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Research article

Matrix-Valued hypergeometric Appell-Type polynomials

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Abstract: In recent years, much attention has been paid to the role of special matrix polynomials of a real or complex variable in mathematical physics, especially in boundary value problems. In this article, we define a new type of matrix-valued polynomials, called the first Appell matrix polynomial of two complex variables. The properties of the newly definite matrix polynomial involving, generating matrix functions, recurrence relations, Rodrigues' type formula and integral representation are investigated. Further, relevant connections between the first Appell matrix polynomial and various matrix functions are reported. The current study may open the door for further investigations concerning the practical applications of matrix polynomials associated with a system of differential equations.

Keywords: matrix-valued polynomials; first Appell matrix functions; generating matrix relations; functional relations

1. Introduction

In 1880, Appell introduced in [1] sequences of polynomials $\{\mathbf{P}_n(\eta)\}_{n\geq 0}$ satisfying the relation

$$\frac{d}{d\eta}\mathbf{P}_n(\eta) = n\mathbf{P}_{n-1}(\eta), \quad \mathbf{P}_0(\eta) \neq 0.$$

If $\mathbf{P}(\tau)$ is a formal power series of the form

$$\mathbf{P}(\tau) = \sum_{n=0}^{\infty} a_n \frac{\tau^n}{n!}, \quad a_0 \neq 0,$$

Then Appell sequences can also be defined by means of their exponential generating function $\mathcal{G}(\eta,\tau)$ given by

$$\mathcal{G}(\eta,\tau)=\mathbf{P}(\tau)e^{\eta\tau}=\sum_{n=0}^{\infty}\mathbf{P}_n(\eta)\,\frac{\tau^n}{n!}.$$

Several members belonging to the Appell family can be obtained by properly choosing $\mathbf{P}(\tau)$, such as Bernoulli polynomials, Euler polynomials, hypergeometric Bernoulli polynomials, Genocchi polynomials and Hermit polynomials (cf. [2]).

In [3], the new Appell polynomial family $\mathfrak{A}_n^k(m,\eta)$ are presented in the terms of the generalized hypergeometric function as follows

$$\mathfrak{A}_n^k(m,\eta)=\eta^n\ {}_{k+p}\mathbf{F}_q\left[\begin{matrix}\alpha_1,\alpha_2,...,\alpha_p,-\frac{n}{k},-\frac{n-1}{k},...,-\frac{n-k+1}{k}\\ \\ \beta_1,\beta_2,.....,\beta_q\end{matrix};\frac{m}{\eta^k}\right],$$

where the generalized hypergeometric function is written in the following manner

$${}_pF_q\left[\begin{matrix}\alpha_1 & \alpha_2 & \cdots & \alpha_p\\ \\ \beta_1 & \beta_2 & \cdots & \beta_q\end{matrix};\eta\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{\eta^n}{n!}\quad \eta\in\mathbb{C},$$

and $\alpha_1,\cdots\alpha_p$ are complex parameters with $\beta_1,\cdots\beta_p\in\mathbb{C}\setminus\mathbb{Z}_0^-$ ($\mathbb{Z}_0^-=\{0\}\cup\mathbb{Z}^-$), and

$$(\alpha_1)_\ell=\frac{\Gamma(\alpha_1+\ell)}{\Gamma(\alpha_1)}=\begin{cases}\alpha_1(\alpha_1+1)...(\alpha_1+\ell-1), & \ell\in\mathbb{N},\ \ \alpha_1\in\mathbb{C}\\ \\ 1, & \ell=0;\ \vartheta_1\in\mathbb{C}\setminus\{0\},\end{cases}\tag{1.1}$$

is the Pochhammer symbol and $\Gamma(.)$ is gamma function.

The class of Gauss hypergeometric polynomials is defined by Bajpai and Arora [4] in the following equivalent formulas

$$\begin{aligned}\mathcal{A}_n^{\vartheta_2,\vartheta_3}(\xi)&=\xi^n\ {}_2F_1\left(\begin{matrix}-n,\vartheta_2\\ \\ \vartheta_3\end{matrix};\frac{-1}{\xi}\right)\quad (n\in\mathbb{N}_0=\{0\}\cup\mathbb{N})\\ &=\frac{(\vartheta_2)_n}{(\vartheta_3)_n}\,{}_2F_1\left(\begin{matrix}-n,1-\vartheta_3-n\\ \\ 1-\vartheta_2-n\end{matrix};-\xi\right).\end{aligned}\tag{1.2}$$

Also, the authors [4] investigated several results of the polynomials $\mathcal{A}_n^{\alpha,\beta}(\xi)$ included, semi-orthogonality, integral formulas, finite sums and some relevant connection with Foxs H-function and

Jacobi polynomials. Moreover, generating function and orthogonality property of a class of of Gauss hypergeometric polynomials occurring in quantum mechanics presented by Bajpai in [5]. Khanna and Srinivasa Bhagavan [6] derived the generating function of $\mathcal{A}_n^{\alpha,\beta}(\xi)$ using the representations of the Lie group $SL(2,\mathbb{C})$. However, mathemtical proplerties of generalized hypergeometric polynomial are reported by Khan [8]. Morover, A new real-valued Appell-type polynomial family by mean of a generalized hypergeometric function are presented by Bedratyuk and Luno [7]. Further, some finite summation formulas involving multivariable hypergeometric polynomials are archived by Djordjevic et al. [9]. Additionaly, the zeros of a class of generalized hypergeometric polynomials with applications are investegated in [10, 11]. Furthermore, Appell polynomials turn into the Gould-Hopper polynomials in [12]. Later, Çekim and Aktaş [13] introduced the matrix generalization of the Gould-Hopper polynomials by means of the generating function. Recently, the Gould-Hopper-Laguerre-Appell matrix polynomials using operational methods and to investigate their properties are introduced by Nahid and Khan [14]. More recent, they are derived the matrix recurrence relations, the matrix differential, matrix integro-differential and matrix partial differential equations for the Gould-Hopper-Laguerre-Appell matrix polynomials in [15]. In a similar vein, Defez et al. presented Bernoulli matrix polynomials and discussed some its properties in [16].

In recent years, a prominent study problem that gained attraction is matrix-valued polynomials and their applications (see, e.g., [17–19]). Matrix-valued Gegenbauer-type polynomials and their applications are discussed by Koelink et al. [20]. While Ismail et al. [21], presented matrix valued Hermite polynomials and some properties. Further, results on Jacobi, Gegenbauer, Legendre, generalized Bessel and Laguerre matrix polynomials have been archived in [22]. Later on, the researchers in [23] established some results on the two complex variables first Appell hypergeometric matrix function F_1 considering the convergence domain $\{(z,w),\in \mathbb{C}^2 : |z| < 1,\ |w| < 1\}$. They derived new formulas involving the contiguous relations, finite sums, the generating matrix functions, and several recursion formulae.

The purpose of the current manuscript is to introduce a new matrix-valued Appell-type polynomial by using first Appell hypergeometric matrix function F_1 and discuss certain mathematical properties for the novel defined matrix polynomial.

The paper is organized as follows. First, in Section 2, some basic concepts and and notations needed in the results are recorded. In Section 3, we introduce the first appell matrix polynomial in terms of the first Appell hypergeometric matrix function F_1 and discusses some limit formulas and auxiliary matrix polynomials of the first appell matrix polynomial. In Section 4, we prove some different generating matrix functions for the first appell matrix polynomials. Section 5 explore different recurrence relations to simplify the computation of the first appell matrix polynomials. We establish in Section 6 Rodrigues' type formula for the first appell matrix polynomial. Certain integral representation for the first appell matrix polynomial is derived in Section 7. Matrix partial differential equations satisfied by the first appell matrix polynomial in Section 8. Finally, in Section 9, we shows the concluding remarks.

2. Preliminaries

Here, we will summarize basic concepts and notations that will be largely exploited in this work (see, e.g., [17, 18, 22]).

Let $\mathfrak{M}_j(C)$ denotes the complex vector space constituted of all square matrices with j rows and j columns with entries in complex space. For any matrix $T \in \mathfrak{M}_j(C)$, $\sigma(T)$ (spectrum of T) denotes the set of all eigenvalues of T,

$$\mu(T) = \max\{\Re(\eta) : \eta \in \sigma(T)\}, \quad \widetilde{\mu}(T) = \min\{\Re(\eta) : \eta \in \sigma(T)\}, \tag{2.1}$$

where $\mu(T)$ is referred to as the spectral abscissa (the largest of the real parts of its eigenvalues) and $\mu(-T) = -\widetilde{\mu}(T)$. The square matrix T is said to be positive stable if and only if $\widetilde{\mu}(T) > 0$. I and **0** stand for the identity matrix and the zero matrix in $\mathfrak{M}_j(C)$, respectively.

If $\Phi_1(\eta)$ and $\Phi_2(\eta)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and T is a matrix in $\mathfrak{M}_j(C)$ with $\sigma(T) \subset \Omega$, then from the properties of the matrix functional calculus, ([18]) we have

$$\Theta_1(T)\Theta_2(T) = \Theta_2(T)\Theta_1(T).$$

If $R,\ T$ in $\mathfrak{M}_j(C)$ and $RT = TR$, then

$$\Theta_1(R)\Theta_2(T) = \Theta_2(R)\Theta_1(T).$$

Definition 2.1. [26] If T is a positive stable matrix in $\mathfrak{M}_j(C)$, then $\Gamma(T)$ is defined by

$$\Gamma(T) = \int_0^\infty e^{-x}\, x^{T-I} dx; \qquad x^{T-I} = \exp((T-I)\log x). \tag{2.2}$$

The reciprocal Gamma function denoted by $\Gamma^{-1}(\eta) = 1/\Gamma(\eta)$ is an entire function of the complex variable η and for any matrix T in $\mathfrak{M}_j(C)$, the image of $\Gamma^{-1}(\eta)$ acting on T, denoted by $\Gamma^{-1}(T)$, is also well defined [26]. Furthermore, if

$$T + mI \quad \text{is invertible for all integers } m \in \mathbb{N}_0, \tag{2.3}$$

then $\Gamma(T)$ is invertible, its inverse coincides with $\Gamma^{-1}(T)$ and one gets the formula

$$(T)_m = T(T+I)\dots(T+(m-1)I) = \Gamma(T+mI)\,\Gamma^{-1}(T);\ m \in \mathbb{N}_0. \tag{2.4}$$

From (2.4), it easily follows, for a nonzero scalar ε and $T \in \mathfrak{M}_j(C)$, that

$$\lim_{\varepsilon\to 0} \varepsilon^m \left(\frac{T}{\varepsilon}\right)_m = T^m, \quad \lim_{\varepsilon\to 0} \varepsilon^{-m} \left[\left(\frac{T}{\varepsilon}\right)_m\right]^{-1} = [T^m]^{-1} \quad m \in \mathbb{N}_0. \tag{2.5}$$

Also, using (2.4), we have

$$(T)_{k+m} = (T)_m(T+mI)_k = (T)_k(T+kI)_m. \tag{2.6}$$

From [18], we observe that

$$\frac{(-1)^k}{(n-k)!} I = \frac{(-n)_k}{n!} I = \frac{(-nI)_k}{n!};\ \ 0 \le k \le n. \tag{2.7}$$

Recently, authors [23] defined the first Appell matrix hypergeometric functions as follows:

Definition 2.2. For $|z| < 1$ and $|w| < 1$, we have

$$F_1\left(\begin{array}{c}\phi,\vartheta,\omega\\ \\ \varphi\end{array};z,w\right)=\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}(\phi)_{s+r}(\vartheta)_s(\omega)_r[(\varphi)_{s+r}]^{-1}\,\frac{z^s w^r}{s!r!}$$

$$=\sum_{s=0}^{\infty}(\phi)_s(\varphi)_s[(\varphi)_s]^{-1}\,\frac{z^s}{s!}\,{}_2F_1\left(\begin{array}{c}\phi+sI,\omega\\ \\ \varphi+sI\end{array};w\right), \tag{2.8}$$

where ϕ,ϑ,ω and φ are commutative matrices in $\mathfrak{M}_j(C)$ such that $\varphi+nI$ is invertible for all integer $n\in\mathbb{N}_0$.

Immediately, one observes the following simple identities ([18, 23]):

$$F_1\left(\begin{array}{c}\phi,\vartheta,\omega\\ \\ \varphi\end{array};z,0\right)=F_2\left(\begin{array}{c}\phi,\vartheta,\omega\\ \\ \varphi,\theta\end{array};z,0\right)=F_3\left(\begin{array}{c}\psi,\phi,\varphi,\omega\\ \\ \varphi\end{array};z,0\right)$$

$$=F_4\left(\begin{array}{c}\phi,\vartheta\\ \\ \theta,\varphi\end{array};z,0\right)={}_2F_1\left(\begin{array}{c}\phi,\vartheta\\ \\ \varphi\end{array};z\right), \tag{2.9}$$

and

$$F_1\left(\begin{array}{c}\phi,\vartheta,\mathbf{0}\\ \\ \varphi\end{array};z,w\right)=F_2\left(\begin{array}{c}\phi,\vartheta,\mathbf{0}\\ \\ \varphi,\theta\end{array};z,w\right)=F_3\left(\begin{array}{c}\phi,\vartheta,\omega,\mathbf{0}\\ \\ \varphi\end{array};z,w\right), \tag{2.10}$$

where F_2, F_3 and F_4 are matrix versions of the classical Appell hypergeometric functions (see, e.g., [23–25]) and ${}_2F_1$ is matrix version of the classical Gauss hypergeometric function [26] under conditions, $\phi,\vartheta,\omega,\varphi$ and θ are commutative matrices in $\mathfrak{M}_j(C)$ such that $\varphi+nI$ and $\theta+nI$ are invertible for all integer $n\in\mathbb{N}_0$.

The confluent Appell matrix functions or Humbert matrix functions [27]

$$\Phi_1\left(\begin{array}{c}\phi,\vartheta\\ \\ \varphi\end{array};z,w\right)=\sum_{m,n=0}^{\infty}\frac{(\phi)_{m+n}(\vartheta)_n[(\varphi)_{m+n}]^{-1}}{m!n!}z^m w^n, \tag{2.11}$$

$$\Phi_2\left(\begin{array}{c}\phi,\vartheta\\ \\ \varphi\end{array};z,w\right)=\sum_{m,n=0}^{\infty}\frac{(\phi)_m(\vartheta)_n[(\varphi)_{m+n}]^{-1}}{m!n!}z^m w^n, \tag{2.12}$$

$$\Phi_3\left(\begin{array}{c}\phi\\ \\ \varphi\end{array};z,w\right)=\sum_{m,n=0}^{\infty}\frac{(\phi)_m[(\varphi)_{m+n}]^{-1}}{m!n!}z^m w^n. \tag{2.13}$$

3. Matrix-valued polynomial analogue for Appell series F_1

In this section, we define a new matrix-valued Appell-type polynomial in terms of the first Appell hypergeometric matrix function F_1 and discuss limit (confluence) formulas as follows:

Definition 3.1. Let ϕ, θ and φ be positive stable and commuting matrices in $\mathfrak{M}_j(C)$, such that φ satisfies the condition (2.3). Then, for $n \in \mathbb{N}_0$, the first Appell matrix polynomial $\mathbb{F}_n$ is defined by

$$\mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]=\frac{(\varphi)_n}{n!}\,F_1\left(\begin{array}{c}-nI,\phi,\theta\\ \\ \varphi\end{array};z,w\right). \tag{3.1}$$

where F_1 is defined in (2.8).

Remark 3.1. Note that the polynomial $\mathbb{F}_n$ is generalize of the Gauss hypergeometric polynomials (1.2) and a matrix version of the class polynomial [28].

Remark 3.2. Clearly, using the relations (2.9) and (2.10), we obtain other formulas of the Appell matrix polynomial $\mathbb{F}_n$, for example,

$$\mathbb{F}_n\left[\begin{array}{c}\phi,\mathbf{0}\\ \varphi\end{array};z,w\right]=\frac{(\varphi)_n}{n!}\,F_2\left(\begin{array}{c}-nI;\phi,\mathbf{0}\\ \\ \varphi,\theta\end{array};z,w\right), \tag{3.2}$$

and

$$\mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,z\right]=\frac{(\varphi)_n}{n!}\,{}_2F_1\left(\begin{array}{c}-nI;\phi+\theta;\\ \varphi\end{array};z\right). \tag{3.3}$$

Upon using (2.5) with (3.2), the following limit (confluence) formulas can be stated

$$\lim_{\varepsilon\to 0}\, n![(\varphi)_n]^{-1}\,\mathbb{F}_n\left[\begin{array}{c}\phi,\frac{1}{\varepsilon}I\\ \varphi\end{array};z,\varepsilon w\right]=\Phi_1\left(\begin{array}{c}-nI,\phi\\ \varphi\end{array};z,w\right),$$

$$\lim_{\varepsilon\to 0}\, n![(\varphi)_n]^{-1}\,\mathbb{F}_n\left[\begin{array}{c}\frac{1}{\varepsilon}I,\theta\\ \varphi\end{array};\varepsilon z,w\right]=\Phi_1\left(\begin{array}{c}-nI,\theta\\ \varphi\end{array};z,w\right)$$

and

$$\lim_{\varepsilon\to 0}\, n!\;[(\varphi)_n]^{-1}\,\mathbb{F}_n\left[\begin{array}{c}\frac{\phi}{\varepsilon},\frac{\theta}{\varepsilon}\\ \varphi\end{array};\varepsilon z,\varepsilon w\right]=\Phi_3\left(\begin{array}{c}-nI\\ \varphi\end{array};z,w\right).$$

From the relation (2.5) and the equations (2.11)–(2.13) with above limit formulas, various confluence formulas can be easily obtained.

4. A family of generating matrix functions

Generating matrix relations play an important role in the introduce of matrix-valued polynomials and discuss its properties (see, e.g., [17, 22]). In this section, we investigate various generating matrix relations for the first Appell matrix polynomials as follows:

Theorem 4.1. *The generating matrix function of* $\mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]$ *is as*

$$\begin{aligned}&\sum_{n=0}^{\infty}(\lambda)_n[(\varphi)_n]^{-1}\,\mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]t^n\\&=(1-t)^{-\lambda}\,F_1\left(\begin{array}{c}\lambda I;\phi,\theta;\\ \\ \varphi\end{array};\frac{-zt}{1-t},\frac{-wt}{1-t}\right),\end{aligned}\tag{4.1}$$

where ϕ,θ *and* φ *are positive stable and commuting matrices in* $\mathfrak{M}_j(C)$ *such that* $\varphi+nI$ *satisfies the spectral condition (2.3) with* $\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $|t|<1$, $|z/(1-t)|<1$ *and* $|w/(1-t)|<1$.

Proof. Assume that the left-hand side of (4.1) is denoted by $\mathcal{L}$. Upon using the series expression of (4.1) with (2.6) and (2.7) to $\mathcal{L}$, we observe that

$$\begin{aligned}\mathcal{L}&=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\sum_{p+q\leq 0}\frac{(-n)_{p+q}(\phi)_p(\theta)_q[(\varphi)_{p+q}]^{-1}}{p!q!}z^pw^qt^n\\&=\sum_{p,q=0}\frac{(\lambda)_{p+q}(\phi)_p(\theta)_q[(\varphi)_{p+q}]^{-1}}{p!q!}(-zt)^p(-wt)^q\sum_{n=0}^{\infty}\frac{(\lambda+p+q)_n}{n!}t^n.\end{aligned}\tag{4.2}$$

Changing the order of summations in (4.2) and make use of the identity

$$\sum_{n=0}^{\infty}\frac{(\lambda+p+q)_n}{n!}t^n=(1-t)^{-(\lambda+p+q)},\quad |t|<1,$$

then, with a little simplification and the definition (2.8) we arrive at the right-hand side of (4.1).

□

As a consequence of Theorem 4.1, we obtain the following corollaries.

Corollary 4.1. *Let* ϕ,θ *and* φ *be positive stable and commuting matrices in* $\mathfrak{M}_j(C)$ *such that* $\varphi+nI$ *satisfies the spectral condition (2.3) with* $|t|<1$, $|z|<1$, $|w|<1$, $|t+w|<1$ *and* $|t+z|<1$. *The following generating matrix function holds true:*

$$\sum_{n=0}^{\infty}\mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]t^n=(1-t)^{\phi+\theta-\varphi}\,(1-t+zt)^{-\phi}\,(1-t+wt)^{-\theta}.\tag{4.3}$$

Corollary 4.2. *Let* ϕ,θ *and* φ *be positive stable and commuting matrices in* $\mathfrak{M}_j(C)$ *such that* $\varphi+nI$ *satisfies the spectral condition (2.3). Then we have*

$$\sum_{n=0}^{\infty}[(\varphi)_n]^{-1}\,\mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]t^n=e^t\,\Phi_2\left(\begin{array}{c}\phi,\theta\\ \varphi\end{array};-zt,-wt\right),\tag{4.4}$$

where Φ_2 *is defined in (2.12) with* $|zt|<1$ *and* $|wt|<1$.

Corollary 4.3. *Let ϕ, θ and φ be positive stable and commuting matrices in $\mathfrak{M}_j(C)$ such that $\varphi + nI$ satisfies the spectral condition (2.3) with $|t| < 1$ and $|z/(1-t)| < 1$, the following generating matrix function holds true:*

$$\sum_{n=0}^{\infty}(\lambda)_n[(\varphi)_n]^{-1}\ \mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,z\right]t^n = (1-t)^{-\lambda}\ {}_2F_1\left(\begin{array}{c}\lambda I,\phi+\theta;\\ \\ \varphi\end{array};\frac{-zt}{1-t}\right). \tag{4.5}$$

Theorem 4.2. *Let ϕ, θ, ω and φ be positive stable and commuting matrices in $\mathfrak{M}_j(C)$ such that $\varphi + nI$ satisfies the spectral condition (2.3) with $|t| < \min\{\frac{1}{(1+|z|)(1+|\eta|)}, \frac{1}{(1+|z|)(1+|\eta|)}\}$, $|\eta| < 1$, $|z| < 1$ and $|w| < 1$. The following bilinear generating matrix relation holds true:*

$$\begin{aligned}&\sum_{n=0}^{\infty}\ {}_2F_1\left(\begin{array}{c}-nI,\omega\\ \varphi\end{array};\eta\right)\mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]t^n\\ &=(1-t)^{\omega+\phi+\theta-\varphi}(1-t+zt)^{-\phi}(1-t+wt)^{-\theta}(1-t+\eta t)^{-\omega}\\ &\times\ F_1\left(\begin{array}{c}\omega;\phi,\theta\\ \\ \varphi\end{array};\frac{z\eta t}{(1-t+zt)(1-t+\eta t)},\frac{w\eta t}{(1-t+\eta t)(1-t+wt)}\right).\end{aligned} \tag{4.6}$$

Proof. Let S be the left-hand side of (4.6). Then we have

$$\begin{aligned}\mathrm{S}&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(\omega)_k[(\varphi)_k]^{-1}(-\eta)^k\ \mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]t^n\\ &=\sum_{k=0}^{\infty}(\omega)_k[(\varphi)_k]^{-1}(-\eta t)^k\sum_{n=0}^{\infty}\binom{n+k}{k}\ \mathbb{F}_{n+k}\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]t^n\\ &=(1-t)^{\phi+\theta-\varphi}\,(1-t+zt)^{-\phi}\,(1-t+wt)^{-\theta}\\ &\times\ \sum_{k=0}^{\infty}(\omega)_k[(\varphi)_k]^{-1}(\frac{-\eta t}{1-t})^k\ \mathbb{F}_k\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};\frac{z}{1-t+zt},\frac{w}{1-t+wt}\right],\end{aligned}$$

which, in view of (4.1), we obtain the coveted result. □

If $w = 0$ in Theorem 3.1, we have the following result:

Corollary 4.4. *Let ϕ, θ, ω and φ be positive stable and commuting matrices in $\mathfrak{M}_j(C)$ such that $\varphi + nI$ satisfies the spectral condition (2.3) with $|t| < \min\{\frac{1}{(1+|\eta|)}, \frac{1}{(1+|z|)}\}$, $|\eta| < 1$, and $|z| < 1$. Then we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\varphi)_n}{n!}\ {}_2F_1\left(\begin{array}{c}-nI,\omega\\ \varphi\end{array};\eta\right)\ {}_2F_1\left(\begin{array}{c}-nI,\phi\\ \varphi\end{array};z\right)t^n\\ &=(1-t)^{\omega+\phi-\varphi}(1-t+zt)^{-\phi}(1-t+\eta t)^{-\omega}\\ &\times\ {}_2F_1\left(\begin{array}{c}\omega,\phi\\ \varphi\end{array};\frac{\eta z t}{(1-t+zt)(1-t+\eta t)}\right).\end{aligned} \tag{4.7}$$

5. Recurrence relations

In this section, we consider some recurrence relations for the first Appell matrix polynomial.

Theorem 5.1. *Let* $\mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]$ *be given in (3.1), the following recurrence relation holds true:*

$$\begin{aligned} &z\,\mathbf{D}_z\,\mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]+w\,\mathbf{D}_w\,\mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]-n\,\mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]\\ &=-(\theta+(n-1)I)\,\mathbb{F}_{n-1}\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right], \end{aligned} \tag{5.1}$$

where $n \geq 1$ *and* φ,ϕ *and* θ *are positive stable and commuting matrices in* $\mathfrak{M}_j(C)$ *such that* $\varphi+nI$ *is invertible for all integer* $n \geq 0$, *such that* $\mathbf{D}_z=\frac{\partial}{\partial z},\mathbf{D}_w=\frac{\partial}{\partial w}$.

Proof. To prove (5.1) we consider

$$\Lambda = e^t\,\Phi_2\left(\begin{array}{c}\phi,\theta\\ \varphi\end{array};-zt,-wt\right), \tag{5.2}$$

where Φ_2 is given in (2.12). Then

$$\mathbf{D}_z\Lambda=-t\varphi^{-1}\phi\,e^t\,\Phi_2\left(\begin{array}{c}\phi+I,\theta\\ \varphi+I\end{array};-zt,-wt\right), \tag{5.3}$$

$$\mathbf{D}_w\Lambda=-t\varphi^{-1}\theta\,e^t\,\phi_2\left(\begin{array}{c}\phi,\theta+I\\ \varphi+I\end{array};-zt,-wt\right) \tag{5.4}$$

and

$$\begin{aligned} \mathbf{D}_t\Lambda &= e^t\,\Phi_2\left(\begin{array}{c}\phi,\theta\\ \varphi\end{array};-zt,-wt\right)\\ &-z\varphi^{-1}\phi\,e^t\,\Phi_2\left(\begin{array}{c}\phi+I,\theta\\ \varphi+I\end{array};-zt,-wt\right)\\ &-w\varphi^{-1}\theta\,e^t\,\Phi_2\left(\begin{array}{c}\phi,\theta+I\\ \varphi+I\end{array};-zt,-wt\right), \end{aligned} \tag{5.5}$$

we eliminate Φ_2 from (5.3), (5.4) and (5.5) as follows

$$z\,\mathbf{D}_z\Lambda+w\,\mathbf{D}_w\Lambda-t\,\mathbf{D}_t\Lambda=-t\Lambda. \tag{5.6}$$

Since

$$\Lambda=\sum_{n=0}^{\infty}[(\varphi)_n]^{-1}\,\mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]t^n$$

and using (5.6), we get

$$\begin{aligned} & z \sum_{n=0}^{\infty}[(\varphi)_n]^{-1}\ t^n\ \mathbf{D}_z\ \mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right] + w \sum_{n=0}^{\infty}[(\varphi)_n]^{-1}\ t^n\ \mathbf{D}_w\ \mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right] \\ & - \sum_{n=0}^{\infty}[(\varphi)_n]^{-1} n\ \ \mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]t^n = -\sum_{n=1}^{\infty}[(\varphi)_{n-1}]^{-1}t^n\ \mathbb{F}_{n-1}\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right], \end{aligned} \tag{5.7}$$

from which it follows that

$$\begin{aligned} & z\ \mathbf{D}_z\ \mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right] + w\ \mathbf{D}_w\ \mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right] - n\ \mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right] \\ & = -(\varphi + (n-1)I)\ \mathbb{F}_{n-1}\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]. \end{aligned} \tag{5.8}$$

This finishes the proof of Theorem 5.1. □

Theorem 5.2. *For $n \geq 1$, the recurrence relation for the first Appell matrix polynomials is as*

$$\begin{aligned} & n\ \mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right] = (\varphi + 2(n-1)I)\ \mathbb{F}_{n-1}\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right] \\ & \times -z\ \phi\ \mathbb{F}_{n-1}\left[\begin{array}{c}\phi+I,\theta\\ \varphi\end{array};z,w\right] - w\theta\ \mathbb{F}_{n-1}\left[\begin{array}{c}\phi,\theta+I\\ \varphi\end{array};z,w\right] \\ & -(\varphi + (n-2)I)\ \mathbb{F}_{n-2}\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right] \end{aligned} \tag{5.9}$$

where ϕ, θ and φ be positive stable and commuting matrices in $\mathfrak{M}_j(C)$ such that $\varphi + nI$ is invertible for all integer $n \in \mathbb{N}_0$.

Proof. To prove (5.9), assume that

$$Q = (1-t)^{-\varphi}\ (1 + \frac{zt}{1-t})^{-\phi}(1 + \frac{wt}{1-t})^{-\theta}. \tag{5.10}$$

Then, we get

$$(1-t)\mathbf{D}_z Q = -t\phi(1-t)^{-\varphi}\ (1 + \frac{zt}{1-t})^{-(\phi+I)}(1 + \frac{wt}{1-t})^{-\theta}, \tag{5.11}$$

and

$$(1-t)\mathbf{D}_w Q = -t\theta(1-t)^{-\varphi}\ (1 + \frac{zt}{1-t})^{-\phi}(1 + \frac{wt}{1-t})^{-(\theta+I)}, \tag{5.12}$$

But

$$Q = \sum_{n=0}^{\infty}\ \mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]t^n, \tag{5.13}$$

it follows that

$$\mathbf{D}_z\, \mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]=\mathbf{D}_z\, \mathbb{F}_{n-1}\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]-\phi\, \mathbb{F}_{n-1}\left[\begin{array}{c}\phi,\theta\\ \varphi+I\end{array};z,w\right], \tag{5.14}$$

and

$$\mathbf{D}_w\, \mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]=\mathbf{D}_w\, \mathbb{F}_{n-1}\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]-\theta\, \mathbb{F}_{n-1}\left[\begin{array}{c}\phi+I,\theta\\ \varphi\end{array};z,w\right]. \tag{5.15}$$

Elimination of the derivatives from (5.8), (5.14) and (5.15) leads us to the recurrence relation

$$\begin{aligned}
&n\, \mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]=(\varphi+2(n-1)I)\, \mathbb{F}_{n-1}\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]\\
&\times -z\,\phi\, \mathbb{F}_{n-1}\left[\begin{array}{c}\phi+I,\theta\\ \varphi\end{array};z,w\right]-w\theta\, \mathbb{F}_{n-1}\left[\begin{array}{c}\phi,\theta+I\\ \varphi\end{array};z,w\right]\\
&-(\varphi+(n-2)I)\, \mathbb{F}_{n-2}\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right].
\end{aligned} \tag{5.16}$$

This completes the proof of Theorem 5.2. □

6. Rodrigues' type formula

Here, we present Rodrigues' type formula of the first Appell matrix polynomial in the following theorem:

Theorem 6.1. *For $n,k\in\mathbb{N}_0$, the following Rodrigues' type formula holds true:*

$$\begin{aligned}
&\mathbb{F}_k\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};\frac{zt}{zt-1},\frac{wt}{wt-1}\right]\\
&=\frac{t^{I-\varphi}(1-zt)^{\phi}(1-wt)^{\theta}}{n!}\,\mathbf{D}_t^n\Big[t^{\varphi+(n-1)I}(1-zt)^{-\phi}(1-wt)^{-\theta}\Big],
\end{aligned} \tag{6.1}$$

where ϕ,θ and φ be positive stable and commuting matrices in $\mathfrak{M}_j(C)$ such that $\varphi+nI$ is invertible for all integer $n\in\mathbb{N}_0$ with $|t|<1, |zt|<1,\ |wt|<1,\ |\frac{zt}{zt-1}|<1$ and $|\frac{wt}{wt-1}|<1$.

Proof. From the Binomial matrix formula, see [18]

$$\sum_{m=0}^{\infty}\frac{(\phi)_m}{m!}u^m=(1-u)^{-\phi},\quad |u|<1,$$

and the definition of first Appell matrix polynomial $\mathbb{F}_k$, see (3.1), we easily have

$$\begin{aligned}
&\mathbf{D}_t^n\Big[t^{\varphi+(n-1)I}(1-zt)^{-\phi}(1-wt)^{-\theta}\Big]\\
&=\sum_{r=0}^{n}\binom{n}{r}\,\mathbf{D}_t^{n-r}[t^{\varphi+(n-1)I}]\,\mathbf{D}_t^{r}\Big[(1-zt)^{-\phi}(1-wt)^{-\theta}\Big]\\
&=\sum_{r=0}^{n}\binom{n}{r}\binom{r}{k}(\varphi)_n(\phi)_{r-k}(\theta)_k[(\varphi)_r]^{-1}\, z^{r-k}\, w^k\\
&\times t^{\varphi+(n-1)I}(1-zt)^{-\phi-(r-k)I}(1-wt)^{-(\theta+kI)}\\
&=t^{\varphi-I}(1-zt)^{-\phi}(1-wt)^{-\theta}\\
&\times\sum_{k+s\le n}\frac{(-n)_{k+s}(\varphi)_n(\phi)_k(\theta)_s[(\varphi)_{k+s}]^{-1}}{k!s!}\Big(\frac{zt}{zt-1}\Big)^k\Big(\frac{wt}{wt-1}\Big)^s\\
&=n!t^{\varphi-I}(1-zt)^{-\phi}(1-wt)^{-\theta}\,\mathbb{F}_k\left[\begin{matrix}\phi,\theta\\ \varphi\end{matrix};\frac{zt}{zt-1},\frac{wt}{wt-1}\right].
\end{aligned}$$

The above equation gives the proof of Theorem 6.1. □

7. Integral formula

In this section, we show certain integral representation for the first Appell matrix polynomial by the product of (3.3) as follows

$$\begin{aligned}
&\mathbb{F}_m\left[\begin{matrix}\phi_1,\theta_1\\ \varphi_1\end{matrix};z,z\right]\mathbb{F}_n\left[\begin{matrix}\phi_2,\theta_2\\ \varphi_2\end{matrix};w,w\right]\\
&=\frac{(\varphi_2)_n\,(\varphi_2)_m}{m!\,n!}\,{}_2F_1\left(\begin{matrix}-mI,\phi_1+\theta_1\\ \varphi_1\end{matrix};z\right)\,{}_2F_1\left(\begin{matrix}-nI,\phi_2+\theta_2\\ \varphi_2\end{matrix};w\right)\\
&=\Gamma(\varphi_1+mI)\,\Gamma(\varphi_2+nI)\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{\Gamma(m+n-r-s+1)}{\Gamma(m-r+1)\,\Gamma(n-s+1)}\\
&\times\Gamma(\varphi_1+\varphi_2+(r+s-1)I)\,\Gamma^{-1}(\varphi_1+rI)\Gamma^{-1}(\varphi_2+sI)\\
&\times\frac{1}{\Gamma(m+n-r-s+1)}\,\Gamma^{-1}(\varphi_1+\varphi_2+(r+s-1)I)\\
&\times(-1)^{r+s}\,(\phi_1+\theta_1)_r\,(\phi_2+\theta_2)_s\,\frac{z^r\,w^s}{r!\,s!}.
\end{aligned}\tag{7.1}$$

Changing the order of summations in (7.1) and applying the integral formula

$$\begin{aligned}
&\Gamma(A+B+I)\Gamma^{-1}(A+I)\Gamma^{-1}(B+I)\\
&=\frac{2^{A+B}}{\pi}\int_{-\pi/2}^{\pi/2}\exp\Big((A-B)\alpha i\Big)\,\cos^{A+B}\alpha\,d\alpha,\;\;A,B\in\mathfrak{M}_j(C),\;\;\widetilde{\mu}(A+B)>-1,
\end{aligned}$$

then, with a little simplification, we arrive at

$$
\begin{aligned}
&\mathbb{F}_m\left[\begin{array}{c}\phi_1,\theta_1\\ \varphi_1\end{array};z,z\right]\mathbb{F}_n\left[\begin{array}{c}\phi_2,\theta_2\\ \varphi_2\end{array};w,w\right]\\
&=\frac{2^{\varphi_1+\varphi_2+(m+n-2)I}}{\pi^2}\;\Gamma^{-1}(\varphi_1+\varphi_2+(m+n-1)I)\,\Gamma(\varphi_1+mI)\,\Gamma(\varphi_2+nI)\\
&\times\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\exp\Big((\varphi_1-\varphi_2)\alpha i+(m-n)I\beta i\Big)\;\cos^{\varphi_1+\varphi_2-2I}\alpha\;\cos^{m+n}\beta\\
&\times\mathbb{F}_{m+n}\left[\begin{array}{c}\phi_1+\theta_1,\phi_2+\theta_2\\ \\ \varphi_1+\varphi_2-I\end{array};z\,e^{(\alpha-\beta)i}\;\cos\alpha\;\sec\beta,\;w\,e^{-(\alpha-\beta)i}\;\cos\alpha\;\sec\beta\right]d\alpha\,d\beta.
\end{aligned}
\tag{7.2}
$$

Therefore the following theorem can be investigated.

Theorem 7.1. *Assume that $\phi_1,\ \theta_1,\ \varphi_1\ \phi_2,\ \theta_2$ and φ_2 are positive stable and commutative matrices in $\mathfrak{M}_j(C)$, such that $\varphi_1,\ \varphi_2$ satisfies the condition* (2.3) *and $\widetilde{\mu}(\varphi_1+\varphi_2)>-1$. Then, the integral representation (7.2) holds true.*

Remark 7.1. Note that when $j=1$, the provided formulas are reduced to those of the classical Appell polynomial F_1 (cf. [28]).

8. Matrix partial differential equations

Suppose that

$$
\begin{aligned}
\mathcal{H}=\;F_1\left(\begin{array}{c}-nI,\phi,\theta\\ \\ \varphi\end{array};z,w\right)&=\frac{n!}{(\varphi)_n}\,\mathbb{F}_n\left[\begin{array}{c}\phi,\theta\\ \varphi\end{array};z,w\right]\\
&=\sum_{s=0}^{n}\sum_{r=0}^{n}\;U_{s,r}(z,w).
\end{aligned}
\tag{8.1}
$$

Denoting the partial differential operator by

$$
\mathcal{D}=\theta_1+\theta_2,
\tag{8.2}
$$

where $\theta_1=z\frac{\partial}{\partial z}$ and $\theta_2=w\frac{\partial}{\partial w}$. This operator has the particularly pleasant property

$$
\mathcal{D}z^sw^r=(s+r)z^sw^r.
$$

Now, the following partial differential equation for the first Appell hypergeometric matrix polynomial of two complex variables can be deduced

$$
\begin{aligned}
&\mathcal{D}\left[I\mathcal{D}+\varphi-I\right]\mathcal{H}\\
=&\sum_{s,r=0}^{n}(s+r)\left((s+r)I+\varphi-I\right)(-nI)_{s+r}(\phi)_s(\theta)_r[(\varphi)_{s+r}]^{-1}\frac{z^s w^r}{s!r!}\\
=&\sum_{s,r=0}^{n}(s+r)\,(-nI)_{s+r}(\phi)_s(\theta)_r[(\varphi)_{s+r-1}]^{-1}\frac{z^s w^r}{s!r!}\\
=&z\sum_{s,r=0}^{\infty}(\phi+sI)\,(-nI+(s+r)I)\,U_{s,r}(z,w)\\
+&w\sum_{s,r=0}^{n}(\theta+rI)\,(-nI+(s+r)I)\,U_{s,r}(z,w)\\
=&z(I\mathcal{D}-nI)\,(\theta_1 I+\phi)\,\mathcal{H}+w(I\mathcal{D}-nI)\,(\theta_2 I+\theta)\,\mathcal{H}.
\end{aligned}
$$

We readily see that the first Appell hypergeometric matrix polynomial F_1 should be a solution of a partial differential equation given by

$$
\left\{\mathcal{D}\,[I\mathcal{D}+\varphi-I]-(I\mathcal{D}-nI)\Big[z(\theta_1 I+\phi)+w(\theta_2 I+\theta)\Big]\right\}\mathcal{H}=0. \tag{8.3}
$$

Remark 8.1. Similarly, we can indicate other matrix partial differential equations as special cases of (8.3). For example,

$$
\Big\{I\mathcal{D}-Iz\theta_1-Iw\theta_2-(z\phi+w\theta)\Big\}\,F_1\left(\begin{array}{c}-nI,\phi,\theta\\ \\ -nI\end{array};z,w\right)=0.
$$

9. Concluding remarks

The area of matrix polynomial theory has been fast developing and is presently being applied in many fields such as probability theory, physics, scattering theory, statistics, engineering and chemical applications (see, e.g., [17–19]). In particular, the 2D special matrix polynomials are very advantageous in several areas of mathematics, prediction theory and spectral analysis. These polynomials allow the derivation of a number of useful identities in a fairly straight forward way and help in introducing new families of special polynomials. For example, Abdalla and Hidan [29] have established the properties of 2D Jacobi matrix polynomials associated with applications. Khan and Raza have proposed many interesting results on two variable Hermite generalized matrix polynomials in [30]. Also, Fuli [31] introduced 2D Shivleys matrix polynomials and studied some its properties. Very recently, results for certain 2D hybrid families related to the Appell matrix polynomials are derived by Nahid and Khan [14, 15].

Motivated by the previous works, In this paper, we introduce the first Appell matrix polynomials of two complex variables and discuss many its properties. It is interesting to note that the matrix polynomials here lead to the generalization of several matrix polynomials into the two variable forms of the hypergeometric, Laguerre, Hermite, Bernoulli, Jacobi, Legendre and truncated exponential matrix polynomials (see, [22]). In addition, this approach allows to derive several new results that can be used in theoretical and applicable aspects and for some numerical algorithm.

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Conflict of interest

The authors declare there is no conflicts of interest.

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