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Research article

Some elementary properties of Laurent phenomenon algebras

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Abstract: Let Σ be a Laurent phenomenon (LP) seed of rank n, $\mathcal{A}(\Sigma)$, $\mathcal{U}(\Sigma)$, and $\mathcal{L}(\Sigma)$ be its corresponding Laurent phenomenon algebra, upper bound and lower bound respectively. We prove that each seed of $\mathcal{A}(\Sigma)$ is uniquely defined by its cluster and any two seeds of $\mathcal{A}(\Sigma)$ with $n-1$ common cluster variables are connected with each other by one step of mutation. The method in this paper also works for (totally sign-skew-symmetric) cluster algebras. Moreover, we show that $\mathcal{U}(\Sigma)$ is invariant under seed mutations when each exchange polynomials coincides with its exchange Laurent polynomials of Σ. Besides, we obtain the standard monomial bases of $\mathcal{L}(\Sigma)$. We also prove that $\mathcal{U}(\Sigma)$ coincides with $\mathcal{L}(\Sigma)$ under certain conditions.

Keywords: Laurent phenomenon algebra; cluster algebra; seed; upper bound; lower bound

1. Introduction

Cluster algebras were introduced by Fomin and Zelevinsky in [1]. The core idea to define cluster algebra of rank n is that one should have a *cluster seed* and an operator on cluster seeds, called *mutation*. Roughly, a cluster seed Σ_{t_0} is a collection of variables $x_{1;t_0}, \cdots, x_{n;t_0}$ (cluster variables) and binomials $F_{1;t_0}, \cdots, F_{n;t_0}$ (exchange polynomials). One can apply mutation to a cluster seed to produce a new seed, i.e., new variables and new binomials. Note that the exchange polynomial in cluster algebra is always a binomial. One of the main results in cluster algebras is that they have the Laurent phenomenon [1].

In the theory of cluster algebras, the following are interesting conjectures on seeds of cluster algebras: in a cluster algebra of rank n, (1) each seed is uniquely defined by its cluster; (2) any two seeds with $n-1$ common cluster variables are connected with each other by one step of mutation. One can refer to [2, 3] for detailed proof.

Significant notations of the upper cluster algebra, upper bound and lower bound associated with the cluster seed were introduced by Berenstein, Fomin and Zelevinsky to study the structure of cluster algebras in [4]. There are some theorems of upper bounds and lower bounds: (a) under the condition of coprimeness, the upper bound is invariant under seed mutations; (b) the standard monomials in $x_1, x'_1, \ldots, x_n, x'_n$ are linearly independent over $\mathbb{ZP}$ if and only if the cluster seed is acyclic; (c) under the conditions of acyclicity and coprimeness, then the upper bound coincides with the lower bound.

Muller showed that locally acyclic cluster algebras coincide with their upper cluster algebras in [5]. Gekhtman, Shapiro and Vainshte in [3] proved (a) for generalized cluster algebras, then Bai, Chen, Ding and Xu demonstrated (c) and the sufficiency of (b) in [6]. Besides, Bai discovered that acyclic generalized cluster algebras coincide with their generalized upper cluster algebras.

Laurent phenomenon (LP) algebras were introduced by Lam and Pylyavskyy in [7], which generalize cluster algebras from the perspective of exchange relations. The exchange polynomials in LP algebras were allowed to have arbitrarily many monomials, rather than being just binomials. It turns out that the Laurent phenomenon also appears in LP algebras [7].

One should note that our method also works for cluster algebras and generalized cluster algebras. We do not talk much about generalized cluster algebras in this paper, and one can refer to [2, 6, 8–10] for details.

In this paper, we first affirm the conjectures on seeds of cluster algebras with respect to LP algebras.

Theorem 1.1. *In a LP algebra of rank n,*

1) *(Theorem 3.1) each LP seed is uniquely defined by its cluster.*

2) *(Theorem 3.7) any two LP seeds with $n-1$ common cluster variables are connected with each other by one step of mutation.*

Second, we affirm theorems of upper bounds and lower bounds with respect to LP algebras under some conditions, by using the similar methods developed in [4].

Condition 1.2. *Let M_k be the lexicographically first monomial in the irreducible polynomial F_k and $f_k(x_i)$ be the polynomial on x_i in $R[x_2, \ldots, \hat{x}_i, \ldots, \hat{x}_k, \ldots, x_n](x_i)$ without constant terms in F_k for any $i \neq k$. Assume that for a LP seed $(\mathbf{x}, \mathbf{F})$ of rank n, $\forall k \in [1, n]$, F_k satisfies the following conditions:*

(i) $\hat{F}_k = F_k$.

(ii) *M_k is of the form* $\mathbf{x}^{\mathbf{v_k}} = \begin{cases} x_{k+1}^{v_{k+1,k}} \cdots x_n^{v_{n,k}} & k \in [1, n-1] \\ 1 & k = n \end{cases}$, *where* $\mathbf{v_k} \in \mathbb{Z}_{\geq 0}^{n-k}$ *for* $k \in [1, n-1]$.

(iii) *when $x_1 \in F_k$ for $k \neq 1$, $F_k = M_k + f_k(x_1)$.*

(iv) *when $x_1 \notin F_k$ for $k \neq 1$ or 2, if there exist an index i in $[2, k-1]$ such that $x_k \in M_i$, then $F_k = M_k + f_k(x_i)$.*

Theorem 1.3. *(a) (Theorem 4.9) Under (i) of Condition 1.2, the upper bound is invariant under LP mutations.*

(b) (Theorem 4.16) Under (i) and (ii) of Condition 1.2,the standard monomials in $x_1, x'_1, \ldots, x_n, x'_n$ form an R-basis for $\mathcal{L}(\Sigma)$.

(c) (Theorem 4.23) Under Condition 1.2, the upper bound coincides with the lower bound.

This paper is organized as follows: In Section 2, some basic definitions are given. In Section 3, we prove Theorem 1.1, and we give the corresponding results and applications in cluster algebras. In Section 4, we affirm Theorem 1.3.

2. Preliminaries

2.1. Laurent phenomenon algebra

Let a, b be positive integers satisfying $a \leq b$, write $[a, b]$ for $\{a, a+1, \ldots, b\}$.

Let R be a unique factorization domain over $\mathbb{Z}$, and the **ambient field** $\mathcal{F}$ be the rational function field in n independent variables over the field of fractions $\mathrm{Frac}(R)$. Recall that an element f of R is **irreducible** if it is non-zero, not a unit, and not be expressed as the product $f = gh$ of two elements $g, h \in R$ which are non-units.

Definition 2.1. A **Laurent phenomenon (LP) seed** of rank n in $\mathcal{F}$ is a pair $(\mathbf{x}, \mathbf{F})$, in which

(i) $\mathbf{x} = \{x_1 \cdots, x_n\}$ is a transcendence basis for $\mathcal{F}$ over $\mathrm{Frac}(R)$, where $\mathbf{x}$ is called the **cluster** of $(\mathbf{x}, \mathbf{F})$ and $x_1 \cdots, x_n$ are called **cluster variables**.

(ii) $\mathbf{F} = \{F_1, \cdots, F_n\}$ is a collection of irreducible polynomials in $R[x_1, \cdots, x_n]$ such that for each $i, j \in [1, n]$, $x_j \nmid F_i$ (F_i is not divisible by x_j) and F_i does not depend on x_i, where $F_1, \cdots, F_n$ are called the **exchange polynomials** of $(\mathbf{x}, \mathbf{F})$.

The following notations, definitions and propositions can refer to [7, 11].

Let F, N be two rational functions in $x_1, \cdots, x_n$. Denote by $F|_{x_j \leftarrow N}$ the expression obtained by substituting x_j in F by N. And if F involves the variable x_i, then we write $x_i \in F$. Otherwise, we write $x_i \notin F$.

Definition 2.2. Let $(\mathbf{x}, \mathbf{F})$ be a LP seed in $\mathcal{F}$. For each $F_j \in \mathbf{F}$, define a Laurent polynomial $\hat{F}_j = \frac{F_j}{x_1^{a_1} \cdots x_{j-1}^{a_{j-1}} x_{j+1}^{a_{j+1}} \cdots x_n^{a_n}}$, where $a_k \in \mathbb{Z}_{\geq 0}$ is maximal such that $F_k^{a_k}$ divides $F_j|_{x_k \leftarrow F_k/x'_k}$ as an element in $R[x_1, \cdots, x_{k-1}, (x'_k)^{-1}, x_{k+1}, \cdots, x_n]$. The Laurent polynomials in $\hat{\mathbf{F}} := \{\hat{F}_1, \cdots, \hat{F}_n\}$ are called the **exchange Laurent polynomials**.

From the definition of exchange Laurent polynomials, we know that $F_j/\hat{F}_j$ is a monomial in $R[x_1, \cdots, \hat{x}_j, \cdots, x_n]$, where $\hat{x}_j$ means x_j vanishes in the $\{x_1, \cdots, x_n\}$. And $\hat{F}_j|_{x_k \leftarrow F_k/x'_k}$ is not divisible by F_k.

Proposition 2.3. *(Lemma 2.4 of [7]) Let* $(\mathbf{x}, \mathbf{F})$ *be a LP seed in* $\mathcal{F}$*, then* $\mathbf{F} = \{F_1, \cdots, F_n\}$ *and* $\hat{\mathbf{F}} = \{\hat{F}_1, \cdots, \hat{F}_n\}$ *determine each other uniquely.*

Proposition 2.4. *(Lemma 2.7 of [7]) If* $x_k \in F_i$*, then* $x_i \notin F_k/\hat{F}_k$*. In particular,* $x_k \in F_i$ *implies that* $\hat{F}_k|_{x_i \leftarrow 0}$ *is well defined and* $\hat{F}_k|_{x_i \leftarrow 0} \in R[x_1^{\pm 1}, \cdots, \hat{x}_i, \cdots, \hat{x}_k, \cdots, x_n^{\pm 1}]$.

Definition 2.5. Let $(\mathbf{x}, \mathbf{F})$ be a LP seed in $\mathcal{F}$ and $k \in [1, n]$. Define a new pair

$$(\{x'_1, \cdots, x'_n\}, \{F'_1, \cdots, F'_n\}) := \mu_k(\mathbf{x}, \mathbf{F}),$$

where $x'_k = \hat{F}_k/x_k$ and $x'_i = x_i$ for $i \neq k$. And the exchange polynomials change as follows:

(1) $F'_k := F_k$;

(2) If $x_k \notin F_i$, then $F'_i := r_i F_i$, where r_i is a unit in R;

(3) If $x_k \in F_i$, then F'_i is obtained from the following three steps:

(i) Define $G_i := F_i|_{x_k \leftarrow N_k}$, where $N_k = \frac{\hat{F}_k|_{x_i \leftarrow 0}}{x'_k}$. Then we have

$$G_i \in R[x_1^{\pm 1}, \cdots, \hat{x}_i, \cdots, x_k'^{-1}, \cdots, x_n^{\pm 1}] = R[x_1'^{\pm 1}, \cdots, \hat{x}'_i, \cdots, x_k'^{-1}, \cdots, x_n'^{\pm 1}].$$

(ii) Define H_i to be G_i with all common factors (in $R[x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_k, \cdots, x_n]$) removed. Note that H_i is unique up to a unit in R and $H_i \in R[x_1'^{\pm 1}, \cdots, \hat{x}'_i, \cdots, x_k'^{-1}, \cdots, x_n'^{\pm 1}]$.

(iii) Let M be a Laurent monomial in $x'_1, \cdots, \hat{x}'_i, \cdots, x'_n$ with coefficient a unit in R such that $F'_i := MH_i \in R[x'_1, \cdots, x'_n]$ and is not divisible by any variable in $\{x'_1, \cdots, x'_n\}$. Thus

$$F'_i \in R[x'_1, \cdots, \hat{x}'_i, \cdots, x'_k, \cdots, x'_n].$$

Then we say that the new pair $\mu_k(\mathbf{x}, \mathbf{F})$ is obtained from the LP seed $(\mathbf{x}, \mathbf{F})$ by the **LP mutation** in direction k.

Example 2.6. Let $R = \mathbb{Z}$ and $\mathcal{F} = \mathbb{Q}(a, b, c)$. Consider the LP seed $(\mathbf{x}, \mathbf{F})$, where $\mathbf{x} = \{a, b, c\}$ and $\mathbf{F} = \{b + 1,\ a + c,\ b + 1\}$. From the definition of exchange Laurent polynomials, we can get $\hat{F}_a = \frac{F_a}{c},\ \hat{F}_b = F_b,\ \hat{F}_c = \frac{F_c}{a}$.

Let $(\mathbf{x}', \mathbf{F}') = \mu_a(\mathbf{x}, \mathbf{F})$, then we have $a' = \frac{\hat{F}_a}{a} = \frac{b+1}{ac},\ b' = b,\ c' = c$. From the definition of the LP mutation, the exchange polynomial F_a does not change. Since $a \notin F_c$, we have $F'_c = b + 1$ (or up to a unit). Since F_b depends on a, to compute F'_b, we need to procedure the above three steps. By (i), we get $N_a = \frac{1}{a'c}$ and $G_b = \frac{1}{a'c} + c$. By (ii), we get $H_b = G_b$ up to a unit in R. By (iii), $M = a'c$ and $F'_b = MH_b = a'c^2 + 1$. Thus the new seed can be chosen to be

$$(\mathbf{x}', \mathbf{F}') = \{(a', b + 1), (b, a'c^2 + 1), (c, b + 1)\}.$$

It is not clear a priori that the LP mutation $\mu_k(\mathbf{x}, \mathbf{F})$ of a LP seed $(\mathbf{x}, \mathbf{F})$ is still a LP seed because of the irreducibility requirement for the new exchange polynomials. But it can be seen from the following proposition that $\mu_k(\mathbf{x}, \mathbf{F})$ is still a LP seed in $\mathcal{F}$.

Proposition 2.7. *(Proposition 2.15 of [7]) Let* $(\mathbf{x}, \mathbf{F})$ *be a LP seed in* $\mathcal{F}$*, then* $\mu_k(\mathbf{x}, \mathbf{F})$ *is also a LP seed in* $\mathcal{F}$*.*

Proposition 2.8. *(Proposition 2.16 of [7]) If* $(\mathbf{x}', \mathbf{F}')$ *is obtained from* $(\mathbf{x}, \mathbf{F})$ *by LP mutation at k, then* $(\mathbf{x}, \mathbf{F})$ *can be obtained from* $(\mathbf{x}', \mathbf{F}')$ *by LP mutation at k. In this sense, LP mutation is an involution.*

Remark 2.9. It is important to note that because of (ii), F'_i is defined up to a unit in R. And this is the motivation to consider LP seeds up to an equivalence relation.

Definition 2.10. Let $\Sigma_{t_1} = (\mathbf{x}_{t_1}, \mathbf{F}_{t_1})$ and $\Sigma_{t_2} = (\mathbf{x}_{t_2}, \mathbf{F}_{t_2})$ be two LP seeds in $\mathcal{F}$. Σ_{t_1} and Σ_{t_2} are **equivalent** if for each $i \in [1, n]$, there exist r_i, r'_i which are units in R such that $x_{i;t_2} = r_i x_{i;t_1}$ and $F_{i;t_2} = r'_i F_{i;t_1}$.

Denote by $[\Sigma_t]$ the equivalent class of Σ_t, that is, $[\Sigma_t]$ is the set of LP seeds which are equivalent to Σ_t.

Proposition 2.11. *(Lemma 3.1 of [7]) Let* $\Sigma_{t_1} = (\mathbf{x}_{t_1}, \mathbf{F}_{t_1})$ *and* $\Sigma_{t_2} = (\mathbf{x}_{t_2}, \mathbf{F}_{t_2})$ *be two LP seeds in* $\mathcal{F}$*, and* $\Sigma_{t_u} = \mu_k(\Sigma_{t_2})$, $\Sigma_{t_v} = \mu_k(\Sigma_{t_1})$*. If* $[\Sigma_{t_1}] = [\Sigma_{t_2}]$*, then* $[\Sigma_{t_v}] = [\Sigma_{t_u}]$.

Let $\Sigma_t = (\mathbf{x}_t, \mathbf{F}_t)$ be a LP seed in $\mathcal{F}$. By the above proposition, it is reasonable to define LP mutation of $[\Sigma_t]$ at k given by $\mu_k([\Sigma_t]) := [\mu_k(\Sigma_t)]$.

Definition 2.12. A **Laurent phenomenon (LP) pattern** $\mathcal{S}$ in $\mathcal{F}$ is an assignment of each LP seed $(\mathbf{x}_t, \mathbf{F}_t)$ to a vertex t of the n-regular tree $\mathbb{T}_n$, such that for any edge $t \overset{k}{\text{———}} t'$, $(\mathbf{x}_{t'}, \mathbf{F}_{t'}) = \mu_k(\mathbf{x}_t, \mathbf{F}_t)$.

We always denote by $\mathbf{x}_t = \{x_{1;t}, \cdots, x_{n;t}\}$ and $\mathbf{F}_t = \{F_{1;t}, \cdots, F_{n;t}\}$.

Definition 2.13. Let $\mathcal{S}$ be a LP pattern, the **Laurent phenomenon (LP) algebra** $\mathcal{A}(\mathcal{S})$ (of rank n) associated with $\mathcal{S}$ is the R-subalgebra of $\mathcal{F}$ generated by all the cluster variables in the seeds of $\mathcal{S}$.

If $\Sigma = (\mathbf{x}, \mathbf{F})$ is any seed in $\mathcal{F}$, we shall write $\mathcal{A}(\Sigma)$ to mean the LP algebra $\mathcal{A}(\mathcal{S})$ associated with $\mathcal{S}$ containing the seed Σ .

Theorem 2.14. *(Theorem 5.1 of [7], the Laurent phenomenon) Let* $\mathcal{A}(\mathcal{S})$ *be a LP algebra, and* $(\mathbf{x}_{t_0}, \mathbf{F}_{t_0})$ *be a LP seed of* $\mathcal{A}(\mathcal{S})$*. Then any cluster variable* $x_{i;t}$ *of* $\mathcal{A}(\mathcal{S})$ *is in the Laurent polynomial ring* $R(t_0^{\pm 1}) := R[x_{1;t_0}^{\pm 1}, \cdots, x_{n;t_0}^{\pm 1}]$.

Definition 2.15. Let $\Sigma = (\mathbf{x}, \mathbf{F})$ be a LP seed of rank n and $k \in [1, n]$. A new seed $\Sigma^* = (\mathbf{x}^*, \mathbf{F}^*)$ of rank $n-1$ is defined as follows:

1) let $R^* = R[x_k^{\pm 1}]$.

2) $\mathbf{x}^* = \mathbf{x} - \{x_k\}$.

3) let $\mathbf{F}^* = \{F_j^* = F_j x_k^a \mid j \in [1, n] - k,\ a\ is\ the\ power\ of\ x_k\ in\ \hat{F}_j/F_j\}$.

The seed Σ^* is in fact a LP seed, then Σ^* is called **the freezing of the LP seed** Σ at x_k. $\mathcal{A}(\Sigma^*) \subset \mathcal{F} = \mathrm{Frac}(R^*[x_1, \ldots, \hat{x}_k, \ldots, x_n])$ is defined to be the subalgebra generated by all the cluster variables from LP seeds mutation-equivalent to Σ^*. Then $\mathcal{A}(\Sigma^*)$ is called the **freezing of the LP algebra** $\mathcal{A}(\Sigma)$ at x_k.

Example 2.16. Consider the LP seed $\Sigma = (\mathbf{x}, \mathbf{F}) = \{(a, b+1), (b, a+c), (c, b+1)\}$ over $R = \mathbb{Z}$ from Example 2.6. We produce the freezing of $(\mathbf{x}, \mathbf{F})$ at c as follows: first, remove $(c, b+1)$; next, since the powers of c in $\hat{F}_a$ and $\hat{F}_b$ are -1 and 0 respectively, we have $F_a^* = F_a c^{-1} = \frac{b+1}{c}$ and $F_b^* = F_b$. Then the LP seed Σ^* are $\{(a, \frac{b+1}{c}), (b, a+c)\}$ over $\mathbb{Z}[c^{\pm 1}]$.

Proposition 2.17. *(Proposition 3.7 of [7]) The algebra* $\mathcal{A}(\Sigma^*)$ *is a LP algebra.*

Corollary 2.18. The freezing of the LP seed at x_i is compatible with the mutation in direction j for $j \neq i$.

2.2. Cluster algebra

Recall that an integer matrix $B_{n\times n} = (b_{ij})$ is called **skew-symmetrizable** if there is a positive integer diagonal matrix D such that DB is skew-symmetric, where D is said to be the **skew-symmetrizer** of B. $B_{n\times n} = (b_{ij})$ is **sign-skew-symmetric** if $b_{ij}b_{ji} < 0$ or $b_{ij} = b_{ji} = 0$ for any $i, j \in [1, n]$. A sign-skew-symmetric B is **totally sign-skew-symmetric** if any matrix B' obtained from B by a sequence of mutations is sign-skew-symmetric. It is known that skew-symmetrizable integer matrices are always totally sign-skew-symmetric.

The **diagram** for a sign-skew-symmetric matrix $B_{n\times n}$ is the directed graph $\Gamma(B)$ with the vertices $1, 2, \cdots, n$ and the directed edges from i to j if $b_{ij} > 0$. $B_{n\times n}$ is called **acyclic** if $\Gamma(B)$ has no oriented cycles. As shown in [12], an acyclic sign-skew-symmetric integer matrix B is always totally sign-skew-symmetric.

Let $\mathbb{P}$ be the coefficient group, its group ring $\mathbb{ZP}$ is a domain [1]. We take an ambient field $\mathcal{F}$ to be the field of rational functions in n independent variables with coefficients in $\mathbb{ZP}$.

Definition 2.19. A **cluster seed** in $\mathcal{F}$ is a triplet $\Sigma = (\mathbf{x}, \mathbf{y}, B)$ such that

(i) $\mathbf{x} = \{x_1, \cdots, x_n\}$ is a transcendence basis for $\mathcal{F}$ over Frac($\mathbb{ZP}$). $\mathbf{x}$ is called the **cluster** of $(\mathbf{x}, \mathbf{y}, B)$ and $\{x_1 \cdots, x_n\}$ are called **cluster variables**.

(ii) $\mathbf{y} = \{y_1, \cdots, y_n\}$ is a subset of $\mathbb{P}$, where $\{y_1, \cdots, y_n\}$ are called **coefficients**.

(iii) $B = (b_{ij})$ is a $n \times n$ totally sign-skew-symmetric matrix, called an **exchange matrix**.

Let $(\mathbf{x}, \mathbf{y}, B)$ be a cluster seed in $\mathcal{F}$, one can associate binomials $\{F_1, \cdots, F_n\}$ defined by

$$F_j = \frac{y_j}{1 \oplus y_j} \prod_{b_{ij}>0} x_i^{b_{ij}} + \frac{y_j}{1 \oplus y_j} \prod_{b_{ij}<0} x_i^{-b_{ij}}.$$

$\{F_1, \cdots, F_n\}$ are called the **exchange polynomials** of $(\mathbf{x}, \mathbf{y}, B)$.

Note that the coefficients and the exchange matrices in a cluster algebra are used for providing the exchange polynomials and explaining how to produce new exchange polynomials when doing a mutation (see Definition 2.20) on a cluster seed.

Definition 2.20. Let $\Sigma = (\mathbf{x}, \mathbf{y}, B)$ be a cluster seed in $\mathcal{F}$. Define the **mutation** of Σ in the direction $k \in [1, n]$ as a new triple $\Sigma' = (\mathbf{x}', \mathbf{y}', B') := \mu_k(\Sigma)$ in $\mathcal{F}$, where

$$x'_i = \begin{cases} F_k/x_k & i = k \\ x_i & i \neq k. \end{cases}, \quad y'_i = \begin{cases} y_k^{-1} & i = k \\ y_i y_k^{max(b_{ki},0)}(1 \bigoplus y_k)^{-b_{ki}} & i \neq k. \end{cases},$$

$$\text{and } b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + sgn(b_{ik}) max(b_{ik}b_{kj}, 0) & otherwise \end{cases}.$$

It can be seen that $\mu_k(\Sigma)$ is also a cluster seed and the mutation of a cluster seed is an involution, that is, $\mu_k(\mu_k(\Sigma)) = \Sigma$.

Definition 2.21. A **cluster pattern** $\mathcal{S}$ is an assignment of a seed $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ to every vertex t of the n-regular tree $\mathbb{T}_n$, such that for any edge $t \overset{k}{—} t'$, $\Sigma'_{t'} = (\mathbf{x}_{t'}, \mathbf{y}_{t'}, B_{t'}) = \mu_k(\Sigma_t)$.

We always denote by $\mathbf{x}_t = (x_{1;t}, \cdots, x_{n;t})$, $\mathbf{y}_t = (y_{1;t}, \cdots, y_{n;t})$, $B_t = (b^t_{ij})$.

Definition 2.22. Let $\mathcal{S}$ be a cluster pattern, the **cluster algebra** $\mathcal{A}(\mathcal{S})$ (of rank n) associated with the given cluster pattern $\mathcal{S}$ is the $\mathbb{ZP}$-subalgebra of the field $\mathcal{F}$ generated by all cluster variables of $\mathcal{S}$.

Theorem 2.23. *(Theorem 3.1 of [1], the Laurent phenomenon) Let $\mathcal{A}(\mathcal{S})$ be a cluster algebra, and $\Sigma_{t_0} = (\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be a cluster seed of $\mathcal{A}(\mathcal{S})$. Then any cluster variable $x_{i;t}$ of $\mathcal{A}(\mathcal{S})$ is in the Laurent polynomial ring $\mathbb{ZP}(t_0^{\pm 1}) := \mathbb{ZP}[x_{1;t_0}^{\pm 1}, \cdots, x_{n;t_0}^{\pm 1}]$.*

Example 2.24. Let $B = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$, then the exchange polynomials of the cluster seed $(\mathbf{x}, B)$ are the following two polynomials

$$F_1 = x_2^3 + 1 = (x_2 + 1)(x_2^2 - x_2 + 1),$$
$$F_2 = x_1^3 + 1 = (x_1 + 1)(x_1^2 - x_1 + 1).$$

It is easy to see that the exchange polynomials F_1, F_2 of $(\mathbf{x}, B)$ are both reducible in the above example. Thus the cluster $\mathbf{x}$ and the exchange polynomial $\mathbf{F}$ of $(\mathbf{x}, B)$ can not define a LP seed. From [7], we know that sometimes a cluster algebra defines a LP algebra indeed.

Theorem 2.25. *(Theorem 4.5 of [7]) Every cluster algebra with principal coefficients is a Laurent phenomenon algebra.*

3. LP seeds determined by either clusters or mutations

3.1. On Theorem 1.1 (1)

Theorem 3.1. *Let $\mathcal{A}(\mathcal{S})$ be a LP algebra of rank n, and $(\mathbf{x}_{t_1}, \mathbf{F}_{t_1}), (\mathbf{x}_{t_2}, \mathbf{F}_{t_2})$ be two LP seeds of $\mathcal{A}(\mathcal{S})$.*

1) *If there exists a permutation σ of $[1, n]$ and a unit $r_i \in R$ such that $x_{i;t_2} = r_i x_{\sigma(i);t_1}$ for $i \in [1, n]$, then $F_{i;t_2} = r_i' F_{\sigma(i);t_1}$ as polynomials for a certain unit r_i' in R.*
2) *each LP seed is uniquely defined by its cluster.*

Proof. For any fixed $k \in [1, n]$, let $(\mathbf{x}_u, \mathbf{F}_u) = \mu_k(\mathbf{x}_{t_2}, \mathbf{F}_{t_2})$ and $(\mathbf{x}_v, \mathbf{F}_v) = \mu_{\sigma(k)}(\mathbf{x}_{t_1}, \mathbf{F}_{t_1})$, we consider the Laurent expansion of $x_{k;u}$ with respect to $\mathbf{x}_v$ and the Laurent expansion of $x_{\sigma(k);v}$ with respect to $\mathbf{x}_u$.

From the definition of the LP mutation, we know

$$x_{i;u} = \begin{cases} x_{i;t_2} & \text{if } i \neq k \\ \frac{\hat{F}_{k;t_2}}{x_{k;t_2}} & \text{if } i = k \end{cases} \text{ and } x_{i;v} = \begin{cases} x_{i;t_1} & \text{if } i \neq \sigma(k) \\ \frac{\hat{F}_{\sigma(k);t_1}}{x_{\sigma(k);t_1}} & \text{if } i = \sigma(k) \end{cases}. \tag{3.1}$$

Since $x_{i;t_2} = r_i x_{\sigma(i);t_1}$ for $i \in [1, n]$, we have $x_{i;u} = r_i x_{\sigma(i);v}$ for $i \neq k$. By (3.1), we get

$$\begin{aligned} x_{k;u} &= \hat{F}_{k;t_2}(x_{1;t_2}, \cdots, \hat{x}_{k;t_2}, \cdots, x_{n;t_2})/x_{k;t_2} \\ &= \hat{F}_{k;t_2}(r_1 x_{\sigma(1);t_1}, \cdots, \hat{x}_{\sigma(k);t_1}, \cdots, r_n x_{\sigma(n);t_1})/(r_k x_{\sigma(k);t_1}) \\ &= \hat{F}_{k;t_2}(r_1 x_{\sigma(1);v}, \cdots, \hat{x}_{\sigma(k);v}, \cdots, r_n x_{\sigma(n);v})/(r_k x_{\sigma(k);t_1}); \\ x_{\sigma(k);v} &= \hat{F}_{\sigma(k);t_1}(x_{1;t_1}, \cdots, \hat{x}_{\sigma(k);t_1}, \cdots, x_{n;t_1})/x_{\sigma(k);t_1} \\ &= \hat{F}_{\sigma(k);t_1}(x_{1;v}, \cdots, \hat{x}_{\sigma(k);v}, \cdots, x_{n;v})/x_{\sigma(k);t_1}. \end{aligned}$$

Thus $\frac{x_{k;u}}{x_{\sigma(k);v}} = \frac{\hat{F}_{k;t_2}(r_1 x_{\sigma(1);v}, \cdots, \hat{x}_{\sigma(k);v}, \cdots, r_n x_{\sigma(n);v})}{r_k \hat{F}_{\sigma(k);t_1}(x_{1;v}, \cdots, \hat{x}_{\sigma(k);v}, \cdots, x_{n;v})}$ and we get that

$$x_{k;u} = x_{\sigma(k);v} \frac{\hat{F}_{k;t_2}(r_1 x_{\sigma(1);v}, \cdots, \hat{x}_{\sigma(k);v}, \cdots, r_n x_{\sigma(n);v})}{r_k \hat{F}_{\sigma(k);t_1}(x_{1;v}, \cdots, \hat{x}_{\sigma(k);v}, \cdots, x_{n;v})}. \tag{3.2}$$

From the definition of the exchange Laurent polynomial, we know the above equation has the form of

$$x_{k;u} = x_{\sigma(k);v} \frac{F_{k;t_2}(r_1 x_{\sigma(1);v}, \cdots, \hat{x}_{\sigma(k);v}, \cdots, r_n x_{\sigma(n);v})}{r_k F_{\sigma(k);t_1}(x_{1;v}, \cdots, \hat{x}_{\sigma(k);v}, \cdots, x_{n;v})} M, \tag{3.3}$$

where the Laurent monomial M is of the form $x_{1;v}^{m_1}\cdots x_{\sigma(k)-1;v}^{m_{\sigma(k)-1}}x_{\sigma(k)+1;v}^{m_{\sigma(k)+1}}\cdots x_{n;v}^{m_n}$ and m_j is integer for $j\in[1,n]-\sigma(k)$. Thus Eq (3.3) is the Laurent expansion of $x_{k;u}$ with respect to $\mathbf{x}_v$.

Similarly, the following equation is the Laurent expansion of $x_{\sigma(k);v}$ with respect to $\mathbf{x}_u$.

$$x_{\sigma(k);v}=x_{k;u}\frac{F_{\sigma(k);t_1}(r_1^{-1}x_{\sigma^{-1}(1);u},\cdots,\hat{x}_{k;u},\cdots,r_n^{-1}x_{\sigma^{-1}(n);u})}{r_k^{-1}F_{k;t_2}(x_{1;u},\cdots,\hat{x}_{k;u},\cdots,x_{n;u})}M^{-1},\tag{3.4}$$

where M^{-1} is also a Laurent monomial in $R[x_{1;u}^{\pm1},\cdots,\hat{x}_{k;u},\cdots,x_{n;u}^{\pm1}]$ since $x_{i;u}=r_ix_{\sigma(i);v}$ for $i\neq k$.

We know that both $\frac{F_{\sigma(k);t_1}(r_1^{-1}x_{\sigma^{-1}(1);u},\cdots,\hat{x}_{k;u},\cdots,r_n^{-1}x_{\sigma^{-1}(n);u})}{F_{k;t_2}(x_{1;u},\cdots,\hat{x}_{k;u},\cdots,x_{n;u})}=\frac{F_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v})}{F_{k;t_2}(r_1x_{\sigma(1);v},\cdots,\hat{x}_{\sigma(k);v},\cdots,r_nx_{\sigma(n);v})}$ and $\frac{F_{k;t_2}(r_1x_{\sigma(1);v},\cdots,\hat{x}_{\sigma(k);v},\cdots,r_nx_{\sigma(n);v})}{F_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v})}$ are Laurent polynomials in

$$R[x_{1;v}^{\pm1},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v}^{\pm1}]=R[x_{1;u}^{\pm1},\cdots,\hat{x}_{k;u},\cdots,x_{n;u}^{\pm1}]$$

by the Laurent phenomenon.

Thus both $\frac{F_{k;t_2}(r_1x_{\sigma(1);v},\cdots,\hat{x}_{\sigma(k);v},\cdots,r_nx_{\sigma(n);v})}{F_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v})}$ and $\frac{F_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v})}{F_{k;t_2}(r_1x_{\sigma(1);v},\cdots,\hat{x}_{\sigma(k);v},\cdots,r_nx_{\sigma(n);v})}$ are units in $R[x_{1;v}^{\pm1},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v}^{\pm1}]$.

Because both $F_{k;t_2}$ and $F_{\sigma(k);t_1}$ are irreducible and $x_{j;t_2}\nmid F_{k;t_2}$, $x_{j;t_1}\nmid F_{\sigma(k);t_1}$ for each $j\in[1,n]$, so that both $\frac{F_{k;t_2}(r_1x_{\sigma(1);v},\cdots,\hat{x}_{\sigma(k);v},\cdots,r_nx_{\sigma(n);v})}{F_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v})}$ and $\frac{F_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v})}{F_{k;t_2}(r_1x_{\sigma(1);v},\cdots,\hat{x}_{\sigma(k);v},\cdots,r_nx_{\sigma(n);v})}$ are units in R.

Hence

$$F_{k;t_2}(r_1x_{\sigma(1);v},\cdots,\hat{x}_{\sigma(k);v},\cdots,r_nx_{\sigma(n);v})=r'_kF_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v}),$$

for some unit r'_k in R, i.e., $F_{k;t_2}(x_{1;u},\cdots,\hat{x}_{k;u},\cdots,x_{n;u})=r'_kF_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v})$. Thus $F_{k;t_2}=r'_kF_{\sigma(k);t_1}$ as polynomials, for each $k\in[1,n]$. □

Remark 3.2. From the proof of the above theorem, we can see that the choice of the unit r_i such that $x_{i;t_2}=r_ix_{\sigma(i);t_1}$ does not matter when proving $F_{i;t_2}/F_{\sigma(i);t_1}$ is a unit in R. Similarly, in the proof which is based on the Laurent phenomenon and need to use the ratio of two exchange polynomials (for example, the proof of Lemma 3.6), we can assume that $r_i=1$.

Next, we will give the proof of the conjecture for cluster algebras that each seed is uniquely defined by its cluster, and main points of proof that are different from the previous one.

Theorem 3.3. *Let $\mathcal{A}(\mathcal{S})$ be a cluster algebra, and $\Sigma_{t_l}=(\mathbf{x}_{t_l},\mathbf{y}_{t_l},B_{t_l})$, $l=1,2$ be two cluster seeds of $\mathcal{A}(\mathcal{S})$. If there exists a permutation σ of $[1,n]$ such that $x_{i;t_2}=x_{\sigma(i);t_1}$ for $i\in[1,n]$, then*

(i) Either $y_{k;t_2}=y_{\sigma(k);t_1}$, $b_{ik}^{t_2}=b_{\sigma(i)\sigma(k)}^{t_1}$ or $y_{k;t_2}=y_{\sigma(k);t_1}^{-1}$, $b_{ik}^{t_2}=-b_{\sigma(i)\sigma(k)}^{t_1}$ for $i,k\in[1,n]$.

(ii) In both cases, $F_{i;t_2}=F_{\sigma(i);t_1}$ as polynomials for $i\in[1,n]$.

Proof. By the same method with the proof of Theorem 3.1, the version of the equation (3.2) for the cluster algebra is just

$$x_{k;u} = x_{\sigma(k);v}\frac{F_{k;t_2}(x_{\sigma(1);v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{\sigma(n);v})}{F_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v})}, \tag{3.5}$$

and note that $x_{i;u} = x_{\sigma(i);v}$ for any $i \neq k$, we also have

$$x_{\sigma(k);v} = x_{k;u}\frac{F_{\sigma(k);t_1}(x_{\sigma^{-1}(1);u},\cdots,\hat{x}_{k;u},\cdots,x_{\sigma^{-1}(n);u})}{F_{k;t_2}(x_{1;u},\cdots,\hat{x}_{k;u},\cdots,x_{n;u})}. \tag{3.6}$$

We know that Eq (3.5) is the Laurent expansion of $x_{k;u}$ with respect to $\mathbf{x}_v$ and Eq (3.6) is the Laurent expansion of $x_{\sigma(k);v}$ with respect to $\mathbf{x}_u$. Then by the Laurent phenomenon, both $\frac{F_{k;t_2}(x_{\sigma(1);v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{\sigma(n);v})}{F_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v})}$ and $\frac{F_{\sigma(k);t_1}(x_{\sigma^{-1}(1);u},\cdots,\hat{x}_{k;u},\cdots,x_{\sigma^{-1}(n);u})}{F_{k;t_2}(x_{1;u},\cdots,\hat{x}_{k;u},\cdots,x_{n;u})}$ are Laurent polynomials, and this implies that $\frac{F_{k;t_2}(x_{\sigma(1);v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{\sigma(n);v})}{F_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v})}$ is a Laurent monomial in $\mathbb{ZP}[x_{1;v}^{\pm 1},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v}^{\pm 1}]$. We know that

$$F_{k;t_2}(x_{\sigma(1);v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{\sigma(n);v}) = \frac{y_{k;t_2}}{1\oplus y_{k;t_2}}\prod_{b_{ik}^{t_2}>0} x_{\sigma(i);v}^{b_{ik}^{t_2}} + \frac{1}{1\oplus y_{k;t_2}}\prod_{b_{ik}^{t_2}<0} x_{\sigma(i);v}^{-b_{ik}^{t_2}},$$

$$F_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v}) = \frac{y_{\sigma(k);t_1}}{1\oplus y_{\sigma(k);t_1}}\prod_{b_{i\sigma(k)}^{t_1}>0} x_{i;v}^{b_{i\sigma(k)}^{t_1}} + \frac{1}{1\oplus y_{\sigma(k);t_1}}\prod_{b_{i\sigma(k)}^{t_1}<0} x_{i;v}^{-b_{i\sigma(k)}^{t_1}}.$$

Because $\frac{F_{k;t_2}(x_{\sigma(1);v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{\sigma(n);v})}{F_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v})}$ is a Laurent monomial, we must have either $y_{k;t_2} = y_{\sigma(k);t_1},\ b_{ik}^{t_2} = b_{\sigma(i)\sigma(k)}^{t_1}$ or $y_{k;t_2} = y_{\sigma(k);t_1}^{-1},\ b_{ik}^{t_2} = -b_{\sigma(i)\sigma(k)}^{t_1}$. In both cases, we have

$$F_{k;t_2}(x_{\sigma(1);v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{\sigma(n);v}) = F_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v}),$$

i.e., $F_{k;t_2}(x_{1;u},\cdots,\hat{x}_{k;u},\cdots,x_{n;u}) = F_{\sigma(k);t_1}(x_{1;v},\cdots,\hat{x}_{\sigma(k);v},\cdots,x_{n;v})$. Thus $F_{k;t_2} = F_{\sigma(k);t_1}$ as polynomials. □

Lemma 3.4. *Let $\mathcal{A}(\mathcal{S})$ be a skew-symmetrizable cluster algebra with skew-symmetrizer D, and $(\mathbf{x}_{t_1},\mathbf{y}_{t_1},B_{t_1}),(\mathbf{x}_{t_2},\mathbf{y}_{t_2},B_{t_2})$ be two cluster seeds of $\mathcal{A}(\mathcal{S})$. If there exists a permutation σ of $[1,n]$ such that $x_{i;t_2} = x_{\sigma(i);t_1}$ for $i \in [1,n]$, then $b_{ik}^{t_2} = \frac{d_k}{d_{\sigma(k)}}b_{\sigma(i)\sigma(k)}^{t_1}$.*

Proof. Let P_σ be the permutation matrix define by the permutation σ. By the cluster formula (see Theorem 3.5 of [2]), we have $P_\sigma(B_{t_1}D^{-1})P_\sigma^\top = B_{t_2}D^{-1}$. Then $B_{t_2} = (P_\sigma B_{t_1}P_\sigma^\top)(P_\sigma D^{-1}P_\sigma^\top)D$. The result follows.

By the proof of the first statement and the definition of equivalence for two cluster seeds, we conclude the second statement. □

From Theorem 3.3 and Lemma 3.4, we can affirm a conjecture for skew-symmetrizable cluster algebra proposed by Fomin and Zelevinsky in [13], which says every seed of a cluster algebra is uniquely determined by its cluster.

Corollary 3.5. Let $\mathcal{A}(\mathcal{S})$ be a skew-symmetrizable cluster algebra with skew-symmetrizer D, and $(\mathbf{x}_{t_1},\mathbf{y}_{t_1},B_{t_1}),(\mathbf{x}_{t_2},\mathbf{y}_{t_2},B_{t_2})$ be two cluster seeds of $\mathcal{A}(\mathcal{S})$. If there exists a permutation σ of $[1,n]$ such that $x_{i;t_2} = x_{\sigma(i);t_1}$ for $i \in [1,n]$, then $y_{k;t_2} = y_{\sigma(k);t_1},\ b_{ik}^{t_2} = b_{\sigma(i)\sigma(k)}^{t_1}$, $d_k = d_{\sigma(k)}$ for any i and k.

The result (1) of Theorem 3.1 shows that when $x_{i;t_2} = r_i x_{\sigma(i);t_1}$ where $i \in [1, n]$ and $r_i \in R$, then $F_{i;t_2} = r'_i F_{\sigma(i);t_1}$ where $r'_i \in R$. In fact, the proof of result (1) also works for any generalized cluster algebra and any (totally sign-skew-symmetric) cluster algebra with coefficients. The reason is that the proof mainly relies on the Laurent phenomenon and is independent of the form of exchange polynomials. In the meantime, the unit r_i such that $x_{i;t_2} = r_i x_{\sigma(i);t_1}$ can be chosen to be 1 (see Remark 3.2). Although LP algebras, cluster algebras and generalized cluster algebras have different forms of exchange polynomials, and they are not included in each other, they all have the Laurent phenomenon. Thus for cluster algebras or generalized cluster algebras, it is the same method in proving that two clusters up to a permutation imply their corresponding exchange polynomials to be the same up to a permutation.

For cluster algebras or generalized cluster algebras, the equivalence for seeds is defined as two clusters and their corresponding exchange matrices up to a permutation. We know that exchange polynomials are defined by exchange matrices. And exchange polynomials up to a permutation can not imply exchange matrices to be equivalent up to a permutation. So in order to prove the conjecture (1) that each seed is uniquely defined by its cluster, we need to prove that the corresponding exchange matrices are equivalent up to a permutation. For cluster algebras, based on the result that exchange polynomials are the same up to a permutation, we give the proof in Corollary 3.5. For generalized cluster algebras, we will not discuss them more in this article, but will further discuss them in the next work.

3.2. *On Theorem 1.1 (2)*

Let $\mathcal{A}(\mathcal{S})$ be a LP algebra, if there is a seed $(\mathbf{x}_{t_0}, \mathbf{F}_{t_0})$ of $\mathcal{A}(\mathcal{S})$ such that the exchange polynomials in $\mathbf{F}_{t_0}$ are all nontrivial, we say that $\mathcal{A}(\mathcal{S})$ is a LP algebra having no trivial exchange relations.

Note that if there is a trivial exchange polynomial in a LP seed $(\mathbf{x}_{t_0}, \mathbf{F}_{t_0})$, from the definition of LP mutation, this trivial exchange polynomial remain invariant under any sequence of LP mutations. So if $\mathcal{A}(\mathcal{S})$ is a LP algebra having no trivial exchange relations, then each exchange polynomial of $\mathcal{A}(\mathcal{S})$ is a nontrivial polynomial.

Lemma 3.6. *Let $\mathcal{A}(\mathcal{S})$ be a LP algebra having no trivial exchange relations, and $\Sigma_t = (\mathbf{x}_t, \mathbf{F}_t)$, $\Sigma_{t_0} = (\mathbf{x}_{t_0}, \mathbf{F}_{t_0})$ be two LP seeds of $\mathcal{A}(\mathcal{S})$ with $x_{i;t} = r_i x_{i;t_0}$, where r_i is a unit in R for any $i \neq k$. If $x_{k;t} = M x_{k;t_0}$ for some Laurent monomial M in $R[x_{1;t_0}^{\pm 1}, \cdots, \hat{x}_{k;t_0}, \cdots, x_{n;t_0}^{\pm 1}]$, then M is a unit in R, and $[\Sigma_t] = [\Sigma_{t_0}]$.*

Proof. Without loss of generality, we assume that $r_i = 1$ for $i \neq k$. It does not make difference to the proof.

Assume that $M = r \prod_{i \neq k} x_{i;t_0}^{a_i} = r \prod_{i \neq k} x_{i;t}^{a_i}$, where r is a unit in R. If there exists some $j \neq k$ such that $a_j < 0$, then we consider the LP seed $(\mathbf{x}_w, \mathbf{F}_w) = \mu_j(\Sigma_{t_0})$. From the definition of LP mutation, we know that $x_{i;w} = x_{i;t_0}$ for $i \neq j$ and $x_{j;w} x_{j;t_0} = \hat{F}_{j;t_0}(x_{1;t_0}, \cdots, \hat{x}_{j;t_0}, \cdots, x_{n;t_0})$. Then we have

$$x_{k;t} = (r \prod_{i \neq k} x_{i;t_0}^{a_i}) x_{k;t_0} = \frac{(r \prod_{i \neq j,k} x_{i;w}^{a_i}) x_{j;w}^{-a_j}}{\hat{F}_{j;t_0}^{-a_j}(x_{1;w}, \cdots, \hat{x}_{j;w}, \cdots, x_{n;w})} x_{k;w},$$

which can be written as the following equation, from the definition of the exchange Laurent polynomial.

$$x_{k;t} = \frac{(r\prod_{i\neq j,k} x_{i;w}^{a_i})x_{j;w}^{-a_j}L}{F_{j;t_0}^{-a_j}(x_{1;w},\cdots,\hat{x}_{j;w},\cdots,x_{n;w})}x_{k;w}, \tag{3.7}$$

where L is a Laurent monomial in $R[x_{1;w}^{\pm1},\cdots,\hat{x}_{j;w},\cdots,x_{n;w}^{\pm1}]$. Thus Eq (3.7) is the expansion of $x_{k;t}$ with respect to $\mathbf{x}_w$. Because $\mathcal{A}(\mathcal{S})$ has no trivial exchange relations, $F_{j;t_0}$ is a nontrivial polynomial. And we know that $F_{j;t_0}$ is irreducible and $x_s \nmid F_{j;t_0}$ for each $s \in [1,n]$, thus Eq (3.7) will contradict the Laurent phenomenon. So each a_j is nonnegative.

Similarly, by considering that $x_{k;t_0} = M^{-1}x_{k;t} = (r\prod_{i\neq k} x_{i;t}^{-a_i})x_{k;t}$, we can get each $-a_j$ is nonnegative. Thus each a_j is 0, thus $M = r$ is a unit in R. Then by Theorem 3.1, we have $[\Sigma_t] = [\Sigma_{t_0}]$. □

Theorem 3.7. *Let $\mathcal{A}(\mathcal{S})$ be a LP algebra of rank n having no trivial exchange relations, and $\Sigma_{t_1} = (\mathbf{x}_{t_1},\mathbf{F}_{t_1}),\Sigma_{t_2} = (\mathbf{x}_{t_2},\mathbf{F}_{t_2})$ be two LP seeds of $\mathcal{A}(\mathcal{S})$. If $x_{i;t_1} = r_i x_{i;t_2}$ holds for any $i \neq k$, where r_i is a unit in R, then $[\Sigma_{t_1}] = [\Sigma_{t_2}]$ or $[\Sigma_{t_1}] = \mu_k[\Sigma_{t_2}]$, that is, any two LP seeds with $n > 1$ common cluster variables are connected with each other by one step of mutation.*

Proof. Without loss of generality, we assume that $r_i = 1$ for $i \neq k$. It does not make difference to the proof.

By the Laurent phenomenon, we assume that $x_{k;t_2} = f(x_{1;t_1},\cdots,x_{n;t_1})$ and $x_{k;t_1} = g(x_{1;t_2},\cdots,x_{n;t_2})$, where $f \in R[x_{1;t_1}^{\pm1},\cdots,x_{n;t_1}^{\pm1}]$ and $g \in R[x_{1;t_2}^{\pm1},\cdots,x_{n;t_2}^{\pm1}]$. Since $x_{i;t_1} = x_{i;t_2}$ for any $i \neq k$, we know that $x_{k;t_1}$ entries f with exponent 1 or -1; Thus $x_{k;t_2}$ has the form of $x_{k;t_2} = L_1x_{k;t_1}^{\pm1} + L_0$, where $L_1 \neq 0$ and L_0 are Laurent polynomials in

$$R[x_{1;t_1}^{\pm1},\cdots,\hat{x}_{k;t_1},\cdots,x_{n;t_1}^{\pm1}] = R[x_{1;t_2}^{\pm1},\cdots,\hat{x}_{k;t_2},\cdots,x_{n;t_2}^{\pm1}].$$

Let $(\mathbf{x}_u,\mathbf{F}_u) = \mu_k(\Sigma_{t_2})$ and $(\mathbf{x}_v,\mathbf{F}_v) = \mu_k(\Sigma_{t_1})$. From the definition of the LP mutation, we know

$$x_{i;u} = \begin{cases} x_{i;t_2} & \text{if } i \neq k \\ \hat{F}_{k;t_2}/x_{k;t_2} & \text{if } i = k \end{cases} \text{ and } x_{i;v} = \begin{cases} x_{i;t_1} & \text{if } i \neq k \\ \hat{F}_{k;t_1}/x_{k;t_1} & \text{if } i = k \end{cases}.$$

Thus $x_{k;u} = \hat{F}_{k;t_2}(x_{1;t_2},\cdots,\hat{x}_{k;t_2},\cdots,x_{n;t_2})/x_{k;t_2} = \frac{\hat{F}_{k;t_2}(x_{1;t_1},\cdots,\hat{x}_{k;t_1},\cdots,x_{n;t_1})}{L_1x_{k;t_1}^{\pm1}+L_0}$. From the definition of the exchange Laurent polynomial, we know the above equation has the form of

$$x_{k;u} = \frac{F_{k;t_2}(x_{1;t_1},\cdots,\hat{x}_{k;t_1},\cdots,x_{n;t_1})}{L_1x_{k;t_1}^{\pm1} + L_0}M, \tag{3.8}$$

where M is a Laurent monomial in $R[x_{1;t_1}^{\pm1},\cdots,\hat{x}_{k;t_1},\cdots,x_{n;t_1}^{\pm1}]$. The above equation is just the expansion of $x_{k;u}$ with respect to $\mathbf{x}_{t_1}$. By the Laurent phenomenon, and the fact $x_{k;t_1} \notin F_{k;t_2}(x_{1;t_1},\cdots,\hat{x}_{k;t_1},\cdots,x_{n;t_1})$, we obtain that $L_0 = 0$ and $\frac{F_{k;t_2}(x_{1;t_1},\cdots,\hat{x}_{k;t_1},\cdots,x_{n;t_1})}{L_1}$ is a Laurent polynomial in $R[x_{1;t_1}^{\pm1},\cdots,\hat{x}_{k;t_1},\cdots,x_{n;t_1}^{\pm1}]$. Thus we have that $x_{k;t_2} = L_1x_{k;t_1}^{\pm1}$ and $x_{k;u}$ has the form of $x_{k;u} = \tilde{M}x_{k;t_1}^{\mp1}$, where $\tilde{M}$ is a Laurent polynomial in $R[x_{1;t_1}^{\pm1},\cdots,\hat{x}_{k;t_1},\cdots,x_{n;t_1}^{\pm1}]$.

We claim that $\frac{F_{k;t_2}(x_{1;t_1},\cdots,\hat{x}_{k;t_1},\cdots,x_{n;t_1})}{L_1}$ is actually a Laurent monomial, i.e., $\tilde{M}$ is a Laurent monomial in $R[x_{1;t_1}^{\pm1},\cdots,\hat{x}_{k;t_1},\cdots,x_{n;t_1}^{\pm1}]$.

Case (i): $x_{k;t_2} = L_1 x_{k;t_1}$. Then $x_{k;t_1} = L_1^{-1} x_{k;t_2}$, which is the expansion of $x_{k;t_1}$ with respect to $\mathbf{x}_{t_2}$. By the Laurent phenomenon, we can get that L_1 is a Luarent monomial in

$$R[x_{1;t_1}^{\pm 1}, \cdots, \hat{x}_{k;t_1}, \cdots, x_{n;t_1}^{\pm 1}].$$

Then by Lemma 3.6, L_1 is a unit in R and $[\Sigma_{t_1}] = [\Sigma_{t_2}]$.

Case (ii): $x_{k;t_2} = L_1 x_{k;t_1}^{-1}$, in this case, $x_{k;u} = \tilde{M} x_{k;t_1}$. By the same argument in case (i), we can get that $\tilde{M}$ is a Luarent monomial in $R[x_{1;t_1}^{\pm 1}, \cdots, \hat{x}_{k;t_1}, \cdots, x_{n;t_1}^{\pm 1}]$. Then by Lemma 3.6, $\tilde{M}$ is a unit in R and $[\Sigma_{t_1}] = [(\mathbf{x}_u, \mathbf{F}_u)] = [\mu_k(\Sigma_{t_2})] = \mu_k([\Sigma_{t_2}])$. □

Remark 3.8. The same method also works for cluster algebras and one can get the similar result.

4. On upper and lower bound of LP algebras

The following definitions are natural generalizations of the corresponding notions of cluster algebras in [4].

For $i \in [1, n]$, we define the adjacent cluster $\mathbf{x}_i$ by $\mathbf{x}_i = (\mathbf{x} - \{x_i\}) \cup \{x_i'\}$ where the cluster variables x_i and x_i' are related by the exchange Laurent polynomial $\hat{F}_i$. Let $R[\mathbf{x}^{\pm 1}]$ be the ring of Laurent polynomials in $x_1, \ldots, x_n$ with coefficients in R.

Definition 4.1. The **upper bound** $\mathcal{U}(\Sigma)$ and **lower bound** $\mathcal{L}(\Sigma)$ associated with a LP seed $\Sigma = (\mathbf{x}, \mathbf{F})$ is defined by

$$\mathcal{U}(\Sigma) = R[\mathbf{x}^{\pm 1}] \cap R[\mathbf{x}_1^{\pm 1}] \cap \cdots \cap R[\mathbf{x}_n^{\pm 1}],\ \mathcal{L}(\Sigma) = R[x_1, x_1', \ldots, x_n, x_n']$$

Thus, $\mathcal{L}(\Sigma)$ is the R-subalgebra of $\mathcal{F}$ generated by the union of $n+1$ clusters $\mathbf{x}^{\pm 1}, \mathbf{x}_1^{\pm 1}, \ldots, \mathbf{x}_n^{\pm 1}$. Note that $\mathcal{L}(\Sigma) \subseteq \mathcal{A}(\Sigma) \subseteq \mathcal{U}(\Sigma)$.

Remark 4.2. The method of the proof of results for LP algebras in this section is a little similar to those for cluster algebras in [4]. The concepts of LP algebras and cluster algebras are essentially different, since LP algebras and cluster algebras are not included with each other. In general, the calculation for LP algebras is more complicated. Now, we give the following three points to explain the specific differences between LP algebras and cluster algebras.

(1) For cluster algebras, the exchange polynomials are binomials. While for LP algebras, those are multinomials, so that the calculation using the exchange polynomials becomes complicated.

(2) For cluster algebras, coprimeness is necessary for the proof of properties for the upper bound and lower bound, and it is easy to check that the coprimeness keeps under mutations for a certain seed. While for LP algebras, the concept of coprimeness is not yet defined. In order to obtain a LP seed with coprimeness, we assume that a LP seed satisfies the condition that $\hat{F}_k = F_k$ for any k. But it is not obvious whether the condition keeps under mutations (see Example 4.12 which shows that the condition does not keep under mutations), so that in the proof that involves mutations and requires that condition, we need to show that the condition keeps under mutations.

(3) In cluster algebras, the acyclic seed has good property that the lexicographically first monomial of its any exchange polynomial F_j is a monomial in $\{x_i | i > j\}$. While in LP algebras, the concept of the acyclic seed is not yet defined. In order to obtain a LP seed with such good property, we assume that the LP seed satisfies certain conditions, see Condition 1.2.

4.1. Upper bound as invariant under LP mutation

For any LP seed $\Sigma = (\mathbf{x}, \mathbf{F})$, the following lemma and corollary hold parallel to the corresponding results in [4].

Lemma 4.3.

$$\mathcal{U}(\Sigma) = \bigcap_{j=1}^{n} R[x_1^{\pm 1}, \ldots, x_{j-1}^{\pm 1}, x_j, x_j', x_{j+1}^{\pm 1}, \ldots, x_n^{\pm 1}]. \tag{4.1}$$

Proof. It is sufficient to show that

$$R[\mathbf{x}^{\pm 1}] \cap R[\mathbf{x}_1^{\pm 1}] = R[x_1, x_1', x_2^{\pm 1}, \ldots, x_n^{\pm 1}].$$

The inclusion $\supseteq$ is clear, we only need to prove the converse inclusion.

For any $y \in R[\mathbf{x}^{\pm 1}] \cap R[\mathbf{x}_1^{\pm 1}]$, y is of the form $y = \sum\limits_{m=-M}^{N} c_m x_1^m$, where $M, N \in \mathbb{Z}_{\geq 0}$ and $c_m \in R[x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. If $M \geq 0$, it is easy to see that

$$y \in R[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \subseteq R[x_1, x_1', x_2^{\pm 1}, \ldots, x_n^{\pm 1}].$$

If $MN \neq 0$, from the definition of LP seeds, $x_1 \notin \hat{F}_1$, then

$$y|_{x_1 \leftarrow \frac{\hat{F}_1}{x_1'}} = \sum\nolimits_{m=-M}^{N} c_m (\frac{\hat{F}_1}{x_1'})^m = \sum\nolimits_{m=1}^{M} c_{-m} \hat{F}_1^{-m} x_1'^m + \sum\nolimits_{m=0}^{N} c_m \hat{F}_1^m x_1'^{-m}.$$

Since $y \in R[\mathbf{x}_1^{\pm 1}]$, y can be written as $\sum\limits_{p=M'}^{N'} c_p x_1^p$ where $c_p \in R[x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, then we have $c_{-m}\hat{F}_1^{-m} \in R[x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. Thus,

$$y = \sum_{m=1}^{M} c_{-m} \hat{F}_1^{-m} x_1'^m + \sum_{m=0}^{N} c_m x_1^m \in R[x_1, x_1', x_2^{\pm 1}, \ldots, x_n^{\pm 1}].$$

If $N = 0$, by similar discussion, we have $y \in R[x_1', x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \subseteq R[x_1, x_1', x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. □

Corollary 4.4. For $j \in [1, n]$, $y \in R[x_1^{\pm 1}, \ldots, x_{j-1}^{\pm 1}, x_j, x_j', x_{j+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$ if and only if y is of the form $y = \sum\limits_{m=-M}^{N} c_m x_j^m$ where $M, N \in \mathbb{Z}_{\geq 0}$, $c_m \in R[x_1^{\pm 1}, \ldots, \hat{x}_j^{\pm 1}, \ldots, x_n^{\pm 1}]$ and c_{-m} is divisible by $\hat{F}_j^m$ in $R[x_1^{\pm 1}, \ldots, \hat{x}_j^{\pm 1}, \ldots, x_n^{\pm 1}]$ for $m \in [1, M]$.

Lemma 4.5. *Suppose that* $\hat{F}_j = F_j$ *for* $j \in [1, 2]$, *then* $R[x_1, x_2^{\pm 1}] \cap R[x_1^{\pm 1}, x_2, x_2'] = R[x_1, x_2, x_2']$.

Proof. The inclusion $\supseteq$ is clear, we only need to prove the converse inclusion. For $y \in R[x_1, x_2^{\pm 1}] \cap R[x_1^{\pm 1}, x_2, x_2']$, y is of the form $y = \sum\limits_{m \in \mathbb{Z}} x_1^m (c_m + c_m'(x_2) + c_m''(x_2'))$, where $c_m \in R$, $c_m'(x_2)$ and $c_m''(x_2')$ are polynomials over R without constant terms.

Let M be the smaller integer such that $c_M + c_M'(x_2) + c_M''(x_2') \neq 0$. If $M \geq 0$, then it is easy to see that $y \in R[x_1, x_2, x_2']$.

Otherwise, the Laurent expression of y is $\sum_{m\in\mathbb{Z}} x_1^m(c_m + c'_m(x_2) + c''_m(\frac{F_2}{x_2}))$ by the assumption. Let r_2 be the sum of monomials in F_2 without x_1. Then there are nonzero terms with smallest power of x_1 in the Laurent expression of y, which are $x_1^M(c_M + c'_M(x_2) + c''_M(\frac{r_2}{x_2})) \neq 0$, which contradicts the condition that $y \in R[x_1, x_2^{\pm 1}]$. □

Lemma 4.6. *Suppose that* $\hat{F}_j = F_j$ *for* $j \in [1, 2]$, *then*

$$R[x_1, x'_1, x_2^{\pm 1}] = R[x_1, x'_1, x_2, x'_2] + R[x_1, x_2^{\pm 1}].$$

Proof. The inclusion $\supseteq$ is clear, we only need to prove the converse inclusion. It is enough to show that $\forall M, N > 0$, $x_1'^N x_2^{-M} \in R[x_1, x'_1, x_2, x'_2] + R[x_1, x_2^{\pm 1}]$.

By the assumption, we have $x_2x'_2 = \hat{F}_2 = F_2 = g(x_1) + r_2$, where $g(x_1) = \sum_{i=1}^{m} g_i x_1^i$, $g_i \in R$ and $r_2 \neq 0 \in R$ since F_2 is not divisible by x_1. If $g(x_1) = 0$, then $x_2^{-1} = r_2^{-1}x'_2$, which implies that $x_1'^N x_2^{-M} \in R[x_1, x'_1, x_2, x'_2]$.

Otherwise, let $p(x_1) = -\frac{g(x_1)}{r_2} \in R[x_1]$, then $x_2x'_2 = g(x_1) + r_2$ can be written as

$$x_2^{-1} = r_2^{-1}x'_2 + p(x_1)x_2^{-1}.$$

Repeatedly substituting x_2^{-1} in the RHS of the above equation by $r_2^{-1}x'_2 + p(x_1)x_2^{-1}$, we obtain $x_2^{-1} = P(x_1, x'_2) + p^N(x_1)x_2^{-1}$, where $P(x_1, x'_2) = r_2^{-1}x'_2 \sum_{i=0}^{N-1} p^i(x_1) \in R[x_1, x'_2]$.

Then we have

$$\begin{aligned} x_1'^N x_2^{-M} &= x_1'^N P^M(x_1, x'_2) + x_1'^N p^{MN}(x_1)x_2^{-M} \\ &\quad + x_1'^N \sum_{i=1}^{M-1} \binom{M}{i} (P(x_1, x'_2))^{M-i}(p(x_1)^N x_2^{-1})^i, \end{aligned} \tag{4.2}$$

where the first term of (4.2) that is, $x_1'^N P^M(x_1, x'_2) \in R[x_1, x'_1, x'_2]$.

For $p(x_1) = -\frac{1}{r_2}g(x_1) = -\frac{1}{r_2}\sum_{i=1}^{m} g_i x_1^i$, the smallest power of x_1 in $p^N(x_1)$ is N and the greatest is Nm. Thus we can rewrite $p^N(x_1)$ in the form $x_1^N(\sum_{i=0}^{N(m-1)} p_i x_1^i)$ where $p_i \in R$, implying that for any integer $K > 0$, we have $p^{NK}(x_1) \in x_1^N R[x_1]$. Since $x_1x'_1 = \hat{F}_1 = F_1 \in R[x_2]$, we have $x_1'^N p^{NK}(x_1) \in R[x_1, x_2]$.

Then the middle term of (4.2) is obvious in $R[x_1, x_2^{\pm 1}]$, and the last term of (4.2) is equal to $x_1'^N \sum_{i=1}^{M-1} \binom{M}{i}(P(x_1, \frac{F_2}{x_2}))^{M-i}(p(x_1)^N x_2^{-1})^i \in R[x_1, x_2^{\pm 1}]$.

Thus we finish the proof. □

Proposition 4.7. *Suppose that* $n \geq 2$ *and* $\hat{F}_j = F_j$ *for* $j \in [1, n]$, *then*

$$\mathcal{U}(\Sigma) = \bigcap_{j=2}^{n} R[x_1, x'_1, x_2^{\pm 1}, \ldots, x_{j-1}^{\pm 1}, x_j, x'_j, x_{j+1}^{\pm 1}, \ldots, x_n^{\pm 1}]. \tag{4.3}$$

Proof. Comparing (4.1) with (4.3), it is sufficient to show that

$$R[x_1, x_1', x_2, x_2', x_3^{\pm 1}, \dots, x_n^{\pm 1}] = R[x_1, x_1', x_2^{\pm 1}, \dots, x_n^{\pm 1}] \cap R[x_1^{\pm 1}, x_2, x_2', x_3^{\pm 1}, \dots, x_n^{\pm 1}].$$

Freeze the cluster variables $x_3, \dots, x_n$ and view $R[x_3^{\pm 1}, \dots, x_n^{\pm 1}]$ as the new ground ring R, then the above equality reduces to

$$R[x_1, x_1', x_2, x_2'] = R[x_1, x_1', x_2^{\pm 1}] \cap R[x_1^{\pm 1}, x_2, x_2']. \tag{4.4}$$

Suppose $F_1 = f(x_2) + r_1$, $F_2 = g(x_1) + r_2$, where $r_1 \neq 0, r_2 \neq 0 \in R$ and $f(x_2), g(x_1)$ are polynomials over R without constant terms. It is easy to see that Lemma 4.5 and Lemma 4.6 hold for four cases which are: *(C1)* $x_2 \notin F_1$ and $x_1 \notin F_2$, that is, $f(x_2) = 0$ and $g(x_1) = 0$; *(C2)* $x_2 \in F_1$ and $x_1 \in F_2$; *(C3)* $x_2 \notin F_1$ but $x_1 \in F_2$; *(C4)* $x_2 \in F_1$ but $x_1 \notin F_2$. Combining Lemma 4.5 and Lemma 4.6 with the fact that $R[x_1, x_1', x_2, x_2'] \subseteq R[x_1^{\pm 1}, x_2, x_2']$, we obtain:

$$\begin{aligned} R[x_1, x_1', x_2^{\pm 1}] \cap R[x_1^{\pm 1}, x_2, x_2'] &= (R[x_1, x_1', x_2, x_2'] + R[x_1, x_2^{\pm 1}]) \cap R[x_1^{\pm 1}, x_2, x_2'] \\ &= R[x_1, x_1', x_2, x_2'] + (R[x_1, x_2^{\pm 1}] \cap R[x_1^{\pm 1}, x_2, x_2']) \\ &= R[x_1, x_1', x_2, x_2'] \end{aligned}$$

Thus we have (4.4). □

Lemma 4.8. *For a LP seed* $(\mathbf{x}, \mathbf{F})$*, let* x_2' *and* x_2'' *be the cluster variables exchanged with* x_2 *in the LP seeds* $\mu_2(\mathbf{x}, \mathbf{F})$ *and* $\mu_2\mu_1(\mathbf{x}, \mathbf{F})$ *respectively, then*

$$R[x_1, x_1', x_2, x_2', x_3^{\pm 1}, \dots, x_n^{\pm 1}] = R[x_1, x_1', x_2, x_2'', x_3^{\pm 1}, \dots, x_n^{\pm 1}]. \tag{4.5}$$

Proof. We can freeze the cluster variables $x_3, \dots, x_n$ and view $R[x_3^{\pm 1}, \dots, x_n^{\pm 1}]$ as the new ground ring R. Then we will prove the following equality can be reduced from (4.5):

$$R[x_1, x_1', x_2, x_2'] = R[x_1, x_1', x_2, x_2''].$$

We first show that $x_2'' \in R[x_1, x_1', x_2, x_2']$.

In *(C1)*, it is easy to see that $x_2'' = rx_2'$ for certain $r \in R$, which implies $x_2'' \in R[x_1, x_1', x_2, x_2']$.

In *(C2)*, let $(\mathbf{x}', \mathbf{F}') = \mu_1(\mathbf{x}, \mathbf{F})$, then x_2'' is obtained by $x_2 x_2'' = \hat{F}_2'$. Recall that $x_1 x_1' = \hat{F}_1 = F_1 = f(x_2) + r_1$ and $x_2 x_2' = \hat{F}_2 = F_2 = g(x_1) + r_2$, where $g(x_1) = \sum_{i=1}^{m} g_i x_1^i$, $g_i \in R$ and $r_2 \neq 0 \in R$. Because F_2 depends on x_1, from the definition of LP mutations, we have:

1) $G_2 = F_2|_{x_1 \leftarrow N_2} = g(\frac{r_1}{x_1'}) + r_2$.

2) $H_2 = G_2/c$, where c is the product of all common factors of $g_i r_1^i$ for $i \in [1, m]$ and r_2.

3) $F_2' = MH_2 = x_1'^m H_2 = h(x_1') + r_3$,
where $r_3 = \frac{g_m r_1^m}{c}$, $h(x_1') = \sum_{i=1}^{m} h_i x_1'^i$, $h_j = \begin{cases} g_{m-i} r_1^{m-i}/c & j \in [1, m-1], \\ r_2/c & j = m \end{cases}$.

By Proposition 2.4, there exist x_2 in $F'_1 = F_1$ in $(\mathbf{x}', \mathbf{F}')$, so that there is no x'_1 in $F'_2/\hat{F}'_2$, thus we have $\hat{F}'_2 = F'_2$. It follows that

$$\begin{aligned} x_2x''_2 &= \frac{1}{c}r_2x'^m_1 + \sum_{j=1}^{m-1} h_jx'^j_1 + r_3 \\ &= \frac{1}{c}(x_2x'_2 - g(x_1))x'^m_1 + \sum_{j=1}^{m-1} h_jx'^j_1 + r_3 \\ &= x_2(\frac{1}{c}x'_2x'^m_1) - (\frac{1}{c}g(x_1)x'^m_1 - (\sum_{j=1}^{m-1} h_jx'^j_1) - r_3), \end{aligned}$$

where $\frac{1}{c}g(x_1)x'^m_1 = (\sum\limits_{i=1}^{m} \frac{g_i}{c}x^i_1)x'^m_1 = \sum\limits_{i=1}^{m} \frac{g_i}{c}(x_1x'_1)^ix'^{m-i}_1 = \sum\limits_{i=1}^{m} \frac{g_i}{c}(f(x_2) + r_1)^ix'^{m-i}_1$. Recall that $f(x_2)$ is a polynomial in x_2 without constant terms, then $\frac{g_m}{c}(f(x_2)+r_1)^m$ can be written as $x_2P_m + \frac{g_mr^m_1}{c} = x_2P_m + r_3$, where P_m is a polynomial in x_2.

For $i \in [1, m-1]$, we have $\frac{g_i}{c}(f(x_2) + r_1)^ix'^{m-i}_1 = x_2P_i + \frac{g_ir^i_1}{c}x'^{m-i}_1 = x_2P_i + h_{m-i}x'^{m-i}_1$, where P_i is a polynomial in x_2.

Then $\frac{1}{c}g(x_1)x'^m_1 - (\sum\limits_{j=1}^{m-1} h_jx'^j_1) - r_3 = x_2(\sum\limits_{i=1}^{m} P_i)$, which implies that

$$x_2x''_2 = x_2(\frac{1}{c}x'_2x'^m_1 - \sum_{i=1}^{m} P_i).$$

Thus $x''_2 \in R[x_1, x'_1, x_2, x'_2]$.

For (*C3*) and (*C4*), it is enough to show for (*C3*) by symmetry. At this time, F'_2 is the same as that in (*C2*). Since $f(x_2) = 0$, $(F'_1 = F_1)|_{x_2 \leftarrow F'_2/x''_2} = r_1$ is not divisible by F'_2, so that $\hat{F}'_2 = F'_2$. As a consequence, we have $x_2x''_2 = x_2(\frac{1}{c}x'_2x'^m_1)$. Thus $x''_2 \in R[x_1, x'_1, x_2, x'_2]$.

On the other hand, we can prove similarly that $x'_2 \in R[x_1, x'_1, x_2, x''_2]$. Then, (4.5) follows truely. □

Theorem 4.9. *Assume that a LP seed $\Sigma = (\mathbf{x}, \mathbf{F})$ satisfied $\hat{F}_j = F_j$ for $j \in [1, n]$ and $\Sigma' = (\mathbf{x}', \mathbf{F}')$ is the LP seed obtained from the LP seed Σ by mutation in direction k. Then the corresponding upper bounds coincide, that is, $\mathcal{U}(\Sigma) = \mathcal{U}(\Sigma')$.*

Proof. Without loss of generality, we assume that $k = 1$. Combining Proposition 4.7 and Lemma 4.8, we finish the proof. □

Proposition 4.10. *If the exchange polynomials of a LP seed satisfy $\hat{F}_k = F_k$ for any $k \in [1, n]$, then $F_i \neq F_k$ for any $i \neq k$. Furthermore, any two of the exchange polynomials $\{F_k | k \in [1, n]\}$of a LP seed are coprime.*

Proof. We will prove by contradiction. If $F_i = F_k$, then

$$\hat{F}_i|_{x_k \leftarrow F_k/x'_k} = F_i|_{x_k \leftarrow F_k/x'_k} = F_k|_{x_k \leftarrow F_k/x'_k} = F_k$$

for $x_k \notin F_k$, implying that F_k divides $\hat{F}_i|_{x_k \leftarrow F_k/x'_k}$, which contradicts the definition of exchange Laurent polynomials.

Besides, since the irreducibility of exchange polynomials for LP seeds, we conclude that the exchange polynomials of a LP seed are coprime under the condition that $\hat{F}_k = F_k$ for any $k \in [1, n]$. □

Remark 4.11. When a cluster seed is a LP seed, the coprimeness of the cluster seed is equivalent to the condition that $\hat{F}_k = F_k$ for any $k \in [1, n]$.

Example 4.12. Consider the LP seed $(\mathbf{x}, \mathbf{F}) = \{(a, b + c), (b, a + c), (c, a + (a + 1)b)\}$ over $R = \mathbb{Z}$, which satisfies the condition that $\hat{F}_k = F_k$ for any $k \in \{a, b, c\}$. Then the LP seed obtained by mutation at b is

$$(\mathbf{x}', \mathbf{F}') = \{(a, 1 + d), (d, a + c), (c, a + d + 1)\}$$

where $d = \frac{a+C}{b}$. It is easy to see that $\hat{F}'_c = \frac{F'_c}{a}$, meaning that the condition that $\hat{F}_k = F_k$ for any k for a LP seed does not keep under mutations.

Definition 4.13. Let $\Sigma = (\mathbf{x}, \mathbf{F})$ be a LP seed, the **upper LP algebra** $\overline{\mathcal{A}}(\Sigma)$ defined by Σ is the intersection of the subalgebras $\mathcal{U}(\Sigma')$ for all LP seeds Σ' mutation-equivalent to Σ .

Theorem 4.9 has the following direct implication.

Corollary 4.14. Assume that all LP seeds mutation equivalent to a LP seed $\Sigma = (\mathbf{x}, \mathbf{F})$ satisfy the condition that $\hat{F}_j = F_j$ for $j \in [1, n]$, then the upper bound $\mathcal{U}(\Sigma)$ is independent of the choice of LP seeds mutation-equivalent to Σ , and so is equal to the upper LP algebra $\overline{\mathcal{A}}(\Sigma)$.

4.2. On lower bound

4.2.1. A basis for lower bound

Definition 4.15. Let $(\mathbf{x}, \mathbf{F})$ be a LP seed. A **standard monomial** in $\{x_i, x'_i | i \in [1, n]\}$ is a monomial that contains no product of the form $x_i x'_i$.

Let $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \ldots x_n^{a_n}$ be a Laurent monomial where $\mathbf{a} \in \mathbb{Z}^n$. For a Laurent polynomial in $x_1, \ldots, x_n$, we order the each two terms $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{a}'}$ lexicographically as follows:

$$\mathbf{a} \prec \mathbf{a}' \text{ if the first nonzero difference } a'_j - a_j \text{ is positive.} \tag{4.6}$$

We set the term with the smallest lexicographical order as the first term in a Laurent polynomial.

Theorem 4.16. *Assume that a LP seeds* $\Sigma = (\mathbf{x}, \mathbf{F})$ *satisfies*

1) $\hat{F}_j = F_j$ *for* $j \in [1, n]$.

2) in any F_j*, the lexicographically first monomial is of the form*

$$\mathbf{x}^{\mathbf{v_j}} = \begin{cases} x_{j+1}^{v_{j+1,j}} \cdots x_n^{v_{n,j}} & j \in [1, n-1] \\ 1 & j = n \end{cases}$$

where $\mathbf{v_j} \in \mathbb{Z}_{\geq 0}^{n-j}$ *for* $j \in [1, n-1]$.

Then the standard monomials in $x_1, x'_1, \ldots, x_n, x'_n$ *form an R-basis for* $\mathcal{L}(\Sigma)$.

Proof. The proof is using the same technique as in [4]. We denote the standard monomials in $x_1, x'_1, \ldots, x_n, x'_n$ by $\mathbf{x}^{(\mathbf{a})} = x_1^{(a_1)} \cdots x_n^{(a_n)}$, where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ and

$$x_i^{(a_i)} = \begin{cases} x_i^{a_i}, & a_i \geq 0 \\ x_i'^{-a_i}, & a_i < 0. \end{cases}$$

Note that $\mathbf{x}^{(\mathbf{a})}$ is a Laurent polynomial in $x_1, \ldots, x_n$ and for any i, we have $x_i^{(-1)} = x_i' = x_i^{-1}\hat{F}_i = x_i^{-1}F_i$. By the assumption for F_i, it follows that the lexicographically first monomial in $x_i^{(-1)}$ is $x_i^{-1}\mathbf{x}^{\mathbf{v_i}}$, then the power of x_i in $x_i^{(a_i)}$ is a_i and there is no $x_1, \ldots, x_{i-1}$ in $x_i^{(a_i)}$. Then the lexicographically first monomial in $\mathbf{x}^{(\mathbf{a})}$ is the product of $x_i^{a_i}(a_i > 0)$ and $x_i^{a_i}(\mathbf{x}^{\mathbf{v_i}})^{-a_i}(a_i < 0)$.

We assume that $\mathbf{a} \prec \mathbf{a}'$ such that $a_i = a_i'$ for $i \in [1, k-1]$ and $a_k < a_k'$. Let P, M and Q be the lexicographically first monomial of $\prod_{j=1}^{k-1} x_j^{(a_j)}$, $x_k^{(a_k)}$ and $\prod_{j=k+1}^{n} x_j^{(a_j)}$ respectively. Then the lexicographically first monomial of $\mathbf{x}^{(\mathbf{a})}$ is PMQ, similarly that of $\mathbf{x}^{(\mathbf{a}')}$ is $P'M'Q'$.

Since $a_i = a_i'$ for $i \in [1, k-1]$, we have $P = P' = \prod_{j=1}^{n} x_j^{p_j}$.

Since $a_k < a_k'$ and the power of x_k in $x_k^{(a_k)}$ is a_k and there is no x_i ($i \in [1, k-1]$) in $x_k^{(a_k)}$, we obtain $M = x_k^{a_k} \prod_{j=k+1}^{n} x_j^{m_j}$ and $M' = x_k^{a_k'} \prod_{j=k+1}^{n} x_j^{m_j'}$.

And $Q = (\prod_{j=k+1, a_j>0}^{n} x_j^{a_j})(\prod_{j=k+1, a_j<0}^{n} x_j^{a_j}(\mathbf{x}^{\mathbf{v_j}})^{-a_j}) = \prod_{j=k+1}^{n} x_j^{q_j}$, similarly $Q' = \prod_{j=k+1}^{n} x_j^{q_j'}$.

It follows that

$$PMQ = (\prod_{j=1}^{k-1} x_j^{p_j})(x_k^{p_k+a_k})(\prod_{j=k+1}^{n} x_j^{p_j+m_j+q_j}),\ P'M'Q' = (\prod_{j=1}^{k-1} x_j^{p_j})(x_k^{p_k+a_k'})(\prod_{j=k+1}^{n} x_j^{p_j+m_j'+q_j'}).$$

Thus $PMQ \prec P'M'Q'$, implying that

$$\text{if } \mathbf{a} \prec \mathbf{a}', \text{ the lexicographically first monomial of } \mathbf{x}^{(\mathbf{a})} \prec \text{ that of } \mathbf{x}^{(\mathbf{a}')}. \tag{4.7}$$

The linearly independence of standard monomials over R follows at once from (4.7). Since the product $x_i x_i'$ for any i equals to $\hat{F}_i = F_i$, which is the linear combination of standard monomials in $x_1, x_1', \ldots, x_n, x_n'$. Thus they form a basis for $\mathcal{L}(\Sigma)$. □

4.2.2. Lower and upper bound

In the following statements, we always assume that $\Sigma = (\mathbf{x}, \mathbf{F})$ is a LP seed of rank n satisfying Condition 1.2.

Notation 4.17. We denote by $\boldsymbol{\varphi} : R[x_2, x_2', \ldots, x_n, x_n'] \to R[x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ the algebra homomorphism defined as the composition $\varphi_2 \circ \varphi_1$, where

$$\varphi_1 : R[x_2, x_2', \ldots, x_n, x_n'] \to R[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \text{ by } x_i \mapsto x_i \text{ and } x_i' \mapsto F_i/x_i.$$
$$\varphi_2 : R[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \to R[x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \text{ by } x_1 \mapsto 0 \text{ and } x_i^{\pm 1} \mapsto x_i^{\pm 1}.$$

We denote by $R^{st}[x_2, x_2', \ldots, x_n, x_n']$ (resp. $R^{st}[x_1, x_2, x_2', \ldots, x_n, x_n']$) the R-linear span (resp. $R[x_1]$-linear span) of the standard monomials in $x_2, x_2', \ldots, x_n, x_n'$.

Lemma 4.18. $R[x_2, x_2', \ldots, x_n, x_n'] = ker(\varphi) \oplus R^{st}[x_2, x_2', \ldots, x_n, x_n']$.

Proof. For any $y \in R[x_2, x_2', \ldots, x_n, x_n']$, replace $x_i x_i' \in y$ with F_i, then $y \in R^{st}[x_1, x_2, x_2', \ldots, x_n, x_n']$. Thus we have $R[x_2, x_2', \ldots, x_n, x_n'] \subseteq R^{st}[x_1, x_2, x_2', \ldots, x_n, x_n']$. It follows that

$$R[x_2, x_2', \ldots, x_n, x_n'] = ker(\varphi) + R^{st}[x_2, x_2', \ldots, x_n, x_n'].$$

Similarly using the tool of the proof of Theorem 4.16, For $\mathbf{x}^{(\mathbf{a})} \in R^{st}[x_2, x_2', \ldots, x_n, x_n']$, the lexicographically first monomial of $\varphi(x_j^{(a_j)})$ is a Laurent monomial in $x_j, x_{j+1}, \ldots, x_n$ whose the power of x_j is a_j, implying that if $\mathbf{a} \prec \mathbf{a}'$, then the lexicographically first monomial of $\varphi(\mathbf{x}^{(\mathbf{a})})$ precedes the one of $\varphi(\mathbf{x}^{(\mathbf{a}')})$.

Then the restriction of φ to $R^{st}[x_2, x_2', \ldots, x_n, x_n']$ is injective. □

Notation 4.19. Given a Laurent polynomial $y \in R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we denote by $LT(y)$ as the sum of all Laurent monomials with the smallest power of x_1 in the Laurent expansion of y with nonzero coefficient.

The following results parallel to Lemmas 6.4 and 6.5 in [4] can be obtained similarly.

Lemma 4.20. *Suppose that* $y = \sum_{m=a}^{b} c_m x_1^m$ *where* $c_m \in R^{st}[x_2, x_2', \ldots, x_n, x_n']$ *and* $c_a \neq 0$, *then* $LT(y) = \varphi(c_a)x_1^a$.

Lemma 4.21. $R[x_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \cap R[x_1^{\pm 1}, x_2, x_2', \ldots, x_n, x_n'] = R[x_1, x_2, x_2', \ldots, x_n, x_n']$.

Lemma 4.22. $Im(\varphi) = R[x_2, x_2^{(-)}, \ldots, x_n, x_n^{(-)}]$, *where* $x_j^{(-)} = \begin{cases} x_j', & \text{if } x_1 \notin F_j \\ x_j^{-1}, & \text{otherwise} \end{cases}$.

Proof. By Condition 1.2, we have $\varphi(x_j') = \begin{cases} x_j', & \text{if } x_1 \notin F_j \\ x_j^{-1}M_j, & \text{otherwise} \end{cases}$. Thus the inclusion ⊆ is clear.

Let J be the set of indexes $j \in [2, n]$ satisfying $x_1 \notin F_j$. We set $W_j = x_j^{-1}M_j$. To prove the converse inclusion, it is enough to show that $x_j^{-1} \in \mathrm{Im}(\varphi)$ for $j \in [2, n] - J$.

For $\mathbf{m} = (m_2, \ldots, m_n), \mathbf{l} = (l_2, \ldots, l_n) \in \mathbb{Z}^{n-1}$, let $\mathbf{x}^{\mathbf{m}}$ be a Laurent monomial in $R[x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$. Moreover, we set $\mathbf{W}^{\mathbf{l}} = \prod_{j=2}^{n} W_j^{l_j}$. Then we have any $\mathbf{x}^{\mathbf{m}}$ can be written as $\mathbf{W}^{\mathbf{l}}$ satisfying

$$m_j = -l_j + \sum_{2 \le i < j} v_{ji} l_i.$$

Define the multiplicative monoid $\mathcal{W} = \{\mathbf{x}^{\mathbf{m}} = \mathbf{W}^{\mathbf{l}} | l_i \geq 0 \; for \; i \in [2, n] \; and \; m_j \geq 0 \; for \; j \in J\}$. Then $\mathbf{W}^{\mathbf{l}} \in \mathcal{W}$ if and only if

$$(a)\ l_k \geq 0 \text{ for } k \in [2, n], \quad (b) \sum_{2 \le i < j} v_{ji} l_i \geq l_j \text{ for } j \in J. \tag{4.8}$$

By the equivalence condition (4.8) of $\mathbf{W}^{\mathbf{l}} \in \mathcal{W}$, we obtain $x_j^{-1} \in \mathcal{W}$ for $j \in [2, n] - J$, implying that it suffices to show that $\mathcal{W} \subseteq \mathrm{Im}(\varphi)$.

For $W = \mathbf{W}^{\mathbf{l}} \in \mathcal{W}$, we prove that $W \in \mathrm{Im}(\varphi)$ by induction on the degree of W. When $deg(W) = 0$, we have $W = 1 \in R \subseteq \mathrm{Im}(\varphi)$. Assume that $deg(W) > 0$ and for any $W' \in \mathcal{W}$ such that $deg(W') < deg(W)$, then $W' \in \mathrm{Im}(\varphi)$.

Let $j = max\{i | l_i > 0 \; in \; W\}$, then we have $W/W_j \in \mathcal{W}$ by the equivalence condition (4.8) of $\mathbf{W}^{\mathbf{l}} \in \mathcal{W}$. As a consequence, $W/W_j \in \mathrm{Im}(\varphi)$ under the induction assumption. If $j \in [2, n] - J$, then $W_j \in \mathrm{Im}(\varphi)$ so that $W = (W/W_j)W_j \in \mathrm{Im}(\varphi)$.

Otherwise, since $l_j > 0$, there exist $i \in [2, j-1]$ such that $v_{ji} l_i > 0$, where $v_{ji} \neq 0$ implies that $x_j \in M_i$. Fix such an index i. By (iv) of Condition 1.2, $F_j = f_j(x_i) + M_j$ and $f_j(x_i) = \sum_{t=1}^{s_j} r_t c_t$, where s_j is the number of terms of $f_j(x_i)$, $r_t \in R$ and $c_t = \prod_{p \in [2,n]-j} x_p^{\gamma_{pt}}$ satisfying $\gamma_{pt} \in \mathbb{Z}_{\geq 0}$ and $\gamma_{it} \neq 0$.

From the definition of LP mutations and Condition 1.2(i)(ii), we have $x'_j = x_j^{-1} f_j(x_i) + W_j$. By multiplying both sides of that equation by W/W_j, we have

$$(W/W_j)x'_j = x_j^{-1} \sum_{t=1}^{s_j} r_t c_t(W/W_j) + W.$$

Since $(W/W_j)x'_j \in \mathrm{Im}(\varphi)$, we only need to show that for $t \in [1, s_j]$, $x_j^{-1} c_t(W/W_j) \in \mathrm{Im}(\varphi)$.

Define $W' = W_i^{l'_i} \cdots W_j^{l'_j}$, where $l'_i = 1$ and $l'_p = min\{l_p, \sum_{i \leq q < p} v_{pq} l'_q\}$ for $p \in [i+1, j]$. Because $W/W' = W_2^{l_2} \cdots W_{i-1}^{l_{i-1}} W_i^{l_i - 1} W_{i+1}^{l_{i+1} - l'_{i+1}} \cdots W_j^{l_j - l'_j}$, the equivalence condition (4.8) of $W/W' \in \mathcal{W}$ can be written as

(a) for $k \in [i, j]$, $l_k - l'_k \geq 0$;

(b) for $k \in J$, $l_k - l'_k \leq \sum_{2 \leq h < k} v_{kh}(l_h - l'_h) \Leftrightarrow -l'_k + \sum_{2 \leq h < k} v_{kh} l'_h \leq -l_k + \sum_{2 \leq h < k} v_{kh} l_h$.

The inequalities of (a) are immediate from the definition of W' and the choice of i. And for inequality (b), we discuss in several cases:

1) if $k \in [2, i-1]$, (b) is equivalent to $0 \leq -l_k + \sum_{2 \leq h < k} v_{kh} l_h$.

2) if $k = i$, we have $\sum_{2 \leq h < i} v_{kh} l'_h = 0$, (b) is equivalent to $-1 \leq -l_k + \sum_{2 \leq h < k} v_{kh} l_h$.

3) if $k \in [i+1, n]$, when $l'_k = l_k$, (b) is equivalent to $\sum_{2 \leq h < k} v_{kh} l'_h \leq \sum_{2 \leq h < k} v_{kh} l_h$, when $l'_k \leq l_k$, $l'_k = \sum_{2 \leq h < k} v_{kh} l'_h$, then LHS of (b) is zero.

Since $W \in \mathcal{W}$ and inequalities of (a) hold, we have inequality (b) holds for W/W'. Thus W/W' belongs to $\mathcal{W}$ with $deg(W/W') < deg(W)$, so that $W/W' \in \mathrm{Im}(\varphi)$.

Then we have

$$\begin{aligned} x_j^{-1} c_t(W/W_j) &= W \cdot \prod\nolimits_{p \in [2,n]-j} x_p^{\gamma_{pt}} / \mathbf{x}^{\mathbf{v_j}} \\ &= (W/W') \cdot (W' \cdot (x_2^{\gamma_{2t}} \cdots x_{j-1}^{\gamma_{j-1,t}}) \cdot (x_{j+1}^{\gamma_{j+1,t} - v_{j+1,j}} \cdots x_n^{\gamma_{nt} - v_{nj}})) \\ &= (W/W') \cdot P \end{aligned}$$

The claim $x_j^{-1} c_t(W/W_j) \in \mathrm{Im}(\varphi)$ is a consequence of the statement that $P \in R[x_2, \cdots, x_n]$. Indeed, $R[x_2, \cdots, x_n] \subseteq \mathrm{Im}(\varphi)$.

The only variable with negative power (namely, -1) in W' is x_i, since

$$\begin{aligned} W' &= W_i^1 W_{i+1}^{l'_{i+1}} \cdots W_j^{l'_j} \\ &= x_i^{-1} \cdot (x_{i+1}^{v_{i+1,i} - l'_{i+1}} \cdot x_{i+2}^{(\sum_{i \leq h < i+2} v_{i+2,h} l'_h) - l'_{i+2}} \cdots x_j^{(\sum_{i \leq h < j} v_{j,h} l'_h) - l'_j}) \cdot (x_{j+1}^{\sum_{i \leq h \leq j} v_{j+1,h} l'_h} \cdots x_n^{\sum_{i \leq h \leq j} v_{nh} l'_h}) \\ &= x_i^{-1} \cdot Q \cdot (x_{j+1}^{\delta_{j+1,t}} \cdots x_n^{\delta_{nt}}) \end{aligned}$$

where $\delta_{pt} = \sum_{i \leq h \leq j} v_{ph} l'_h$ for $p \in [j+1, n]$. Then we have

$$P = Q \cdot (\prod_{q \in [2, i-1] \cup [i+1, j-1]} x_q^{\gamma_{qt}}) \cdot x_i^{\gamma_{it} - 1} \cdot (\prod_{p \in [j+1, n]} x_p^{\delta_{pt} + \gamma_{pt} - v_{pj}}).$$

For i is the fixed index such that $\gamma_{it} \in \mathbb{Z}_{>0}$, $\gamma_{it} - 1 > 0$, then the power of x_i is nonnegative.

For $p \in [j+1, n]$, we have

$$\delta_{pt} + \gamma_{pt} - v_{pj} = v_{pi}l'_i + \cdots + v_{pj}l'_j + \gamma_{pt} - v_{pj} \geq v_{pj}(l'_j - 1).$$

From the definition of W', we obtain $l'_j = min\{l_j, \sum_{i \leq q < j} v_{jq}l'_q\}$, and it is easy to see that $l_j \geq 1$ and $\sum_{i \leq q < j} v_{jq}l'_q = v_{ji} + \sum_{i < q < j} v_{jq}l'_q \geq v_{ji} \geq 1$ by the choice of i and j. Then $l'_j \geq 1$. Thus the power of x_p is nonnegative. Hence the power of any cluster variable is nonnegative. It follows that $P \in R[x_2, \cdots, x_n]$. □

By the same technique as in [4], we give the following theorem.

Theorem 4.23. *If a LP seed* $\Sigma = (\mathbf{x}, \mathbf{F})$ *satisfying Condition 1.2,* $\mathcal{L}(\Sigma) = \mathcal{U}(\Sigma)$.

Proof. We apply the induction on n, that is, the rank of the LP seed. When $n = 1$, by Lemma 4.3, we have $\mathcal{L}(\Sigma) = R[x_1, x'_1] = \mathcal{U}(\Sigma)$. Assume that $n \geq 2$ and the statement holds for all algebras of rank 2 to $n-1$. Then we consider about rank n.

By Lemma 4.3, we have

$$\mathcal{U}(\Sigma) = \bigcap_{j=2}^{n} R[x_1^{\pm 1}, \ldots, x_{j-1}^{\pm 1}, x_j, x'_j, x_{j+1}^{\pm 1}, \ldots, x_n^{\pm 1}] \bigcap R[x_1, x'_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}].$$

For the seed Σ' obtained from Σ by freezing at x_1, by the induction assumption, we have $\mathcal{L}(\Sigma') = \mathcal{U}(\Sigma')$, that is, $\bigcap_{j=2}^{n} R[x_1^{\pm 1}, \ldots, x_{j-1}^{\pm 1}, x_j, x'_j, x_{j+1}^{\pm 1}, \ldots, x_n^{\pm 1}] = R[x_1^{\pm 1}, x_2, x'_2, \ldots, x_n, x'_n]$. Then it is enough to show that

$$R[x_1, x'_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}] \cap R[x_1^{\pm 1}, x_2, x'_2, \ldots, x_n, x'_n] = R[x_1, x'_1, \ldots, x_n, x'_n]. \tag{4.9}$$

The inclusion $\supseteq$ is clear, we only need to prove the converse inclusion.

For $\forall y \in$ LHS of (4.9), let a be the smallest power of x_1 in $y|_{x_i x'_i \leftarrow F_i}$. Then y can be written as $\sum_{m=a}^{b} c_m x_1^m$ where $c_m \in R^{st}[x_2, x'_2, \ldots, x_n, x'_n]$. By Lemma 4.20, we have

$$LT(y) = \varphi(c_a)x_1^a \in R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}].$$

If $a \geq 0$, by Lemma 4.21, we have $y \in R[x_1, \ldots, x_n, x'_n] \subseteq$ the RHS of (4.9).

Otherwise, we apply the induction on $|a|$. Since $y \in R[x_1, x'_1, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, by Lemma 4.4 we have $\varphi(c_a)$ is divisible by $F_1^{|a|}$, that is $\varphi(c_a) = F_1^{|a|} z_a$ for certain $z_a \in R[x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$.

When $J = \emptyset$, we have $\mathrm{Im}(\varphi) = R[x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ according to Lemma 4.22. Then $z_a \in \mathrm{Im}(\varphi)$.

When $J \neq \emptyset$, we consider the LP seed Σ^* obtained from Σ by freezing at $\{x_j | j \in [2, n] - J\}$ and removing x_1. In view of Lemma 4.22, we have $\mathcal{L}(\Sigma^*)$ =$\mathrm{Im}(\varphi)$. Besides, by the induction assumption, we have $\mathcal{L}(\Sigma^*) = \mathcal{U}(\Sigma^*)$. Using Lemma 4.3, we obtain $\mathrm{Im}(\varphi) = \bigcap_{j \in J} R[x_2^{\pm 1}, \ldots, x_j, x'_j, \ldots, x_n^{\pm 1}]$.

For certain $j \in J$, z_a can be written as $\sum_{s \in \mathbb{Z}} c_s x_j^s$, where $c_s \in R[x_2^{\pm 1}, \ldots, \hat{x}_j, \ldots, x_n^{\pm 1}]$. Since $F_1^{|a|} z_a = \sum_{s \in \mathbb{Z}} (c_s F_1^{|a|}) x_j^s \in \mathrm{Im}(\varphi) \subseteq R[x_2^{\pm 1}, \ldots, x_j, x'_j, \ldots, x_n^{\pm 1}]$, by Corollary 4.4 $c_s F_1^{|a|}$ is divisible by $F_j^{|s|}$ for $s < 0$.

By Proposition 4.10, F_1 and F_j are coprime, implying that c_s is divisible by $F_j^{|s|}$. Using Corollary 4.4 again, we have $z_a \in R[x_2^{\pm 1}, \ldots, x_j, x'_j, \ldots, x_n^{\pm 1}]$.

By the arbitrariness of $j \in J$, we obtain $z_a \in \mathrm{Im}(\varphi)$.

Then there exist $c'_a \in R[x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that $z_a = \varphi(c'_a)$. It implies that

$$LT(y) = \varphi(c_a)x_1^a = F_1^{|a|} z_a x_1^a = \varphi(c'_a)F_1^{|a|}x_1^a = \varphi(c'_a)x_1'^{|a|}.$$

Then we have $y = \sum\limits_{m=a}^{b} c_m x_1^m = \sum\limits_{m=a}^{-1} c'_m x_1'^{|m|} + \sum\limits_{m=0}^{b} c_m x_1^m \in R[x_1, x'_1, \ldots, x_n, x'_n]$. □

Corollary 4.24. If a LP seed $\Sigma = (\mathbf{x}, \mathbf{F})$ satisfied Condition 1.2, then the standard monomials in $x_1, x'_1, \ldots, x_n, x'_n$ form an R-basis of the LP algebra $\mathcal{A}(\Sigma)$.

Proof. It is immediately from Theorem 4.16 and Theorem 4.23. □

Example 4.25. Consider the LP seed $(\mathbf{x}, \mathbf{F}) = \{(a, bcd + 1), (b, a + cd), (c, bd + 1), (d, 1 + abc)\}$, it is easy to see that Condition 1.2 (i) (ii) (iii) hold. Since $M_c = 1$ and $b|(F_c - M_c)$, (iv) of Condition 1.2 holds. Besides, $\varphi: \ b' \mapsto \frac{cd}{b},\ c' \mapsto \frac{bd+1}{c} = c',\ d' \mapsto \frac{1}{d}$, Then it is clear that $d^{-1} \in \mathrm{Im}(\varphi)$ and $b^{-1} = \varphi(b'c'd' - d) \in \mathrm{Im}(\varphi)$. Thus by Theorem 4.23, we have $\mathcal{L}(\Sigma) = \mathcal{U}(\Sigma)$.

Note that this LP seed is not a cluster seed or a generalized cluster seed for $c \in F_a$ since $a \notin F_c$.

The cluster seed is acyclic if and only if there exist a permutation σ such that for $i > j$, $b_{\sigma(i),\sigma(j)} \geq 0$. Renumbering if necessary the indexes of the initial acyclic cluster, we assume that for $i > j$, $b_{ij} \geq 0$. Then by the exchange polynomials for cluster algebras, we conclude that the cluster seed is acyclic if and only if for any j, $F_j = \frac{y_j}{1 \oplus y_j} \prod\limits_{i>j} x_i^{b_{ij}} + \frac{y_j}{1 \oplus y_j} \prod\limits_{i<j} x_i^{-b_{ij}}$.

Proposition 4.26. *Condition 1.2 is equivalent to acyclicity and coprimeness of exchange polynomials for a cluster seed which is a also LP seed.*

Proof. When a cluster seed is a LP seed, recall that (i) in Condition 1.2 is equivalent to coprimeness of exchange polynomials by Remark 4.11.

When a cluster seed satisfies the conditions (i) and (ii), for any $j \in [2, n-1]$, since F_j is a binomial, we have $F_j = \mathbf{x}^{\mathbf{v_j}} + \mathbf{x}^{\mathbf{b_j}}$. If $x_i \in \mathbf{x}^{\mathbf{b_j}}$ for some $i > j$, then $x_j \in \mathbf{x}^{\mathbf{v_i}}$ for $b_{ji}b_{ij} < 0$, which contradicts to the condition (ii). For $j = n$, $F_n = 1 + \mathbf{x}^{\mathbf{b_n}}$. From the definition of exchange polynomials for cluster algebras, we have $\mathbf{x}^{\mathbf{b_n}}$ is of the form $x_1^{|b_{1n}|} \cdots x_{n-1}^{|b_{n-1,n}|}$. For $j = 1$, since for any $j > 1$, we have $b_{j1} > 0$ by the above discussion, so that we obtain $F_1 = 1 + x_2^{|b_{12}|} \cdots x_n^{|b_{1n}|}$. Then the cluster seed satisfied the conditions (i) and (ii) is acyclic. Besides, it is easy to see that when a cluster seed is acyclic, it satisfies the conditions (i) and (ii).Thus the conditions (i) and (ii) are equivalent to acyclicity of exchange polynomials.

Under the conditions of acyclicity and coprimeness, the cluster seed in fact satisfies (iii) and (iv) in Condition 1.2. □

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Conflict of interest

The authors declare there is no conflicts of interest.

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