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Research article

Some almost-Schur type inequalities and applications on sub-static manifolds

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Abstract: In this paper, we establish some almost-Schur type inequalities on sub-static manifolds, naturally arising in General Relativity. In particular, our results generalize those in [1] of Li-Xia for r-th mean curvatures of closed sub-static hypersurfaces in space forms and k-scalar curvatures for closed locally conformally flat sub-static manifolds. Moreover, our results also generalize those of Cheng [2].

Keywords: almost-Schur type inequalities; sub-static manifolds; mean curvature; Ricci curvature

1. Introduction

Let (M, g) be an n-dimensional smooth Riemannian manifold. We denote ∇,Δ and ∇^2 the gradient, Laplacian and Hessian operator on M with respect to g, respectively. Ric and R denote Ricci curvature and scalar curvature, respectively. An n-dimensional Riemannian manifold (M, g) is called to be Einstein if its traceless Ricci tensor $\mathring{\mathrm{Ric}} = \mathrm{Ric} - \frac{R}{n}g$ is identically zero. The classical Schur's lemma states that the scalar curvature of an Einstein manifold of dimension $n \geq 3$ must be constant. Recently, De Lellis and Topping [3] proved (and independently by Andrews, cf. [4, Corollary B. 20]) the following almost-Schur lemma as they called:

Theorem 1.1. *(see [3]) If (M, g) is a closed Riemannian manifold of dimension $n \geq 3$, with non-negative Ricci curvature, then*

$$\int_M (R - \overline{R})^2 \, dv_g \leq \frac{4n(n-1)}{(n-2)^2} \int_M \left|\mathrm{Ric} - \frac{R}{n}g\right|^2 dv_g, \tag{1.1}$$

and equivalently,

$$\int_M \left|\mathrm{Ric} - \frac{\overline{R}}{n}g\right|^2 dv_g \leq \frac{n^2}{(n-2)^2} \int_M \left|\mathrm{Ric} - \frac{R}{n}g\right|^2 dv_g, \tag{1.2}$$

where $\overline{R}$ denotes the average of R over M. Moreover the equality in (1.1) *or* (1.2) *holds if and only if M is Einstein.*

As it is customary, we say that M is called a closed manifold if it is compact and without boundary. In the case of dimension $n = 3, 4$, Ge and Wang [5, 6] proved that Theorem 1.1 holds under the weaker condition of non-negative scalar curvature. However, as pointed out by De Lellis and Topping in [3], the coefficient in inequality (1.1) is optimal and the non-negativity of Ricci curvature can not be removed for $n \geq 5$. Generalizing De Lellis and Topping's results in [3], Cheng [7] proved an almost-Schur lemma for the closed manifolds with Ricci curvature bounded from below by a negative constant. That is, she obtained a similar inequality with the coefficient depending on not only the lower bound of Ricci curvature but also the value of the first non-zero eigenvalue of Laplace operator. Later, Cheng [2] also generalized her own previous results to a class of symmetric $(0, 2)$-tensors and gave the applications for r-th mean curvatures of closed hypersurfaces in space forms and k-scalar curvatures for closed locally conformally flat manifolds. For the recent research in this direction, see [5, 8–15] and the references therein.

The aim of the present paper is to extend the above results to the vast context of sub-static manifolds.

Let us provide the rigorous definitions we are going to employ for sub-static manifolds (see [1]).

Definition 1.1. (sub-static) Let (M, g) be a Riemannian manifold endowed with a positive twice differentiable function V. We say that it is sub-static if

$$(\Delta V)g - \nabla^2 V + V\mathrm{Ric} \geq 0 \tag{1.3}$$

on the whole of M. In this case, we say that V is the sub-static potential of (M, g).

In addition to the sub-static warped products considered in [16], we mention that complete noncompact manifolds with nonnegative Ricci curvature and without boundary fulfil the assumptions above. The definition of sub-static Riemannian manifolds comes naturally from and has roots in the study of static spacetimes in general relativity. The sub-static condition appear more generally in General Relativity in relation with the so called null convergence condition [17]. Discussing in details the deep connections with Mathematical General Relativity is far out the scope of this contribution, and so we refer the interested reader to the comprehensive thesis of Borghini [18] and the references therein. For the recent research in this direction, see [1, 19, 20] and the references therein.

In this paper, we give new almost-Schur type inequalities for symmetric $(0, 2)$-tensors on sub-static manifolds. Applying such unified inequalities for symmetric $(0, 2)$-tensors, we may obtain inequalities besides those in the papers mentioned above. For this purpose, we prove the following.

Theorem 1.2. *Let V be a positive twice differentiable function on an n-dimensional closed Riemannian manifold* (M^n, g) $(n \geq 3)$ *with*

$$(\Delta V)g - \nabla^2 V + V\mathrm{Ric} \geq -(n-1)Kg,$$

where K is a nonnegative constant. Let T be a symmetric $(0, 2)$*-tensor satisfying* $\mathrm{div}T = c\nabla B$*, where* $B = \mathrm{tr}T$ *denotes its trace and c is a constant. Then*

$$(nc-1)^2 \int_M V(B - \overline{B}^V)^2 \, dv_g \leq n(n-1)\left(1 + \frac{nK}{\lambda_1}\right) \int_M V \left|T - \frac{B}{n}g\right|^2 dv_g \tag{1.4}$$

and, equivalently,

$$\begin{aligned}(nc-1)^2\int_M V\left|T-\frac{\overline{B}^V}{n}g\right|^2 dv_g \\ \le \left[(nc-1)^2+(n-1)\left(1+\frac{nK}{\lambda_1}\right)\right]\int_M V\left|T-\frac{B}{n}g\right|^2 dv_g,\end{aligned} \tag{1.5}$$

where $\overline{B}^V = \frac{\int_M BV\,dv_g}{\int_M V\,dv_g}$ *and* λ_1 *denotes the first nonzero eigenvalue of problem* $\Delta f - \frac{\Delta V}{V}f = -\lambda_1\frac{f}{V}$ *on* (M^n, g).

Assume $(\Delta V)g - \nabla^2 V + V\mathrm{Ric} > 0$ *on* (M^n, g). *If* $c \neq \frac{1}{n}$, *statements* (i), (ii) *and* (iii) *below are equivalent. If* $c = \frac{1}{n}$, *then* (i) *and* (ii) *are equivalent.*

(i) *Equality holds in* (1.4) *and* (1.5).

(ii) $T = \frac{B}{n}g$ *on* (M^n, g).

(iii) $T = \frac{\overline{B}^V}{n}g$ *on* (M^n, g).

Remark 1.1. We define $\mathcal{L}f = \mathrm{div}\left(V^2\nabla\left(\frac{f}{V}\right)\right)$. It is easy to check that

$$\mathrm{div}\left(V^2\nabla\left(\frac{f}{V}\right)\right) = V\Delta f - f\Delta V.$$

Hence, the problem $\Delta f - \frac{\Delta V}{V}f = -\lambda\frac{f}{V}$ is equivalent to the eigenvalue problem of $\mathcal{L}f = -\lambda f$. In particular, if $V = 1$, then Theorem 1.2 reduces to [2, Theorem 1.7]. Hence, Theorem 1.2 is a generalization of Cheng's Theorem. On the other hand, if $K = 0$, then (M^n, g) appears as a sub-static manifold and we obtain some almost-Schur type inequalities for general tensors on sub-static manifolds.

We now focus on using Theorem 1.2 to obtain two other applications on closed sub-static manifolds as follows.

As the first application of Theorem 1.2, we give an almost-Schur type inequality for r-th mean curvatures of closed sub-static hypersurfaces in space forms. Let us first recall the definition of the r-th mean curvatures, which was first introduced by Reilly [21] and has been intensively studied by many mathematicians. Assume (Σ, g) is a connected oriented closed sub-static hypersurface immersed in a space form with induced metric g. Associated with the second fundamental form A of Σ, we have r-th mean curvatures H_r of Σ and the Newton transformations P_r, $0 \le r \le n$ (see their definition and related notation in Section 3).

In Section 3, we prove the following.

Theorem 1.3. *Let* $(N^{n+1}, \widetilde{g})$ *be a space form,* $n \ge 3$. *Assume that* (Σ, g) *is a smooth connected oriented closed sub-static hypersurface immersed in N with induced metric g and V is the sub-static potential of* (Σ, g). *Then, for* $2 \le r \le n$,

$$(n-r)^2\int_\Sigma V(s_r - \overline{s}_r^V)^2\,dv_g \le n(n-1)\int_\Sigma V\left|P_r - \frac{(n-r)s_r}{n}g\right|^2 dv_g, \tag{1.6}$$

where $s_r = \mathrm{tr}P_r = \frac{n!}{r!(n-r)!}H_r$ *and* $\overline{s}_r^V = \frac{\int_M s_r V\, dv_g}{\int_M V\, dv_g}$. *Equivalently,*

$$\int_\Sigma V \left| P_r - \frac{(n-r)\overline{s}_r^V}{n} g \right|^2 dv_g \leq n \int_\Sigma V \left| P_r - \frac{(n-r)s_r}{n} g \right|^2 dv_g. \tag{1.7}$$

If the strict sub-static inequality holds on (Σ, g)*, then statements* (i)*,* (ii) *and* (iii) *below are equivalent.*

(i) *Equality holds in* (1.6) *and* (1.7).

(ii) $P_r = \frac{(n-r)s_r}{n} g$ *on* (Σ, g).

(iii) $P_r = \frac{(n-r)\overline{s}_r^V}{n} g$ *on* (Σ, g).

Remark 1.2. If $V = 1$, Theorem 1.3 becomes Theorem 1.10 of Cheng in [2]. If $r = 1$, $P_1 = s_1 I - A = HI - A$. So (1.6) turns into

$$\int_\Sigma V(H - \overline{H}^V)^2\, dv_g \leq \frac{n}{n-1} \int_\Sigma V \left| A - \frac{H}{n} g \right|^2 dv_g. \tag{1.8}$$

Equation (1.8) was proved in [1, Theorem 1.9], if Σ is a horo-convex (resp. convex) hypersurface in the hyperbolic space $\mathbf{H}^n$ (resp. the hemi-sphere $\mathbf{S}_n^+$). This is because a horo-convex (resp. convex) hypersurface in $\mathbf{H}^n$ (resp. $\mathbf{S}_+^n$) is a closed sub-static manifold. Hence, Theorem 1.3 is a generalization of Li and Xia's Theorem 1.9 in [1].

As the second application of Theorem 1.2, we give an almost-Schur type inequality for the k-scalar curvatures of locally conformally flat closed sub-static manifolds (see the definition in Section 4). Since they were first introduced in [22], k-scalar curvatures have been much studied. When M is a locally conformally flat closed sub-static manifold, we obtain an almost-Schur type lemma for k-scalar curvatures, $k \geq 2$, as follows.

Theorem 1.4. *Let* (M^n, g) *be a closed locally conformally flat sub-static Riemannian manifold of dimension* $n \geq 3$ *and* V *is the sub-static potential of* (M^n, g)*. Then, for* $2 \leq k \leq n$*, the* k*-scalar curvature* $\sigma_k(\mathcal{S})$ *and the Newton transformation* T_k *associated with the Schouten tensor* $\mathcal{S}$ *satisfy*

$$(n-k)^2 \int_M V(\sigma_k(\mathcal{S}) - \overline{\sigma}_k^V(\mathcal{S}))^2\, dv_g \leq n(n-1) \int_M V \left| T_k - \frac{(n-k)\sigma_k(\mathcal{S})}{n} g \right|^2 dv_g \tag{1.9}$$

where $\overline{\sigma}_k^V(\mathcal{S}) = \frac{\int_M \sigma_k(\mathcal{S}) V\, dv_g}{\int_M V\, dv_g}$. *Equivalently,*

$$\int_M V \left| T_k - \frac{(n-k)\overline{\sigma}_k^V(\mathcal{S})}{n} g \right|^2 dv_g \leq n \int_M V \left| T_k - \frac{(n-k)\sigma_k(\mathcal{S})}{n} g \right|^2 dv_g. \tag{1.10}$$

If the strict sub-static inequality holds on (M^n, g)*, then statements* (i)*,* (ii) *and* (iii) *below are equivalent.*

(i) *Equality holds in* (1.9) *and* (1.10).

(ii) $T_k = \frac{(n-k)\sigma_k(\mathcal{S})}{n} g$ *on* (M^n, g).

(iii) $T_k = \frac{(n-k)\overline{\sigma}_k^V(\mathcal{S})}{n} g$ *on* (M^n, g).

Remark 1.3. If $V = 1$, Theorem 1.4 becomes Theorem 1.11 of Cheng in [2] with $K = 0$. Hence, Theorem 1.4 is a generalization of Cheng's Theorem.

The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.2. In Section 3, we prove Theorem 1.3 by applying Theorem 1.2. In Section 4, we prove Theorem 1.4 by applying Theorem 1.2.

2. Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

First we give some notation. Assume (M, g) is an n-dimensional closed Riemannian manifold. Let ∇ denote the Levi-Civita connection on (M, g) and also the induced connections on tensor bundles on M. Let T be a symmetric $(0, 2)$-tensor field on M and denotes by B the trace of T. Denote by $\overline{B}^V = \frac{\int_M BV\,dv_g}{\int_M V\,dv_g}$ and set $\mathring{T} = T - \frac{B}{n}g$.

For the proof of Theorem 1.2, the following lemmas will be used.

Lemma 2.1. *([1, Theorem 1.1]) Let (M^n, g) be an n-dimensional closed Riemannian manifold. Let V be a positive twice differentiable function on (M^n, g). Then for any $f \in C^\infty(M)$, the following integral identity holds:*

$$\begin{aligned}&\int_M V\left[\left(\Delta f - \frac{\Delta V}{V}f\right)^2 - \left|\nabla^2 f - \frac{\nabla^2 V}{V}f\right|^2\right] dv_g\\ &= \int_M ((\Delta V)g - \nabla^2 V + V\mathrm{Ric})\left(\nabla f - \frac{\nabla V}{V}f, \nabla f - \frac{\nabla V}{V}f\right) dv_g.\end{aligned} \tag{2.1}$$

Proof of Theorem 1.2. We note that if $c = \frac{1}{n}$, then

$$(nc-1)^2 \int_M V(B - \overline{B}^V)^2\,dv_g = 0$$

and inequality (1.4) or (1.5) follows trivially. Hence, it suffices to prove the case $c \neq \frac{1}{n}$.

Now, let us suppose that $c \neq \frac{1}{n}$. By the assumption $\mathrm{div}T = c\nabla B$,

$$\mathrm{div}\mathring{T} = \mathrm{div}\left(T - \frac{B}{n}g\right) = \frac{nc-1}{n}\nabla B. \tag{2.2}$$

We let $f : M \to \mathbb{R}$ be the unique solution to

$$\Delta f - \frac{\Delta V}{V}f = B - \overline{B}^V, \text{ on } M. \tag{2.3}$$

The existence and uniqueness of Eq (2.3) is due to the Fredholm alternative (for detail, see Remark 1.1 or page 512 in [1]).

For notation simplicity, we denote by

$$C_{ij} = f_{ij} - \frac{f}{V}V_{ij}, \;\; \mathrm{tr}(C_{ij}) = \Delta f - \frac{\Delta V}{V}f, \;\; \mathring{C}_{ij} = C_{ij} - \frac{\mathrm{tr}(C_{ij})}{n}g_{ij}$$

under the local coordinates $\{x_i\}$ on M, where f_{ij} and V_{ij} represent the Hessian components of f and V, respectively. Moreover, we have

$$\mathrm{tr}(\mathring{C}_{ij}) = g^{ij}\mathring{C}_{ij} = 0.$$

Then by using (2.3) and the divergence theorem, we have

$$\begin{aligned}
&\int_M V(B-\overline{B}^V)^2\, dv_g \\
&= \int_M (B-\overline{B}^V)(V\Delta f - f\Delta V)\, dv_g \\
&= \int_M \left(-\langle\nabla(VB),\nabla f\rangle + \langle\nabla(fB),\nabla V\rangle\right)\, dv_g \\
&= -\int_M \langle\nabla B, V\nabla f - f\nabla V\rangle\, dv_g.
\end{aligned} \tag{2.4}$$

Using (2.2) and (2.4), we have

$$\begin{aligned}
&\int_M V(B-\overline{B}^V)^2\, dv_g \\
&= -\frac{n}{nc-1}\int_M \mathring{T}_{ij,j}(Vf_i - fV_i)\, dv_g \\
&= \frac{n}{nc-1}\int_M \mathring{T}_{ij}(Vf_i - fV_i)_j\, dv_g \\
&= \frac{n}{nc-1}\int_M \mathring{T}_{ij}(V_jf_i - f_jV_i + Vf_{ij} - fV_{ij})\, dv_g \\
&= \frac{n}{nc-1}\int_M V\mathring{T}_{ij}\left(f_{ij} - \frac{f}{V}V_{ij}\right)\, dv_g \\
&= \frac{n}{nc-1}\int_M V\mathring{T}_{ij}C_{ij}\, dv_g \\
&= \frac{n}{nc-1}\int_M V\mathring{T}_{ij}\mathring{C}_{ij}\, dv_g \\
&\le \frac{n}{|nc-1|}\left(\int_M V|\mathring{T}_{ij}|^2\, dv_g\right)^{\frac{1}{2}}\left(\int_M V|\mathring{C}_{ij}|^2\, dv_g\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.5}$$

Note that

$$|\mathring{C}_{ij}|^2 = |C_{ij}|^2 - \frac{1}{n}\left(\Delta f - \frac{\Delta V}{V}f\right)^2$$

which gives

$$\int_M V|\mathring{C}_{ij}|^2\, dv_g = \int_M V|C_{ij}|^2\, dv_g - \frac{1}{n}\int_M V\left(\Delta f - \frac{\Delta V}{V} f\right)^2\, dv_g. \tag{2.6}$$

Since $(\Delta V)g - \nabla^2 V + V\mathrm{Ric} \geq -(n-1)K$ on (M^n, g), by (2.1) we have

$$\begin{aligned}\int_M V|C_{ij}|^2\, dv_g &= \int_M V\left|\nabla^2 f - \frac{f}{V}\nabla^2 V\right|^2\, dv_g \\ &= -\int_M ((\Delta V)g - \nabla^2 V + V\mathrm{Ric})\left(\nabla f - \frac{\nabla V}{V} f, \nabla f - \frac{\nabla V}{V} f\right)\, dv_g \\ &\quad + \int_M V\left(\Delta f - \frac{\Delta V}{V} f\right)^2\, dv_g \\ &\leq \int_M V\left(\Delta f - \frac{\Delta V}{V} f\right)^2\, dv_g + (n-1)K\int_M V^2\left|\nabla \frac{f}{V}\right|^2\, dv_g,\end{aligned} \tag{2.7}$$

where we note that $\nabla f - \frac{\nabla V}{V} f = V\nabla\frac{f}{V}$. Therefore,

$$\int_M V|\mathring{C}_{ij}|^2\, dv_g \leq \frac{n-1}{n}\int_M V\left(\Delta f - \frac{\Delta V}{V} f\right)^2\, dv_g + (n-1)K\int_M V^2\left|\nabla\frac{f}{V}\right|^2\, dv_g. \tag{2.8}$$

Using the Rayleigh-Reitz principle, we note that the first nonzero eigenvalue λ_1 of $\Delta f - \frac{\Delta V}{V} f = -\lambda\frac{f}{V}$ on (M^n, g) can be characterized by

$$\lambda_1 = \inf_{f\in C^\infty(M)} \frac{\int_M V^2\left|\nabla\frac{f}{V}\right|^2\, dv_g}{\int_M V\left(\frac{f}{V}\right)^2\, dv_g}.$$

Then,

$$\begin{aligned}\int_M V^2\left|\nabla\frac{f}{V}\right|^2\, dv_g &= -\int_M V\left(\frac{f}{V}\right)\left(\Delta f - \frac{\Delta V}{V} f\right)\, dv_g \\ &\leq \left(\int_M V\left(\frac{f}{V}\right)^2\, dv_g\right)^{\frac{1}{2}}\left(\int_M V\left(\Delta f - \frac{\Delta V}{V} f\right)^2\, dv_g\right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{\lambda_1}\int_M V^2\left|\nabla\frac{f}{V}\right|^2\, dv_g\right)^{\frac{1}{2}}\left(\int_M V\left(\Delta f - \frac{\Delta V}{V} f\right)^2\, dv_g\right)^{\frac{1}{2}},\end{aligned} \tag{2.9}$$

where in the first equality we have used

$$\mathrm{div}\left(V^2\nabla\left(\frac{f}{V}\right)\right) = V\Delta f - f\Delta V.$$

Consequently

$$\int_M V^2 \left|\nabla \frac{f}{V}\right|^2 dv_g \le \frac{1}{\lambda_1} \int_M V\left(\Delta f - \frac{\Delta V}{V} f\right)^2 dv_g = \frac{1}{\lambda_1} \int_M V\left(B - \overline{B}^V\right)^2 dv_g. \tag{2.10}$$

Inserting (2.10) into (2.8), we have

$$\begin{aligned}\int_M V|\mathring{C}_{ij}|^2 dv_g \le& \frac{n-1}{n} \int_M V\left(\Delta f - \frac{\Delta V}{V} f\right)^2 dv_g + \frac{(n-1)K}{\lambda_1} \int_M V\left(\Delta f - \frac{\Delta V}{V} f\right)^2 dv_g \\ =& \frac{n-1}{n}\left(1 + \frac{nK}{\lambda_1}\right) \int_M V\left(\Delta f - \frac{\Delta V}{V} f\right)^2 dv_g \\ =& \frac{n-1}{n}\left(1 + \frac{nK}{\lambda_1}\right) \int_M V\left(B - \overline{B}^V\right)^2 dv_g.\end{aligned} \tag{2.11}$$

Therefore, combining (2.11) with (2.5) concludes the proof of (1.4). From the identity

$$\left|T - \frac{\overline{B}^V}{n} g\right|^2 = \left|T - \frac{B}{n} g\right|^2 + \frac{1}{n}\left(B - \overline{B}^V\right)^2,$$

we obtain the inequality (1.5).

Next, we consider the case of equality of (1.4) or (1.5). The idea in the proof of the case of equality of (1.4) or (1.5) is similar to the one used by Cheng in [2]. We will assume that $(\Delta V)g - \nabla^2 V + V\mathrm{Ric} > 0$ on (M^n, g)

Fisrt, let us suppose that $c = \frac{1}{n}$. Obviously, if $T = \frac{B}{n} g$ on (M^n, g), the equalities in (1.4) and (1.5) hold. On the other hand, suppose the equality in (1.4) (or, equivalently, (1.5)) holds. It is obvious that $T = \frac{B}{n} g$. Hence conclusions (i) and (ii) are equivalent.

Now, let us suppose that $c \neq \frac{1}{n}$. We may take $K = 0$. By the proof of Theorem 1.2, the equality in (1.4) holds if and only if

$$T_{ij} - \frac{B}{n} g_{ij} = \mathring{C}_{ij} \tag{2.12}$$

and

$$((\Delta V)g - \nabla^2 V + V\mathrm{Ric})\left(\nabla \frac{f}{V}, \nabla \frac{f}{V}\right) = 0 \tag{2.13}$$

on (M^n, g). Note that (2.13) and $(\Delta V)g - \nabla^2 V + V\mathrm{Ric} > 0$ on (M^n, g). We have that $\nabla \frac{f}{V} \equiv 0$ on (M^n, g). Then $\frac{f}{V}$ is constant. It follows then from (2.12) that $T_{ij} = \frac{B}{n} g_{ij}$ on (M^n, g).

On the other hand, if $T_{ij} = \frac{B}{n} g_{ij}$ on (M^n, g), then B is constant on (M^n, g) (see [2, Proposition 2.1]). Thus $B = \overline{B}^V$ on (M^n, g). The equalities in (1.4) and (1.5) hold. Hence conclusions (i) and (ii) are equivalent. Obviously (iii) implies (ii). If (ii) holds, by the above argument, (ii) implies $B = \overline{B}^V$ on (M^n, g). Thus (iii) also holds. Hence conclusions (ii) and (iii) are equivalent. Therefore, we complete the proof of Theorem 1.2.

3. High-order mean curvatures of sub-static hypersurfaces in space forms

Assume $(N, \widetilde{g})$ is an $(n+1)$-dimensional Riemannian manifold, $n \geq 3$. Suppose (Σ, g) is a smooth connected oriented closed hypersurface immersed in $(N, \widetilde{g})$ with induced metric g. Let ν denote the outward unit normal to Σ, and A the second fundamental form, denoted by $A(X, Y) = -\langle \widetilde{\nabla}_X Y, \nu \rangle$, where $X, Y \in T_p\Sigma$, $p \in \Sigma$, and $\widetilde{\nabla}$ denotes the Levi-Civita connection of $(N, \widetilde{g})$. A determines an equivalent $(1,1)$-tensor, called the shape operator given by $AX = \widetilde{\nabla}_X \nu$. Σ is called totally umbilical if A is a multiple of its metric g at every point of Σ.

The r-th symmetric functions s_r on hypersurface (Σ, g) of $(N, \widetilde{g})$ is related to the Newton transformation P_r by

$$\mathrm{tr} P_r = (n-r) s_r,$$

where

$$P_r = \sum_{j=0}^{r} (-1)^j s_{r-j} A^j, \quad r = 1, \cdots, n.$$

The r-th mean curvature H_r of Σ at p is defined by $s_r = \frac{n!}{r!(n-r)!} H_r$, $0 \leq r \leq n$. When the ambient space is a space form N_a^{n+1}, we have $\mathrm{div} P_r = 0$, for $0 \leq r \leq n$. For detail, see [2,21] and the references therein.

Proof of Theorem 1.3. Since $\mathrm{tr} P_r = (n-r)s_r$ and $\mathrm{div} P_r = 0$, taking $T = P_r$ and $B = (n-r)s_r$ in Theorem 1.2, we have

$$(n-r)^2 \int_\Sigma V (s_r - \overline{s}_r^V)^2 \, dv_g \leq n(n-1) \int_\Sigma V \left| P_r - \frac{(n-r)s_r}{n} g \right|^2 dv_g,$$

equivalently,

$$\int_\Sigma V \left| P_r - \frac{(n-r)\overline{s}_r^V}{n} g \right|^2 dv_g \leq n \int_\Sigma V \left| P_r - \frac{(n-r)s_r}{n} g \right|^2 dv_g,$$

which are (1.6) and (1.7), respectively.

Now we prove the case of equalities in (1.6) and (1.7). If the strict sub-static inequality holds on (Σ, g), by Theorem 1.2, statements (i), (ii), and (iii) are equivalent and $s_r = \overline{s}_r^V$ is constant on (Σ, g). Therefore, we complete the proof of Theorem 1.3.

4. k-scalar curvature of locally conformal flat sub-static manifolds

We first recall the definition of the k-scalar curvatures of a Riemannian manifold. If (M^n, g) is an n-dimensional Riemannian manifold, $n \geq 3$, the Schouten tensor of M is

$$\mathcal{S} = \frac{1}{n-2} \left(\mathrm{Ric} - \frac{1}{2(n-1)} R g \right).$$

By definition, $\mathcal{S} : TM \to TM$ is a symmetric $(1,1)$-tensor field. The k-scalar curvatures $\sigma_k(\mathcal{S})$ on a Riemannian manifold (M^n, g) is related to the k-th Newton tensor T_k is defined by

$$(T_k)_i^j = \frac{1}{k!} \sum \delta_{i_1 \ldots i_k i}^{j_1 \ldots j_k j} \mathcal{S}_{i_1}^{j_1} \ldots \mathcal{S}_{i_k}^{j_k}.$$

Moreover, $\text{tr}T_k = (n-k)\sigma_k(\mathcal{S})$. When (M^n, g) is locally conformally flat, then, for $1 \leq k \leq n$, $\text{div}T_k = 0$. For detail, see [2, 22] and the references therein.

Proof of Theorem 1.4. Taking $T = T_k$ and $B = (n-k)\sigma_k(\mathcal{S})$ in Theorem 1.2, then we complete the proof of Theorem 1.4.

Finally, it is worth noticing that Theorem 1.2 has been stated in a quite general form, that can be adapted to analyse different versions of almost-Schur type inequalities for different interesting geometric tensors related, such as Lovelock curvatures and Q-curvatures, which have been in the center of plenty of resent research in geometric analysis.

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Conflict of interest

The author declares no conflict of interest.

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