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Research article

Existence and stability of normalized solutions to the mixed dispersion nonlinear Schrödinger equations

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Abstract: We study the existence and orbital stability of normalized solutions of the biharmonic equation with the mixed dispersion and a general nonlinear term

$$\gamma\Delta^2 u - \beta\Delta u + \lambda u = f(u), \quad x \in \mathbb{R}^N$$

with a priori prescribed L^2-norm constraint $S_a := \left\{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = a\right\}$, where $a > 0$, $\gamma > 0, \beta \in \mathbb{R}$ and the nonlinear term f satisfies the suitable L^2-subcritical assumptions. When $\beta \geq 0$, we prove that there exists a threshold value $a_0 \geq 0$ such that the equation above has a ground state solution which is orbitally stable if $a > a_0$ and has no ground state solution if $a < a_0$. However, for $\beta < 0$, this case is more involved. Under an additional assumption on f, we get the similar results on the existence and orbital stability of ground state. Finally, we consider a specific nonlinearity $f(u) = |u|^{p-2}u + \mu|u|^{q-2}u, 2 < q < p < 2 + 8/N, \mu < 0$ under the case $\beta < 0$, which does not satisfy the additional assumption. And we use the example to show that the energy in the case $\beta < 0$ exhibits a more complicated nature than that of the case $\beta \geq 0$.

Keywords: biharmonic nonlinear Schrödinger equations; normalized solution; orbital stability; general nonlinearity

1. Introduction

We consider the biharmonic nonlinear Schrödinger equation (NLS) with the mixed dispersion and a general nonlinear term

$$\mathrm{i}\partial_t\psi - \gamma\Delta^2\psi + \beta\Delta\psi + f(\psi) = 0,\ \psi(0,x) = \psi_0(x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{1.1}$$

where $N \geq 1$, i denotes the imaginary unit, $\gamma > 0, \beta \in \mathbb{R}$ and the nonlinear term f satisfies the following conditions throughout this paper:

(F1) $f \in C(\mathbb{C},\mathbb{C}), f(0) = 0$.

(F2) $f(s) \in \mathbb{R}$ for $s \in \mathbb{R}, f\left(e^{i\theta}z\right) = e^{i\theta}f(z)$ for $\theta \in \mathbb{R}, z \in \mathbb{C}$.

(F3) $\lim_{z\to 0} f(z)/|z| = 0$.

(F4) $\lim_{|z|\to\infty} f(z)/|z|^{l-1} = 0$, where $l := 2 + 8/N$.

(F5) There exists $s_0 > 0$ such that $F(s_0) > 0$, where $F(z) = \int_0^{|z|} f(\tau)d\tau$ for $z \in \mathbb{C}$.

In nonlinear optics, the NLS is usually derived from the nonlinear Helmholtz equation for the electric field by separating the fast oscillations from the slowly varying amplitude. In the so-called paraxial approximation, the NLS appears in the limit as the equation solved by the dimensionless electric-field amplitude, see [1, Section 2]. The fact that its solutions may blow up in finite time suggests that some small terms neglected in the paraxial approximation play an important role in preventing this phenomenon. Therefore, a small fourth-order dispersion term was proposed in [1] (see also [2–4]) as a nonparaxial correction, which eventually gives rise to (1.1). For more background, see [5–7] and references therein.

Under these conditions (F1)-(F5), for a solution u of (1.1), it has been established that the following conservations laws:

$$|\psi(t,\cdot)|_2 = |\psi(0,\cdot)|_2, \quad I(\psi(t,\cdot)) = I(\psi(0,\cdot)) \text{ for any } t \in \mathbb{R},$$

where $L^q(\mathbb{R}^N)$ is the usual Lebesgue space with norm $|u|_q^q := \int_{\mathbb{R}^N} |u|^q dx$, $1 \leq q < \infty$, and I is the energy functional associated with (1.1) defined by

$$I(\psi) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta\psi|^2\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla\psi|^2\,dx - \int_{\mathbb{R}^N} F(\psi)dx$$

for $\psi \in H^2(\mathbb{R}^N)$.

If ψ is a standing wave, i.e., $\psi(t,x) = e^{\mathrm{i}\lambda t}u(x)$, then $u \in H^2(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$ satisfy the following equation:

$$\gamma\Delta^2 u - \beta\Delta u + \lambda u = f(u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

In this paper, we look for solutions (u,λ) with a priori prescribed L^2-norm. For a given $a > 0$, we put

$$S_a := \left\{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = a\right\}. \tag{1.3}$$

In this way, the parameter λ is unknown and appears as a Lagrange multiplier. We remark that this is natural, from a physical point view, to search for the normalized solutions which have prescribed mass.

When $\gamma = 0, \beta \neq 0$, the problem (1.2)-(1.3) has attracted much attention in the last twenty years. The presence of the L^2-constraint makes several methods developed to deal with unconstrained variational problems unavailable, and new phenomena arise. If we set $f(u) = |u|^{p-2}u$, then a new critical exponent appears, i.e., the L^2-critical exponent

$$\bar{p} = 2 + \frac{4}{N}.$$

This is the threshold exponent for many dynamical properties such as global existence vs. blow-up, and the stability or instability of ground states. From the variational point of view, if the problem is purely L^2-subcritical, then I is bounded from below on S_a. Thus, for every $a > 0$, a ground state can be found as a global minimizer of $I|_{S_a}$, and moreover, the minimizer would be orbitally stable, see [8, 9] for homogeneous nonlinear term and [10, 11] for general nonlinear term. And multiplicity results of normalized solutions in the L^2-subcritical case can be referred to [12, 13] and the references therein. In the purely L^2-supercritical case, on the contrary, $I|_{S_a}$ is unbounded from below; however, exploiting the mountain pass lemma and a smart compactness argument, L. Jeanjean [14] could show that a normalized ground state does exist for every $a > 0$ also in this case. The associated standing wave is strongly unstable [15, 16] for homogeneous nonlinear term, due to the supercritical character of the equation. We point out that, in [14, 17–19], more general nonlinearities are considered. With regard to the combined nonlinearities, we refer the reader to [20, 21] for the existence and stability results.

For the case $\gamma \neq 0, \beta \neq 0$, there is only a few papers about the normalized solutions. As we know, this kind of problem would give rise to a new L^2-critical exponent, i.e.,

$$l = 2 + 8/N.$$

When $\gamma > 0, \beta > 0$, Bonheure et al. [5] have dealt with the L^2-subcritical case and obtained the existence of normalized solutions as energy minimizers, while for the L^2-critical and supercritical case, [6] is concerned with several questions including the existence of ground states and of positive solutions and the multiplicity of radial solutions, and the stability of the standing waves of the associated dispersive equation have also been discussed. Recently, in [22] the authors have improved some results to [5] and [6]. When $\gamma > 0, \beta < 0$, the problem is more involved, see [7, 23] for the L^2-subcritical case and [24] for the L^2-supercritical case. We remark that all the aforementioned papers have only considered the homogeneous nonlinearity, i.e., $f(u) = |u|^{p-2}u$. For the general nonlinear term, as far as we know, the results are not there yet. With regard to the point, we attempt to study this kind of problem in this paper. We point out that, when dealing with general nonlinearity, we will face some extra difficulties. Such as the loss of homogeneity, which often plays an important role to use the scaling transformations. Besides, some inequalities about energy would be more difficult to obtain. But these inequalities are the key to obtain the compactness of the minimizing sequences. Finally, other types of normalized solution problems can be referred to [25–35] and the references therein.

In what follows, we give some notations. In $H^2(\mathbb{R}^N)$, when $\beta > 0$, we define its norm by

$$||u||_{H^2} := \left(\int_{\mathbb{R}^N} \left[|\Delta u|^2 + |\nabla u|^2 + |u|^2 \right] dx \right)^{\frac{1}{2}},$$

and for $\beta \leq 0$, by

$$\|u\|_{H^2} := \left(\int_{\mathbb{R}^N} \left[|\Delta u|^2 + |u|^2 \right] dx \right)^{\frac{1}{2}}.$$

Recalling that the following interpolation inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \leq \left(\int_{\mathbb{R}^N} |\Delta u|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |u|^2 \, dx \right)^{\frac{1}{2}}, \quad \forall u \in H^2(\mathbb{R}^N), \tag{1.4}$$

we easily see these two norms above are equivalent in $H^2(\mathbb{R}^N)$. We define the energy functional associated with (1.2) by

$$I(u) = \frac{\gamma}{2} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{\beta}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} F(u) dx$$

for $u \in H^2(\mathbb{R}^N)$, and we consider a constrained variational problem as follows:

$$m_a = \inf_{u \in S_a} I(u). \tag{1.5}$$

Denote the set of minimizers, called ground states for (1.1),

$$\mathcal{M}_a = \{ u \in S_a : I(u) = m_a \}.$$

In this paper, we will study the orbital stability of standing waves for (1.1), in the following sense:

Definition 1.1. *The set $\mathcal{M}_a$ is said to be orbitally stable if any given $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial data ψ_0 satisfying*

$$\inf_{u \in \mathcal{M}_a} \|\psi_0 - u\|_{H^2} < \delta,$$

the corresponding solution $\psi(t, x)$ of the Cauchy problem (1.1) *satisfies*

$$\inf_{u \in \mathcal{M}_a} \|\psi(t, \cdot) - u\|_{H^2} < \varepsilon \quad \textit{for all } t \geq 0.$$

According to the sign of β, we consider the following two cases respectively: (I) $\beta \geq 0$, (II) $\beta < 0$.

(I): $\beta \geq 0$. In this case, we define

$$a_0 = \inf\{ a > 0; m_a < 0 \}, \tag{1.6}$$

see Lemma 3.3 and (3.6) for more details about a_0. For the existence and stability of the minimizer of m_a, we have

Theorem 1.2. *Under the case $\beta \geq 0$, suppose (F1)–(F5) and that a constant $a_0 \geq 0$ which satisfies* (1.6) *is uniquely determined. If $a > a_0$,*

(i) There exists a global minimizer with respect to m_a, i.e., $\mathcal{M}_a \neq \emptyset$.

(ii) Assume the local existence of the Cauchy problem (1.1)*, then $\mathcal{M}_a$ is orbitally stable, i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any solution u of* (1.1) *with $dist(u(0, \cdot), \mathcal{M}_a) < \delta$, it holds that*

$$dist(u(t, \cdot), \mathcal{M}_a) < \varepsilon \quad \textit{for any } t \in \mathbb{R},$$

where $dist(\phi, \mathcal{M}_a) = \inf_{\psi \in \mathcal{M}_a} \|\phi - \psi\|_{H^2}$.

If $0 < a < a_0$*, there is no global minimizer with respect to* m_a*.*

Remark 1.3. *Note that under assumption (F1)–(F5) it is not known if* (1.1) *is locally well posed. Thus, we need to assume the local existence of the Cauchy problem* (1.1) *in Theorem 1.2, a similar assumption also appears in Theorem 1.7. However, when* $f(u) = |u|^{p-2}u, 2 < p < 2 + 8/N$*, the local (even global) existence of the Cauchy problem* (1.1) *has been known, see [1] for* $\beta \geq 0$ *and [40] for* $\beta < 0$*.*

We briefly outline the proof of Theorem 1.2. As the celebrated paper [8] to prove the orbital stability of ground state, the main method we use is the Concentration Compactness Principle. However, our situation is far more complex because we deal with the operator $\gamma\Delta^2 - \beta\Delta$ and the general nonlinearity. To rule out the vanishing case of the minimizing sequences, we need to know when the condition $m_a < 0$ holds for which mass a. This is the reason why we define the value of a_0 in (1.6). Besides, the second difficulty we face is to exclude the dichotomy. And we prove a strict subadditivity (conditional) inequality for m_a to overcome this obstacle, see Lemma 3.3. In addition, since we often use the scaling transformations of functions, the general nonlinearity also causes some extra difficulties.

Next, we give the characterization of m_a. And it is a direct consequence of the definition of a_0 and Lemma 3.3.

Corollary 1.4. *(i) If* $a_0 = 0$*, then* $m_a < 0$ *for any* $a > 0$*.*

(ii) If $a_0 > 0$*, then* $m_a = 0$ *for any* $a \in (0, a_0]$*, and* $m_a < 0$ *for any* $a > a_0$*.*

It is a natural question that "When $a_0 > 0$ holds". To answer the question, the behavior of f near 0 is important. We can show that the following results:

Theorem 1.5. *If* $\beta = 0$ *and we assume* f *satisfies (F1)-(F5).*

(i) If $\liminf_{s\to 0} F(s)/|s|^l = \infty$ *holds, then* $a_0 = 0$ *holds.*

(ii) If $\limsup_{s\to 0} F(s)/|s|^l < \infty$ *holds, then* $a_0 > 0$ *holds.*

Theorem 1.6. *If* $\beta > 0$ *and we assume* f *satisfies (F1)-(F5).*

(i) If $\liminf_{s\to 0} F(s)/|s|^{2+4/N} = \infty$ *holds, then* $a_0 = 0$ *holds.*

(ii) If $\limsup_{s\to 0} F(s)/|s|^{2+4/N} < \infty$ *holds, then* $a_0 > 0$ *holds.*

Let us explain why the conditions in Theorems 1.5 and 1.6 are different. For $\beta = 0$, the main effect for the integral $\int_{\mathbb{R}^N} F(u)dx$ is the semi-norm $|\Delta u|_2^2$. While for $\beta > 0$, both $|\Delta u|_2^2$ and $|\nabla u|_2^2$ affect the integral $\int_{\mathbb{R}^N} F(u)dx$. In particular, whether m_a is negative or not greatly depends on the "small" u. For u small, compared with $|\Delta u|_2^2$, the gradient norm $|\nabla u|_2^2$ (when it exists) dominates the effect.

(II): $\beta < 0$. Under this case, the problem (1.2)-(1.3) is more involved since the term $\beta|\nabla u|_2^2$ in the energy functional I can't be a part of H^2-norm but acts as an independent part which effects the behavior of the energy. At present, except for (F1)-(F5), we also assume f satisfies:

(F6) Assume that $F(s) \geq 0$ for every $s \geq 0$ and there exists a constant $\eta > 2$ such that $F(\tau s) \geq \tau^\eta F(s)$ for every $\tau \geq 1, s \geq 0$.

We set

$$a_1 := \inf\left\{a > 0 : m_a < -\frac{\beta^2}{8\gamma}a\right\}, \tag{1.7}$$

see Lemma 4.4 and (4.2) for more details about a_1. For the existence and stability of the minimizer of m_a, we have

Theorem 1.7. *Under the case $\beta < 0$, suppose (F1)–(F6) and that a constant $a_1 \geq 0$ which satisfies* (1.7) *is uniquely determined. If $a > a_1$, we have*

(i) there exists a global minimizer with respect to m_a, i.e., $\mathcal{M}_a \neq \emptyset$.

(ii) assume the local existence of the Cauchy problem (1.1) *, then $\mathcal{M}_a$ is orbitally stable, i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any solution u of* (1.1) *with $dist(u(0,\cdot), \mathcal{M}_a) < \delta$, it holds that*

$$dist(u(t,\cdot), \mathcal{M}_a) < \varepsilon \quad \text{for any } t \in \mathbb{R},$$

where $dist(\phi, \mathcal{M}_a) = \inf_{\psi \in \mathcal{M}_a} \|\phi - \psi\|_{H^2}$.

If $0 < a \leq a_1$, there is no global minimizer with respect to m_a.

Remark 1.8. *A typical example of the nonlinear term satisfying (F1)–(F6) is $f(u) = |u|^{p-2}u + \mu|u|^{q-2}u, 2 < q < p < 2 + 8/N, \mu \geq 0$.*

In the proof of Theorem 1.7, the situation is more involved compared with the case $\beta \geq 0$. To rule out the vanishing case of minimizing sequences, we need to analyse the spectrum of the operator $\gamma\Delta^2 u - \beta\Delta u$. Thanks to a result in [23] (see Lemma 4.2), we can infer the behavior of m_a and overcome the difficulty by defining the value of a_1 in (1.7). Besides, the dichotomy case is more hard to deal with. To this aim, we use suitable scaling transformations of functions to get the subadditivity condition for the minimizing energy and hence exclude this case. Once we get the precompactness of minimizing sequences, we can prove the existence and orbital stability of normalized solutions.

Next, we give the characterization of m_a. And it is a direct consequence of the definition of a_1 and Lemma 4.4.

Corollary 1.9. *(i) If $a_1 = 0$, then $m_a < -\frac{\beta^2}{8\gamma}a$ for any $a > 0$.*

(ii) If $a_1 > 0$, then $m_a = -\frac{\beta^2}{8\gamma}a$ for any $a \in (0, a_1]$, and $m_a < -\frac{\beta^2}{8\gamma}a$ for any $a > a_1$.

Finally, in the case $\beta < 0$, we consider the nonlinearity $f(u) = |u|^{p-2}u + \mu|u|^{q-2}u, 2 < q < p < 2 + 8/N, \mu < 0$. It is easy to see that f satisfies the conditions (F1)–(F5), but does not satisfy (F6). However, with regard to the value of minimizing energy m_a, we can still give its partial characterization as follows.

Theorem 1.10. *Let $\beta < 0$ and $f(u) = |u|^{p-2}u + \mu|u|^{q-2}u, 2 < q < p < 2 + 8/N, \mu < 0$. Then $m_a \leq -\frac{\beta^2}{8\gamma}a$ for any $a > 0$. Moreover, there exist two constants $a_*, a^* \in [0, \infty)$ with $a^* \geq a_*$ such that*

(i) if $a^* = 0$, then $m_a < -\frac{\beta^2}{8\gamma}a$ for any $a > 0$.

(ii) if $a^* > 0$ and $a_* > 0$, then $m_a = -\frac{\beta^2}{8\gamma}a$ for any $a \in (0, a_*]$, and $m_a < -\frac{\beta^2}{8\gamma}a$ for any $a > a^*$.

(iii) if $a^* > 0$ and $a_* = 0$, then $m_a < -\frac{\beta^2}{8\gamma}a$ for any $a > a^*$.

Compared with the results of Corollary 1.9, the present situation is more involved and the behavior of m_a is more difficult to figure out. Based on this point, we think the condition (F6) may be crucial to determine the behavior of m_a and hence the existence of minimizers for m_a. Although we assume it holds that $a^* > a_* > 0$, we can't infer the behavior of m_a in $(a_*, a^*]$ due to the combined effect of the nonlinear terms $|u|^{p-2}u$ and $\mu|u|^{q-2}u$. On the other hand, for $a > a^*$, we have $m_a < -\frac{\beta^2}{8\gamma}a$, but we still don't know whether m_a can be achieved. At this case, ruling out the vanishing case of the minimizing sequence is easy, however, the dichotomy case is difficult to deal with because the strict subadditivity condition is unclear. Thus, we can't deduce the precompactness of the minimizing sequence. All the facts above show that there is a sharp contrast between the conditions containing (F6) and these conditions lacking (F6).

This paper is organized as follows. In Section 2, we give some preliminaries which will be used later. Section 3 is devoted to studying the existence and orbital stability of normalized solutions which belong to the ground state set $\mathcal{M}_a$ under the case $\beta \geq 0$. The main method we use is the concentration compactness principle. And we rule out the vanishing case according to the negative of energy and exclude the dichotomy by proving a strict additivity inequality for m_a, see Lemma 3.4. Also in this section, we give the proofs of Theorems 1.2, 1.5 and 1.6. In Section 4, we consider the case $\beta < 0$. This situation is more involved. To obtain the existence and orbital stability of normalized solutions, we propose an extra condition on f, i.e., (F6). But it is more hard to rule out the vanishing case. We make use of the spectral analysis for the operator $\gamma\Delta^2 u - \beta\Delta u$ which was given in [23] to rule out the vanishing case. The proof of Theorem 1.7 is finished in this section. Finally, to reveal the effect of the condition (F6), we consider a special nonlinearity which does not satisfy (F6) and investigate the behavior of the energy m_a, i.e., Theorem 1.10.

2. Preliminaries

We begin by recall two well-known Gagliardo-Nirenberg interpolation inequalities for functions $u \in H^2(\mathbb{R}^N)$, namely,

$$|u|_p \leq B_{N,p}|\Delta u|_2^{\frac{(p-2)N}{4p}} |u|_2^{1-\frac{(p-2)N}{4p}}, \tag{2.1}$$

where

$$\begin{cases} 2 \leq p & \text{for } N \leq 4, \\ 2 \leq p \leq \frac{2N}{N-4} & \text{for } N > 4, \end{cases}$$

and

$$|u|_p \leq C_{N,p}|\nabla u|_2^{\frac{(p-2)N}{2p}} |u|_2^{1-\frac{(p-2)N}{2p}}, \tag{2.2}$$

where

$$\begin{cases} 2 \leq p & \text{for } N \leq 2, \\ 2 \leq p \leq \frac{2N}{N-2} & \text{for } N > 2. \end{cases}$$

See, for instance, [36, 37].

In what follows, we give a concentration-compactness lemma for the sequence in $H^2(\mathbb{R}^N)$.

Lemma 2.1. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^2(\mathbb{R}^N)$ which satisfies*

$$\sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n|^2 dx \to 0 \quad \textit{as } n \to \infty.$$

Then, for $p \in (2, 4^*)$,

$$|u_n|_p \to 0 \quad as\ n \to \infty$$

holds, where $4^* = 2N/(N-2)_+$ *is the critical Sobolev exponent.*

Proof. For the proof, we can take a similar argument as that of the classical concentration-compactness lemma by Lions and omit the details. See, for instance, [38, Lemma 1.21]. □

3. Existence and stability under the case $\beta \geq 0$

Throughout this section, unless otherwise stated, we always assume f satisfies (F1)–(F5). First, we show that m_a is bounded from blow for any $a > 0$.

Lemma 3.1. *(i) Let* $\{u_n\}_{n\in\mathbb{N}}$ *be a bounded sequence in* $H^2(\mathbb{R}^N)$. *If either* $\lim_{n\to\infty} |u_n|_2 = 0$ *or* $\lim_{n\to\infty} |u_n|_l = 0$ *holds, then it is true that* $\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx = 0$.

(ii) There exists a positive constant $C = C(f, N, a, \gamma)$ *depending* f, N *and* a *such that*

$$I(u) \geq \frac{\gamma}{4} \int_{\mathbb{R}^N} |\Delta u|^2\, dx + \frac{\beta}{2} \int_{\mathbb{R}^N} |\nabla u|^2\, dx - C \tag{3.1}$$

holds for any $u \in S_a$. *Specifically,* $m_a \geq -C > -\infty$.

Proof. (i): By the assumptions $(F1)$–$(F4)$, for any $\varepsilon > 0$, there exists a positive constant $C(f, \varepsilon)$ which depends on ε and f such that

$$|F(u)| \leq C(f, \varepsilon)|u|^2 + \varepsilon |u|^l, \quad |F(u)| \leq \varepsilon |u|^2 + C(f, \varepsilon)|u|^l,$$

where $l = 2 + 8/N$. For $u \in H^2(\mathbb{R}^N)$, we have

$$\left| \int_{\mathbb{R}^N} F(u)dx \right| \leq C(f, \varepsilon)|u|_2^2 + \varepsilon |u|_l^l, \tag{3.2}$$

$$\left| \int_{\mathbb{R}^N} F(u)dx \right| \leq \varepsilon |u|_2^2 + C(f, \varepsilon)|u|_l^l. \tag{3.3}$$

The Gagliardo-Nirenberg inequality implies that

$$|u|_l^l \leq B_N |\Delta u|_2^2 |u|_2^{\frac{8}{N}},$$

where B_N is a positive constant which depends on N. Thus, we obtain

$$\left| \int_{\mathbb{R}^N} F(u)dx \right| \leq C(f, \varepsilon)|u|_2^2 + \varepsilon B_N |\Delta u|_2^2 |u|_2^{\frac{8}{N}}. \tag{3.4}$$

We take the case where $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^2(\mathbb{R}^N)$ satisfying $\lim_{n\to\infty} |u_n|_2 = 0$. By (3.4), we have $\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx = 0$. Alternatively, we can take the case where $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^2(\mathbb{R}^N)$ satisfying $\lim_{n\to\infty} |u_n|_l = 0$. By (3.3), we have

$$\limsup_{n\to\infty} \left| \int_{\mathbb{R}^N} F(u)dx \right| \leq \varepsilon |u|_2^2.$$

Since we can choose $\varepsilon > 0$ arbitrary, we obtain $\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx = 0$.

(ii): In (3.4), we choose $\varepsilon > 0$ satisfying $B_N a^{\frac{4}{N}} \varepsilon = \frac{\gamma}{4}$. Then, for $u \in S_a$, we have

$$\int_{\mathbb{R}^N} F(u)dx \le C + \frac{\gamma}{4} \int_{\mathbb{R}^N} |\Delta u|^2\, dx,$$

where $C = C(f, N, a, \gamma)$ is a positive constant which depends on f, N, γ and a. This implies (3.1). □

Lemma 3.2. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^2(\mathbb{R}^N)$ satisfying $\lim_{n\to\infty} |u_n|_2^2 = a > 0$. Let $\alpha_n = \sqrt{a}/|u_n|_2$ and $\tilde{u}_n = \alpha_n u_n$. Then the following holds:*

$$\tilde{u}_n \in S_a, \quad \lim_{n\to\infty} \alpha_n = 1, \quad \lim_{n\to\infty} |I(\tilde{u}_n) - I(u_n)| = 0.$$

Proof. Clearly, $\tilde{u}_n \in S_a$ and $\lim_{n\to\infty} \alpha_n = 1$ hold. We can compute

$$\begin{aligned} I(\tilde{u}_n) - I(u_n) &= \frac{\gamma(\alpha_n^2 - 1)}{2} \int_{\mathbb{R}^N} |\Delta u_n|^2\, dx + \frac{\beta(\alpha_n^2 - 1)}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2\, dx \\ &\quad - \int_{\mathbb{R}^N} [F(\alpha_n u_n) - F(u_n)]\, dx \\ &= \frac{\gamma(\alpha_n^2 - 1)}{2} \int_{\mathbb{R}^N} |\Delta u_n|^2\, dx + \frac{\beta(\alpha_n^2 - 1)}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2\, dx \\ &\quad - \int_{\mathbb{R}^N} [F(|\alpha_n u_n|) - F(|u_n|)]\, dx \\ &= \frac{\gamma(\alpha_n^2 - 1)}{2} \int_{\mathbb{R}^N} |\Delta u_n|^2\, dx + \frac{\beta(\alpha_n^2 - 1)}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2\, dx \\ &\quad - \int_{\mathbb{R}^N} \left(\int_0^1 f(|u_n| + (|\alpha_n - 1|)\theta |u_n|)(|\alpha_n| - 1)|u_n| d\theta \right) dx \\ &= \frac{\gamma(\alpha_n^2 - 1)}{2} \int_{\mathbb{R}^N} |\Delta u_n|^2\, dx + \frac{\beta(\alpha_n^2 - 1)}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2\, dx \\ &\quad - (|\alpha_n| - 1) \int_{\mathbb{R}^N} \left(\int_0^1 f(|u_n| + (|\alpha_n - 1|)\theta |u_n|)|u_n| d\theta \right) dx. \end{aligned}$$

We have $0 \le |u_n| + (|\alpha_n - 1|)\theta|u_n| \le (|\alpha_n + 2|)\,|u_n|$. Under the assumptions (F1)–(F4), we have $|f(s)| \le |s| + C|s|^{l-1}$. Hence, we obtain

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \left(\int_0^1 f(|u_n| + (|\alpha_n - 1|)\theta |u_n|)|u_n| d\theta \right) dx \right| \\ &\quad \le \int_{\mathbb{R}^N} \left(\int_0^1 |(\alpha_n + 2)|u_n|^2 + C(\alpha_n + 2)^{l-1} |u_n|^l d\theta \right) dx \\ &\quad = \int_{\mathbb{R}^N} (\alpha_n + 2)|u_n|^2 + C(\alpha_n + 2)^{l-1} |u_n|^l dx. \end{aligned}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^2(\mathbb{R}^N)$, we achieve our conclusion. □

In what follows, we give some properties on m_a.

Lemma 3.3. *(i)* $m_a \leq 0$ *for any* $a > 0$.

(ii) $m_{a+b} \leq m_a + m_b$ *for any* $a, b > 0$.

(iii) $a \mapsto m_a$ *is nonincreasing.*

(iv) For sufficiently large a, $m_a < 0$ *holds.*

(v) $a \mapsto m_a$ *is continuous.*

Proof. (i): Let $u \in S_a$. For $\tau > 0$, we set $u_\tau(x) = \tau^{N/2}u(\tau x)$, giving $u_\tau \in S_a$. Moreover, $|u_\tau|_l^l = \tau^4|u|_l^l \to 0$ as $\tau \to 0$. By Lemma 3.1(i), we have

$$\lim_{\tau\to 0}\int_{\mathbb{R}^N} F(u_\tau)dx = 0.$$

As

$$|\Delta u_\tau|_2^2 = \tau^4|\Delta u|_2^2, \quad |\nabla u_\tau|_2^2 = \tau^2|\nabla u|_2^2,$$

we see that $\lim_{\tau\to 0} I(u_\tau) = 0$ holds. By the definition of m_a, we have $m_a \leq I(u_\tau)$. Thus, we obtain $m_a \leq 0$.

(ii): We fix $\varepsilon > 0$. By the definition of m_a and m_b, there exist $u \in S_a \cap C_0^\infty(\mathbb{R}^N)$ and $v \in S_b \cap C_0^\infty(\mathbb{R}^N)$ such that

$$I(u) \leq m_a + \varepsilon, \quad I(v) \leq m_b + \varepsilon.$$

Since u and v have compact support, by using parallel translation, we can assume supp $u \cap$ supp $v = \emptyset$. Therefore, we have $u + v \in S_{a+b}$. Thus, we find

$$m_{a+b} \leq I(u+v) = I(u) + I(v) \leq m_a + m_b + 2\varepsilon.$$

Since ε is arbitrary, we have $m_{a+b} \leq m_a + m_b$.

(iii): By (i) and (ii), we have

$$m_{a+b} \leq m_a + m_b \leq m_a$$

for any $a, b > 0$. This gives (iii).

(iv): We set

$$M_0 := \sup_{s\in[0,s_0]} |F(s)|,$$

where s_0 is a constant determined in $(F5)$. By $(F5)$, we know $M_0 > 0$. And then we choose a constant $\alpha > 1$ such that

$$M_0(\alpha^N - 1) = \frac{F(s_0)}{2}.$$

We take a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\varphi(x) = \begin{cases} s_0, & |x| \leq 1, \\ 0, & |x| \geq \alpha, \end{cases}$$

For $R > 0$, we set $\varphi_R(x) = \varphi(x/R)$, then there exist two constants $C_1, C_2 > 0$ such that

$$|\nabla\varphi_R| \leq \frac{C_1}{R}, \quad |\Delta\varphi_R| \leq \frac{C_2}{R^2}.$$

We write $|S^{N-1}|$ for the surface area of the unit sphere. If $N = 1$, set $|S^0| = 2$. Now we estimate $I(\varphi_R)$ as follows:

$$\begin{aligned} I(\varphi_R) &= \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta\varphi_R|^2\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla\varphi_R|^2\,dx - \int_{\mathbb{R}^N} F(\varphi_R)dx \\ &= \int_{R\le|x|\le\alpha R}\left[\frac{\gamma}{2}|\Delta\varphi_R|^2 + \frac{\beta}{2}|\nabla\varphi_R|^2 - F(\varphi_R)\right]dx - \int_{|x|\le R} F(s_0)dx \\ &\le \int_{R\le|x|\le\alpha R}\left[\frac{\gamma C_2^2}{2R^4} + \frac{\beta C_1^2}{2R^2} + M_0\right]dx - F(s_0)\frac{|S|^{N-1}R^N}{N} \\ &= \frac{(\alpha^N-1)|S|^{N-1}}{2N}\left[\gamma C_2^2 R^{N-4} + \beta C_1^2 R^{N-2}\right] + \frac{|S|^{N-1}R^N}{N}\left[M_0(\alpha^N-1) - F(s_0)\right] \\ &= \frac{|S|^{N-1}R^N F(s_0)}{2N}\left[\frac{\gamma C_2^2}{2M_0R^4} + \frac{\beta C_1^2}{2M_0R^2} - 1\right]. \end{aligned}$$

Since

$$\frac{\gamma C_2^2}{2M_0R^4} + \frac{\beta C_1^2}{2M_0R^2} - 1 \to -1 \quad \text{as } R \to \infty,$$

for a sufficiently large R, we have $I(\varphi_R) < 0$. By choosing such a R and setting $a_R = |\varphi_R|_2^2$, we obtain $m_{a_R} \le I(\varphi_R) < 0$. By (iii), we have $m_b \le m_{a_R} < 0$ if $b \ge a_R$.

(v): We fix $a > 0$. By (iii), m_{a-h} and m_{a+h} are monotonic and bounded as $h \to 0^+$, so therefore they has limits. Moreover, $m_{a-h} \ge m_a \ge m_{a+h}$ holds due to (iii). Thus, we obtain

$$\lim_{h\to 0^+} m_{a-h} \ge m_a \ge \lim_{h\to 0^+} m_{a+h}.$$

Claim: $\lim_{h\to 0^+} m_{a-h} \le m_a$.

By (i), this is clear if $m_a = 0$. So we consider the case $m_a < 0$. Take $u \in S_a$ and let $u_h(x) = \sqrt{1-h/a}\,u(x)$ for $0 < h << 1$. Since $|u_h|_2^2 = (1-h/a)a = a-h$, we have $u_h \in S_a$. On the other hand, we have

$$\|u_h - u\|_{H^2} = \left(1 - \sqrt{1-\frac{h}{a}}\right)\|u\|_{H^2} \to 0 \quad \text{as } h \to 0^+.$$

Thus, we obtain $\lim_{h\to 0^+} I(u_h) = I(u)$. By $m_{a-h} \le I(u_h)$, we have

$$\lim_{h\to 0^+} m_{a-h} \le \lim_{h\to 0^+} I(u_h) = I(u).$$

As we choose $u \in S_a$ arbitrarily, we obtain $\lim_{h\to 0^+} m_{a-h} \le m_a$.

Claim: $\lim_{h\to 0^+} m_{a+h} \ge m_a$.

Since the left hand side converges, it is sufficient to consider the case $h = 1/n$, where $n \in \mathbb{N}$. Choose a $u_n \in S_{a+1/n}$ which satisfies $I(u_n) \le m_{a+1/n} + 1/n$ for each $n \in \mathbb{N}$. By (i), $I(u_n) \le 1/n$. Lemma 3.1(ii) asserts that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^2(\mathbb{R}^N)$. By the definition of u_n, we have

$$\lim_{n\to\infty} m_{a+1/n} \le \lim_{n\to\infty} I(u_n) \le \lim_{n\to\infty} m_{a+1/n} + 1/n,$$

which implies

$$\lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} m_{a+1/n} = \lim_{h\to 0^+} m_{a+h}. \tag{3.5}$$

Let $v_n = u_n/\sqrt{1 + 1/(an)}$ for $n \in \mathbb{N}$. Then, $\{v_n\}_{n\in\mathbb{N}}$ is also a bounded sequence in $H^2(\mathbb{R}^N)$. Moreover, we have

$$|v_n|_2^2 = \frac{|u_n|_2^2}{1 + 1/(an)} = \frac{a + 1/n}{1 + 1/(an)} = a.$$

Hence, $v_n \in S_a$ holds. By Lemma 3.2, we obtain

$$m_a \le I(v_n) = I(u_n) + o(1) \text{as } n \to \infty.$$

By (3.5), the claim holds. □

We define

$$a_0 = \inf\{a > 0; m_a < 0\}. \tag{3.6}$$

By Lemma 3.3, a_0 is well-defined. Moreover, if $a_0 > 0$, by Lemma 3.3 (v), we know

$$m_{a_0} = 0. \tag{3.7}$$

Under certain conditions, we can further prove the strict subadditivity for m_a.

Lemma 3.4. *(i) Assume that there exists a global minimizer $u \in S_a$ with respect to m_a for some $a > 0$. Then $m_b < m_a$ for any $b > a$. In particular, we have $m_b < 0$ for any $b > a$.*

(ii) Assume that there exist global minimizers $u \in S_a$ and $v \in S_b$ with respect to m_a and m_b respectively for some $a, b > 0$. Then $m_{a+b} < m_a + m_b$.

Proof. (i): By Lemma 3.3(i), we have $I(u) \le 0$. Now setting $\tau = b/a > 1$ and $\tilde{u}(x) = u(\tau^{-1/N}x)$, by the assumption, we have $|\tilde{u}|_2^2 = b$ and

$$I(\tilde{u}) = \tau\left(\frac{\gamma\tau^{-4/N}}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx + \frac{\beta\tau^{-2/N}}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx - \int_{\mathbb{R}^N}F(u)dx\right) < \tau I(u),$$

Noticing that $I(u) = m_a$ and the definition of m_b, we obtain $m_b \le I(\tilde{u}) < \tau m_a \le m_a$.

(ii): By the assumption and the argument as above, we have

$$m_{\eta a} < \eta m_a \quad \text{for any } \eta > 1,$$
$$m_{\tau a} \le \tau m_a \quad \text{for any } \tau \ge 1.$$

Noting that we can assume $0 < b \le a$ without loss of generality, taking $\eta = (a + b)/a$ and $\tau = a/b$, we obtain

$$m_{a+b} < \frac{a+b}{a}m_a = m_a + \frac{b}{a}m_{\frac{a}{b}\cdot b} \le m_a + m_b.$$

It completes the lemma. □

With regard to the minimizing sequence for m_a, we have

Theorem 3.5. *Suppose (F1)–(F5) and that $a > 0$. If $\{u_n\}_{n\in\mathbb{N}} \subset S_a$ is a minimizing sequence with respect to m_a, then one of the following holds:*

(i)

$$\limsup_{n\to\infty}\sup_{z\in\mathbb{R}^N}\int_{B(z,1)}|u_n|^2dx = 0. \tag{3.8}$$

(ii) Taking a subsequence if necessary, there exist $u \in S_a$ and a family $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $u_n(\cdot - y_n) \to u$ in $H^2(\mathbb{R}^N)$ as $n \to \infty$. Specifically, u is a global minimizer.

Proof. Suppose that $\{u_n\}_{n\in\mathbb{N}} \subset S_a$ is a minimizing sequence which does not satisfy (3.8). It is sufficient to show that (ii) holds. Since (3.8) does not hold and $\{u_n\}_{n\in\mathbb{N}} \subset S_a$, we have

$$0 < \limsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n|^2 dx \le \alpha < \infty.$$

Taking a subsequence if necessary, there exists a family $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$, such that

$$0 < \lim_{n\to\infty} \int_{B(0,1)} |u_n(x - y_n)|^2 dx < \infty. \tag{3.9}$$

Since $\{u_n\}_{n\in\mathbb{N}} \subset S_a$ is a minimizing sequence, Lemma 3.1(ii) asserts that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^2(\mathbb{R}^N)$. Hence $\{u_n(\cdot - y_n)\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^2(\mathbb{R}^N)$. Using the weak compactness of a Hilbert space and the Rellich compactness, for some subsequence, there exists $u \in H^2(\mathbb{R}^N)$ such that

$$u_n(\cdot - y_n) \rightharpoonup u \quad \text{weakly in } H^2(\mathbb{R}^N), \tag{3.10}$$

$$u_n(\cdot - y_n) \to u \quad \text{in } L^2_{loc}(\mathbb{R}^N), \tag{3.11}$$

$$u_n(\cdot - y_n) \to u \quad \text{a.e. in } \mathbb{R}^N. \tag{3.12}$$

Equations (3.9) and (3.11) assert that $|u|_2 > 0$. We put $v_n = u_n(\cdot - y_n) - u$. By (3.10), $v_n \rightharpoonup 0$ weakly in $H^2(\mathbb{R}^N)$. Thus, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\Delta u + \Delta v_n|^2\, dx &= \int_{\mathbb{R}^N} |\Delta u|^2\, dx + \int_{\mathbb{R}^N} |\Delta v_n|^2\, dx + 2\Re \int_{\mathbb{R}^N} \Delta u \overline{\Delta v_n} dx \\ &= \int_{\mathbb{R}^N} |\Delta u|^2\, dx + \int_{\mathbb{R}^N} |\Delta v_n|^2\, dx + o(1) \quad \text{as } n \to \infty,\end{aligned} \tag{3.13}$$

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u + \nabla v_n|^2\, dx &= \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \int_{\mathbb{R}^N} |\nabla v_n|^2\, dx + 2\Re \int_{\mathbb{R}^N} \nabla u \cdot \overline{\nabla v_n} dx \\ &= \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \int_{\mathbb{R}^N} |\nabla v_n|^2\, dx + o(1) \quad \text{as } n \to \infty,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\int_{\mathbb{R}^N} |u + v_n|^2\, dx &= \int_{\mathbb{R}^N} |u|^2\, dx + \int_{\mathbb{R}^N} |v_n|^2\, dx + 2\Re \int_{\mathbb{R}^N} u \overline{v_n} dx \\ &= \int_{\mathbb{R}^N} |u|^2\, dx + \int_{\mathbb{R}^N} |v_n|^2\, dx + o(1) \quad \text{as } n \to \infty.\end{aligned} \tag{3.15}$$

Using (3.12), the Brezis-Lieb theorem(see [39] or [13, Lemma 3.2]) implies that

$$\int_{\mathbb{R}^N} F(u + v_n) dx = \int_{\mathbb{R}^N} F(u) dx + \int_{\mathbb{R}^N} F(v_n) dx + o(1) \quad \text{as } n \to \infty.$$

Since $I(u_n) = I(u_n(\cdot - y_n)) = I(u + v_n)$, we can obtain

$$\begin{aligned} &I(u_n) = I(u) + I(v_n) + o(1), \\ &|u_n|_2^2 = |u|_2^2 + |v_n|_2^2 + o(1) \quad \text{as } n \to \infty. \end{aligned} \tag{3.16}$$

We will show the following claim.
Claim.

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |v_n|^2 dx = 0. \tag{3.17}$$

Suppose that (3.17) does not hold. Since $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $H^2(\mathbb{R}^N)$, similarly as above, for some subsequence, there exist a family $\{z_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $v \in H^2(\mathbb{R}^N)$ satisfying $|v|_2 > 0$ such that

$$\begin{aligned} v_n(\cdot - z_n) &\rightharpoonup v \quad \text{weakly in } H^2(\mathbb{R}^N), \\ v_n(\cdot - z_n) &\to v \quad \text{in } L^2_{loc}(\mathbb{R}^N), \\ v_n(\cdot - z_n) &\to v \quad \text{a.e. in } \mathbb{R}^N. \end{aligned}$$

We put $w_n = v_n(\cdot - z_n) - v$. Then, similarly as above, we can obtain

$$\begin{aligned} &I(v_n) = I(v + w_n) = I(v) + I(w_n) + o(1), \\ &|v_n|_2^2 = |v|_2^2 + |w_n|_2^2 + o(1) \quad \text{as } n \to \infty. \end{aligned}$$

Consequently, we have

$$I(u_n) = I(u) + I(v) + I(w_n) + o(1) \quad \text{as } n \to \infty, \tag{3.18}$$

$$|u_n|_2^2 = |u|_2^2 + |v|_2^2 + |w_n|_2^2 + o(1) \quad \text{as } n \to \infty. \tag{3.19}$$

Here, we set $\eta = |u|_2^2, \zeta = |v|_2^2$ and $\delta = a - \eta - \zeta$. Then, we have $\lim_{n\to\infty} |w_n|_2^2 = \delta \geq 0$. We will consider cases $\delta > 0$ and $\delta = 0$.

In the case $\delta > 0$, we set $\tilde{w}_n = \alpha_n w_n$ and $\alpha_n = \sqrt{\delta}/|w_n|_2$. By Lemma 3.2, we have $\tilde{w}_n \in S_\delta$ and $I(w_n) = I(\tilde{w}_n) + o(1)$. Thus, by (3.18) and the definition of m_δ, we have

$$\begin{aligned} I(u_n) &= I(u) + I(v) + I(w_n) + o(1) \\ &= I(u) + I(v) + I(\tilde{w}_n) + o(1) \\ &\geq I(u) + I(v) + m_\delta + o(1) \quad \text{as } n \to \infty. \end{aligned}$$

As $n \to \infty$, Lemma 3.3 implies that

$$m_a \geq I(u) + I(v) + m_\delta \geq m_\eta + m_\zeta + m_\delta \geq m_{\eta+\zeta+\delta} = m_a. \tag{3.20}$$

Hence u and v are global minimizers with respect to m_η and m_ζ respectively. Here, we can apply Lemma 3.4 (ii) to obtain

$$m_{\eta+\zeta} < m_\eta + m_\zeta.$$

It contradicts to (3.20).

In the case $\delta = 0$, the equations $a = \eta + \zeta$ and $\lim_{n\to\infty} |w_n|_2 = 0$ hold. By Lemma 3.1(i), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(w_n)dx = 0.$$

Thus, we obtain

$$\liminf_{n\to\infty} I(w_n) \geq 0.$$

As $n \to$ in (3.18), we have

$$m_a \geq I(u) + I(v) \geq m_\eta + m_\zeta \geq m_{\eta+\zeta} = m_a.$$

Hence u and v are global minimizers with respect to m_η and m_ζ respectively. Here, we can apply Lemma 3.4 (ii) to obtain

$$m_a = m_{\eta+\zeta} < m_\eta + m_\zeta,$$

which is a contradiction. It completes the proof of the claim.

By (3.17) and Lemma 2.1, we have $\lim_{n\to\infty} |v_n|_l = 0$. Lemma 3.1(i) asserts that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(v_n)dx = 0. \tag{3.21}$$

Next, we estimate the L^2 norm of v_n.
Claim. $\lim_{n\to\infty} |v_n|_2 = 0$. In particular, $|u|_2^2 = a$.
By (3.16) and $\eta = |u|_2^2$, it is sufficient to show that $\eta = a$. Otherwise, $\eta < a$ holds because $\eta \leq a$. By (3.21), we have

$$\liminf_{n\to\infty} I(v_n) \geq \liminf_{n\to\infty} - \int_{\mathbb{R}^N} F(v_n)dx = 0.$$

Taking the limit in (3.16), we obtain $m_a \geq I(u)$. Using Lemma 3.3(iii) along with $u \in S_\eta$, we have

$$m_a \geq I(u) \geq m_\eta \geq m_a. \tag{3.22}$$

This requires $m_\eta = m_a$. Moreover, u is a global minimizer with respect to m_η. By Lemma 3.4(i), we obtain $m_\eta > m_a$ because $\eta < a$. It contradicts to (3.22).

Finally, we estimate the H^2-norm of v_n. Using the above claim, $u \in S_a$. This gives $I(u) \geq m_a$. Therefore, we have

$$I(u_n) = I(u) + I(v_n) + o(1) \geq m_a + I(v_n) + o(1) \quad \text{as } n \to \infty.$$

As $n \to \infty$, we obtain

$$\limsup_{n\to\infty} I(v_n) \leq 0,$$

while (3.21) asserts that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \left[\frac{\gamma}{2}|\Delta v_n|^2 + \frac{\beta}{2}|\nabla v_n|^2\right] dx \leq \limsup_{n\to\infty} I(v_n) + \limsup_{n\to\infty} \int_{\mathbb{R}^N} F(v_n)dx \leq 0.$$

Since $\lim_{n\to\infty} |v_n|_2 = 0$, we have $\lim_{n\to\infty} \|v_n\|_{H^2} = 0$. Hence $\lim_{n\to\infty} u_n(\cdot - y_n) = u$ in $H^2(\mathbb{R}^N)$. □

For any minimizing sequence of m_a, Theorem 3.5 shows that the dichotomy case can't occur. To rule out the vanishing case, we will use the condition $m_a < 0$. Thus, for $a > a_0$ (a_0 is given by (3.6)), we can obtain the compactness of the minimizing sequence for m_a.

Proposition 3.6. *Suppose that $a > a_0$. If $\{u_n\}_{n\in\mathbb{N}} \subset S_a$ is a minimizing sequence with respect to m_a, i.e., $\lim_{n\to\infty} I(u_n) = m_a$. Then, taking a subsequence if necessary, there exist a family $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $u \in S_a$ such that $\lim_{n\to\infty} u_n(\cdot - y_n) = u$ in $H^2(\mathbb{R}^N)$. In particular, u is a global minimizer, i.e., $u \in \mathcal{M}_a$.*

Proof. By the assumption of the proposition and (3.6), we have $m_a < 0$. Let $\{u_n\}_{n\in\mathbb{N}} \subset S_a$ be a minimizing sequence with respect to m_a. It is sufficient to show that $\{u_n\}_{n\in\mathbb{N}}$ satisfies (ii) in Theorem 3.5. Otherwise, by Theorem 3.5, $\{u_n\}_{n\in\mathbb{N}}$ satisfies (3.8). By Lemma 3.1(ii), $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^2(\mathbb{R}^N)$, so (3.8) and Lemma 2.1 imply that $u_n \to 0$ in $L^l(\mathbb{R}^N)$. By Lemma 3.1(i), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx = 0.$$

Since $I(u_n) \geq -\int_{\mathbb{R}^N} F(u_n)dx$, we can obtain

$$m_a = \lim_{n\to\infty} I(u_n) \geq \liminf_{n\to\infty} -\int_{\mathbb{R}^N} F(u_n)dx = 0,$$

contradicting to $m_a < 0$. □

After the above preparations have been done, we are now in position to prove our main results.

Proof of Theorem 1.2. First, we consider the case $0 < a < a_0$ and suppose by contradiction that there exists a global minimizer with respect to m_a. By the assumption, we have $m_a = 0$. Here, Lemma 3.4 (i) asserts that

$$0 = m_a > m_{a_0}.$$

It contradicts to (3.7).

Next, we consider the case $a > a_0$. Proposition 3.6 asserts Theorem 1.2 (i). For (ii), we assume it does not hold by contradiction. Then there exists $\varepsilon_0 > 0$ such that for a sequence of solutions u_n of (1.1) with $\operatorname{dist}(u_n(0,\cdot), \mathcal{M}_a) < 1/n$, it holds that

$$\operatorname{dist}(u_n(t_n,\cdot), \mathcal{M}_a) \geq \varepsilon_0,$$

which implies that

$$|u_n(t_n,\cdot)|_2^2 = |u_n(0,\cdot)|_2^2 \to a, \quad I(u_n(t_n,\cdot)) = I(u_n(0,\cdot)) \to m_a.$$

Let $\alpha_n = \sqrt{a}/|u_n(t_n,\cdot)|_2$ and $\tilde{u}_n(x) = \alpha_n u_n(t_n, x)$. Then by Lemma 3.2 the following holds:

$$\tilde{u}_n \in S_a, \quad \lim_{n\to\infty} \alpha_n = 1, \quad \lim_{n\to\infty} I(\tilde{u}_n) \to m_a.$$

By Proposition 3.6, there exist a family $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $u \in \mathcal{M}_a$ such that $\lim_{n\to\infty} \tilde{u}_n(\cdot - y_n) = u$ in $H^2(\mathbb{R}^N)$. Thus, we also get $\lim_{n\to\infty} \|u_n(t_n, \cdot - y_n) - u\|_{H^2} = 0$. We can deduce a contradiction from the following inequalities:

$$\operatorname{dist}(u_n(t_n,\cdot), \mathcal{M}_a) \leq \|u_n(t_n,\cdot) - u(\cdot - y_n)\|_{H^2} = \|u_n(t_n, \cdot - y_n) - u\|_{H^2} \to 0 \quad \text{as } n \to \infty.$$

□

In what follows, we prove Theorems 1.5 and 1.6, which answer the question that "When $a_0 > 0$ holds".

Proof of Theorem 1.5. (i): We fix $a > 0$ and take some function $u \in S_a \cap C_0^\infty(\mathbb{R}^N) \setminus \{0\}$. For $\tau > 0$, let $u_\tau(x) = \tau^{N/2} u(\tau x)$. Then, we see that $u_\tau \in S_a$. By the assumption of (i), there exists a positive constant δ such that

$$F(s) \geq C|s|^l \quad \text{if } |s| < \delta,$$

where C is a constant determined by

$$C = \gamma \int_{\mathbb{R}^N} |\Delta u|^2 dx \bigg/ \int_{\mathbb{R}^N} |u|^l dx \,.$$

Hence $F(u_\tau) \geq C|u_\tau|^l$ holds for a sufficiently small τ. Thus we have

$$I(u_\tau) \leq \frac{\gamma}{2} \int_{\mathbb{R}^N} |\Delta u_\tau|^2 dx - C \int_{\mathbb{R}^N} |u_\tau|^l dx = -\frac{\gamma\tau^4}{2} \int_{\mathbb{R}^N} |\Delta u|^2 dx.$$

It concludes that $m_a \leq I(u_\tau) < 0$ for any $a > 0$.

(ii): By the assumption of (ii), there exists a positive constant $C = C(f)$ such that $F(s) \leq C|s|^l$ holds for any $s \geq 0$. For $u \in S_a$, using the Gagliardo-Nirenberg inequality, we have

$$\int_{\mathbb{R}^N} F(u) dx \leq C|u|_l^l \leq CB_N |\Delta u|_2^2 a^{4/N}.$$

For a sufficiently small $a > 0$, it can be shown that $CB_N a^{4/N} \leq \gamma/2$ holds. After choosing an appropriately small a, we have

$$I(u) \geq \frac{\gamma}{2} \int_{\mathbb{R}^N} |\Delta u|^2 dx - \frac{\gamma}{2} \int_{\mathbb{R}^N} |\Delta u|^2 dx = 0.$$

This together with Lemma 3.3 (i) implies $m_a = 0$ for a small $a > 0$. Hence, we obtain $a_0 > 0$. □

Proof of Theorem 1.6. (i): We fix $a > 0$ and take some function $u \in S_a \cap C_0^\infty(\mathbb{R}^N) \setminus \{0\}$. For $\tau > 0$, let $u_\tau(x) = \tau^{N/2} u(\tau x)$. Then, we see that $u_\tau \in S_a$. By the assumption of (i), there exists a positive constant δ such that

$$F(s) \geq C|s|^{2+4/N} \quad \text{if } |s| < \delta,$$

where C is a constant determined by

$$C = \beta \int_{\mathbb{R}^N} |\nabla u|^2 dx \bigg/ \int_{\mathbb{R}^N} |u|^{2+4/N} dx \,.$$

Hence $F(u_\tau) \geq C|u_\tau|^{2+4/N}$ holds for a sufficiently small τ. Thus we have

$$\begin{aligned} I(u_\tau) &\leq \frac{\gamma}{2} \int_{\mathbb{R}^N} |\Delta u_\tau|^2 dx + \frac{\beta}{2} \int_{\mathbb{R}^N} |\nabla u_\tau|^2 dx - C \int_{\mathbb{R}^N} |u_\tau|^{2+4/N} dx \\ &= \frac{\gamma\tau^4}{2} \int_{\mathbb{R}^N} |\Delta u|^2 dx - \frac{\beta\tau^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx. \end{aligned}$$

If necessary, we take a smaller τ, then we conclude that $m_a \leq I(u_\tau) < 0$ for any $a > 0$.

(ii): By (F3)-(F4), there exist two positive constants $C_1 = C_1(f)$ and $C_2 = C_2(f)$ such that $F(s) \leq C_1|s|^{2+4/N} + C_2|s|^l$ holds for any $s \geq 0$. For $u \in S_a$, using the Gagliardo-Nirenberg inequality, we have

$$\int_{\mathbb{R}^N} F(u)dx \leq C_1|u|_{2+4/N}^{2+4/N} + C_2|u|_l^l \leq C_1C_N|\nabla u|_2^2 a^{2/N} + C_2B_N|\Delta u|_2^2 a^{4/N}.$$

For a sufficiently small $a > 0$, it can be shown that $C_1C_Na^{2/N} \leq \beta/2$ and $C_2B_Na^{4/N} \leq \gamma/2$ hold. After choosing an appropriately small a, we have

$$I(u) = \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2 dx + \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(u)dx \geq 0.$$

This together with Lemma 3.3 (i) implies $m_a = 0$ for a small $a > 0$. Hence, we obtain $a_0 > 0$. □

4. Existence and stability under the case $\beta < 0$

Throughout this section, unless otherwise stated, we always assume f satisfies (F1)-(F6).

Lemma 4.1. *(i) Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^2(\mathbb{R}^N)$. If either $\lim_{n\to\infty} |u_n|_2 = 0$ or $\lim_{n\to\infty} |u_n|_l = 0$ holds, then it is true that $\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx = 0$.*

(ii) There exist two positive constants $C_1 = C_1(a, \beta)$ and $C_2 = C_2(f, N, a, \gamma)$ such that

$$I(u) \geq \frac{\gamma}{4}|\Delta u|_2^2 - C_1|\Delta u|_2 - C_2 \tag{4.1}$$

holds for any $u \in S_a$. Specifically, $m_a > -\infty$.

Proof. (i): the proof can be proceeded as that of Lemma 3.1 (i) and is omitted.

(ii): First, notice that by (1.4), we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \leq \sqrt{a}|\Delta u|_2, \quad \forall u \in S_a,$$

and we set $C_1 = -\beta\sqrt{a}/2$. In addition, recalling that (3.4) in the proof of Lemma 3.1, we choose $\varepsilon > 0$ satisfying $B_N a^{\frac{4}{N}}\varepsilon = \frac{\gamma}{4}$. Then, for $u \in S_a$, we have

$$\int_{\mathbb{R}^N} F(u)dx \leq C_2 + \frac{\gamma}{4}\int_{\mathbb{R}^N} |\Delta u|^2 dx,$$

where $C_2 = C_2(f, N, a, \gamma)$ is a positive constant which depends on f, N, γ and a. Together with the two inequalities above, we get (4.1). □

To character the properties of m_a, we will use some results from [23].

Lemma 4.2. *(see [23])*

(i) For any $\gamma > 0, \beta \in \mathbb{R}$ and $u \in H^2(\mathbb{R}^N)$, it follows that

$$\gamma|\Delta u|_2^2 - \beta|\nabla u|_2^2 + \frac{\beta^2}{4\gamma}|u|_2^2 \geq 0.$$

Thus

$$\inf_{u\in H^2(\mathbb{R}^N)} \left(\frac{\gamma|\Delta u|_2^2 - \beta|\nabla u|_2^2}{|u|_2^2}\right) \geq -\frac{\beta^2}{4\gamma}.$$

(ii) When $\beta < 0$, we introduce

$$J(u) := \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx,$$

and consider the constrained minimization problem

$$m_a^J := \inf_{u\in S_a} J(u).$$

Then for all $a > 0$, it follows that:

(J1) $m_a^J = -\frac{\beta^2}{8\gamma}a$.

(J2) m_a^J *is never achieved.*

(J3) All minimizing sequences present vanishing, i.e., if $\{u_n\}_{n\in\mathbb{N}} \subset S_a$ is a minimizing sequence with respect to m_a^J, then $\{u_n\}_{n\in\mathbb{N}}$ satisfies (3.8).

Lemma 4.3. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^2(\mathbb{R}^N)$ satisfying $\lim_{n\to\infty}|u_n|_2^2 = a > 0$. Let $\alpha_n = \sqrt{a}/|u_n|_2$ and $\tilde{u}_n = \alpha_n u_n$. Then the following holds:*

$$\tilde{u}_n \in S_a, \quad \lim_{n\to\infty}\alpha_n = 1, \quad \lim_{n\to\infty}|I(\tilde{u}_n) - I(u_n)| = 0.$$

Proof. Since the proof is similar as that of Lemma 3.2, we omit it. □

In what follows, we give some properties about m_a.

Lemma 4.4. *(i)* $m_a \le -\frac{\beta^2}{8\gamma}a$ *for any $a > 0$.*

(ii) $m_{a+b} \le m_a + m_b$ *for any $a, b > 0$.*

(iii) $a \mapsto m_a$ *is decreasing.*

(iv) $m_{a\tau} \le \tau m_a$ *for any $a > 0$ and $\tau \ge 1$.*

(v) For sufficiently large a, $m_a < -\frac{\beta^2}{8\gamma}a$ *holds.*

(vi) $a \mapsto m_a$ *is continuous.*

Proof. (i): By (F5) and (F6), we know $F(z) = F(|z|) \ge 0$ for any $z \in \mathbb{C}$. Thus, we get $I(u) \le J(u)$ for any $u \in S_a$. By Lemma 4.2 (ii), it holds that $m_a \le m_a^J = -\frac{\beta^2}{8\gamma}a$.

(ii): The proof is similar as that of Lemma 3.3 (ii) and omitted.

(iii): For any $0 < a < b$, we get from (ii) that $m_b \le m_a + m_{b-a}$. By (i), we have $m_{b-a} \le -\frac{\beta^2}{8\gamma}(b-a) < 0$, which implies $m_b < m_a$.

(iv): For any $u \in S_a$ and $\tau \ge 1$, we set $u_\tau(x) = \tau^{1/2}u(x)$, then $u_\tau \in S_{a\tau}$. Moreover, by (F5)-(F6), we obtain

$$\begin{aligned} I(u_\tau) &= \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u_\tau|^2\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u_\tau|^2\,dx - \int_{\mathbb{R}^N} F(u_\tau)dx \\ &= \tau\left(\frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right) - \int_{\mathbb{R}^N} F(\tau^{1/2}|u(x)|)dx \end{aligned}$$

$$\leq \tau\left(\frac{\gamma}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx+\frac{\beta}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)-\tau^{\eta/2}\int_{\mathbb{R}^N}F(|u(x)|)dx$$
$$\leq \tau I(u).$$

Since u is arbitrary, we get (iv).

(v): By (F5), we can choose a function $u\in H^2(\mathbb{R}^N)$ with $|u|_2^2=1$ such that $\int_{\mathbb{R}^N}F(u)dx>0$. In fact, we take a cut-off function $\varphi\in C_0^\infty(\mathbb{R}^N)$ such that

$$\varphi(x)=\begin{cases} s_0, & |x|\leq 1,\\ 0, & |x|\geq 2,\end{cases}$$

where $s_0>0$ is a constant given by (F5). For $R>0$, we set $\varphi_R(x)=\varphi(x/R)$, then $|\varphi_R|_2^2=R^N|\varphi|_2^2$. Thus, we can choose a $R_0>0$ such that $|\varphi_{R_0}|_2=1$. Now we take $u(x):=\varphi_{R_0}(x)$, then, by (F5)-(F6), we see

$$\int_{\mathbb{R}^N}F(u)dx=\int_{\mathbb{R}^N}F(\varphi_{R_0}(x))dx\geq\int_{|x|\leq R_0}F(s_0)dx>0.$$

For the u above, we set $u_a(x)=a^{1/2}u(x), a\geq 1$, then $u_a\in S_a$. Moreover, by (F5)-(F6), we obtain

$$\begin{aligned} I(u_a)&=\frac{\gamma}{2}\int_{\mathbb{R}^N}|\Delta u_a|^2\,dx+\frac{\beta}{2}\int_{\mathbb{R}^N}|\nabla u_a|^2\,dx-\int_{\mathbb{R}^N}F(u_a)dx\\ &=a\left(\frac{\gamma}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx+\frac{\beta}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)-\int_{\mathbb{R}^N}F(a^{1/2}u)dx\\ &\leq a\left(\frac{\gamma}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx+\frac{\beta}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)-a^{\frac{\eta}{2}}\int_{\mathbb{R}^N}F(u)dx\\ &=a\left(\frac{\gamma}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx+\frac{\beta}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx-a^{\frac{\eta}{2}-1}\int_{\mathbb{R}^N}F(u)dx\right). \end{aligned}$$

Since $\eta>2$, we have

$$\frac{\gamma}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx+\frac{\beta}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx-a^{\frac{\eta}{2}-1}\int_{\mathbb{R}^N}F(u)dx\to-\infty\quad\text{as } a\to\infty.$$

For $a>0$ large enough, we deduce that

$$m_a\leq I(u_a)<-\frac{\beta^2}{8\gamma}a.$$

(vi): The proof is similar as that of Lemma 3.3 (v) and omitted. □

Next we define

$$a_1:=\inf\left\{a>0:m_a<-\frac{\beta^2}{8\gamma}a\right\}. \tag{4.2}$$

By Lemma 4.4 (v), we see that $a_1<\infty$. And again by Lemma 4.4 (iv) and (vi), we know $m_a<-\frac{\beta^2}{8\gamma}a$ for $a>a_1$. Moreover, if $a_1>0$, then it concludes from Lemma 4.4 (i) and (vi) that

$$m_a=-\frac{\beta^2}{8\gamma}a,\quad 0<a\leq a_1. \tag{4.3}$$

Under certain conditions, we can further prove the strict subadditivity for m_a.

Lemma 4.5. *(i) Assume $\mathcal{M}_a \neq \emptyset$ for some $a > 0$. Then $m_{\tau a} < \tau m_a$ for any $\tau > 1$.*

(ii) Assume that there exists a global minimizer $u \in S_a$ with respect to m_a for some $a > 0$ and let $b > 0$. Then $m_{a+b} < m_a + m_b$.

Proof. (i): First, if $u \in \mathcal{M}_a$, then we claim that $\int_{\mathbb{R}^N} F(u)dx > 0$. Otherwise, by Lemma 4.2(ii), $m_a = I(u) = \int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx \geq -\frac{\beta^2}{8\gamma}a$. Thus, we conclude together with Lemma 4.4 (i) that $m_a = -\frac{\beta^2}{8\gamma}a$, which implies u is also a minimizer for m_a^J. This contradicts to (J2) of Lemma 4.2(ii).

Next, for $u \in \mathcal{M}_a$, we set $u_\tau(x) = \tau^{1/2}u(x), \tau > 1$, then $u_\tau \in S_{a\tau}$. Moreover, by (F5)-(F6), we obtain

$$\begin{aligned} I(u_\tau) &= \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u_\tau|^2\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u_\tau|^2\,dx - \int_{\mathbb{R}^N} F(u_\tau)dx \\ &= \tau\left(\frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right) - \int_{\mathbb{R}^N} F(\tau^{1/2}|u(x)|)dx \\ &\leq \tau\left(\frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx\right) - \tau^{\eta/2}\int_{\mathbb{R}^N} F(|u(x)|)dx \\ &< \tau I(u). \end{aligned}$$

Therefore, we get $m_{\tau a} \leq I(u_\tau) < \tau I(u) = \tau m_a$.

(ii): Assume first that $0 < b \leq a$. Then, by Lemma 4.4(iv) and Lemma 4.5 (i), we have

$$\begin{aligned} m_{a+b} &< \frac{a+b}{a}m_a = m_a + \frac{b}{a}m_a = m_a + \frac{b}{a}m_{\frac{a}{b}b} \\ &\leq m_a + \frac{b}{a}\frac{a}{b}m_b = m_a + m_b. \end{aligned}$$

If $0 < a < b$, by again Lemma 4.4(iv) and Lemma 4.5 (i), we obtain

$$\begin{aligned} m_{a+b} &\leq \frac{a+b}{b}m_b = m_b + \frac{a}{b}m_b = m_b + \frac{a}{b}m_{\frac{b}{a}a} \\ &< m_b + \frac{a}{b}\frac{b}{a}m_a = m_a + m_b. \end{aligned}$$

□

With regard to the minimizing sequence for m_a, we have

Theorem 4.6. *Suppose (F1)–(F6) and that $a > 0$. If $\{u_n\}_{n\in\mathbb{N}} \subset S_a$ is a minimizing sequence with respect to m_a, then one of the following holds:*

(i)

$$\limsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n|^2 dx = 0. \tag{4.4}$$

(ii) Taking a subsequence if necessary, there exist $u \in S_a$ and a family $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $u_n(\cdot - y_n) \to u$ in $H^2(\mathbb{R}^N)$ as $n \to \infty$. Specifically, u is a global minimizer.

Proof. Suppose that $\{u_n\}_{n\in\mathbb{N}} \subset S_a$ is a minimizing sequence which does not satisfy (4.4). It is sufficient to show that (ii) holds. Since (4.4) does not hold and $\{u_n\}_{n\in\mathbb{N}} \subset S_a$, we have

$$0 < \limsup_{n\to\infty} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |u_n|^2 dx \leq \alpha < \infty.$$

Taking a subsequence if necessary, there exists a family $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$, such that

$$0 < \lim_{n\to\infty} \int_{B(0,1)} |u_n(x-y_n)|^2 dx < \infty. \tag{4.5}$$

Since $\{u_n\}_{n\in\mathbb{N}} \subset S_a$ is a minimizing sequence, Lemma 4.1(ii) asserts that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^2(\mathbb{R}^N)$. Hence $\{u_n(\cdot - y_n)\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^2(\mathbb{R}^N)$. Using the weak compactness of a Hilbert space and the Rellich compactness, for some subsequence, there exists $u \in H^2(\mathbb{R}^N)$ such that

$$u_n(\cdot - y_n) \rightharpoonup u \quad \text{weakly in } H^2(\mathbb{R}^N), \tag{4.6}$$

$$u_n(\cdot - y_n) \to u \quad \text{in } L^2_{loc}(\mathbb{R}^N), \tag{4.7}$$

$$u_n(\cdot - y_n) \to u \quad \text{a.e. in } \mathbb{R}^N. \tag{4.8}$$

Equations (4.5) and (4.7) assert that $|u|_2 > 0$. We put $v_n = u_n(\cdot - y_n) - u$. By (4.6), $v_n \rightharpoonup 0$ weakly in $H^2(\mathbb{R}^N)$. Thus, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |\Delta u + \Delta v_n|^2 \, dx &= \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \int_{\mathbb{R}^N} |\Delta v_n|^2 \, dx + 2\mathfrak{R} \int_{\mathbb{R}^N} \Delta u \overline{\Delta v_n} dx \\ &= \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \int_{\mathbb{R}^N} |\Delta v_n|^2 \, dx + o(1) \quad \text{as } n \to \infty,\end{aligned} \tag{4.9}$$

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u + \nabla v_n|^2 \, dx &= \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + 2\mathfrak{R} \int_{\mathbb{R}^N} \nabla u \cdot \overline{\nabla v_n} dx \\ &= \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |\nabla v_n|^2 \, dx + o(1) \quad \text{as } n \to \infty,\end{aligned} \tag{4.10}$$

$$\begin{aligned}\int_{\mathbb{R}^N} |u + v_n|^2 \, dx &= \int_{\mathbb{R}^N} |u|^2 \, dx + \int_{\mathbb{R}^N} |v_n|^2 \, dx + 2\mathfrak{R} \int_{\mathbb{R}^N} u\overline{v_n} dx \\ &= \int_{\mathbb{R}^N} |u|^2 \, dx + \int_{\mathbb{R}^N} |v_n|^2 \, dx + o(1) \quad \text{as } n \to \infty.\end{aligned} \tag{4.11}$$

Using (4.8), the Brezis-Lieb theorem(see [39] or [13, Lemma 3.2]) implies that

$$\int_{\mathbb{R}^N} F(u + v_n) dx = \int_{\mathbb{R}^N} F(u) dx + \int_{\mathbb{R}^N} F(v_n) dx + o(1) \quad \text{as } n \to \infty.$$

Since $I(u_n) = I(u_n(\cdot - y_n)) = I(u + v_n)$, we can obtain

$$\begin{aligned}&I(u_n) = I(u) + I(v_n) + o(1), \\ &|u_n|_2^2 = |u|_2^2 + |v_n|_2^2 + o(1) \quad \text{as } n \to \infty.\end{aligned} \tag{4.12}$$

Claim: $|v_n|_2^2 \to 0$ as $n \to \infty$.

In order to prove this, we set $\zeta = |u|_2^2 > 0$. By (4.12), if we show that $\zeta = a$, the claim follows. We assume by contradiction that $\zeta < a$ and we define

$$\tilde{v}_n = \frac{\sqrt{a-\zeta}}{|v_n|_2} v_n.$$

By Lemma 4.3 and (4.12), it follows that

$$I(u_n) = I(u) + I(v_n) + o(1) = I(u) + I(\tilde{v}_n) + o(1) \geq I(u) + m_{a-\zeta} + o(1).$$

Let $n \to \infty$, and by Lemma 4.4 (ii), we have

$$m_a \geq I(u) + m_{a-\zeta} \geq m_\zeta + m_{a-\zeta} \geq m_a, \tag{4.13}$$

and so, $I(u) = m_\zeta$, i.e., $u \in S_\zeta$ is a global minimizer with respect to m_ζ. Thus, by Lemma 4.5 (ii), we get

$$m_a < m_\zeta + m_{a-\zeta},$$

which contradicts (4.13). Hence, the claim follows and $|u|_2^2 = a$.

At this point, since $\{v_n\}$ is a bounded sequence in $H^2(\mathbb{R}^N)$, it follows from (1.4) that $|\nabla v_n|_2 \to 0$ as $n \to \infty$. By Lemma 4.1 (i), we obtain that

$$\liminf_{n\to\infty} I(v_n) = \liminf_{n\to\infty} \frac{\gamma}{2}|\Delta v_n|_2^2 \geq 0. \tag{4.14}$$

On the other hand, since $|u|_2^2 = a$, we deduce from (4.12) that

$$I(u_n) = I(u) + I(v_n) + o(1) \geq m_a + I(v_n) + o(1),$$

and so, that

$$\limsup_{n\to\infty} I(v_n) \leq 0 \tag{4.15}$$

From (4.14) and (4.15) we deduce that $|\Delta v_n|_2^2 \to 0$ as $n \to \infty$ and so, that $u_n(\cdot - y_n) \to u$ in $H^2(\mathbb{R}^N)$. $\square$

For any minimizing sequence of m_a, Theorem 4.6 shows that the dichotomy case can't occur. To rule out the vanishing case, we will use the condition $m_a < -\frac{\beta^2}{8\gamma}a$. Thus, for $a > a_1$ (a_1 is given by (4.2)), we can obtain the compactness of the minimizing sequence for m_a.

Proposition 4.7. *Suppose that $a > a_1$. If $\{u_n\}_{n\in\mathbb{N}} \subset S_a$ is a minimizing sequence with respect to m_a, i.e., $\lim_{n\to\infty} I(u_n) = m_a$. Then, taking a subsequence if necessary, there exist a family $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $u \in S_a$ such that $\lim_{n\to\infty} u_n(\cdot - y_n) = u$ in $H^2(\mathbb{R}^N)$. In particular, u is a global minimizer, i.e., $u \in \mathcal{M}_a$.*

Proof. By the assumption of the proposition and (4.2), we have $m_a < -\frac{\beta^2}{8\gamma}a$. Let $\{u_n\}_{n\in\mathbb{N}} \subset S_a$ be a minimizing sequence with respect to m_a. It is sufficient to show that $\{u_n\}_{n\in\mathbb{N}}$ satisfies (ii) in Theorem 4.6. Otherwise, by Theorem 4.6, $\{u_n\}_{n\in\mathbb{N}}$ satisfies (4.4). By Lemma 4.1(ii), $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^2(\mathbb{R}^N)$, so (4.4) and Lemma 2.1 imply that $u_n \to 0$ in $L^l(\mathbb{R}^N)$. By Lemma 4.1(i), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx = 0.$$

Since $u_n \in S_a$, by Lemma 4.2(ii), we can obtain

$$m_a = \lim_{n\to\infty} I(u_n) \geq \liminf_{n\to\infty} \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u_n|^2\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx \geq m_a^J = -\frac{\beta^2}{8\gamma}a,$$

contradicting to $m_a < -\frac{\beta^2}{8\gamma}a$ for $a > a_1$. □

With the above preparations at hand, we prove our Theorem 1.7.

Proof of Theorem 1.7. First, we consider the case $0 < a \leq a_1$ and suppose by contradiction that there exists a global minimizer u with respect to m_a. By (4.3), we have

$$I(u) = m_a = -\frac{\beta^2}{8\gamma}a, \tag{4.16}$$

From (4.16), we deduce that $u \neq 0$. We choose a sequence $\{u_n\}$ in $H^2(\mathbb{R}^N)$ such that $u_n \to u$ in $H^2(\mathbb{R}^N)$. Then $|u_n|_2 \to |u|_2$ as $n \to \infty$. We define

$$\tilde{u}_n(x) = \frac{|u|_2}{|u_n|_2}u_n(x),$$

then we easily see that

$$\tilde{u}_n \in S_a, \quad \tilde{u}_n \to u \quad \text{in } H^2(\mathbb{R}^N) \quad \text{and } I(\tilde{u}_n) \to I(u)$$

as $n \to \infty$. Thus, $\tilde{u}_n$ is a minimizing sequence of m_a. By (J1) of Lemma 4.2 (ii), we know $m_a = m_a^J$ for $0 < a \leq a_1$. So $\tilde{u}_n$ is also a minimizing sequence of m_a^J. By (J3) of Lemma 4.2 (ii), it must be vanishing, i.e., it satisfies (4.4). Combining with Lemma 2.1, we infer that $u = 0$ a.e. in $\mathbb{R}^N$. This contradicts to (4.16).

Next, we consider the case $a > a_1$. Proposition 4.7 asserts Theorem 1.7 (i).

(ii): The proof is similar as that of Theorem 1.2 (ii) and omitted. □

Finally, in the case $\beta < 0$, we consider the nonlinearity $f(u) = |u|^{p-2}u + \mu|u|^{q-2}u, 2 < q < p < 2 + 8/N, \mu < 0$. We give the partial characterization of the value of minimizing energy m_a.

Proof of Theorem 1.10. Let $\{u_n\} \in S_a$ be a minimizing sequence for m_a^J, then by (J3) of Lemma 4.2 (ii), it must be vanishing. Thus, we have

$$\limsup_{n\to\infty} I(u_n) \leq \liminf_{n\to\infty} \frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u_n|^2\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx = m_a^J = -\frac{\beta^2}{8\gamma}a,$$

which implies $m_a \leq -\frac{\beta^2}{8\gamma}a$ for any $a > 0$. On the other hand, let $u \in S_1$ and $u_a(x) = \sqrt{a}u(x)$, then we see

$$\begin{aligned} I(u_a) &= \frac{\gamma a}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{\beta a}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{a^{p/2}}{p}\int_{\mathbb{R}^N} |u|^p - \frac{a^{q/2}\mu}{q}\int_{\mathbb{R}^N} |u|^q \\ &= a\left(\frac{\gamma}{2}\int_{\mathbb{R}^N} |\Delta u|^2\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N} |\nabla u|^2\,dx - \frac{a^{p/2-1}}{p}\int_{\mathbb{R}^N} |u|^p - \frac{a^{q/2-1}\mu}{q}\int_{\mathbb{R}^N} |u|^q\right) \end{aligned}$$

Since $p > q > 2$, we have

$$\frac{\gamma}{2}\int_{\mathbb{R}^N}|\Delta u|^2\,dx + \frac{\beta}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx - \frac{a^{p/2-1}}{p}\int_{\mathbb{R}^N}|u|^p - \frac{a^{q/2-1}\mu}{q}\int_{\mathbb{R}^N}|u|^q \to -\infty$$

as $a \to \infty$, and we conclude that

$$I(u_a) < -\frac{\beta^2}{8\gamma}a \tag{4.17}$$

for a large enough.

In what follows, we set

$$a_* = \sup\left\{a > 0 : m_\tau = -\frac{\beta^2}{8\gamma}\tau, 0 < \tau \le a\right\}, \tag{4.18}$$

if the a above does not exist, we set $a_* = 0$. Besides, we define

$$a^* = \inf\left\{a > 0 : m_\tau < -\frac{\beta^2}{8\gamma}\tau, \tau \ge a\right\}, \tag{4.19}$$

then we know $a^* < \infty$ from (4.17). Noticing that m_a is continuous (the continuity of m_a can be proved as that of Lemma 3.3 (v)), together with the definitions of a_* and a^*, we get the conclusion. □

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Conflict of interest

The authors declare there is no conflicts of interest.

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