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Research article

The well-posedness for semilinear time fractional wave equations on $\mathbb{R}^N$

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Abstract: This paper is concerned with the semilinear time fractional wave equations on the whole Euclidean space, also known as the super-diffusive equations. Considering the initial data in the fractional Sobolev spaces, we prove the local/global well-posedness results of L^2-solutions for linear and semilinear problems. The methods of this paper rely upon the relevant wave operators estimates, Sobolev embedding and fixed point arguments.

Keywords: fractional derivative; fractional wave equations; well-posedness

1. Introduction

In this paper, we focus on the following time fractional wave equation

$$\partial_t^\beta u - \Delta u = f(u), \quad t > 0, \tag{1.1}$$

supplemented with the initial conditions

$$u(0,x) = \phi(x), \ \partial_t u(0,x) = \psi(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where ∂_t^β stands for the Caputo fractional derivative operator of order $\beta \in (1,2)$, Δ is the Laplacian operator, and f is the semilinear data to be specified later, initial data ϕ, ψ are given in certain fractional Sobolev spaces, likely $(\phi,\psi) \in H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)$ for $s \in \mathbb{R}$.

From the point of view of physics, it's often better for fractional derivatives to fit practical problems than the integer order setting counterpart in many cases, for example, Hamiltonian chaos [1], biophysics [2, 3], control engineering [4, 5], viscoelasticity [6, 7], anomalous diffusion [8–11], switched systems [12], and other problems [13–15]. In particular, the study of the Eq (1.1) has always been an important topic in the mathematical physics as it represents anomalous diffusion phenomena. The time fractional partial differential equation $\partial_t^\beta u = \Delta u$ of order $\beta \in (0, 1)$ models anomalous diffusion phenomena ensuring the behavior of a subdiffusion process driven by a fractional Brownian motion [16]. The case of order $\beta \in (1, 2)$ will govern intermediate processes between diffusion and wave propagation, for example, see [7], and it also ensures the behavior of superdiffusion process, for instance, see [17]. While the cases $\beta \to 1^+$ and $\beta \to 2^-$ respectively correspond to standard diffusion equation (heat equation) and ballistic diffusion (wave equation). Besides, in anomalous diffusion equations of order $\beta \in (0, 1)$ or $\beta \in (1, 2)$, the mean squared displacement of a diffusive particle behavior likes $\langle x^2(t)\rangle \sim t^\beta$, in contrast to normal diffusion behavior (Brownian motion) $\langle x^2(t)\rangle \sim t$.

There are many recent interesting works about time fractional wave equations. One of the most favorable reasons is that the integral kernel in fractional time derivative represents memory of a long-time tail of the power order. The investigation of existence and uniqueness of solutions for a low regularity initial data is a matter of interest in the mathematical analysis. For instance, Kian and Yamamoto [18] investigated a weak solution for semilinear case of (1.1) in bounded domain Ω for dimension $n = 2, 3$. By using the technique of eigenvalue expansion together with the properties of Mittag-Leffler functions, they established the existence and uniqueness results, which the solution shall lie in $L^p(0, T; L^q(\Omega)) \cap C([0, T]; H^{2r}(\Omega))$ for $r = \min(1 - 1/\beta, \gamma)$ with some $1 \le p, q \le \infty, 0 < \gamma < 1$. Following this technique, Alvarez et al. [19] considered the well-posedness for an abstract Cauchy problem in a Hilbert space, where the solutions will lie in $L^{rq}(0, T; L^{2r}(X))$ associated with initial data $(\phi, \psi) \in D(A^\gamma) \times L^2(X)$, $D(A^\gamma)$ are the fractional power spaces with spectrum form of A (nonnegative self-adjoint operator) for some $\beta \in (1, 2)$, $\gamma = 1/\beta$, X is a (relatively) compact metric space and $q \in (1/(\beta - 1), \infty]$, $r > 1$. Moreover, Otarola and Salgado [20] studied the time and space regularities of weak solutions for the space-time fractional wave equations, Zhou and He [21] discussed a well-posedness and time regularity of mild solutions for time fractional damped wave equations, etc. For more details, we refer to see the papers [22–26].

However, few results discuss the unbounded domain case, likely the whole Euclidean space $\mathbb{R}^N$. This is due the difficulty to establish the relevant estimates on solution operators, also the method of eigenvalue expansion is not appropriate to such problem. As we know, Alemida and Precioso [27] investigated the global existence with large initial data in the framework of Besov-Morrey spaces, Alemida and Viana [28] studied existence, stability, self-similarity and symmetries of solutions with initial data in Sobolev-Morrey space. Zhang and Li [29] studied the local existence on $C([0, T], L^q(\mathbb{R}^N))$ for $\beta N(p - 1)/2 < q$ for a special semilinear function $f \sim |u|^p$, also the critical exponents of the global existence and blow-up solutions are determined when $\psi \equiv 0$ and $\psi \not\equiv 0$. Djida et al. [30] worked on a well-posedness result for semilinear space-time fractional wave equations, by adopting the method of Laplace-Fourier transforms, the properties of Mittag-Leffler functions and Fox H-functions.

Our goal in this work is to establish the well-posedness results for time fractional wave equations with initial conditions in certain function spaces. More precisely, we consider linear and semilinear problems on $\mathbb{R}^N$, and present a general assumption in semilinear function and a special case $\lambda|u|^\alpha u$ to deal with the current problem. In addition, this paper is devoted to studying the well-posedness of mild solutions to the current problems. In order to obtain the solution operators for time fractional wave equation and their properties, we shall establish some useful estimates which depend on that of wave operators. We remark that our proofs and results are completely different from the previous mentioned works. We will concern about the local/global well-posedness of solutions. Let us now enlist the main results presented in this paper:

I. The solution operators of Eq (1.1). Let $\varpi = (-\Delta)^{1/2}$, we know that the wave operators can be given by $\cos(\varpi t)$ and $\varpi^{-1}\sin(\varpi t)$. Concerned with the principle of subordination, two inherent relationships between probability density function and wave operators are presented. More precisely, we get

$$\mathcal{C}_\sigma(t) = \int_0^\infty M_\sigma(\theta)\cos(\varpi t^\sigma \theta)d\theta, \quad \mathcal{S}_\sigma(t) = \sigma t^{\sigma-1}\int_0^\infty \theta M_\sigma(\theta)\varpi^{-1}\sin(\varpi t^\sigma \theta)d\theta.$$

To the best of our knowledge, these forms of solution operators are completely different from the existing literatures, relying upon the estimates of wave operators, some useful estimates of solution operators can be derived, which provide very helpful tools for the proofs. Observe that, the probability density function builds a bridge between integer wave equations and the fractional ones.

II. The well-posedness results on $\mathbb{R}^N$. We first establish some estimates on several spaces for the linear problem with initial data $(\phi,\psi) \in H^s(\mathbb{R}^N)\times H^{s-1}(\mathbb{R}^N)$, and then an existence of L^2-solution is established on a space of continuous functions. Next, under the case that the semilinear function $f: L^2(\mathbb{R}^N)\cap L^r(\mathbb{R}^N) \to L^{r'}(\mathbb{R}^N)$, and satisfies

$$\|f(u)-f(v)\|_{L^{r'}(\mathbb{R}^N)} \le C_\alpha(R)\left(\|u\|^\alpha_{L^r(\mathbb{R}^N)} + \|v\|^\alpha_{L^r(\mathbb{R}^N)}\right)\|u-v\|_{L^r(\mathbb{R}^N)},$$

where constants r and α satisfy some restriction requirements, the local well-posedness results of mild solutions are established in the framework of $L^\gamma(L^{\gamma'})$ spaces, here γ and γ' are the conjugate indices. By concerning with a special semilinear data $f \sim \lambda|u|^\alpha u$, we show that the local well-posedness on Besov space $B^s_{r,2}(\mathbb{R}^N)$. Furthermore, when $\psi \equiv 0$, we also show the global existence to the Cauchy problem (1.1)–(1.2).

The rest of this paper is divided into three sections. In Section 2, some basic notations and useful preliminaries are introduced. In Section 3, for the linear problem, we derive the solution operators and establish their some properties. In addition, two existence of L^2-solutions are given. In Section 4, we prove some local/global well-posedness results on Lebesgue and Besov spaces for the semilinear problems.

2. Preliminaries

In this section, some notations and preliminaries related to our work will be introduced.

Denote by $L^q(\mathbb{R}^N)$ $(q \ge 1)$ the Lebesgue space of q-integrable functions with the norm $\|\cdot\|_{L^q(\mathbb{R}^N)}$. Let $\mathcal{S}(\mathbb{R}^N)$ be the Schwartz space and let $\mathcal{S}'(\mathbb{R}^N)$ denote its topological dual, for any $v \in \mathcal{S}(\mathbb{R}^N)$, $\mathcal{F}$ represents the Fourier transform

$$\hat{v}(\xi) = \mathcal{F}(v)(\xi) = (2\pi)^{-N/2}\int_{\mathbb{R}^N} e^{-ix\cdot\xi}v(x)dx,$$

with its inverse

$$\check{v}(x) = \mathcal{F}^{-1}(v)(x) = (2\pi)^{-N/2}\int_{\mathbb{R}^N} e^{ix\cdot\xi}v(x)dx.$$

Define the Sobolev space by

$$H^{s,q}(\mathbb{R}^N) = \{u \in \mathcal{S}'(\mathbb{R}^N):\ \mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}(u)] \in L^q(\mathbb{R}^N)\},$$

equipped with the norm

$$\|u\|_{H^{s,q}(\mathbb{R}^N)} := \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}(u)]\|_{L^q(\mathbb{R}^N)},$$

for $s \in \mathbb{R}$, $1 \le q \le \infty$. We also use the Besov space $B^s_{p,q} := B^s_{p,q}(\mathbb{R}^N)$ and homogeneous Besov space $\dot{B}^s_{p,q} := \dot{B}^s_{p,q}(\mathbb{R}^N)$, for $s \in \mathbb{R}$, $1 \le p, q \le \infty$. For the definitions and properties of Besov spaces, we refer to see [31, 32]. In particular, it yields $H^{0,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$ and $H^s(\mathbb{R}^N) = H^{s,2}(\mathbb{R}^N) = B^s_{2,2}(\mathbb{R}^N)$ for $p \ge 1$ and $s \in \mathbb{R}$. Throughout this paper, we denote the notation $a \lesssim b$ that stands for $a \le Cb$, with a positive generic constant C that does not depend on a, b, the notations $\vee$, $\wedge$ stand for $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$, respectively. Let p and p' be the conjugate indices such that $1/p + 1/p' = 1$.

Let $T > 0$ (or $T = +\infty$) and let X be a usual Banach space. For any $u \in L^1(0, T; X)$ and $v \in L^1(0, T; X)$, denote $*$ the convolution by

$$(u * v)(t) = \int_0^t u(t-s)v(s)ds, \quad t \ge 0,$$

and for $\beta \ge 0$ let the weak singular kernel $g_\beta(\cdot)$ be defined by

$$g_\beta(t) = t^{\beta-1}/\Gamma(\beta), \quad t > 0,$$

where $\Gamma(\cdot)$ is the usual gamma function. We denote $g_0(t) = \delta(t)$, the Dirac measure is concentrated at the origin. Next, let us recall the concepts of fractional calculus and Mittag-Leffler functions. The Riemann-Liouville fractional integral of order $\beta \ge 0$ for a function $v \in L^1(0, T; X)$ is defined as

$$J_t^\beta v(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}v(s)ds = (g_\beta * v)(t),\ t > 0.$$

Definition 2.1. *Let $\beta \in (1, 2)$. Consider a function $v \in L^1(0, T; X)$ such that convolution $g_{2-\beta} * v \in W^{2,1}(0, T; X)$. The representation*

$$\partial_t^\beta v(t) = \partial_{tt}^2(g_{2-\beta} * [v(t) - v(0) - t\,\partial_t v(0)]), \quad t > 0,$$

is called the Caputo fractional derivative of order β.

An important special function for the fractional differential equations involving the Caputo fractional derivative is the Mittag-Leffler function, which is defined by

$$E_{\nu,\mu}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\nu k + \mu)}, \quad z \in \mathbb{C},\ \nu > 0,\ \mu \in \mathbb{R}.$$

Note that, if $w(t) := E_{\nu,1}(at^\nu)$, $\nu \in (0, 2)$, $a \in \mathbb{R}$, then one can check that w is the solution of the equation $\partial_t^\nu w(t) = aw(t)$.

Let the probability density function $M_\upsilon(\cdot)$ (also called Mainardi's Wright function) be defined by

$$M_\upsilon(z) = \sum_{k=0}^{\infty}\frac{(-z)^k}{k!\Gamma(1-\upsilon(k+1))}, \quad \upsilon \in (0, 1),\ z \in \mathbb{C}.$$

For $\theta > 0$, this probability density function has the properties

$$M_\upsilon(\theta) \ge 0, \quad \int_0^\infty \theta^\delta M_\upsilon(\theta)d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+\upsilon\delta)}, \quad for\ -1 < \delta < \infty. \tag{2.1}$$

Next, we see that the probability density function can be viewed as a bridge between the classical and fractional wave equations.

Lemma 2.1. *Let $\beta \in (1,2)$ [21]. Then for $z \in \mathbb{C}$, the following formulas expressing the Mittag-Leffler function in term of probability density function hold:*

$$E_{\beta,1}(-z^2) = \int_0^\infty M_{\beta/2}(\theta)\cos(z\theta)d\theta, \quad E_{\beta,\beta}(-z^2) = \frac{\beta}{2z}\int_0^\infty \theta M_{\beta/2}(\theta)\sin(z\theta)d\theta.$$

Lemma 2.2. *Let $f \in L^1(0,T;\mathbb{R})$ [33]. The unique solution of the fractional order problem*

$$\begin{cases} \partial_t^\beta u(t) + au(t) = f(t), & a \geq 0,\ t \geq 0, \\ u(0) = u_0,\ u'(0) = u_1, \end{cases}$$

is give by

$$u(t) = E_{\beta,1}(-at^\beta)u_0 + tE_{\beta,2}(-at^\beta)u_1 + \int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(-a(t-s)^\beta)f(s)ds.$$

In particular,

$$u(t) = \cos(\sqrt{a}\,t)u_0 + \frac{1}{\sqrt{a}}\sin(\sqrt{a}\,t)u_1 + \frac{1}{\sqrt{a}}\int_0^t \sin(\sqrt{a}\,(t-s))f(s)ds,$$

which is the unique solution to the corresponding classical wave equation, i.e., $\beta = 2$.

3. Local/global solutions of linear problems

In this section, we are concerned with the following linear Cauchy problem

$$\begin{cases} \partial_t^\beta u(t,x) - \Delta u(t,x) = f(t,x), & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = \phi(x),\ \partial_t u(0,x) = \psi(x). \end{cases} \tag{3.1}$$

Without loss of generality, the solutions in this paper are defined as mild solutions associated with the corresponding initial data.

3.1. Solution representation

We first establish the solution representation of linear problem (3.1). Let u satisfies (3.1), taking the Fourier transform of both sides in (3.1) with respect to $x \in \mathbb{R}^N$, we obtain

$$\begin{cases} \partial_t^\beta \hat{u}(t,\xi) + |\xi|^2\hat{u}(t,\xi) = \hat{f}(t,\xi), & t > 0, \\ \hat{u}(0,\xi) = \hat{\phi}(\xi),\ \partial_t\hat{u}(0,\xi) = \hat{\psi}(\xi). \end{cases}$$

It follows from [4, (1.100)] that

$$tE_{\beta,2}(-t^\beta|\xi|^2) = \frac{1}{\Gamma(2-\beta)}\int_0^t (t-s)^{1-\beta}E_{\beta,\beta}(-s^\beta|\xi|^2)s^{\beta-1}ds.$$

Therefore, by virtue of Lemma 2.2, we get

$$\hat{u}(t,\xi) = \hat{\varphi}(t,\xi)\hat{\phi}(\xi) + (\hat{\vartheta}(\cdot,\xi) * g_{2-\beta})(t)\hat{\psi}(\xi) + (\hat{\vartheta}(\cdot,\xi) * \hat{f})(t),$$

where

$$\hat{\varphi}(t,\xi) = E_{\beta,1}(-t^{\beta}|\xi|^2), \quad \hat{\vartheta}(t,\xi) = t^{\beta-1}E_{\beta,\beta}(-t^{\beta}|\xi|^2).$$

By using the inverse Fourier transform, we get

$$\begin{aligned} u(t,x) = &\int_{\mathbb{R}^N} \varphi(t,x-y)\phi(y)dy + \int_0^t \int_{\mathbb{R}^N} g_{2-\beta}(t-s)\vartheta(s,x-y)\psi(y)dyds \\ &+ \int_0^t \int_{\mathbb{R}^N} \vartheta(t-s,x-y)f(s,y)dyds, \end{aligned}$$

where

$$\begin{aligned} \varphi(t,x) &= (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{ix\cdot\xi}E_{\beta,1}(-t^{\beta}|\xi|^2)d\xi, \\ \vartheta(t,x) &= (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{ix\cdot\xi}t^{\beta-1}E_{\beta,\beta}(-t^{\beta}|\xi|^2)d\xi. \end{aligned}$$

On the other hand, set $\sigma = \beta/2 \in (1/2, 1)$, it follows from Lemma 2.1 that

$$\varphi(t,x) = (2\pi)^{-N/2} \int_0^{\infty} \int_{\mathbb{R}^N} e^{ix\cdot\xi}M_{\sigma}(\theta)\cos(t^{\sigma}\theta|\xi|)d\xi d\theta,$$

and

$$\vartheta(t,x) = t^{\sigma-1}(2\pi)^{-N/2} \int_0^{\infty} \int_{\mathbb{R}^N} e^{ix\cdot\xi}\sigma\theta M_{\sigma}(\theta)\frac{\sin(t^{\sigma}\theta|\xi|)}{|\xi|}d\xi d\theta.$$

Let

$$\dot{K}(t)v = \mathcal{F}^{-1}[\cos(t|\xi|)\hat{v}(\xi)], \quad K(t)v = \mathcal{F}^{-1}\left[\frac{\sin(t|\xi|)}{|\xi|}\hat{v}(\xi)\right],$$

and

$$\mathcal{S}_{\sigma}(t)v = \sigma t^{\sigma-1} \int_0^{\infty} \theta M_{\sigma}(\theta)K(t^{\sigma}\theta)vd\theta.$$

Rewriting $u(t)$ for the function $u(t,\cdot)$, we get an equivalent integral representation for problem (3.1) by

$$u(t) = C_{\sigma}(t)\phi + \mathcal{P}_{\sigma}(t)\psi + \int_0^t \mathcal{S}_{\sigma}(t-s)f(s)ds, \tag{3.2}$$

where solution operators $C_{\sigma}(\cdot)$ and $\mathcal{P}_{\sigma}(\cdot)$ are defined by

$$C_{\sigma}(t)\phi = \int_0^{\infty} M_{\sigma}(\theta)\dot{K}(t^{\sigma}\theta)\phi d\theta, \quad \mathcal{P}_{\sigma}(t)\psi = (g_{2-2\sigma} * \mathcal{S}_{\sigma})(t)\psi.$$

3.2. Some properties of solution operators

Let $\alpha(r) = \frac{1}{2} - \frac{1}{r}$ for $r \in [2,\infty]$, and

$$\beta(r) = \frac{N+1}{2}\alpha(r), \quad \gamma(r) = (N-1)\alpha(r), \quad \delta(r) = N\alpha(r).$$

Let us recall the following two results in [34, 35], which play a key role in proving the general results on the solution operators.

Lemma 3.1. *Let $2 \le p < \infty$ and $2\beta(p) \le \nu \le 2\delta(p)$. Then for $t \neq 0$, it follows that*

$$\left\| \mathcal{F}^{-1}\left[|\xi|^{-\nu} e^{it|\xi|}\hat{v}(\xi) \right] \right\|_{L^p(\mathbb{R}^N)} \lesssim |t|^{\nu-2\delta(p)} \|v\|_{L^{p'}(\mathbb{R}^N)}.$$

Lemma 3.2. *Let $2 \le p < \infty$ and $2\beta(p) \le \nu \le 2\delta(p)$, $0 \le \mu \le s+\nu$. Then for $1 \le q \le \infty$, $t \neq 0$, it follows that*

$$\left\| \mathcal{F}^{-1}\left[|\xi|^{-\mu} e^{it|\xi|}\hat{v}(\xi) \right] \right\|_{B^s_{p,q}(\mathbb{R}^N)} \lesssim |t|^{\nu-2\delta(p)} \|v\|_{B^{s+\nu-\mu}_{p',q}(\mathbb{R}^N)}.$$

In the sequel, we set $\varpi = (-\Delta)^{\frac{1}{2}}$ and $U(t) = \exp(i\varpi t) = \mathcal{F}^{-1}[\exp(i|\xi|t)\mathcal{F}]$, so that $K(t) = \varpi^{-1}\sin(\varpi t)$, $\dot{K}(t) = \cos(\varpi t)$. Hence, it follows that

$$K(t) = \varpi^{-1}\frac{U(t) - U(-t)}{2i}, \quad \dot{K}(t) = \frac{U(t) + U(-t)}{2}.$$

Lemma 3.3. *Let $N \ge 2$, $2N/(N-1) \le p \le 2(N+1)/(N-1)$, then $\mathcal{S}_\sigma(t) : L^{p'}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ and moreover*

$$\|\mathcal{S}_\sigma(t)v\|_{L^p(\mathbb{R}^N)} \lesssim t^{2\sigma-2\sigma\delta(p)-1}\|v\|_{L^{p'}(\mathbb{R}^N)}, \quad v \in L^{p'}(\mathbb{R}^N), \ \ t > 0.$$

Proof. Obviously, the condition $\frac{2N}{N-1} \le p \le \frac{2(N+1)}{N-1}$ for $N \ge 2$ implies that $1/2 \le \delta(p) \le N/(N+1)$. Hence, it follows from Lemma 3.1 that

$$\|K(t)v\|_{L^p(\mathbb{R}^N)} \lesssim t^{1-2\delta(p)}\|v\|_{L^{p'}(\mathbb{R}^N)}.$$

By the definition of $\mathcal{S}_\sigma(t)$, and (2.1) we obtain

$$\begin{aligned}
\|\mathcal{S}_\sigma(t)v\|_{L^p(\mathbb{R}^N)} &\le \sigma t^{\sigma-1}\int_0^\infty \theta M_\sigma(\theta)\|K(t^\sigma\theta)v\|_{L^p(\mathbb{R}^N)}d\theta \\
&\lesssim \sigma t^{\sigma-1+\sigma(1-2\delta(p))}\int_0^\infty \theta^{2-2\delta(p)}M_\sigma(\theta)d\theta\|v\|_{L^{p'}(\mathbb{R}^N)} \\
&\lesssim t^{2\sigma-2\sigma\delta(p)-1}\|v\|_{L^{p'}(\mathbb{R}^N)}.
\end{aligned}$$

Consequently, we get the desired inequality. □

Lemma 3.4. *Let $2 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$, $t > 0$.*

If $(-s) \vee (2\delta(p)-1) \vee (2\beta(p)) \le \nu \le 2\delta(p)$, then

$$\|C_\sigma(t)v\|_{B^s_{p,q}(\mathbb{R}^N)} \lesssim t^{\sigma\left(\nu-2\delta(p)\right)}\|v\|_{B^{s+\nu}_{p',q}(\mathbb{R}^N)}.$$

If $(1-s) \vee (2\delta(p)-2) \vee (2\beta(p)) \le \nu \le 2\delta(p)$, then

$$\|\mathcal{S}_\sigma(t)v\|_{B^s_{p,q}(\mathbb{R}^N)} \lesssim t^{\sigma\left(\nu-2\delta(p)+1\right)-1}\|v\|_{B^{s+\nu-1}_{p',q}(\mathbb{R}^N)}.$$

Proof. From Lemma 3.2, for $t > 0$ we have the following estimates

$$\left\|\dot{K}(t)v\right\|_{B^s_{p,q}(\mathbb{R}^N)} \lesssim t^{\nu-2\delta(p)}\|v\|_{B^{s+\nu}_{p',q}(\mathbb{R}^N)}, \quad 0 \le s+\nu,$$

$$\left\|K(t)v\right\|_{B^s_{p,q}(\mathbb{R}^N)} \lesssim t^{\nu-2\delta(p)}\|v\|_{B^{s+\nu-1}_{p',q}(\mathbb{R}^N)}, \quad 1 \le s+\nu.$$

Therefore, using the same argument as employed in Lemma 3.3, we obtain the desired results. □

3.3. The existence results

Let $\sigma = \beta/2$ for $\beta \in (1, 2)$. We now introduce an operator Q_σ defined by

$$(Q_\sigma f)(t) = \int_0^t \mathcal{S}_\sigma(t-s) f(s) ds.$$

In order to obtain the existence of L^2-solutions, we need the following lemma.

Lemma 3.5. *For each $h \in L^p(0, T; X)$ with $1 \le p < +\infty$* [36], *we have*

$$\lim_{\tau \to 0} \int_0^T \|h(t+\tau) - h(t)\|_X^p dt = 0,$$

where we suppose that $h(s) = 0$ for s not belonging to $[0, T]$.

Lemma 3.6. *For any q and μ with $1 < q \le 2$ and $\mu = N(1/q - 1/2 - 1/N)$, $N \ge 1$, let $f \in L^r(0, T; H^{\mu,q}(\mathbb{R}^N))$, then*

$$\|Q_\sigma f\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} \lesssim \|f\|_{L^r(0,T;H^{\mu,q}(\mathbb{R}^N))},$$

for $r > 1/\sigma$. Furthermore, let $q = 2N/(N+2)$ for $N \ge 3$, and $f \in L^r(0, T; L^q(\mathbb{R}^N))$, then

$$\|Q_\sigma f\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} \lesssim \|f\|_{L^r(0,T;L^q(\mathbb{R}^N))}.$$

Moreover the operator Q_σ maps $L^r(0, T; H^{-1}(\mathbb{R}^N))$ into $C([0, T]; L^2(\mathbb{R}^N))$.

Proof. The inequality $|\sin\alpha| \le 1 \wedge |\alpha|$ for any $\alpha \in \mathbb{R}$ implies

$$\begin{aligned} |(\sin(t|\xi|)\langle\xi\rangle)/|\xi|| \le& (1 \wedge (t|\xi|)) \cdot \langle\xi\rangle/|\xi| \le (1+t) \cdot (1 \wedge |\xi|) \cdot \langle\xi\rangle/|\xi| \\ \le& \sqrt{2}(1+t), \end{aligned} \tag{3.3}$$

where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$, for $t \ge 0$, $\xi \in \mathbb{R}^N$. Let $g(t, s, \xi) = \sin((t-s)^\sigma\theta|\xi|)\hat{f}(s, \xi)/|\xi|$ and $y(t, x) = \langle x\rangle^{-1}\hat{f}(s, x)$, for $s \in (0, t)$, $\xi, x \in \mathbb{R}^N$. It is clear by (3.3) that

$$|g(t, s, \xi)| \le \sqrt{2}(1 + (t-s)^\sigma\theta)|y(s, \xi)|, \quad \text{for } s \in (0, t),\ \xi \in \mathbb{R}^N.$$

Then it follows from the Plancherel theorem (see e.g. [37]) that

$$\begin{aligned} \|\check{g}(t, s, \cdot)\|_{L^2(\mathbb{R}^N)} =& \|g(t, s, \cdot)\|_{L^2(\mathbb{R}^N)} \\ \le& \sqrt{2}(1 + (t-s)^\sigma\theta)\|y(s, \cdot)\|_{L^2(\mathbb{R}^N)} = \sqrt{2}(1 + (t-s)^\sigma\theta)\|\check{y}(s, \cdot)\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|(Q_\sigma f)(t)\|_{L^2(\mathbb{R}^N)} \le& \int_0^t \|\mathcal{S}_\sigma(t-s) f(s, \cdot)\|_{L^2(\mathbb{R}^N)} ds \\ \le& \int_0^t \int_0^\infty \sigma(t-s)^{\sigma-1}\theta M_\sigma(\theta)\, \|\check{g}(t, s, \cdot)\|_{L^2(\mathbb{R}^N)}\, d\theta ds \end{aligned}$$

$$\leq \sqrt{2}\int_0^t\int_0^\infty \sigma(t-s)^{\sigma-1}\theta M_\sigma(\theta)(1+(t-s)^\sigma\theta)\|\check{y}(s,\cdot)\|_{L^2(\mathbb{R}^N)}d\theta ds.$$

Due to $\|\check{y}(s,\cdot)\|_{L^2(\mathbb{R}^N)} = \|\mathcal{F}^{-1}[\langle\xi\rangle^{-1}\hat{f}(s,\xi)]\|_{L^2(\mathbb{R}^N)} = \|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)}$, in view of (2.1) and Hölder inequality, for $r > 1/\sigma$ we get

$$\begin{aligned}\|(Q_\sigma f)(t)\|_{L^2(\mathbb{R}^N)} &\leq \sqrt{2}\int_0^t\int_0^\infty \sigma(t-s)^{\sigma-1}\theta M_\sigma(\theta)\|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)}d\theta ds\\ &\quad+\sqrt{2}\int_0^t\int_0^\infty \sigma(t-s)^{2\sigma-1}\theta^2 M_\sigma(\theta)\|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)}d\theta ds\\ &=\sqrt{2}\int_0^t g_\sigma(t-s)\|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)}ds\\ &\quad+\sqrt{2}\int_0^t g_{2\sigma}(t-s)\|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)}ds\\ &\leq C\|f\|_{L^r(0,T;H^{-1}(\mathbb{R}^N))},\end{aligned} \tag{3.4}$$

where C depends on σ, r, N and T.

Now, let us show the first estimate. Due to the embedding $H^{s,p}(\mathbb{R}^N) \hookrightarrow H^{s_1,p_1}(\mathbb{R}^N)$ for $1 < p \leq p_1 < \infty$, $s, s_1 \in \mathbb{R}$, and $s - N/p = s_1 - N/p_1$, (see e.g., [31, 32]) we know that $H^{\mu,q}(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N)$ for $\mu = N(1/q - 1/2 - 1/N)$. This means that

$$\|(Q_\sigma f)(t)\|_{L^2(\mathbb{R}^N)} \lesssim \|f\|_{L^r(0,T;H^{\mu,q}(\mathbb{R}^N))}.$$

Hence, the first estimate holds. On the other hand, since $B^s_{2,2}(\mathbb{R}^N) = H^{s,2}(\mathbb{R}^N)$ for $s \in \mathbb{R}$, recall the embedding $H^{s,q}(\mathbb{R}^N) \hookrightarrow B^s_{q,2}(\mathbb{R}^N)$ for $1 < q \leq 2$ and the embedding (see e.g., [31, 32])

$$B^{s_0}_{p_0,q_0}(\mathbb{R}^N) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^N), \quad \text{for } s_0 - N/p_0 = s_1 - N/p_1, \tag{3.5}$$

for any $s_0, s_1 \in \mathbb{R}$, $1 \leq p_0 \leq p_1 \leq \infty$, $1 \leq q_0 \leq q_1 \leq \infty$, we have

$$B^0_{q,2}(\mathbb{R}^N) \hookrightarrow B^{-1}_{2,2}(\mathbb{R}^N), \quad \text{for } q = \frac{2N}{N+2},$$

by virtue of $H^{0,q}(\mathbb{R}^N) \hookrightarrow B^0_{q,2}(\mathbb{R}^N)$, and $H^{0,q}(\mathbb{R}^N) = L^q(\mathbb{R}^N)$ for $q \geq 1$, we obtain the embedding $L^q(\mathbb{R}^N) \hookrightarrow H^{-1}(\mathbb{R}^N)$ when $q = 2N/(N+2)$. These means that

$$\|(Q_\sigma f)(t)\|_{L^2(\mathbb{R}^N)} \lesssim \|f\|_{L^r(0,T;L^q(\mathbb{R}^N))}.$$

Hence, the second estimate holds.

Finally, in order to obtain the conclusion that the operator Q_σ maps $L^r(0,T;H^{-1}(\mathbb{R}^N))$ into $C([0,T];L^2(\mathbb{R}^N))$, by (3.4) it suffices to prove the continuity of operator Q_σ for any $f \in L^r(0,T;H^{-1}(\mathbb{R}^N))$, i.e., we next to check that for $0 \leq t < t+h \leq T$

$$\|(Q_\sigma f)(t+h) - (Q_\sigma f)(t)\|_{L^2(\mathbb{R}^N)} \to 0, \quad \text{as } h \to 0.$$

In fact, we first have

$$\|(Q_\sigma f)(t+h) - (Q_\sigma f)(t)\|_{L^2(\mathbb{R}^N)}$$

$$
\begin{aligned}
&\leq \Big\| \int_t^{t+h} \mathcal{S}_\sigma(t+h-s) f(s,\cdot) ds \Big\|_{L^2(\mathbb{R}^N)} \\
&\quad + \Big\| \int_0^t \int_0^\infty \sigma((t+h-s)^{\sigma-1} - (t-s)^{\sigma-1}) \theta M_\sigma(\theta) \breve{g}(t+h,s,\cdot) d\theta ds \Big\|_{L^2(\mathbb{R}^N)} \\
&\quad + \Big\| \int_0^t \int_0^\infty \sigma(t-s)^{\sigma-1} \theta M_\sigma(\theta) (\breve{g}(t+h,s,\cdot) - \breve{g}(t,s,\cdot)) d\theta ds \Big\|_{L^2(\mathbb{R}^N)} \\
&:= I_1 + I_2 + I_3.
\end{aligned}
$$

Obviously, applying (2.1) and Hölder inequality, it follows that

$$
\begin{aligned}
I_1 &\leq \sqrt{2} \int_t^{t+h} \int_0^\infty \sigma(t+h-s)^{\sigma-1} \theta M_\sigma(\theta) \|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)} d\theta ds \\
&\quad + \sqrt{2} \int_t^{t+h} \int_0^\infty \sigma(t+h-s)^{2\sigma-1} \theta^2 M_\sigma(\theta) \|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)} d\theta ds \\
&= \sqrt{2} \int_t^{t+h} g_\sigma(t+h-s) \|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)} ds \\
&\quad + \sqrt{2} \int_t^{t+h} g_{2\sigma}(t+h-s) \|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)} ds \\
&\lesssim h^{\sigma-1/r} \|f\|_{L^r(0,T;H^{-1}(\mathbb{R}^N))} + h^{2\sigma-1/r} \|f\|_{L^r(0,T;H^{-1}(\mathbb{R}^N))} \\
&\to 0, \quad \text{as } h \to 0.
\end{aligned}
$$

For the second term I_2, similarly to (3.4) we have

$$
\begin{aligned}
I_2 &\leq \sqrt{2} \int_0^t \int_0^\infty \sigma |(t+h-s)^{\sigma-1} - (t-s)^{\sigma-1}| \theta M_\sigma(\theta) \|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)} d\theta ds \\
&\quad + \sqrt{2} \int_0^t \int_0^\infty \sigma |(t+h-s)^{\sigma-1} - (t-s)^{\sigma-1}| (t+h-s)^\sigma \theta^2 M_\sigma(\theta) \|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)} d\theta ds \\
&= C_h \int_0^t |(t+h-s)^{\sigma-1} - (t-s)^{\sigma-1}| \|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)} ds \\
&\leq C_h \left(\int_0^t |(t+h-s)^{\sigma-1} - (t-s)^{\sigma-1}|^{r/(r-1)} ds \right)^{1-1/r} \|f\|_{L^r(0,T;H^{-1}(\mathbb{R}^N))},
\end{aligned}
$$

where $C_h = \left(\sqrt{2}/\Gamma(\sigma) + \sqrt{2}(T+h)^\sigma/\Gamma(2\sigma) \right)$. Using the Lebesgue dominated convergence theorem and Lemma 3.5 we find that $I_2 \to 0$ as $h \to 0$.

For estimating the third term I_3, by virtue of (3.3), we first have

$$
\left\| \breve{g}(t+h,s,\cdot) - \breve{g}(t,s,\cdot) \right\|_{L^2(\mathbb{R}^N)} \leq 2\sqrt{2}(1 + (t+h-s)^\sigma \theta) \|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)},
$$

which means that

$$
\begin{aligned}
I_3 &\leq \int_0^t \int_0^\infty \sigma(t-s)^{\sigma-1} \theta M_\sigma(\theta) \left\| \breve{g}(t+h,s,\cdot) - \breve{g}(t,s,\cdot) \right\|_{L^2(\mathbb{R}^N)} d\theta ds \\
&\leq 2\sqrt{2} \int_0^t \int_0^\infty \sigma(t-s)^{\sigma-1} \theta M_\sigma(\theta) \|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)} d\theta ds
\end{aligned}
$$

$$+ 2\sqrt{2}\int_0^t\int_0^\infty \sigma(t-s)^{\sigma-1}(t+h-s)^\sigma\theta^2 M_\sigma(\theta)\|f(s,\cdot)\|_{H^{-1}(\mathbb{R}^N)}d\theta ds$$
$$\lesssim \|f\|_{L^r(0,T;H^{-1}(\mathbb{R}^N))}.$$

Hence, by the Lebesgue dominated convergence theorem, we conclude that $I_3 \to 0$ as $h \to 0$. Similarly, for any $0 \le t-h < t \le T$, it is not difficult to verify that $\|(\mathcal{Q}_\sigma f)(t) - (\mathcal{Q}_\sigma f)(t-h)\|_{L^2(\mathbb{R}^N)} \to 0$, as $h \to 0$. Thus, the proof is completed. □

Lemma 3.7. *For any $s \in \mathbb{R}$, operators $C_\sigma(\cdot)$ and $\mathcal{P}_\sigma(\cdot)$ satisfy*

$$\|C_\sigma(\cdot)\phi\|_{C([0,T];H^s(\mathbb{R}^N))} \le \|\phi\|_{H^s(\mathbb{R}^N)}, \quad \|\mathcal{P}_\sigma(\cdot)\psi\|_{C([0,T];H^s(\mathbb{R}^N))} \lesssim \|\psi\|_{H^{s-1}(\mathbb{R}^N)},$$

for any $(\phi,\psi) \in H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)$.

Proof. By virtue of $|\cos(t|x|)| \le 1$ for all $t \ge 0$, $x \in \mathbb{R}^N$, it is easy to get $\|\dot{K}(t)\phi\|_{H^s(\mathbb{R}^N)} \le \|\phi\|_{H^s(\mathbb{R}^N)}$. In fact, for any $\phi \in H^s(\mathbb{R}^N)$, we have

$$\|\dot{K}(t)\phi\|_{H^s(\mathbb{R}^N)} = \|\mathcal{F}^{-1}[\cos(t|\cdot|)\hat{\phi}]\|_{H^s(\mathbb{R}^N)} = \|\mathcal{F}^{-1}[(1+|\cdot|^2)^{s/2}\cos(t|\cdot|)\hat{\phi}]\|_{L^2(\mathbb{R}^N)}.$$

The Plancherel theorem implies

$$\begin{aligned}\|\mathcal{F}^{-1}[(1+|\cdot|^2)^{s/2}\cos(t|\cdot|)\hat{\phi}]\|_{L^2(\mathbb{R}^N)} &= \|(1+|\cdot|^2)^{s/2}\cos(t|\cdot|)\hat{\phi}\|_{L^2(\mathbb{R}^N)}\\ &\le \|(1+|\cdot|^2)^{s/2}\hat{\phi}\|_{L^2(\mathbb{R}^N)}\\ &= \|\mathcal{F}^{-1}[(1+|\cdot|^2)^{s/2}\hat{\phi}]\|_{L^2(\mathbb{R}^N)},\end{aligned}$$

which means that $\|\dot{K}(t)\phi\|_{H^s(\mathbb{R}^N)} \le \|\phi\|_{H^s(\mathbb{R}^N)}$. In addition, by virtue of the inequality $|\sin(t|x|)| \le t|x|$, for all $t \ge 0$, $x \in \mathbb{R}^N$, as repeating the above processes, by (3.3) it is easy to check that $\|K(t)\psi\|_{H^s(\mathbb{R}^N)} \le \sqrt{2}(1+t)\|\psi\|_{H^{s-1}(\mathbb{R}^N)}$ for any $\psi \in H^{s-1}(\mathbb{R}^N)$.

Let us show that $\|C_\sigma(t)\phi\|_{H^s(\mathbb{R}^N)} \le \|\phi\|_{H^s(\mathbb{R}^N)}$. In fact, from the definition of operator $C_\sigma(\cdot)$, we have

$$\|C_\sigma(t)\phi\|_{H^s(\mathbb{R}^N)} \le \int_0^\infty M_\sigma(\theta)\|\dot{K}(t^\sigma\theta)\phi\|_{H^s(\mathbb{R}^N)}d\theta \le \|\phi\|_{H^s(\mathbb{R}^N)},$$

where we have used the identity (2.1). Moreover, from the definition of operator $\mathcal{S}_\sigma(\cdot)$ and the semigroup $(g_a * g_b)(t) = g_{a+b}(t)$ for $a, b > 0$, $t > 0$, we have

$$\begin{aligned}\|\mathcal{P}_\sigma(t)\psi\|_{H^s(\mathbb{R}^N)} &\le \int_0^t g_{2-2\sigma}(t-s)\|\mathcal{S}_\sigma(s)\psi\|_{H^s(\mathbb{R}^N)}ds\\ &\le \int_0^t\int_0^\infty g_{2-2\sigma}(t-s)\sigma s^{\sigma-1}\theta M_\sigma(\theta)\|K(s^\sigma\theta)\psi\|_{H^s(\mathbb{R}^N)}d\theta ds\\ &\le \sqrt{2}\int_0^t\int_0^\infty g_{2-2\sigma}(t-s)\sigma s^{\sigma-1}(1+s^\sigma\theta)\theta M_\sigma(\theta)\|\psi\|_{H^{s-1}(\mathbb{R}^N)}d\theta ds\\ &= \sqrt{2}((g_{2-2\sigma} * g_\sigma)(t) + (g_{2-2\sigma} * g_{2\sigma})(t))\|\psi\|_{H^{s-1}(\mathbb{R}^N)}\\ &\lesssim \|\psi\|_{H^{s-1}(\mathbb{R}^N)},\end{aligned}$$

where $(g_{2-2\sigma} * g_\sigma)(t) + (g_{2-2\sigma} * g_{2\sigma})(t) \le g_{2-\sigma}(T) + g_2(T)$. Hence, it yields $\|\mathcal{P}_\sigma(t)\psi\|_{H^s(\mathbb{R}^N)} \lesssim \|\psi\|_{H^{s-1}(\mathbb{R}^N)}$. To end this proof, it suffices to check the continuity of $C_\sigma(t)\phi$ and $\mathcal{P}_\sigma(t)\psi$.

By the continuity of $\cos(t|\xi|)$, for $h > 0$ and $0 \leq t < t+h \leq T$, by the Plancherel theorem we know that

$$\begin{aligned}\|\dot{K}((t+h)^{\sigma}\theta)\phi - \dot{K}(t^{\sigma}\theta)\phi\|_{H^s(\mathbb{R}^N)} =&\|\mathcal{F}^{-1}[\langle\cdot\rangle^s(\cos((t+h)^{\sigma}\theta|\cdot|) - \cos(t^{\sigma}\theta|\cdot|))\hat{\phi}]\|_{L^2(\mathbb{R}^N)}\\ =&\|\langle\cdot\rangle^s(\cos((t+h)^{\sigma}\theta|\cdot|) - \cos(t^{\sigma}\theta|\cdot|))\hat{\phi}\|_{L^2(\mathbb{R}^N)}\\ \leq&2\|\langle\cdot\rangle^s\hat{\phi}\|_{L^2(\mathbb{R}^N)} = 2\|\phi\|_{H^s(\mathbb{R}^N)}.\end{aligned}$$

Hence, passing to the Fourier representation and the Lebesgue dominated convergence theorem, we have the pointwise convergence

$$\|\dot{K}((t+h)^{\sigma}\theta)\phi - \dot{K}(t^{\sigma}\theta)\phi\|_{H^s(\mathbb{R}^N)} \to 0, \quad \text{as } h \to 0, \text{ a.e. } \theta \in (0,\infty).$$

On the other hand, by (2.1) we have

$$M_{\sigma}(\theta)\|\dot{K}((t+h)^{\sigma}\theta)\phi - \dot{K}(t^{\sigma}\theta)\phi\|_{H^s(\mathbb{R}^N)} \leq M_{\sigma}(\theta)\|\phi\|_{H^s(\mathbb{R}^N)}$$

is integrable for a.e. $\theta \in (0,\infty)$, hence Lebesgue dominated convergence theorem implies

$$\begin{aligned}\|C_{\sigma}(t+h)\phi - C_{\sigma}(t)\phi\|_{H^s(\mathbb{R}^N)} =&\left\|\int_0^{\infty} M_{\sigma}(\theta)(\dot{K}((t+h)^{\sigma}\theta)\phi - \dot{K}(t^{\sigma}\theta)\phi d\theta\right\|_{H^s(\mathbb{R}^N)}\\ \to&0, \quad \text{as } h \to 0.\end{aligned}$$

This means that $C_{\sigma}(\cdot)\phi \in C([0,T], H^s(\mathbb{R}^N))$. Furthermore, as repeating the above processes, we also get the continuity of $\mathcal{P}_{\sigma}(t)\psi$. Hence, $\mathcal{P}_{\sigma}(\cdot)\psi \in C([0,T], H^s(\mathbb{R}^N))$. The proof is completed. □

Remark 3.1. *Obviously, in view of the inequality* $|\sin(t|x|)| \leq t|x|$, *for all* $t \geq 0$, $x \in \mathbb{R}^N$, *from the same way as in Lemma 3.7, we get,*

$$\|Q_{\sigma}f\|_{C([0,T];L^2(\mathbb{R}^N))} \lesssim \|f\|_{L^{\infty}(0,T;L^2(\mathbb{R}^N))}.$$

By using Lemma 3.6 and Lemma 3.7, it is not difficult to obtain the existence theorem to linear problem (3.1).

Theorem 3.1. *Let* $T > 0$. *Given* $(\phi,\psi) \in H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)$ *for* $s \in \mathbb{R}$, *let* $r > 2/\beta$ *and* $f \in L^r(0,T;H^{\mu,q}(\mathbb{R}^N))$, *for any* q *and* μ *satisfying* $1 < q \leq 2$ *and* $\mu = N(1/q - 1/2 - 1/N)$, $N \geq 1$, *then there exists a unique solution* $u \in C([0,T];L^2(\mathbb{R}^N))$ *to the linear problem (3.1), and moreover*

$$\|u\|_{L^{\infty}(0,T;L^2(\mathbb{R}^N))} \lesssim \|\phi\|_{H^s(\mathbb{R}^N)} + \|\psi\|_{H^{s-1}(\mathbb{R}^N)} + \|f\|_{L^r(0,T;H^{\mu,q}(\mathbb{R}^N))}.$$

Let $q = 2N/(N+2)$ *for* $N \geq 3$, *if* $f \in L^r(0,T;L^q(\mathbb{R}^N))$, *then*

$$\|u\|_{L^{\infty}(0,T;L^2(\mathbb{R}^N))} \lesssim \|\phi\|_{H^s(\mathbb{R}^N)} + \|\psi\|_{H^{s-1}(\mathbb{R}^N)} + \|f\|_{L^r(0,T;L^q(\mathbb{R}^N))}.$$

In the sequel, we consider the global existence to the linear Cauchy problem (3.1).

Theorem 3.2. *Given* $\phi \in H^s(\mathbb{R}^N)$, $\psi \equiv 0$ *for* $s \in \mathbb{R}$. *Let* $N \geq 2$, $\sigma = \beta/2$ *for* $\beta \in (1,2)$ *and* $f \in C_{\sigma}([0,\infty);H^{s-1}(\mathbb{R}^N))$, *where the Banach space*

$$C_{\sigma}([0,\infty);H^{s-1}(\mathbb{R}^N)) = \{u \in C([0,\infty);H^{s-1}(\mathbb{R}^N)) : \ t^{\sigma}\|u(t)\|_{H^{s-1}(\mathbb{R}^N)} < \infty, \ t \geq 0\},$$

equipped with its natural norm $\|u\|_{\sigma} = \sup_{t\in[0,\infty)} t^{\sigma}\|u(t)\|_{H^{s-1}(\mathbb{R}^N)}$. *Then there exists a unique solution* $u \in C([0,\infty);H^s(\mathbb{R}^N))$ *to the linear problem (3.1), and moreover*

$$\|u\|_{L^{\infty}(0,\infty;H^s(\mathbb{R}^N))} \lesssim \|\phi\|_{H^s(\mathbb{R}^N)} + \|f\|_{C_{\sigma}([0,\infty);H^{s-1}(\mathbb{R}^N))}. \tag{3.6}$$

Proof. Observe that, for $t \geq 0$, from Lemma 3.7, $\|C_\sigma(t)\phi\|_{H^s(\mathbb{R}^N)} \leq \|\phi\|_{H^s(\mathbb{R}^N)}$. Moreover, for any $v \in H^{s-1}(\mathbb{R}^N)$, Lemma 3.4 implies

$$\|\mathcal{S}_\sigma(t)v\|_{B^s_{2,2}(\mathbb{R}^N)} \lesssim t^{\sigma-1}\|v\|_{B^{s-1}_{2,2}(\mathbb{R}^N)},$$

which means that $\|\mathcal{S}_\sigma(t)v\|_{H^s(\mathbb{R}^N)} \lesssim t^{\sigma-1}\|v\|_{H^{s-1}(\mathbb{R}^N)}$. Hence, we have

$$\begin{aligned}\|C_\sigma(t)\phi\|_{H^s(\mathbb{R}^N)} + \|(Q_\sigma f)(t)\|_{H^s(\mathbb{R}^N)} \lesssim& \|\phi\|_{H^s(\mathbb{R}^N)} + \int_0^t \|\mathcal{S}_\sigma(t-s)f(s,\cdot)\|_{H^s(\mathbb{R}^N)}ds \\ \lesssim& \|\phi\|_{H^s(\mathbb{R}^N)} + \int_0^t (t-s)^{\sigma-1}\|f(s)\|_{H^{s-1}(\mathbb{R}^N)}ds \\ \lesssim& \|\phi\|_{H^s(\mathbb{R}^N)} + \|f\|_{C_\sigma([0,\infty);H^{s-1}(\mathbb{R}^N))},\end{aligned}$$

where we have used the semigroup $(g_a * g_b)(t) = g_{a+b}(t)$ for $a, b > 0$. Similarly to Lemma 3.6 and Lemma 3.7, the continuity is easy to check, where we can use the Plancherel theorem and Lebesgue dominated convergence theorem. Consequently, there exists a solution satisfying (3.2) and its values lie in $C([0,\infty); H^s(\mathbb{R}^N))$, and then (3.6) holds. Moreover, the uniqueness follows (3.6). The proof is completed. □

4. Results of semilinear problems

In this section, we focus on the well-posedness results of the semilinear problem, we first establish a local well-posed result of L^2-solutions that also belong to the setting of $L^\gamma(0,T;L^r(\mathbb{R}^N))$, furthermore, by a similar way in Theorem 3.2, another conclusion will be given in the framework of $L^\infty(0,T;H^s(\mathbb{R}^N))$. In the sequel, for a given semilinear function, we obtain the well-posedness results in the setting of Besov space $B^s_{r,2}$.

Theorem 4.1. *Let $N \geq 2$ and $(\phi,\psi) \in H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)$ for any $s \in (1/2, N/2)$. Assume that*

$$f : L^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N) \to L^{r'}(\mathbb{R}^N),$$

where index r satisfying

$$\frac{2N}{N-1} \leq r \leq \frac{2(N+1)}{N-1}\Big|_{N\geq 2} \wedge \frac{2N}{N-2s}\Big|_{N>2s} \wedge \frac{2N}{N-2}\Big|_{N\geq 3}.$$

For every $R > 0$, there exist $\alpha \geq 0$ and constant $C_\alpha(R) \in (0,\infty)$ such that

$$\|f(u) - f(v)\|_{L^{r'}(\mathbb{R}^N)} \leq C_\alpha(R)\left(\|u\|^\alpha_{L^r(\mathbb{R}^N)} + \|v\|^\alpha_{L^r(\mathbb{R}^N)}\right)\|u-v\|_{L^r(\mathbb{R}^N)},$$

for all $u, v \in L^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ with $\|u\|_{L^2(\mathbb{R}^N)}, \|v\|_{L^2(\mathbb{R}^N)} \leq R$. Let $\gamma > 0$ be an element to the admissible set

$$\{\gamma \in \mathbb{R}^+ : \ \gamma > 2+\alpha, \ \gamma(\beta-2)+2 > 0, \ \gamma(\sigma_N - 1)+1 > 0\},$$

for $\sigma_N = \beta(1-\delta(r))$. Then there exists a unique mild solution $u \in C([0,T]; L^2(\mathbb{R}^N)) \cap L^\gamma(0,T;L^r(\mathbb{R}^N))$ of the problem (1.1)–(1.2) for some $T > 0$. Moreover, u depends continuously on ϕ, ψ in the following sense. If $\phi_m \to \phi$ in $H^s(\mathbb{R}^N)$ and $\psi_m \to \psi$ in $H^{s-1}(\mathbb{R}^N)$ and if u_m denotes the solution of problem (1.1)-(1.2) with the initial value ϕ_m, ψ_m, then for all sufficiently large m, u_m converges to u in $C([0,T]; L^2(\mathbb{R}^N)) \cap L^\gamma(0,T;L^r(\mathbb{R}^N))$.

Proof. Let $\sigma = \beta/2$ for $\beta \in (1, 2)$. We want to construct a local (in t) solution to the integral equation

$$(\Phi u)(t) = C_\sigma(t)\phi + \mathcal{P}_\sigma(t)\psi + (Q_\sigma f)(u)(t).$$

By applying Lemma 3.6, Lemma 3.7, and Remark 3.1, it is clear that Φ is well defined. We next shall use a fixed point argument to verify this proof. Fixed $T, R > 0$ and set a space by

$$U_T = \{u \in L^\infty(0, T; L^2(\mathbb{R}^N)) \cap L^\gamma(0, T; L^r(\mathbb{R}^N)); d(u, 0) \le R\},$$

where $d(\cdot, \cdot)$ is the distance to the space U_T given by

$$d(u, v) = \|u - v\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} + \|u - v\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))}, \quad \text{for } u, v \in U_T.$$

Obviously, (U_T, d) is a complete metric space.

For $u \in U_T$ and every $R > 0$, from the assumption of nonlinearity f, by the trigonometric inequality we first have

$$\|f(u)\|_{L^{r'}(\mathbb{R}^N)} \le \|f(u) - f(0)\|_{L^{r'}(\mathbb{R}^N)} + \|f(0)\|_{L^{r'}(\mathbb{R}^N)} \le C_\alpha(R)\|u\|^{\alpha+1}_{L^r(\mathbb{R}^N)} + \|f(0)\|_{L^{r'}(\mathbb{R}^N)}.$$

Therefore, for $\gamma > \alpha + 2$, Hölder inequality yields

$$\begin{aligned}\|f(u)\|_{L^{\gamma'}(0,T;L^{r'}(\mathbb{R}^N))} \le & C_\alpha(R)\|u\|^{\alpha+1}_{L^{(\alpha+1)\gamma'}(0,T;L^r(\mathbb{R}^N))} + T^{\frac{1}{\gamma'}}\|f(0)\|_{L^{r'}(\mathbb{R}^N)} \\ \le & C_\alpha(R)T^{\frac{\gamma-(\alpha+2)}{\gamma}}\|u\|^{\alpha+1}_{L^\gamma(0,T;L^r(\mathbb{R}^N))} + T^{\frac{\gamma-1}{\gamma}}\|f(0)\|_{L^{r'}(\mathbb{R}^N)}.\end{aligned}$$

Similarly, for $u, v \in U_T$, we have

$$\begin{aligned}&\|f(u) - f(v)\|_{L^{\gamma'}(0,T;L^{r'}(\mathbb{R}^N))} \\ \le & C_\alpha(R)\left(\int_0^T \left(\|u(t)\|^\alpha_{L^r(\mathbb{R}^N)} + \|v(t)\|^\alpha_{L^r(\mathbb{R}^N)}\right)^{\gamma'} \|u(t) - v(t)\|^{\gamma'}_{L^r(\mathbb{R}^N)} dt\right)^{1/\gamma'},\end{aligned}$$

which, by Hölder inequality and Minkowski inequality, leads to

$$\begin{aligned}\|f(u) - f(v)\|_{L^{\gamma'}(0,T;L^{r'}(\mathbb{R}^N))} \le & C_\alpha(R)\Big(\Big\|\|u\|^\alpha_{L^r(\mathbb{R}^N)} + \|v\|^\alpha_{L^r(\mathbb{R}^N)}\Big\|_{L^{\frac{\gamma}{\gamma-2}}(0,T)}\Big)\|u - v\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))} \\ \le & 2C_\alpha(R)T^{\frac{\gamma-(\alpha+2)}{\gamma}}R^\alpha\|u - v\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))}.\end{aligned} \tag{4.1}$$

Next, we obtain the existence and uniqueness results for T small. By Lemma 3.3, for $2N/(N - 1) \le r \le 2(N + 1)/(N - 1)$ and $(\sigma_N - 1)\gamma + 1 > 0$, we have

$$\begin{aligned}\|(Q_\sigma f)(u)(t)\|_{L^r(\mathbb{R}^N)} \lesssim & \int_0^t (t - s)^{\sigma_N - 1}\|f(u)(s)\|_{L^{r'}(\mathbb{R}^N)} ds \\ \lesssim & t^{\sigma_N - 1 + \frac{1}{\gamma}}\|f(u)\|_{L^{\gamma'}(0,T;L^{r'}(\mathbb{R}^N))}.\end{aligned}$$

Observe that the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $s \in [0, N/2)$, $2 \le r < 2N/(N - 2s)$, we have

$$\|\Phi(u)(t)\|_{L^r(\mathbb{R}^N)} \lesssim \|C_\sigma(t)\phi\|_{H^s(\mathbb{R}^N)} + \|\mathcal{P}_\sigma(t)\psi\|_{H^s(\mathbb{R}^N)} + \|(Q_\sigma f)(u)(t)\|_{L^r(\mathbb{R}^N)}.$$

Therefore, we have

$$\begin{aligned}\|\Phi(u)\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))} &\lesssim T^{\frac{1}{\gamma}}\|\phi\|_{H^s(\mathbb{R}^N)} + T^{\frac{1}{\gamma}}\|\psi\|_{H^{s-1}(\mathbb{R}^N)} + \|(Q_\sigma f)(u)(t)\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))}\\ &\lesssim T^{\frac{1}{\gamma}}\|\phi\|_{H^s(\mathbb{R}^N)} + T^{\frac{1}{\gamma}}\|\psi\|_{H^{s-1}(\mathbb{R}^N)} + T^{\sigma_N}\|f(0)\|_{L^{r'}(\mathbb{R}^N)}\\ &\quad + C_\alpha(R)T^{\sigma_N-\frac{\alpha+1}{\gamma}}R^{\alpha+1}.\end{aligned}$$

On the other hand, by Lemma 3.6, for any $u \in U_T$, we have

$$\begin{aligned}\|\Phi(u)\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} &\lesssim \|\phi\|_{H^s(\mathbb{R}^N)} + \|\psi\|_{H^{s-1}(\mathbb{R}^N)} + \|(Q_\sigma f)(u)\|_{L^\infty(0,T;L^2(\mathbb{R}^N))}\\ &\lesssim \|\phi\|_{H^s(\mathbb{R}^N)} + \|\psi\|_{H^{s-1}} + \|f(u)\|_{L^{\gamma'}(0,T;L^{r'}(\mathbb{R}^N))}\\ &\lesssim \|\phi\|_{H^s(\mathbb{R}^N)} + \|\psi\|_{H^{s-1}(\mathbb{R}^N)} + T^{\frac{\gamma-1}{\gamma}}\|f(0)\|_{L^{r'}(\mathbb{R}^N)} + C_\alpha(R)T^{\frac{\gamma-(\alpha+2)}{\gamma}}R^{\alpha+1}.\end{aligned}$$

Therefore, there exists a constant $C \geq 1$ such that

$$\begin{aligned}&\|\Phi(u)\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} + \|\Phi(u)\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))}\\ \leq& C(1+T^{\frac{1}{\gamma}})(\|\phi\|_{H^s(\mathbb{R}^N)} + \|\psi\|_{H^{s-1}(\mathbb{R}^N)}) + C(T^{\frac{\gamma-1}{\gamma}} + T^{\sigma_N})\|f(0)\|_{L^{r'}(\mathbb{R}^N)}\\ &+ CC_\alpha(R)(T^{\frac{\gamma-(\alpha+2)}{\gamma}} + T^{\sigma_N-\frac{\alpha+1}{\gamma}})R^{\alpha+1}.\end{aligned}$$

Fixed $R > 0$ satisfying $C(\|\phi\|_{H^s(\mathbb{R}^N)} + \|\psi\|_{H^{s-1}(\mathbb{R}^N)}) \leq R/4$, let T be small enough such that

$$C_T := CC_\alpha(R)R^\alpha\Big(T^{\frac{\gamma-(\alpha+2)}{\gamma}} + T^{\sigma_N-\frac{\alpha+1}{\gamma}}\Big) < \frac{1}{2}, \tag{4.2}$$

and

$$C\left(\|\phi\|_{H^s(\mathbb{R}^N)} + \|\psi\|_{H^{s-1}(\mathbb{R}^N)}\right)T^{\frac{1}{\gamma}} + \left(T^{\sigma_N} + T^{\frac{\gamma-1}{\gamma}}\right)\|f(0)\|_{L^{r'}(\mathbb{R}^N)} < \frac{R}{2}.$$

Hence, it follows that $\Phi(u) \in U_T$ for any $u \in U_T$. Moreover, from Lemma 3.6 we deduce

$$\begin{aligned}\|\Phi(u)-\Phi(v)\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} =& \|(Q_\sigma f)(u) - (Q_\sigma f)(v)\|_{L^\infty(0,T;L^2(\mathbb{R}^N))}\\ \lesssim& \|f(u) - f(v)\|_{L^{\gamma'}(0,T;L^{r'}(\mathbb{R}^N))},\end{aligned}$$

where $\gamma' > 1/\sigma$, $1 < r' \leq 2$ for $N = 1, 2$ and $2N/(N+2) \leq r' \leq 2$ for $N \geq 3$. Hence, by selecting T small enough so that (4.2) holds, by virtue of (4.1), we have

$$\|\Phi(u)-\Phi(v)\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} \leq \frac{1}{3}d(u,v),$$

for all $u, v \in U_T$. Since

$$\|(Q_\sigma f)(u)(t) - (Q_\sigma f)(v)(t)\|_{L^r(\mathbb{R}^N)} \lesssim t^{\sigma_N-1+\frac{1}{\gamma}}\|f(u)-f(v)\|_{L^{\gamma'}(0,T;L^{r'}(\mathbb{R}^N))},$$

for T small enough so that (4.2) holds, by virtue of (4.1), it follows that

$$\|\Phi(u)-\Phi(v)\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))} = \|(Q_\sigma f)(u) - (Q_\sigma f)(v)\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))} \leq \frac{1}{3}d(u,v).$$

Consequently, Φ is a strict contraction on U_T. From the similar proof of continuity in Theorem 3.1 taking on $\Phi(u)$, it follows that Φ has a fixed point $u \in C([0,T]; L^2(\mathbb{R}^N)) \cap L^\gamma(0,T; L^r(\mathbb{R}^N))$, which is the unique mild solution of problem (1.1)–(1.2).

For the choice of T (independent to $\|\phi\|_{H^s(\mathbb{R}^N)}$ and $\|\psi\|_{H^{s-1}(\mathbb{R}^N)}$), as before, R is determined only by the size of the norm of initial data. Hence T and R are independent of $u_m \in U_T$ for m sufficiently large. Suppose $\phi_m \to \phi$ in $H^s(\mathbb{R}^N)$ and $\psi_m \to \psi$ in $H^{s-1}(\mathbb{R}^N)$ when $m \to \infty$. Now, let m be large enough, then

$$\begin{aligned}\|u - u_m\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} &\lesssim \|\phi - \phi_m\|_{H^s(\mathbb{R}^N)} + \|\psi - \psi_m\|_{H^{s-1}(\mathbb{R}^N)} + \|Q_\sigma(f(u) - f(u_m))\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} \\ &\lesssim \|\phi - \phi_m\|_{H^s(\mathbb{R}^N)} + \|\psi - \psi_m\|_{H^{s-1}(\mathbb{R}^N)} + \|f(u) - f(u_m)\|_{L^{\gamma'}(0,T;L^{r'}(\mathbb{R}^N))}.\end{aligned}$$

By the same argument, one can conclude that

$$\|u - u_m\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))} \lesssim T^{\frac{1}{\gamma}}\|\phi - \phi_m\|_{H^s(\mathbb{R}^N)} + T^{\frac{1}{\gamma}}\|\psi - \psi_m\|_{H^{s-1}(\mathbb{R}^N)} + T^{\sigma_N - 1 + \frac{1}{\gamma}}\|f(u) - f(u_m)\|_{L^{\gamma'}(0,T;L^{r'}(\mathbb{R}^N))}.$$

Then, (4.1) implies that

$$\begin{aligned}d(u, u_m) \lesssim &\left(1 + T^{\frac{1}{\gamma}}\right)\|\phi - \phi_m\|_{H^s(\mathbb{R}^N)} + \left(1 + T^{\frac{1}{\gamma}}\right)\|\psi - \psi_m\|_{H^{s-1}(\mathbb{R}^N)} \\ &+ 2C_\alpha(R)T^{\frac{\gamma-(\alpha+2)}{\gamma}}R^\alpha\left(1 + T^{\sigma_n - 1 + \frac{1}{\gamma}}\right)d(u, u_m).\end{aligned}$$

Let T be chosen so small such that (4.2) holds, then

$$(1 - C_T)d(u, u_m) \lesssim \left(1 + T^{\frac{1}{\gamma}}\right)\left(\|\phi - \phi_m\|_{H^s(\mathbb{R}^N)} + \|\psi - \psi_m\|_{H^{s-1}(\mathbb{R}^N)}\right).$$

Consequently, we deduce that $d(u, u_m) \to 0$ as $m \to \infty$. The proof is completed. □

Remark 4.1. *From the above theorem, we have the following remarks:*

i) The admissible set of γ is not empty. Indeed, for $\alpha = 2$, $r = 3$, $N = 3$ and taking initial values $(\phi, \psi) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, for a suitable semilinear function $f(u)$ satisfying the assumptions in Theorem 4.1, likely $f(u) = \lambda|u|^2u$ for $\lambda > 0$, it follows that $4 < \gamma < 2/(2-\beta)$ for $\beta \in (3/2, 2)$.

ii) Let α, r, N and f satisfy the assumptions of Theorem 4.1, noting that if $\delta(r) \le 1/2$, then the restrictions in the admissible set of γ reduce to $2+\alpha < \gamma < 2/(2-\beta)$, or if $0 \le \alpha < 2(\beta-1)/(2-\beta)$ and $1/2 < \delta(r) < 1 - (1+\alpha)/(\beta(2+\alpha))$, then the restrictions in the admissible set of γ reduce to $2 + \alpha < \gamma < 1/(1 - \sigma_N)$.

iii) By the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ for $s \in [0, N/2)$, the requirement $2N/(N-1) \le r < 2N/(N-2s)$ implies that the conclusion fails for $0 \le s \le 1/2$, and thus the assumption $s > 1/2$ is needed in the initial value conditions.

Noting that, by virtue of the critical embedding $H^s(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, for $2 \le r < \infty$, $s = N/2$, we obtain a weaken requirement of index r in Theorem 4.1.

Corollary 4.1. *Let $N \ge 2$ and $(\phi, \psi) \in H^{N/2}(\mathbb{R}^N) \times H^{N/2-1}(\mathbb{R}^N)$. Assume that*

$$f : L^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N) \to L^{r'}(\mathbb{R}^N),$$

where index r satisfying

$$\frac{2N}{N-1} \le r \le \frac{2(N+1)}{N-1}\Big|_{N\ge2} \wedge \frac{2N}{N-2}\Big|_{N\ge3}.$$

For every $R > 0$, there exist $\alpha \geq 0$ and constant $C_\alpha(R) \in (0, \infty)$ such that

$$\|f(u) - f(v)\|_{L^{r'}(\mathbb{R}^N)} \leq C_\alpha(R) \left(\|u\|^\alpha_{L^r(\mathbb{R}^N)} + \|v\|^\alpha_{L^r(\mathbb{R}^N)} \right) \|u - v\|_{L^r(\mathbb{R}^N)},$$

for all $u, v \in L^2(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ with $\|u\|_{L^2(\mathbb{R}^N)}, \|v\|_{L^2(\mathbb{R}^N)} \leq R$. Let $\gamma > 0$ be an element to the admissible set

$$\{\gamma \in \mathbb{R}^+ : \ \gamma > 2 + \alpha, \ \gamma(\beta - 2) + 2 > 0, \ \gamma(\beta(1 - \delta(r)) - 1) + 1 > 0\}.$$

Then the problem (1.1)–(1.2) is local well-posed on $C([0, T]; L^2(\mathbb{R}^N)) \cap L^\gamma(0, T; L^r(\mathbb{R}^N))$.

By using the embedding

$$H^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad \text{for } s > N/2,$$

one can prove the following result by employing the arguments used in the proof of Theorem 4.1.

Corollary 4.2. *Let $s > N/2$. Assume that for every $R > 0$, there exists $C(R) < \infty$ such that*

$$\|f(u)\|_{H^s(\mathbb{R}^N)} \leq C(R)\|u\|_{H^s(\mathbb{R}^N)},$$

$$\|f(u) - f(v)\|_{L^2(\mathbb{R}^N)} \leq C(R)\|u - v\|_{L^2(\mathbb{R}^N)},$$

for all $u, v \in H^s(\mathbb{R}^N)$ and that of $\|u\|_{L^\infty(\mathbb{R}^N)} \leq R$, $\|v\|_{L^\infty(\mathbb{R}^N)} \leq R$. Then for $(\phi, \psi) \in H^s(\mathbb{R}^N) \times H^{s-1}(\mathbb{R}^N)$, problem (1.1)–(1.2) is local well-posed on $u \in (W_{T,R}, d)$ for some $T, R > 0$, where $(W_{T,R}, d)$ is the metric space given by

$$W_{T,R} = \left\{ u \in L^\infty(0, T; H^s(\mathbb{R}^N)) \ : \ \|u\|_{L^\infty(0,T;H^s(\mathbb{R}^N))} \leq R \right\},$$

equipped with the distance

$$d(u, v) = \|u - v\|_{L^\infty(0,T;L^2(\mathbb{R}^N))}, \quad \textit{for } u, v \in W_{T,R}.$$

In the sequel, we consider a semilinear function of the form $f(u) = \lambda|u|^\alpha u$ for $\lambda \in \mathbb{R}$ and for $\alpha \geq 0$, a well-posed result on Besov setting is also established. In order to complete the proof, we need the following result [38].

Lemma 4.1. *Let $f(u) = \lambda|u|^\alpha u$ for $\lambda \in \mathbb{R}$ and for $\alpha \geq 0$. If $0 < s < N/2$, $2 \leq \rho \leq \rho^* = N(\alpha+2)/(N-2s)$, then*

$$\|f(u)\|_{B^s_{\rho',2}} \lesssim \|u\|^{\alpha+1}_{B^s_{\rho,2}},$$

and

$$\|f(u) - f(v)\|_{L^{\rho'}(\mathbb{R}^N)} \lesssim \left(\|u\|^\alpha_{B^s_{\rho,2}} + \|v\|^\alpha_{B^s_{\rho,2}} \right) \|u - v\|_{L^\rho(\mathbb{R}^N)},$$

for any $u, v \in B^s_{\rho,2}$.

Proof. Noting that

$$\left| |u|^\alpha u - |v|^\alpha v \right| \leq (\alpha + 1)(|u|^\alpha + |v|^\alpha)|u - v|,$$

we obtain by Hölder inequality and the embedding $\dot{B}^s_{\rho,2}(\mathbb{R}^N) \hookrightarrow L^{\rho^*}(\mathbb{R}^N)$ that

$$\|f(u)\|_{L^{\rho'}(\mathbb{R}^N)} \lesssim \|u\|^\alpha_{\dot{B}^s_{\rho,2}} \|u\|_{L^\rho(\mathbb{R}^N)},$$

and

$$\|f(u)-f(v)\|_{L^{\rho'}(\mathbb{R}^N)} \lesssim \Big(\|u\|^{\alpha}_{\dot{B}^s_{\rho,2}} + \|v\|^{\alpha}_{\dot{B}^s_{\rho,2}}\Big)\|u-v\|_{L^{\rho}(\mathbb{R}^N)}.$$

In view of the inequality

$$\|f(u)\|_{\dot{B}^s_{\rho',2}} \lesssim \|u\|^{\alpha+1}_{\dot{B}^s_{\rho,2}},$$

and the interpolation property $B^s_{\rho',2}(\mathbb{R}^N) = L^{\rho'}(\mathbb{R}^N) \cap \dot{B}^s_{\rho',2}(\mathbb{R}^N)$, the desired inequalities follow.

□

Theorem 4.2. *Let $N \geq 2$, $s \in (0, N/2)$ and $f(u) = \lambda|u|^{\alpha}u$ for $\alpha \geq 0$, $\lambda \in \mathbb{R}$. Let*

$$\frac{2N}{N-1} \leq r \leq \frac{2(N+1)}{N-1} \wedge \frac{N(\alpha+2)}{N-2s},$$

given $(\phi, \psi) \in H^{s+1,r'}(\mathbb{R}^N) \times H^{s,r'}(\mathbb{R}^N)$. Moreover, let $\gamma > 0$ being an element to the set

$$\{\gamma \in \mathbb{R}^+ : \ \gamma > \alpha + 2, \ \gamma(\beta(2 - 2\delta(r)) - 2) + 2 > 0\}. \tag{4.3}$$

Then the problem (1.1)–(1.2) is local well-posed on $L^{\gamma}(0, T; B^s_{r,2})$.

Proof. Following the method of proof for Theorem 4.1, we just need to construct a local solution to the operator equation $(\Psi u)(t) = u(t)$ for $t \in [0, T]$ in a suitable ball in $L^{\gamma}(0, T; B^s_{r,2})$, where the ball is defined by

$$\mathcal{B}_M = \{u \in L^{\gamma}(0, T; B^s_{r,2}) : \ \|u\|_{L^{\gamma}(0,T;B^s_{r,2})} \leq M\},$$

with the radius $M > 0$. Note that this space is not trivial. Indeed, for $\phi \in H^{s+1,r'}(\mathbb{R}^N)$, by virtue of Lemma 3.4, and the embedding (3.5) and $H^{s_1,r'}(\mathbb{R}^N) \hookrightarrow H^{s_0,r'}(\mathbb{R}^N)$ for $s_1 \geq s_0$, $s_1, s_0 \in \mathbb{R}$ we have

$$\|C_{\sigma}(t)\phi\|_{B^s_{r,2}} \lesssim \|\phi\|_{B^{s+\delta(r)}_{r',2}} \lesssim \|\phi\|_{H^{s+\delta(r),r'}(\mathbb{R}^N)} \lesssim \|\phi\|_{H^{s+1,r'}(\mathbb{R}^N)},$$

which shows that $\|C_{\sigma}(\cdot)\phi\|_{L^{\gamma}(0,T;B^s_{r,2})} \lesssim \|\phi\|_{H^{s+1,r'}(\mathbb{R}^N)}$. Similarly, by $(g_{2-2\sigma} * g_{\sigma})(t) = g_{2-\sigma}(t) \in L^{\gamma}(0, T; \mathbb{R})$ we also get $\|\mathcal{P}_{\sigma}(\cdot)\psi\|_{L^{\gamma}(0,T;B^s_{r,2})} \lesssim \|\psi\|_{H^{s,r'}(\mathbb{R}^N)}$. Thus, $u(t) = C_{\sigma}(t)\phi$ is in $\mathcal{B}_M$ if $\phi \in H^{s+1,r'}(\mathbb{R}^N)$ and $\|\phi\|_{H^{s+1,r'}(\mathbb{R}^N)}$ is small enough. Endowed with the metric

$$d(u, v) = \|u - v\|_{L^{\gamma}(0,T;L^r(\mathbb{R}^N))},$$

then $(\mathcal{B}_M, d)$ is a complete metric space. Indeed, since $L^{\gamma}(0, T; B^s_{r,2})$ is reflexive, the closed ball of radius M is weakly compact, for details, see [38].

From Lemma 4.1, we get $\|f(u)\|_{B^s_{r',2}} \lesssim \|u\|^{\alpha+1}_{B^s_{r,2}}$, for all $u \in B^s_{r,2}$. Thus, it remains to consider the case $\Phi u \in L^{\gamma}(0, T; B^s_{r,2})$ for some $T > 0$. Noting that from the requirements of index r, it yields $1/2 \leq \delta(r) \leq N/(N+1)$, consider any $u \in B^s_{r,2}$, then it follows from (4.3) that

$$\begin{aligned}
\|(\Phi u)(t)\|_{B^s_{r,2}} \leq & \|C_{\sigma}(t)\phi\|_{B^s_{r,2}} + \|\mathcal{P}_{\sigma}(t)\psi\|_{B^s_{r,2}} + \|(Q_{\sigma}f)(u)(t)\|_{B^s_{r,2}} \\
\lesssim & \|C_{\sigma}(t)\phi\|_{H^{s+1,r'}(\mathbb{R}^N)} + \|\mathcal{P}_{\sigma}(t)\psi\|_{H^{s,r'}(\mathbb{R}^N)} + \int_0^t \|\mathcal{S}_{\sigma}(t-s)f(u)(s)\|_{B^s_{r,2}} ds \\
\lesssim & \|\phi\|_{H^{s+1,r'}(\mathbb{R}^N)} + \|\psi\|_{H^{s,r'}(\mathbb{R}^N)} + \int_0^t (t-s)^{\sigma(2-2\delta(r))-1} \|u(s)\|^{\alpha+1}_{B^s_{r,2}} ds.
\end{aligned}$$

Noting that

$$\||u|^\alpha u\|_{L^{\gamma'}(0,t;B^s_{r,2})} \leq t^{\frac{\gamma-(\alpha+2)}{\gamma}} \|u\|^{\alpha+1}_{L^\gamma(0,t;B^s_{r,2})},$$

we obtain

$$\|\Phi(u)\|_{L^\gamma(0,T;B^s_{r,2})} \lesssim T^{\frac{1}{\gamma}} \|\phi\|_{H^{s+1,r'}(\mathbb{R}^N)} + T^{\frac{1}{\gamma}} \|\psi\|_{H^{s,r'}(\mathbb{R}^N)} + T^{\sigma(2-2\delta(r))-\frac{\alpha+1}{\gamma}} \|u\|^{\alpha+1}_{L^\gamma(0,T;B^s_{r,2})}.$$

On the other hand, from Lemma 4.1, for any $u, v \in \mathcal{B}_M$, we get

$$\|f(u) - f(v)\|_{L^{\gamma'}(0,T;L^{r'}(\mathbb{R}^N))} \lesssim T^{\frac{\gamma-(\alpha+2)}{\gamma}} M^\alpha \|u - v\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))}.$$

Therefore, we have

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))} &= \|(Q_\sigma f)(u) - (Q_\sigma f)(v)\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))} \\
&\lesssim T^{\sigma_N - 1 + \frac{1}{\gamma}} \|f(u) - f(v)\|_{L^{\gamma'}(0,T;L^{r'}(\mathbb{R}^N))} \\
&\lesssim T^{\sigma_N - \frac{\alpha+1}{\gamma}} M^\alpha \|u - v\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))}.
\end{aligned}$$

This means that there exists a constant $C > 0$ such that

$$\|\Phi(u)\|_{L^\gamma(0,T;B^s_{r,2})} \leq CT^{\frac{1}{\gamma}} (\|\phi\|_{H^{s+1,r'}(\mathbb{R}^N)} + \|\psi\|_{H^{s,r'}(\mathbb{R}^N)}) + CT^{\sigma(2-2\delta(r))-\frac{\alpha+1}{\gamma}} M^{\alpha+1},$$

and

$$\|\Phi(u) - \Phi(v)\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))} \leq CT^{\sigma_N - \frac{\alpha+1}{\gamma}} M^\alpha \|u - v\|_{L^\gamma(0,T;L^r(\mathbb{R}^N))}.$$

Hence, fixed $M > 0$, let T be small enough such that

$$CT^{\frac{1}{\gamma}} (\|\phi\|_{H^{s+1,r'}(\mathbb{R}^N)} + \|\psi\|_{H^{s,r'}(\mathbb{R}^N)}) < M/2, \ CM^\alpha (T^{\sigma(2-2\delta(r))-\frac{\alpha+1}{\gamma}} + T^{\sigma_N - \frac{\alpha+1}{\gamma}}) < 1/2.$$

Then Φ maps $\mathcal{B}_M$ into itself and we obtain

$$d(\Phi(u), \Phi(v)) \leq \frac{1}{2} d(u, v).$$

Thus, there exists a unique solution of problem (1.1)–(1.2). The remaining proof is similar to that of Theorem 4.1. So we omit its details. The proof is completed.

□

Remark 4.2. *Observe that, the embedding $H^{s,r'}(\mathbb{R}^N) \hookrightarrow B^s_{r',2}(\mathbb{R}^N)$, Lemma 3.4 yields*

$$\|C_\sigma(t)\phi\|_{B^s_{r,2}} \lesssim t^{-\beta\delta(r)} \|\phi\|_{H^{s,r'}(\mathbb{R}^N)},$$

which belongs to $L^\gamma(0, T; \mathbb{R})$. Hence, it shall possess a decay rate $\beta\delta(r)$, which implies that the solution does not belong to $C([0, T]; B^s_{r,2}) \cup L^\infty(0, T; B^s_{r,2})$.

Concerning with Remark 4.2, in the sequel, we establish the global well-posed result for a special initial data.

Theorem 4.3. *Let $N \geq 2$, $\beta \in (1, 4/3)$, $s \in (0, N/2)$, $\delta(r) = 1/2$ and $f(u) = \lambda|u|^2 u$ for $\lambda \in \mathbb{R}$. Given $\phi \in H^{s+\frac{1}{2},r'}(\mathbb{R}^N)$ with $\|\phi\|_{H^{s+\frac{1}{2},r'}(\mathbb{R}^N)} \leq \epsilon$ for some $\epsilon > 0$ and $\psi \equiv 0$. Then the problem (1.1)–(1.2) is global well-posed on Y_R, where (Y_R, d) $(R > 0)$ is the metric space given by*

$$Y_R = \Big\{u \in C((0, \infty); B^s_{r,2}(\mathbb{R}^N)) \,:\, \sup_{t\in[0,\infty)} t^{\beta/4}\|u(t)\|_{B^s_{r,2}} \leq R\Big\},$$

equipped with the distance

$$d(u, v) = \sup_{t\in[0,\infty)} t^{\beta/4}\|u(t) - v(t)\|_{L^r(\mathbb{R}^N)}, \quad \text{for } u, v \in Y_R.$$

Proof. Let $\sigma = \beta/2$ for $\beta \in (1, 2)$. We next verity the operator Φ maps Y_R into itself. Indeed, for $u \in Y_R$, by virtue of Lemma 3.4, we have

$$\begin{aligned}
\|(\Phi u)(t)\|_{B^s_{r,2}} \lesssim & t^{-\sigma/2}\|\phi\|_{H^{s+\frac{1}{2},r'}(\mathbb{R}^N)} + \int_0^t \|\mathcal{S}_\sigma(t-s)f(u)(s)\|_{B^s_{r,2}}ds \\
\lesssim & t^{-\sigma/2}\epsilon + \int_0^t (t-s)^{\sigma-1}\|f(u)(s)\|^3_{B^{s-\frac{1}{2}}_{r',2}}ds \\
\lesssim & t^{-\sigma/2}\epsilon + \int_0^t (t-s)^{\sigma-1}\|u(s)\|^3_{B^s_{r,2}}ds \\
\lesssim & t^{-\sigma/2}\epsilon + \int_0^t (t-s)^{\sigma-1}s^{-3\sigma/2}dsR^3,
\end{aligned}$$

where we have used the embedding $B^{s_0}_{r',2} \hookrightarrow B^{s_1}_{r',2}$ for $s_0 \geq s_1$, $s_0, s_1 \in \mathbb{R}$. Therefore, there exists a constant $C > 0$ such that

$$t^{\sigma/2}\|(\Phi u)(t)\|_{B^s_{r,2}} \leq C\epsilon + CR^3.$$

Taking $\epsilon \leq R/(2C)$ for $R \leq (1/(2C))^{1/2}$, due to the Fourier representation of operators and the Lebesgue dominated convergence theorem, the proof of the continuity of Φ is similar to Lemma 3.6. Hence, we deduce that $\Phi(u) \in Y_R$ for any $u \in Y_R$.

Next, we show that Φ is a contraction on Y_R. Indeed, for any $u, v \in Y_R$, combined the requirement of $\delta(r) = 1/2$ and Lemma 3.3 imply

$$\begin{aligned}
\|(\Phi u)(t) - (\Phi v)(t)\|_{L^r(\mathbb{R}^N)} \leq & \int_0^t \|\mathcal{S}_\sigma(t-s)(f(u)(s) - f(v)(s))\|_{L^r(\mathbb{R}^N)}ds \\
\lesssim & \int_0^t (t-s)^{\sigma-1}\|f(u)(s) - f(v)(s)\|_{L^{r'}(\mathbb{R}^N)}ds.
\end{aligned}$$

Lemma 4.1 shows that

$$\begin{aligned}
\|(\Phi u)(t) - (\Phi v)(t)\|_{L^r(\mathbb{R}^N)} \lesssim & \int_0^t (t-s)^{\sigma-1}\Big(\|u\|^2_{B^s_{r,2}} + \|v\|^2_{B^s_{r,2}}\Big)\|u - v\|_{L^r(\mathbb{R}^N)}ds \\
\lesssim & \int_0^t (t-s)^{\sigma-1}s^{-3\sigma/2}dsR^2 d(u, v).
\end{aligned}$$

Therefore, there exists a constant $C' > 0$ for $R \leq (2/C')^{1/2} \wedge (1/(2C))^{1/2}$ such that

$$d(\Phi(u), \Phi(v)) \leq C'R^2 d(u, v) \leq \frac{1}{2}d(u, v).$$

Consequently, Φ is a strict contraction on Y_R. This means that Φ has a unique fixed point $u \in Y_R$. The proof is completed. $\square$

5. Conclusions

In this paper, we proved some well-posedness results for linear and semilinear time fractional wave equations, which are also called super-diffusive equations. Under the probability density function and wave operators, we construct the solution operators that are differential to the references therein, which are also useful to establish the local well-posedness of L^2-solutions as well as the local well-posedness on Besov space. Moreover, based on the standard fixed point arguments, the initial data ϕ and ψ are taking in the more regularity fractional Sobolev spaces, respectively. Finally, we also establish the global existence results for a linear and a spacial semilinear fractional Cauchy problems.

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Conflict of interest

The authors declare there is no conflicts of interest.

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