Span\{v \in X | Tv = \lambda v \text{ for some } \lambda \in \mathbb{C} \text{ such that } |\lambda| > 1\}$$

are dense in X. Then the dynamics of T on X is topologically mixing, i.e., for any two nonempty open sets $U, V \subset X$, there exists $N \geq 1$ such that $T^n U \cap V \neq \emptyset$ for all $n \geq N$.

We can now prove Theorem 1.1:

Proof of Theorem 1.1: Let $\lambda_0 \in S_p$ be such that $m_T(\lambda_0) \neq 0$, let $\nu \in \mathbb{C}$ be such that $|\nu| = |m_T(\lambda)|$ and set $\alpha = m_T(\lambda)/\nu \in S^1$. Let $\mathbb{D}_0 = \{z \in \mathbb{C} | |z| < 1\}$ and $\mathbb{D}_\infty = \{z \in \mathbb{C} | |z| > 1\}$. Let $U \subset S_p$ be an open neighbourhood of λ_0, then since $\alpha \in S^1$ and by Lemma 3.2 m_T is a nonconstant holomorphic function, there are nonempty open subsets $U^+, U^- \subset U$ such that $\{m_T(\lambda)/\nu | \lambda \in U^+\} \subset \mathbb{D}_0$ and $\{m_T(\lambda)/\nu | \lambda \in U^-\} \subset \mathbb{D}_\infty$. By Proposition 2.8, for all $\lambda \in U$ and $x \in X$, the function $\phi_{\lambda,x} \in L^p(X)$ is an eigenfunction of the operator $\frac{1}{\nu}T$ with eigenvalue $m_T(\lambda)/\nu$. By Lemma 3.1, the subspaces $V^+ = \{\phi_{\lambda,x} | \lambda \in U^+, x \in X\}$ and $V^- = \{\phi_{\lambda,x} | \lambda \in U^-, x \in X\}$ are dense in $L^p(X)$. It follows from the Godfrey-Shapiro criterion that the dynamics of $\frac{1}{\nu}T$ is topologically mixing.

It remains to show that the periodic points of $\frac{1}{\nu}T$ are dense in $L^p(X)$. Since m_T is a nonconstant holomorphic function and $m_T(\lambda_0)/\nu \in S^1$, we can choose sequences $\{\lambda_n\} \subset U$ and $\{p_n/q_n\} \subset \mathbb{Q}$ such that $m_T(\lambda_n)/\nu = e^{2\pi i p_n/q_n}$ and $\lambda_n \to \lambda_0$ as $n \to \infty$. Then by Lemma 3.1, the subspace $V = Span\{\phi_{\lambda_n,x} | x \in X, n \geq 1\}$ is dense in $L^p(X)$. It thus suffices to show that each element of V is a periodic point of $\frac{1}{\nu}T$. Any element $\phi \in V$ can be written as $\phi = \sum_{j=1}^N a_j \phi_{\lambda_j,x_j}$ for some $N \geq 1, a_1, \ldots, a_N \in \mathbb{C}$ and $x_1, \ldots, x_N \in X$. Since ϕ_{λ_j,x_j} is an eigenvector of $\frac{1}{\nu}T$ with eigenvalue $e^{2\pi i p_j/q_j}$, letting $q = \prod_{j=1}^N q_j$ it follows that $(\frac{1}{\nu}T)^q \phi_{\lambda_j,x_j} = \phi_{\lambda_j,x_j}$ for all j, thus $(\frac{1}{\nu}T)^q \phi = \phi$ and ϕ is a periodic point of $\frac{1}{\nu}T$. $\diamond$

3.2. The heat semigroup

We recall some basic facts about the heat semigroup and heat kernel on a complete Riemannian manifold X. Denote by $\Delta_X = div\ grad$ the Laplacian acting on $C_c^\infty(X) \subset L^2(X)$, then this is an essentially self-adjoint operator, and so its closure $\Delta_{X,2}$ is a self-adjoint operator on $L^2(X)$. Since $\Delta_{X,2}$ is negative, it generates a semigroup $e^{t\Delta_{X,2}}$ on $L^2(X)$ by the spectral theorem for unbounded self-adjoint operators. The operators $e^{t\Delta_{X,2}}$ are positive, leave $L^1(X) \cap L^\infty(X) \subset L^2(X)$ invariant, and may be extended to a positive contraction semigroup $e^{t\Delta_{X,p}}$ on $L^p(X)$ for any $p \in [1, +\infty]$, which is strongly continuous for $p \in [1, +\infty)$ [24]. In the sequel we will write simply $e^{t\Delta}$ for the semigroup $e^{t\Delta_{X,p}}$ on $L^p(X)$. From [25] we have the following:

There exists a C^∞ function $H_t(x, y)$ on $\mathbb{R}^+ \times X \times X$, the *heat kernel*, such that for all $t > 0$ and $x \in X$ the function $H_t(x, .)$ is positive and in L^p for all $p \in [1, +\infty]$, and for all $f \in L^p(X)$,

$$e^{t\Delta} f(x) = \int_X f(y) H_t(x, y) dvol(y)$$

and

$$\frac{\partial}{\partial t} e^{t\Delta} f(x) = \Delta e^{t\Delta} f(x).$$

Moreover, it is shown in [18] that for a X a simply connected harmonic manifold, the heat kernel is radial, i.e., there exists a function h_t radial around the basepoint o such that $H_t(x, y) = (\tau_x h_t)(y)$. Thus the action of the heat semigroup on $L^p(X)$ is given in our case by convolution with the radial L^1 function h_t,

$$e^{t\Delta} f = f * h_t$$

for all $f \in L^p(X)$, so $e^{t\Delta}$ is an L^p-multiplier for all $p \in [1, +\infty]$. The symbol of the multiplier $e^{t\Delta}$ is given by the following proposition:

Proposition 3.4. *For any $t > 0$, the spherical Fourier transform of the heat kernel is given by*

$$\widehat{h_t}^o(\lambda) = e^{-t(\lambda^2+\rho^2)}, \lambda \in S_\infty.$$

Proof: Let $p \in (2, \infty)$ and let $\lambda \in S_p$. Then $\phi_{\lambda,o} \in L^p(X)$, and using the fact that the operators $\Delta, e^{t\Delta}$ on $L^p(X)$ commute and $\Delta\phi_{\lambda,o} = -(\lambda^2 + \rho^2)\phi_{\lambda,o}$, we have

$$\begin{aligned} \frac{\partial}{\partial t} e^{t\Delta}\phi_{\lambda,o} &= \Delta e^{t\Delta}\phi_{\lambda,o} \\ &= e^{t\Delta}\Delta\phi_{\lambda,o} \\ &= -(\lambda^2 + \rho^2)e^{t\Delta}\phi_{\lambda,o}. \end{aligned}$$

Thus $t \mapsto e^{t\Delta}\phi_{\lambda,o} \in L^p(X)$ satisfies the first order linear ODE

$$\frac{\partial}{\partial t} e^{t\Delta}\phi_{\lambda,o} = -(\lambda^2 + \rho^2)e^{t\Delta}\phi_{\lambda,o}$$

and $e^{t\Delta}\phi_{\lambda,o} \to \phi_{\lambda,o}$ in $L^p(X)$ as $t \to 0$, hence

$$e^{t\Delta}\phi_{\lambda,o} = e^{-t(\lambda^2+\rho^2)}\phi_{\lambda,o}$$

for all $t > 0$. Evaluating both sides above at the point o gives

$$\begin{aligned} \widehat{h_t}^o(\lambda) &= \int_X \phi_{\lambda,o}(x)h_t(x)dvol(x) \\ &= e^{t\Delta}\phi_{\lambda,o}(o) \\ &= e^{-t(\lambda^2+\rho^2)}\phi_{\lambda,o}(o) \\ &= e^{-t(\lambda^2+\rho^2)}. \end{aligned}$$

◇

We can now prove the result on the chaotic dynamics of shifted heat semigroups:

Proof of Corollary 1.2: Given $2 < p < \infty$ and $1 < q < 2$ such that $1/p + 1/q = 1$, let $c_p = 4\rho^2/(pq)$. Let $c \in \mathbb{C}$ be such that $\mathrm{Re}\, c > c_p$, and let $t_0 > 0$. Let $T = e^{t_0\Delta}$ and $\nu = e^{-ct_0}$. By Proposition 3.4 above, the symbol of T is given by $m_T(\lambda) = e^{-t_0(\lambda^2+\rho^2)}$. In order to show that the operator $e^{ct_0}e^{t_0\Delta} = \frac{1}{\nu}T$ is chaotic, it suffices by Theorem 1.1 to show that there exists $\lambda \in S_p$ such that $|\nu| = |m_T(\lambda)|$.

Letting $\lambda = s + it \in S_p$, the equality $|\nu| = |m_T(\lambda)|$ is equivalent to

$$s^2 - t^2 + \rho^2 = \mathrm{Re}\, c.$$

Let t be such that $t = (1 - 2/p)\rho - \epsilon$ where $\epsilon > 0$ is small, then we have

$$\mathrm{Re}\, c + t^2 - \rho^2 = (\mathrm{Re}\, c - c_p) + c_p + ((1 - 2/p)^2 - 1)\rho^2 + O(\epsilon)$$

$$
\begin{aligned}
&= (\operatorname{Re} c - c_p) + (4(1/p)(1-1/p) - 4/p + 4/p^2)\rho^2 + O(\epsilon) \\
&= (\operatorname{Re} c - c_p) + O(\epsilon) \\
&> 0
\end{aligned}
$$

for ϵ small enough since $\operatorname{Re} c - c_p > 0$. Thus we can choose t with $0 < t < (1 - 2/p)\rho$ such that $\operatorname{Re} c + t^2 - \rho^2 > 0$, so we can then choose $s \in \mathbb{R}$ such that $s^2 = \operatorname{Re} c + t^2 - \rho^2$, or $s^2 - t^2 + \rho^2 = \operatorname{Re} c$, as required. $\diamond$

Acknowledgments

The authors would like to thank Swagato K. Ray for helpful discussions, and the Indian Statistical Institute Kolkata, at which the authors are faculty.

Conflict of interest

The authors declare there are no conflicts of interest.

References

1. K. Biswas, G. Knieper, N. Peyerimhoff, The Fourier transform on harmonic manifolds of purely exponential volume growth, *J. Geom. Anal.*, **31** (2021), 126–163. https://doi.org/10.1007/s12220-019-00253-9
2. G. Godefroy, J. H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, *J. Funct. Anal.*, **98** (1991), 229–269.
3. F. Bayart, E. Matheron, *Dynamics of Linear Operators*, Cambridge Tracts in Mathematics, 179, Cambridge University Press, Cambridge, 2009.
4. K. G. Grosse-Erdmann, A. Peris Manguillot, *Linear Chaos*, Universitext, Springer, London, 2011.
5. G. Herzog, On a universality of the heat equation, *Math. Nachr.*, **188** (1997), 169–171. https://doi.org/10.1002/mana.19971880110
6. R. deLaubenfels, H. Emamirad, K. G. Grosse-Erdmann, Chaos for semigroups of unbounded operators, *Math. Nachr.*, **261/262** (2003), 47–59. https://doi.org/10.1002/mana.200310112
7. J. A. Conejero, A. Peris, M. Trujillo, Chaotic asymptotic behavior of the hyperbolic heat transfer equation solutions, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, **20** (2010), 2943–2947. https://doi.org/10.1142/S0218127410027489
8. L. Ji, A. Weber, L^p spectral theory and heat dynamics of locally symmetric spaces, *J. Funct. Anal.*, **258** (2010), 1121–1139.
9. L. Ji, A. Weber, Dynamics of the heat semigroup on symmetric spaces, *Ergod. Theory Dyn. Syst.*, **30** (2010), 457–468. https://doi.org/10.1017/S0143385709000133
10. R. P. Sarkar, Chaotic dynamics of the heat semigroup on the Damek-Ricci spaces, *Israel J. Math.*, **198** (2013), 487–508. https://doi.org/10.1007/s11856-013-0035-6

11. E. Damek, F. Ricci, A class of nonsymmetric harmonic Riemannian spaces, *Bull. Amer. Math. Soc., N.S.*, **27** (1992), 139–142. https://doi.org/10.1090/S0273-0979-1992-00293-8

12. M. Pramanik, R. P. Sarkar, Chaotic dynamics of the heat semigroup on Riemannian symmetric spaces, *J. Funct. Anal.*, **266** (2014), 2867–2909. https://doi.org/10.1016/j.jfa.2013.12.026

13. L. Ji, A. Weber, The L^p spectrum and heat dynamics of locally symmetric spaces of higher rank, *Ergod. Theory Dyn. Syst.*, **35** (2015), 1524–1545. https://doi.org/10.1017/etds.2014.3

14. S. K. Ray, R. P. Sarkar, Chaotic behaviour of the Fourier multipliers on Riemannian symmetric spaces of noncompact type, *Preprint, https://arxiv.org/pdf/1805.10048.pdf*, 2017.

15. E. T. Copson, H. S. Ruse, Harmonic Riemannian spaces, *Proc. Roy. Soc. Edinburgh*, **60** (1940), 117–133. https://doi.org/10.1017/S0370164600020095

16. A. C. Walker. On Lichnerowicz's conjecture for harmonic 4-spaces. *J. London Math. Soc.*, **24** (1948), 317–329.

17. Y. Nikolayevsky, Two theorems on harmonic manifolds, *Comment. Math. Helv.*, **80** (2005), 29–50. https://doi.org/10.4171/CMH/2

18. Z. Szabo, The Lichnerowicz conjecture on harmonic manifolds, *J. Differ. Geometry*, **31** (1990), 1–28. https://doi.org/10.4310/jdg/1214444087

19. J. Heber, On harmonic and asymptotically harmonic homogeneous spaces, *Geom. Funct. Anal.*, **16** (2006), pages 869–890. https://doi.org/10.1007/s00039-006-0569-4

20. G. Knieper, N. Peyerimhoff, Noncompact harmonic manifolds, *Oberwolfach Preprints, https://arxiv.org/pdf/1302.3841.pdf*, 2013.

21. K. Biswas, The Fourier transform on negatively curved harmonic manifolds, *Preprint, https://arxiv.org/pdf/1802.07236.pdf*, 2018.

22. G. Knieper, New results on noncompact harmonic manifolds, *Comment. Math. Helv.*, **87** (2012), 669–703. https://doi.org/10.4171/CMH/265

23. M. R. Bridson, A. Haefliger, Metric spaces of non-positive curvature, *Grundlehren der mathematischen Wissenschaften, ISSN 0072-7830; 319*, 1999. https://doi.org/10.1007/978-3-662-12494-9

24. E. B. Davies, Heat kernels and spectral theory, *Cambridge Tracts in Mathematics*, **92**, Cambridge University Press, 1990. https://doi.org/10.1017/CBO9780511566158

25. R. S. Strichartz, Analysis of the Laplacian on the complete Riemannian manifold, *J. Funct. Anal.*, **52** (1983), 48–79. https://doi.org/10.1016/0022-1236(83)90090-3