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Research article

Linear instability of periodic orbits of free period Lagrangian systems

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Abstract: Minimizing closed geodesics on surfaces are linearly unstable. By starting with this classical Poincaré's instability result, in the present paper we prove a result that allows to deduce the linear instability of periodic solutions of autonomous Lagrangian systems admitting an orbit cylinder (condition which is satisfied for instance if the periodic orbit is transversally non-degenerate) in terms of the parity properties of a suitable quantity which is obtained by adding the dimension of the configuration space to a suitably defined spectral index. Such a spectral index coincides with the Morse index of the periodic orbit seen as a critical point of the free period action functional in the case the Lagrangian is Tonelli, namely fibrewise strictly convex and superlinear, and it encodes the functional and symplectic properties of the problem.
The main result of the paper is a generalization of the celebrated Poincaré 's instability result for closed geodesics on surfaces and at the same time extends to the autonomous case several previous results which have been proved by the authors (as well as by others) in the case of non-autonomous Lagrangian systems.

Keywords: periodic orbits; free period Lagrangian systems; linear instability; Maslov index; spectral flow

1. Introduction and description of the problem

A celebrated result proved by Poincaré at the beginning of the last century puts on evidence the relation between the (linear and exponential) instability of an orientation preserving closed geodesic as a critical point of the geodesic energy functional on the free loop space of a surface and the minimization property of such a critical point. The literature on this criterion is quite broad. We refer the interested reader to [1–3] and references therein.

A quite natural question is related to understand the role played by the energy level h on the instability properties of a periodic orbit of an autonomous Lagrangian system. Some years ago, in his expository article [4], Abbondandolo studied the question of the existence of periodic orbits of the free period Lagrangian systems in terms of the Mañé critical values. It is worth noticing that the fixed energy problem for Tonelli Lagrangians as well as for the restricted class of magnetic systems has been intensively studied by many outstanding mathematicians such as Arnold, Novikov, Ginzburg, Taimanov, Contreras, Paternain, etc. by starting on the work of Arnol'd [5] in the 1960s. If on the one hand it is impossible to keep track of all the literature on the topic, some breakthrough papers in the field are [6] and [7].

Very recently, Ureña in [8], studied the instability of closed orbits obtained by minimization, for an autonomous Lagrangian system by combining the classical principle of Jacobi-Maupertuis principle together with a nice reduction argument firstly introduced by Carathéodory. In this way, under some suitable conditions, the dynamics of a free period Lagrangian system can be seen as the dynamics of a non-autonomous and fixed period Lagrangian system lowering by 1 the degrees of freedom.

1.1. Basic definitions and main result

Let $(M, \langle\cdot,\cdot\rangle_g)$ be a smooth n-dimensional Riemannian manifold without boundary, which represents the configuration space of a Lagrangian dynamical system. Elements in the tangent bundle TM will be denoted by (q, v), with $q \in M$ and $v \in T_qM$. Let $L : TM \to \mathbb{R}$ be a smooth autonomous (Lagrangian) function satisfying the following assumptions

- **(N1)** L is non-degenerate on the fibers of TM, that is, for every $(q, v) \in TM$ we have that $d_{vv}L(q, v) \neq 0$ is non-degenerate as a quadratic form, where $d_{vv}L$ denotes the fiberwise second differential of L
- **(N2)** L is *exactly quadratic* in the velocities meaning that the function $L(q, v)$ is a polynomial of degree at most 2 with respect to v.

Remark 1.1. Before stating our main result, we observe that we require that the Lagrangian function is exactly quadratic in the velocity assumption (N2). The smoothness property of L are in general not sufficient for guaranteeing the twice Fréchét differentiability of the Lagrangian action functional. In fact, the functional is twice Fréchét differentiable if and only if L is exactly quadratic in the velocities. In this case, the Lagrangian action functional is actually smooth. This fact depends upon the differentiability properties of Nemitsky operators. We observe that, by using a finite dimensional approximations for the free-period action functional as developed by authors in [7], it should be possible to remove the technical condition (N2).

On the cartesian product $\Lambda^1(M) \times \mathbb{R}^+$, where $\Lambda^1(M)$ denotes the Hilbert manifold of 1-periodic loops on M having Sobolev regularity H^1, we define the free period Lagrangian action functional as

$$\mathbb{E}_h(x, T) := T \int_0^1 \left[L\left(x(s), \frac{x'(s)}{T}\right) + h \right] ds,$$

where h is a real constant playing the role of energy. In fact, since the system is autonomous, the energy is a first integral. By a direct computation of the first variation of the action functional $\mathbb{E}_h$, it follows that (x, T) is a critical point of $\mathbb{E}_h$ if and only if it satisfies the Euler-Lagrangian equations having fixed energy h.

Definition 1.2. Let (x, T) be a critical point of $\mathbb{E}_h$. We term (x, T) *non-null* if

$$\langle d_{vv}L(x(t), x'(t))x'(t), x'(t)\rangle_g \neq 0 \quad \text{for every } t \in [0, 1].$$

Moreover, (x, T) is termed

- *L-Positive* if $\langle d_{vv}L(x(t), x'(t))x'(t), x'(t)\rangle_g > 0$;
- *L-Negative* if $\langle d_{vv}L(x(t), x'(t))x'(t), x'(t)\rangle_g < 0$.

We observe that the notion of L-positivity (resp. L-negativity) provides a sort of generalization of the Legendre convexity (resp. concavity) condition only along the selected orbit. Recall that the classical *Legendre convexity* condition reads

(L1) L is $\mathscr{C}^2$ strictly convex on the fibers of TM, that is, for every $(q, v) \in TM$ we get $d_{vv}L(q, v) > 0$ as a quadratic form.

It is worth to observe that the above conditions have been introduced for avoiding any sign changing of the quanity

$$\langle d_{vv}L(x(t_0), x'(t_0))x'(t_0), x'(t_0)\rangle_g.$$

Otherwise, this provides deep difficulties, like in the definition of the spectral index.

Following authors, in [9, Definition 1.2], we are ready to introduce the notion of orbit cylinder.

Definition 1.3. A critical point (x, T) of $\mathbb{E}_h$ admits an *orbit cylinder* if there exist $\epsilon > 0$ and a smooth (in s) family critical points $\{(x_{h+s}, T_{h+s}), s \in (-\epsilon, \epsilon)\}$ of $\mathbb{E}_{h+s}$ such that $(x_h, T_h) = (x, T)$. Moreover, this orbit cylinder is called non-degenerate if $T'(h) \neq 0$.

Under the above notation our main result reads as follows.

Theorem 1.4. *Let (x, T) be a non-null critical point of the free-period Lagrangian action $\mathbb{E}_h$ admitting an orbit cylinder. If one of the following four statements holds:*

(1) x is **L − Positive** *and*

(OR) *x is orientation preserving and $\iota_{\mathrm{spec}}(x, T) + n$ is even*
(NOR) *x is orientation reversing and $\iota_{\mathrm{spec}}(x, T) + n$ is odd*

(2) x is **L − Negative** *and*

(OR) *x is orientation preserving and $\iota_{\mathrm{spec}}(x, T) + n$ is odd*
(NOR) *x is orientation reversing and $\iota_{\mathrm{spec}}(x, T) + n$ is even*

then x is linearly unstable.

Under the classical Legendre convexity condition, it is possible to prove that the spectral index reduces to the classical Morse index. So, we get the following.

Corollary 1. *Let us assume that (x, T) is a critical point of free period Lagrangian action $\mathbb{E}_h$ admitting an orbit cylinder and we assume that (L1) holds. If one of the following two alternatives holds*

(OR) *x is orientation preserving and $\iota_{\mathrm{Mor}}(x, T) + n$ is even*
(NOR) *x is orientation reversing and $\iota_{\mathrm{Mor}}(x, T) + n$ is odd*

then x is linearly unstable.

From a tecnical viewpoint a key step for proving our main result is based on a spectral flow formula relating the Morse index of a periodic orbit as critical point of the free and fixed time Lagrangian action functional. In [9], authors actually established such a relation under a non-degenerate assumption of the orbit cylider. As already observed, here we give a generalization of the aforementioned results provided in [9] by dropping the non-degeneracy assumption of the orbit cylinder. Such a generalization is essentially based on a quite new results constructed by the second named author in [10].

A crucial intermediate step for proving the main result of this paper is based on a spectral flow formula relating the spectral index to a symplectic invariant known in literature as Maslov index (which is an intersection invariant constructed in the Lagrangian Grassmannian manifold of a symplectic space) that plays a crucial role in detecting the stability properties of a periodic orbit. (For an index formula, we refer the interested reader to [11–14] and references therein). Very recently, new spectral flow formulas have been established and applied for detecting bifurcation of heteroclinic and homoclinic orbits of Hamiltonian systems or bifurcation of semi-Riemannian geodesics. (Cfr. [15–17]).

Notation

For the sake of the reader, let us introduce some common notation that we'll use henceforth throughout the paper.

- $(M, \langle\cdot,\cdot\rangle_g)$ denotes a Riemannian manifold without boundary, TM its tangent bundle and T^*M its cotangent bundle.
- $\Lambda^1(M)$ is the Hilbert manifold of loops on manifold M having Sobolev regularity H^1.
- ω denotes the symplectic structure J the standard symplectic matrix such that $\omega(\cdot,\cdot) = \langle J\cdot,\cdot\rangle$ where $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean product.
- $\iota_{\text{Mor}}(x)$ stands for the Morse index of x, $\iota_{\text{spec}}(x)$ for the spectral index of x, $\iota_{\text{geo}}(x)$ for the geometrical index of x, ι_1 denotes the Maslov-type index or Conley-Zehnder index of a symplectic matrix path, ι_{CLM} denotes the Maslov (intersection) index and finally sf denotes the spectral flow.
- $P(L)$ denotes the set of T-periodic solutions of the Euler-Lagrange Equation, $P(H)$ the set of T-periodic solutions of the Hamiltonian Equation.
- δ_{ij} is the Kronecker symbol. I_X or just I will denote the identity operator on a space X and we set for simplicity $I_k := I_{\mathbb{R}^k}$ for $k \in \mathbb{N}$. $\mathrm{Gr}(\cdot)$ denotes the graph of its argument, Δ denotes the graph of identity matrix I.
- $\mathbb{U}$ is the unit circle of the complex plane.
- $\mathrm{O}(n)$ denotes the orthogonal group, $\mathrm{Sp}(2n,\mathbb{R})$ or just $\mathrm{Sp}(2n)$ denotes the $2n \times 2n$ real symplectic group.
- $\mathfrak{P}$ denotes the linearized Poincaré map.
- $\mathcal{BF}^{sa}$ denotes the set of all bounded selfadjoint Fredholm operators, $\sigma(\cdot)$ denotes the spectrum of the operator in its argument.
- We denote throughout by the symbol $\cdot^{\mathsf{T}}$ (resp. $\cdot^{-\mathsf{T}}$) the transpose (resp. inverse transpose) of the operator $\cdot$. Moreover $\mathrm{rge}\,(\cdot), \ker(\cdot)$ and $\mathrm{rank}(\cdot)$ denote respectively the image, the kernel and the rank of the argument.
- Γ denotes the crossing form and $\mathrm{n}_+/\mathrm{n}_-$ denote respectively the dimensions of the positive/negative spectral spaces and finally $\mathrm{sgn}(\cdot)$ is the signature of the quadratic form (or matrix) in its argument

and it is given by $\mathrm{sgn}(\cdot) = \mathrm{n}_+(\cdot) - \mathrm{n}_-(\cdot)$.

2. Lagrangian dynamics and variational framework

In this preliminary section we fix our basic notation and we explicitly provided the computation of the first and second variation of the free period Lagrangian action functional.

2.1. Free-period Lagrangian action

Let $(M, \langle \cdot, \cdot \rangle_g)$ be a (not necessarily compact or connected) smooth n-dimensional Riemannian manifold without boundary, which represents the configuration space of a Lagrangian dynamical system and we denote by $\|\cdot\|$ the Riemannian norm. Elements in the tangent bundle TM will be denoted by (q, v), with $q \in M$ and $v \in T_qM$. The metric $\langle \cdot, \cdot \rangle_g$ induces: a metric on TM, Levi-Civita connections both on M and TM and finally the isomorphisms

$$T_{(q,v)}TM = T^h_{(q,v)}TM \oplus T^v_{(q,v)}TM \cong T_qM \oplus T_qM,$$

for $T^v_{(q,v)}TM = \ker D\tau(q, v)$ where $\tau : TM \to M$ denotes the canonical tangent projection.

Notation 2.1. We shall denote by ∇_t the covariant derivative of a vector field along a smooth curve x with respect to the metric $\langle \cdot, \cdot \rangle_g$. ∂_q (resp. ∂_v) denotes the partial derivative along the horizontal part (resp. vertical part) given by the Levi-Civita connection in the splitting of TTM and We shall denote by $\partial_{vv}, \partial_{qv}, \partial_{qq}$ the components of the Hessian in the splitting of TTM.

Given a positive number $T \in \mathbb{R}$, we denote by $\mathbb{T}$ the one-dimensional torus $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$. Let $\Lambda^1_T(M)$ be the Hilbert manifold of all loops $y : \mathbb{T} \to M$ having Sobolev class H^1. Setting $x(t) = y(tT), t \in [0, 1]$ we get that the closed curve y will be identified with the pair (x, T). The action of y on time interval $[0, T]$ is given by

$$\int_0^T [L(y(s), y'(s)) + h]\, ds = T \int_0^1 [L(x(t), x'(t)/T) + h]\, dt,$$

where $h \in \mathbb{R}$ is a parameter. For convenience, we denote

$$\mathbb{E}_h(x, T) := T \int_0^1 [L(x(t), x'(t)/T) + h]\, dt. \tag{2.1}$$

In short-hand notation, in what follows we use $\Lambda^1(M)$ instead of $\Lambda^1_1(M)$. In this way, we define a one-to-one correspondente between

$$\bigcup_{T>0} \Lambda^1_T(M) \quad \text{and} \quad \Lambda^1(M) \times \mathbb{R}^+$$

which preserves the action values. Bearing in mind such a correspondence, the free period action functional (2.1) is defined on the manifold $\Lambda^1(M) \times \mathbb{R}^+$.

It is well-known that the tangent space $T_x\Lambda^1(M)$ can be identified in a natural way with the Hilbert space of 1-periodic H^1 (tangent) vector fields along x, i.e.,

$$\mathcal{H}(x) = \left\{ \xi \in H^1(\mathbb{R}/\mathbb{Z}, TM) \,\middle|\, \tau \circ \xi = x \right\}.$$

It is worth noticing that under the (N2) assumption, the Lagrangian functional is of regularity class $\mathscr{C}^2$ (actually it is smooth). Let $\langle\langle\cdot,\cdot\rangle\rangle_1$ denote the Riemannian metric on $\Lambda^1(M)$ defined by

$$\langle\langle\xi,\eta\rangle\rangle_1 := \int_0^1 \left[\langle\nabla_t\xi,\nabla_t\eta\rangle_g + \langle\xi,\eta\rangle_g\right] dt, \qquad \forall\,\xi,\eta\in\mathcal{H}(x).$$

For $(\xi,b)\in\mathcal{H}(x)\times\mathbb{R}$, the first variation of $\mathbb{E}_h$ at $(x,T)\in\Lambda^1(M)\times\mathbb{R}^+$ is given by

$$d\mathbb{E}_h(x,T)[(\xi,0)] = \int_0^1\left[\left\langle T\partial_q L(x(t),x'(t)/T) - \frac{D}{dt}\partial_v L(x(t),x'(t)/T),\xi\right\rangle_g\right] dt + \langle\partial_v L(x(t),x'(t)/T),\xi\rangle_g\Big|_0^1 \quad (2.2)$$

and for $(0,b)\in\mathcal{H}(x)\times\mathbb{R}$, it reduces to

$$d\mathbb{E}_h(x,T)[(0,b)] = \int_0^1\Big[(L(x(t),x'(t)/T) + h - \langle\partial_v L(x(t),x'(t)/T),x'(t)/T\rangle_g)\cdot b\Big]dt. \quad (2.3)$$

By Eqs (2.2) and (2.3), up to standard regularity arguments, it follows that critical points (x,T) of $\mathbb{E}_h$ are 1-periodic solutions of corresponding Euler-Lagrange equation having energy h. So, (x,T) satisfies the following equations:

$$\begin{cases}\dfrac{D}{dt}\Big(\partial_v L(x(t),x'(t)/T)\Big) = T\partial_q L(x(t),x'(t)/T), & t\in(0,1)\\ L(x(t),x'(t)/T) + h - \langle\partial_v L(x(t),x'(t)/T),x'(t)/T\rangle_g = 0\end{cases}$$

Now, being $\mathbb{E}_h$ smooth it follows that the first variation $d\mathbb{E}_h(x,T)$ at (x,T) coincides with the Fréchét differential $D\mathbb{E}_h(x,T)$ and the second variation of $\mathbb{E}_h$ at (x,T) coincides with $D^2\mathbb{E}_h(x,T)$. We set

$$\bar P(t) := \partial_{vv}L(x(t),x'(t)/T), \qquad \bar Q(t) := \partial_{qv}L(x(t),x'(t)/T), \qquad \bar Q^{\mathsf T}(t) := \partial_{vq}L(x(t),x'(t)/T)$$
$$\bar R(t) := \partial_{qq}L(x(t),x'(t)/T), \qquad \bar L(t) = \partial_q L(x(t),x'(t)/T), \qquad \kappa(t) = \langle\bar P(t)x'(t),x'(t)\rangle_g$$

so, we get

$$\begin{aligned} d^2\mathbb{E}_h[(\xi,b),(\eta,d)] = {} & \\ = \int_0^1 & \langle -\frac{D}{dt}\left[\frac{1}{T}\bar P(t)\nabla_t\xi + \bar Q(t)\xi\right] + \bar Q^{\mathsf T}(t)\nabla_t\xi + T\bar R(t)\xi,\eta\rangle_g dt + \left[\left\langle\frac{1}{T}\bar P(t)\nabla_t\xi + \bar Q(t)\xi,\eta\right\rangle_g\right]_{t=0}^1 \\ & + \int_0^1\left\{-\frac{1}{T^2}\left\langle\bar P(t)x'(t),\nabla_t\eta\right\rangle_g\cdot b - \frac{1}{T^2}\left\langle\bar P(t)x'(t),\nabla_t\xi\right\rangle_g\cdot d + \left\langle\bar L(t) - \frac{1}{T}\bar Q(t)x'(t),\xi\right\rangle_g\cdot d\right. \\ & \left. + \left\langle\bar L(t) - \frac{1}{T}\bar Q^{\mathsf T}(t)x'(t),\eta\right\rangle_g\cdot b + \frac{1}{T^3}\kappa(t)\cdot bd\right\} dt. \end{aligned}$$

Remark 2.2. For a complete details on the first and second variations, we refer the interested reader to [18] and references therein.

For a given critical point (x, T) of E_h, let us consider the following fixed period Lagrangian action functional:

$$\mathbb{E}_h^T(x) := T \int_0^1 [L(x(t), x'(t)/T) + h]\, dt, \tag{2.4}$$

where $x \in \Lambda^1(M)$. Actually, $\mathbb{E}_h^T$ is the restriction of $\mathbb{E}_h$ to the submanifold $\Lambda^1(M) \times \{T\}$. By similar calculations, the first variation of $\mathbb{E}_h^T$ at $x \in \Lambda^1(M)$ is given by

$$d\mathbb{E}_h^T(x)[\xi] = \int_0^1 \left[\left\langle T\partial_q L(x(t), x'(t)/T) - \frac{D}{dt}\partial_v L(x(t), x'(t)/T), \xi\right\rangle_g\right] dt + \langle \partial_v L(x(t), x'(t)/T), \xi\rangle_g\Big|_0^1.$$

Critical points x of $\mathbb{E}_h^T$ are 1-periodic solutions of corresponding Euler-Lagrange Equation:

$$\frac{D}{dt}\Big(\partial_v L(x(t), x'(t)/T)\Big) = T\partial_q L(x(t), x'(t)/T), \qquad t \in (0, 1). \tag{2.5}$$

The second variation of $\mathbb{E}_h^T$ at x is given by

$$d^2\mathbb{E}_h^T(x)[\xi, \eta] = \int_0^1 \langle -\frac{D}{dt}[\frac{1}{T}\bar{P}(t)\nabla_t\xi + \bar{Q}(t)\xi] + \bar{Q}^{\mathsf{T}}(t)\nabla_t\xi + T\bar{R}(t)\xi, \eta\rangle_g dt + \left[\langle \frac{1}{T}\bar{P}(t)\nabla_t\xi + \bar{Q}(t)\xi, \eta\rangle_g\right]_0^1.$$

By the previous computation, we get that $\xi \in \ker d^2\mathbb{E}_h^T(x)$ if and only if ξ is a H^2 vector field along x which solves weakly (in the Sobolev sense) the following boundary value problem

$$\begin{cases} -\frac{D}{dt}(\frac{1}{T}\bar{P}(t)\nabla_t\xi + \bar{Q}(t)\xi) + \bar{Q}^{\mathsf{T}}(t)\nabla_t\xi + T\bar{R}(t)\xi = 0, & t \in (0, 1) \\ \xi(0) = \xi(1), \qquad \nabla_t\xi(0) = \nabla_t\xi(1). \end{cases} \tag{2.6}$$

By standard bootstrap arguments, it follows that ξ is also a classical (smooth) solution of Eq (2.6).

Remark 2.3. It is easy to prove that x is a critical point of fixed period Lagrangian system (2.4) provided that (x, T) is a critical point of free period Lagrangian system.

The next result provides an answer to the existence of an orbit cylinder about a T-periodic orbit x. We refer the interested reader to [19, Section 4.1, Proposition 2] for the proof.

Proposition 2.4. *Let us assume that a periodic solution $x(t, E^*)$ of a Hamiltonian vector field X_H on M having energy $E^* = H(x(t, E^*))$ and period T^* has exactly two Floquet multipliers equal* 1. *Then, there exists a unique and smooth one-parameter family $x(t, E)$ of periodic solutions with periods $T(E)$ close to T^* and lying on the energy surfaces $H(x(t, E)) = E$ for $|E - E^*|$ sufficiently small. Moreover, $T(E)$ converges to $T(E^*)$ for $E \to E^*$.*

3. Spectral index, geometrical index and Poincaré map

The goal of this section is to associate at the critical point (x, T) of the free period Lagrangian action given at Eq (2.1) and to a critical point x of the fixed period Lagrangian action given at Eq (2.4) the *spectral index*, defined in terms of the spectral flow of a suitable path of Fredholm quadratic forms. We refer the interested reader to [20, Appendix B] and references therein for a discussion about the spectral flow.

3.1. Spectral index: an intrinsic (coordinate free) definition

Given (x, T) be a critical point of $\mathbb{E}_h$, for any $s \in [0, +\infty)$ we let $\mathcal{I}_s : (\mathcal{H}(x) \times \mathbb{R}) \times (\mathcal{H}(x) \times \mathbb{R}) \to \mathbb{R}$ the bilinear form defined by

$$\mathcal{I}_s[(\xi, b), (\eta, d)] := d^2\mathbb{E}_h(x, T)[(\xi, b), (\eta, d)] + s\alpha(x, T)[(\xi, b), (\eta, d)]$$
$$\text{where } \alpha(x, T)[(\xi, b), (\eta, d)] := \int_0^1 \{\langle \frac{1}{T}\bar{P}(t)\xi, \eta\rangle_g + \frac{1}{T^3}\kappa(t)bd\}\, dt. \quad (3.1)$$

Notation 3.1. In short-hand notation and if no confusion can arise, we set $\mathcal{Q}^h := d^2\mathbb{E}_h(x, T)$.

Proposition 3.2. *For any $s \in [0, +\infty)$ let $\mathcal{Q}_s$ denote the quadratic form associated to $\mathcal{I}_s$ defined at Eq* (3.1). *Then*

> *$s \mapsto \mathcal{Q}_s$ is a smooth path of Fredholm quadratic forms onto $\mathcal{H}(x) \times \mathbb{R}$. In particular $\mathcal{Q}^h$ is a Fredholm quadratic form on $\mathcal{H}(x) \times \mathbb{R}$.*

Proof. The proof is completely analogous to [20, Proposition 3.2]. □

We let (x, T) be a critical point of the Lagrangian action functional (2.1). Then we can define the bilinear form $\mathcal{I}_{s,T} : \mathcal{H}(x) \times \mathcal{H}(x) \to \mathbb{R}$ of x as a critical point of system (2.4) in the same way as (x, T). In fact, $\mathcal{I}_{s,T}$ is given by

$$\mathcal{I}_{s,T}[\xi, \eta] := d^2\mathbb{E}_h^T(x)[\xi, \eta] + s\alpha_T(x)[\xi, \eta] \text{ where } \alpha_T(x)[\xi, \eta] := \int_0^1 \left\langle \frac{1}{T}\bar{P}(t)\xi, \eta \right\rangle_g dt.$$

Set $\mathcal{Q}_T^h := d^2\mathbb{E}_h^T(x)$. Arguing precisely as done in Proposition 3.2, it can be proved $\mathcal{I}_{s,T}$ and $\mathcal{Q}_T^h$ both are Fredholm quadratic forms.

For any $s \in [0, +\infty)$, let $\mathcal{Q}_{s,T}$ denote the quadratic form associated to $\mathcal{I}_{s,T}$. Now we are entitled to define the following *spectral indexes.*

Definition 3.3. Let (x, T) be a non-null critical point of the free period Lagrangian action given at Eq (2.1). We term *spectral indices* of (x, T) and x are respectively the integers given by

$$\iota_{\text{spec}}(x, T) := \text{sf}\,(\mathcal{Q}_s, s \in [0, s_0]) \quad \text{and} \quad \iota_{\text{spec}}^T(x) := \text{sf}\,(\mathcal{Q}_{s,T}, s \in [0, s_0]).$$

where the (RHS) denotes the spectral flow of the path of Fredholm quadratic forms defined on the interval $[0, s_0]$ for a sufficiently arge $s_0 > 0$.

Remark 3.4. It is easy to show that Definition 3.3 is well-posed meaning that the spectral indices are independent on s_0. This fact will be proved in the sequel and it is actually a direct consequence of Lemma 3.6.

Proposition 3.5. *We assume that (L1) holds. Then the Morse indices of* (x, T) *and* x *(i.e., the dimension of the maximal negative subspace of the Hessian of* $\mathcal{Q}^h$ *and* $\mathcal{Q}^{h_T}$*) are both finite and the following equalities hold*

$$\iota_{\text{spec}}(x, T) = \iota_{\text{Mor}}(x, T) \quad \text{and} \quad \iota^T_{\text{spec}}(x) = \iota_{\text{Mor}}(x)$$

Proof. We only consider the free period Lagrangian action, being the fixed period case, completely analogous. So, we start by observing that if L is $\mathscr{C}^2$-strictly convex on TM, then $\alpha(x, T)$ is a positive Fredholm quadratic form and hence $s \mapsto \mathcal{Q}_s$ is a path of essentially positive Fredholm quadratic forms being realized by a path of compact symmetric perturbation of a positive definite Fredholm operator. In particular the Morse index of $\mathcal{Q}_s$ is finite for every $s \in [0, +\infty)$. If s_0 is sufficiently large the form $\mathcal{Q}_{s_0}$ is non-degenerate and positive definite being the quadratic form associated to $\alpha(x, T)$ be Fredholm and positive definite. In particular, its Morse index vanishes. Since the spectral flow for path of essentially positive Fredholm quadratic forms is equal to the difference of the Morse indices at the ends (cf. [20, Appendix B]), we get that

$$\text{sf}(\mathcal{Q}_s, s \in [0, s_0]) = \iota_{\text{Mor}}(\mathcal{Q}_0) - \iota_{\text{Mor}}(\mathcal{Q}_{s_0}) = \iota_{\text{Mor}}(\mathcal{Q}_0).$$

This concludes the proof. □

3.2. Pull-back bundles and push-forward of Fredholm forms

We denote by $\mathbf{E}$ the $\langle\cdot,\cdot\rangle_g$-orthonormal and parallel frame pointwise given by

$$\mathcal{E}(t) = \{e_1(t), \dots, e_n(t)\}.$$

Given a critical point (x, T) of the free period Lagrangian action, we let $\bar{A} : T_{x(0)}M \to T_{x(1)}M \cong T_{x(0)}M$ the $\langle\cdot,\cdot\rangle_g$-orthogonal operator defined by

$$\bar{A}e_j(0) = e_j(1).$$

Such a frame $\mathbf{E}$, induces a trivialization of the pull-back bundle $x^*(TM)$ over $[0, 1]$ through the smooth curve $x : [0, 1] \to M$; namely the smooth one parameter family of isomorphisms

$$[0,1] \ni t \longmapsto E_t \quad \text{where} \quad E_t : \mathbb{R}^n \ni e_i \longmapsto e_i(t) \in T_{x(t)}M \qquad \forall\, t \in [0, 1] \text{ and } i = 1, \dots, n$$
$$\text{are such that } \langle E_t e_i, E_t e_j\rangle_g = \delta_{ij} \text{ and } \nabla_t E_t e_i = 0 \tag{3.3}$$

here $\{e_i\}_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$ and δ_{ij} denotes the Kronecker symbol.

By Eq (3.3) we get that the pull-back by E_t of the metric $\langle\cdot,\cdot\rangle_g$ induces the Euclidean product on $\mathbb{R}^n$ and moreover this pull-back is independent on t, as directly follows by the orthogonality assumption on the frame $\mathbf{E}$.

We set $A := E_0^{-1}\bar{A}^{-1}E_1 \in \mathrm{O}(n)$ and define

$$A_d := \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}. \tag{3.4}$$

Let us now consider the Hilbert space

$$H^1_A([0,1],\mathbb{R}^n) = \left\{ u \in H^1([0,1],\mathbb{R}^n) \;\middle|\; u(0) = Au(1) \right\}$$

equipped with the inner product

$$\langle\langle v, w\rangle\rangle_A := \int_0^1 \left[\langle v'(s), Aw'(s)\rangle + \langle v(s), Aw(s)\rangle\right] ds.$$

Denoting by $\Psi : \mathcal{H}(x) \to H^1_A([0,1],\mathbb{R}^n)$ the map defined by $\Psi(\xi) = u$ where $u(t) = E_t^{-1}(\xi(t))$, it follows that Ψ is a linear isomorphism and it is easy to check that

$$\xi(0) = \xi(1) \quad\iff\quad E_0u(0) = E_1u(1) \quad\iff\quad u(0) = Au(1) \text{ and}$$
$$\nabla_t\xi(0) = \nabla_t\xi(1) \quad\iff\quad u'(0) = Au'(1)$$

where, in the last, we used property of the frame being parallel.

For $i = 1,\dots,n$ and $t \in [0,1]$, we let $e_i(t) := E_te_i$ and we denote by $\langle P(t)\cdot,\cdot\rangle$, $\langle Q(t)\cdot,\cdot\rangle$ and $\langle R(t)\cdot,\cdot\rangle$ respectively the pull-back by E_t of $\langle \bar{P}(t)\cdot,\cdot\rangle_g$, $\langle \bar{Q}(t)\cdot,\cdot\rangle_g$ and $\langle \bar{R}(t)\cdot,\cdot\rangle_g$ and y L the pull-back of $\bar{L}$ by E_t. Thus, we get

$$P(t) := [p_{ij}(t)]^n_{i,j=0}, \quad Q(t) := [q^{ij}(t)]^n_{i,j=0}, \quad R(t) := [r_{ij}(t)]^n_{i,j=0} \quad \text{where}$$
$$p_{ij}(t) := \langle \bar{P}(t)e_i(t), e_j(t)\rangle_g, \quad q_{ij}(t) := \langle \bar{Q}(t)e_i(t), e_j(t)\rangle_g, \quad r_{ij}(t) = \langle \bar{R}(t)e_i(t), e_j(t)\rangle_g.$$

We observe that P and R are symmetric matrices and being $e_i(T) = \sum_{j=1}^n a_{ij}e_j(0)$ we infer that

$$P(0) = AP(T)A^\mathsf{T}, \qquad P'(0) = AP'(T)A^\mathsf{T}, \qquad Q(0) = AQ(T), \qquad R(0) = AR(T)A^\mathsf{T}. \tag{3.5}$$

Now, for every $s \in [0,+\infty)$, the push-forward by Ψ of the index forms $\mathcal{I}_s$ on $\mathcal{H}(x)\times\mathbb{R}$ is given by the symmetric bilinear forms on $H^1_A([0,1],\mathbb{R}^n)\times\mathbb{R}$ defined by

$$\begin{aligned}
&I_s[(u,b),(v,d)]\\
&\quad= \int_0^1 \left\{\left\langle \frac{1}{T}P(t)u'(t), v'(t)\right\rangle + \langle Q(t)u(t), v'(t)\rangle + \left\langle Q^\mathsf{T}(t)u'(t), v(t)\right\rangle + \langle TR(t)u(t), v(t)\rangle\right\} dt\\
&\quad+ \int_0^1 \left\{-\frac{1}{T^2}\langle P(t)x'(t), v'(t)\rangle\cdot b - \frac{1}{T^2}\langle P(t)x'(t), u'(t)\rangle\cdot d + \left\langle L(t) - \frac{1}{T}Q(t)x'(t), u(t)\right\rangle\cdot d\right.\\
&\qquad\left. + \left\langle L(t) - \frac{1}{T}Q^\mathsf{T}(t)x'(t), v(t)\right\rangle\cdot b + \frac{1}{T^3}\kappa(t)\cdot bd\right\} dt + s\alpha(x)[(\xi,b),(\eta,d)]\\
&\qquad\qquad \text{where } \alpha(x,T)[(u,b),(v,d)] := \int_0^1 \left\{\left\langle \frac{1}{T}P(t)u(t), v(t)\right\rangle + \frac{1}{T^3}\kappa(t)bd\right\} dt.
\end{aligned} \tag{3.6}$$

Denoting by q^A_s the quadratic form on $H^1_A([0,1],\mathbb{R}^n)\times\mathbb{R}$ associated to I_s then, as direct consequence of Proposition 3.2, we get that for every $s \in [0,+\infty)$, the quadratic form q^A_s is Fredholm on $H^1_A([0,1],\mathbb{R}^n)\times\mathbb{R}$. The following result is crucial in the well-posedness of the spectral index.

Lemma 3.6. *Under the above notation, there exists $s_0 \in [0,+\infty)$ large enough such that for every $s \geq s_0$, the form I_s given in Eq (3.6) is non-degenerate (in the sense of bilinear forms).*

Proof. We argue by contradiction and we assume that for every $s_0 \geq 0$ there exists $s \geq s_0$ such that I_s is degenerate. Then there exists a $(u, b) \in H^1_A([0, 1], \mathbb{R}^n) \times \mathbb{R}$ such that $I_s((u, b), (v, d)) \equiv 0$ for every $(v, d) \in H^1_A([0, 1], \mathbb{R}^n) \times \mathbb{R}$, namely we have

$$\begin{aligned}
0 &\equiv I_s((u, b), (v, d)) \\
&= \int_0^1 \left\{ \left\langle \frac{1}{T} P(t)u'(t), v'(t) \right\rangle + \langle Q(t)u(t), v'(t)\rangle + \left\langle Q^\top(t)u'(t), v(t)\right\rangle + \langle TR(t)u(t), v(t)\rangle \right\} dt \\
&+ \int_0^1 \left\{ -\frac{1}{T^2} \langle P(t)x'(t), v'(t)\rangle \cdot b - \frac{1}{T^2} \langle P(t)x'(t), u'(t)\rangle \cdot d + \left\langle L(t) - \frac{1}{T} Q(t)x'(t), u(t)\right\rangle \cdot d \right. \\
&\qquad \left. + \left\langle L(t) - \frac{1}{T} Q^\top(t)x'(t), v(t)\right\rangle \cdot b + \frac{1}{T^3}\kappa(t) \cdot bd \right\} dt \\
&\qquad + s \int_0^1 \left\{ \left\langle \frac{1}{T} P(t)u(t), v(t)\right\rangle + \frac{1}{T^3}\kappa(t) bd \right\} dt. \qquad (3.7)
\end{aligned}$$

We let $v(t) := \frac{1}{T} P(t)u(t)$ and we observe that as direct consequence of Eq (3.5), the function v is admissible (meaning that v belongs to H^1_A). Since (x, T) is non-null, we set $d := \operatorname{sign}\kappa(t) \cdot b$ for $\operatorname{sign}\kappa(t) \in \{1, -1\}$.

By replacing v and d into Eq (3.7) respectively by $v = Pu/T$ and $d = \operatorname{sign}\kappa(t) \cdot b$ and dropping the argument t in each function, we get

$$\begin{aligned}
0 &\equiv I_s((u, b), (v, d) \\
&= \int_0^1 \left\{ \left\langle \frac{1}{T} Pu', \frac{1}{T}(P'u + Pu')\right\rangle + \left\langle Qu, \frac{1}{T}(P'u + Pu')\right\rangle + \left\langle Q^\top u', \frac{1}{T} Pu\right\rangle + \left\langle TRu, \frac{1}{T} Pu\right\rangle \right\} dt \\
&+ \int_0^1 \left\{ -\frac{1}{T^2} \left\langle Px', \frac{1}{T}(P'u + Pu')\right\rangle \cdot b - \frac{1}{T^2} \langle Px', u'\rangle \cdot \operatorname{sign}\kappa(t) \cdot b + \left\langle L - \frac{1}{T} Qx', u\right\rangle \cdot \operatorname{sign}\kappa(t) \cdot b \right. \\
&\qquad \left. + \left\langle L - \frac{1}{T} Q^\top x', \frac{1}{T} Pu\right\rangle \cdot b + \frac{1}{T^3}\kappa \cdot b \cdot \operatorname{sign}\kappa(t) \cdot b \right\} dt \\
&\qquad + s \int_0^1 \left\{ \left\langle \frac{1}{T} Pu, \frac{1}{T} Pu\right\rangle + \frac{1}{T^3}\kappa b \cdot \operatorname{sign}\kappa(t) \cdot b \right\} dt. \qquad (3.8)
\end{aligned}$$

For $i = 1, 2, 3, 4$, we let C_i be given as follows:

$$\begin{aligned}
C_1 &= \frac{1}{T^2}\|P'P^{-1}\| + \frac{1}{T}\|QP^{-1}\| + \frac{1}{T}\|Q^\top P^{-1}\|; \quad C_2 = \frac{1}{T}\|QP^{-1}\|\|P'P^{-1}\| + \|RP^{-1}\|; \\
C_3 &= \frac{1}{T^3}\|Px'\| + \frac{1}{T^2}\|Px'\|\|P^{-1}\| + \frac{1}{T}\|L - \frac{1}{T} Q^\top x'\|; \\
C_4 &= \frac{1}{T^3}\|Px'\|\|PP^{-1}\| + \|L - \frac{1}{T} Qx'\|\|P^{-1}\|.
\end{aligned}$$

and for for s sufficiently large we get

$$\begin{aligned}
0 &\equiv I_s((u, b), (v, d) \\
&\geq \int_0^1 \left[\frac{1}{T^2}\|Pu'\|^2 - C_1\|Pu'\|\|Pu\| + \left(\frac{s}{T^2} - C_2\right)\|Pu\|^2 \right.
\end{aligned}$$

$$-C_3\|Pu'\| \cdot b - C_4\|Pu\| \cdot b + \frac{s+1}{T^3}\,|\kappa|\, b^2 \Bigg]\, dt > 0.$$

This concludes the proof. □

Remark 3.7. Here we would like to observe that the NON-NULL ASSUMPTION has been strongly used in the proof of Lemma 3.6 and it is crucial in the previous construction. However, it is interesting to understand if this condition is just technical or if it represent an obstruction to carry over this case Theorem 1.4.

Now, for every $s \in [0, +\infty)$, the push-forward by Ψ of the index form $\mathcal{I}_s$ on $\mathcal{H}(x)$ is given by the symmetric bilinear form on $H^1_A([0,1], \mathbb{R}^n)$ defined by

$$I_{s,T}[u, v] := \int_0^1 \left\{ \left\langle \frac{1}{T} P(t)u'(t), v'(t) \right\rangle + \langle Q(t)u(t), v'(t)\rangle + \left\langle Q^{\mathsf{T}}(t)u'(t), v(t)\right\rangle + \langle TR(t)u(t), v(t)\rangle \right\} dt$$

$$\text{where } \alpha_T(x)[u, v] := \int_0^1 \left\langle \frac{1}{T} P(t)u(t), v(t) \right\rangle dt.$$

By arguing precisely as before, we get that there exists a $s_0 > 0$ large enough such that $I_{s,T}$ is non-degenerate for every $s > s_0$.

Proposition 3.8. *Let* (x, T) *be a non-null critical point of Lagrangian action given at Eq* (2.1). *Then the spectral indexes are well-defined.*

Proof. We start observing that, $s \mapsto q^A_s$ is a path of Fredholm quadratic forms on $H^1_A([0, T], \mathbb{R}^n) \times \mathbb{R}$. Moreover, by Lemma 3.6, there exists $s_0 \in [0, +\infty)$ such that q^A_s is non-degenerate for every $s \geq s_0$ and hence the integer $\mathrm{sf}\{q^A_s, s \in [0, s_0]\}$ is well-defined.

The conclusion follows by observing that q^A_s is the push-forward by Ψ of the Fredholm quadratic form $\mathcal{Q}_s$ and by the fact that the spectral flow of a generalized family of Fredholm quadratic forms on the (trivial) Hilbert bundle $[0, s_0] \times (\mathcal{H}(x) \times \mathbb{R})$ is independent on the trivialization. This concludes the proof. □

3.3. The difference between two spectral indices

This subsection is to provide an abstract formula for computing the difference between the spectral indices defined above.

Let H be a real separable Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$, $W \subset H$ be a dense subspace and let the inclusion map $i : W \to H$ be compact. We let A be an unbounded linear operator on H having domain W and we assume that A is a Fredholm operator. Given a finite dimensional Hilbert space V, we assume that $B : V \to H$ is a bounded linear operator and $C : V \to V$ is a bounded self-adjoint linear operator. We denote by $\mathcal{A} : W \oplus V \to H \oplus V$ the self-adjoint operator defined by

$$\mathcal{A}(w, v) = (Aw + Bv, B^*w + Cv),$$

where B^* is the adjoint operator of B. In matrix form the operator $\mathcal{A}$ can be written as

$$\mathcal{A} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}.$$

Lemma 3.9. *Under above assumptions, we have*

$$m^-\left(\begin{bmatrix} 0 & B \\ B^* & C \end{bmatrix}\right) = m^-(C|_{\ker B}) + \dim(\operatorname{Im} B),$$

where m^- denotes the Morse index.

Proof. For the Hilbert space V, the following splitting holds

$$V = \operatorname{Im} B^* \oplus \ker B = \operatorname{Im} B \oplus \ker B^*.$$

By choosing a suitable basis, the matrix $\begin{bmatrix} 0 & B \\ B^* & C \end{bmatrix}$ has the block form

$$\begin{bmatrix} 0 & 0 & B_{11} & 0 \\ 0 & 0 & 0 & 0 \\ B_{11}^* & 0 & C_{11} & C_{12} \\ 0 & 0 & C_{12}^* & C_{22} \end{bmatrix},$$

where $B_{11} : \operatorname{Im} B^* \to \operatorname{Im} B$ and $B_{11}^* : \operatorname{Im} B \to \operatorname{Im} B^*$ are both invertible. So, in particular

$$\begin{bmatrix} 0 & B \\ B^* & C \end{bmatrix} \quad \text{is similar to} \quad \begin{bmatrix} 0 & 0 & B_{11} & 0 \\ 0 & 0 & 0 & 0 \\ B_{11}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{22} \end{bmatrix}.$$

Now, since

$$m^-\left(\begin{bmatrix} 0 & B_{11} \\ B_{11}^* & 0 \end{bmatrix}\right) = \dim(\operatorname{Im} B) \quad \text{and} \quad m^-(C|_{\ker B}) = m^-(C_{22}|_{\ker B})$$

the thesis readily follows. □

Lemma 3.10. *For $s \in [0, 1]$, we let $\mathcal{A}(s) = \begin{bmatrix} A & (1-s)B \\ (1-s)B^* & (1-s)C \end{bmatrix}$ Then, the following spectral flow formula holds*

$$\operatorname{sf}(\mathcal{A}(s), s \in [0, 1]) = m^-(\mathcal{A}(0)|_{W^\perp}) + \dim(W \cap W^\perp) - \dim(W \cap \ker \mathcal{A}(0)).$$

Proof. We start to consider the splitting $W = (\ker A)^\perp \oplus \ker A$. So, $\mathcal{A}(s)$ can be written in the following block form

$$\mathcal{A}(s) = \begin{bmatrix} A_{11} & 0 & (1-s)B_1 \\ 0 & 0 & (1-s)B_2 \\ (1-s)B_1^* & (1-s)B_2^* & (1-s)C \end{bmatrix},$$

where $A_{11} : (\ker A)^\perp \to (\ker A)^\perp$ is invertible and $B_1 : V \to (\ker A)^\perp$, $B_2 : V \to \ker A$.

For $s \in [0, 1]$, we let

$$\mathcal{B}(s) = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & 0 & (1-s)B_2 \\ 0 & (1-s)B_2^* & (1-s)[C - (1-s)B_1^* A_{11}^{-1} B_1] \end{bmatrix}.$$

First claim. The following equality holds:

$$\mathrm{sf}\{\mathcal{A}(s), s \in [0,1]\} = \mathrm{sf}\{\mathcal{B}(s), s \in [0,1]\}. \tag{3.9}$$

This equality is a direct consequence of the stratum homotopy invariant property of the spectral flow. So, let's start to consider the 2-parameter family of operators pointwise defined by

$$A(s,t) = \begin{bmatrix} A_{11} & 0 & t(1-s)B_1 \\ 0 & 0 & (1-s)B_2 \\ t(1-s)B_1^* & (1-s)B_2^* & (1-s)[C-(1-t)^2(1-s)B_1^*A_{11}^{-1}B_1] \end{bmatrix}$$

and we observe that we have $A(s,t) = K(t)^*A(s)K(t)$ for

$$K(t) = \begin{bmatrix} I & 0 & -(1-t)(1-s)A_{11}^{-1}B_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

By a straightforward calculation it follows that $\dim\ker A(0,t)$ and $\dim\ker A(1,t)$ are both constants. By using the zero axiom of the spectral flow (namely each path is contained in a fixed stratum of the Fredholm Lagrangian Grassmannian), we get that

$$\mathrm{sf}\{A(0,t), t \in [0,1]\} = \mathrm{sf}\{A(1,t), t \in [0,1]\} = 0.$$

By invoking the stratum homotopy invariant property of the spectral flow, we get that

$$\mathrm{sf}\{A(s,0), s \in [0,1]\} = \mathrm{sf}\{A(s,1) s \in [0,1]\}$$

which is precisely the equality appearing at Eq (3.9).

Let

$$\mathcal{C}(s) = \begin{bmatrix} 0 & (1-s)B_2 \\ (1-s)B_2^* & (1-s)[C-(1-s)B_1^*A_{11}^{-1}B_1] \end{bmatrix}, s \in [0,1].$$

By taking into account the additivity property of the spectral flow under direct sum as well as of Eq (3.9), we get that

$$\mathrm{sf}\{\mathcal{B}(s), s \in [0,1]\} = \mathrm{sf}\{\mathcal{C}(s), s \in [0,1]\}.$$

Now, since $\mathcal{C}(1) = 0$, then we have

$$\mathrm{sf}\{\mathcal{C}(s), s \in [0,1]\} = m^-(C(0)).$$

By Lemma 3.9, we get that

$$\mathrm{sf}\{\mathcal{C}(s), s \in [0,1]\} = m^-(C(0)) = m^-((C - B_1^*A_{11}^{-1}B_1)|_{\ker B_2}) + \dim(\operatorname{Im} B_2).$$

In order to conclude, we have to prove that

$$m^-(\mathcal{A}(0)|_{W^\perp}) = m^-((C - B_1^*A_{11}^{-1}B_1)|_{\ker B_2}) \quad \text{and}$$
$$\dim(W \cap W^\perp) - \dim(W \cap \ker \mathcal{A}(0)) = \dim(\operatorname{Im} B_2).$$

Let us consider $(x_1, x_2, 0)^\top \in \ker \mathcal{A}(0) \cap W$. Then for every $(u_1, u_2, v)^\top \in W \oplus V$ we have

$$\left\langle \begin{bmatrix} A_{11} & 0 & B_1 \\ 0 & 0 & B_2 \\ B_1^* & B_2^* & C \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \\ v \end{bmatrix} \right\rangle = \langle A_{11}x_1, u_1\rangle + \langle B_1^* x_1, v\rangle + \langle B_2^* x_2, v\rangle \equiv 0. \tag{3.10}$$

We set $v = 0$. So, $\langle A_{11}x_1, u_1\rangle \equiv 0$ for every $u_1 \in (\ker A)^\perp$ implies that $A_{11}x_1 = 0$. Consequently we have $x_1 = 0$. Now, Eq (3.10) becomes $\langle B_2^* x_2, v\rangle \equiv 0$. Since v is arbitrary, then $B_2^* x_2 = 0$. Hence $W \cap \ker \mathcal{A}(0) = \{(0, x_2, 0)^\top \mid B_2^* x_2 = 0\} = \ker B_2^*$.

If $(x_1, x_2, y)^\top \in W^\perp$, then for every $(u_1, u_2, 0)^\top \in W$ we have

$$\left\langle \begin{bmatrix} A_{11} & 0 & B_1 \\ 0 & 0 & B_2 \\ B_1^* & B_2^* & C \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \right\rangle = \langle A_{11}x_1, u_1\rangle + \langle B_1 y, u_1\rangle + \langle B_2 y, u_2\rangle \equiv 0. \tag{3.11}$$

We set $u_2 = 0$. Then $\langle A_{11}x_1 + B_1 y, u_1\rangle \equiv 0$ for every $u_1 \in (\ker A)^\perp$ implies that $A_{11}x_1 + B_1 y = 0$. Consequently we have $x_1 = -A_{11}^{-1}B_1 y$. Let $u_1 = 0$, Eq (3.11) becomes $\langle B_2 y, u_2\rangle \equiv 0$ for every $u_2 \in \ker A$, then $B_2 y = 0$. Hence $W^\perp = \{(-A_{11}^{-1}B_1 y, x_2, y)^\top \mid B_2 y = 0, x_2 \in \ker A\}$ and $W \cap W^\perp = \{(0, x_2, 0)^\top\} = \ker A$.

Now, we get

$$\dim W \cap W^\perp - \dim W \cap \ker \mathcal{A}(0) = \dim \ker A - \dim \ker B_2^* = \dim \operatorname{Im} B_2.$$

For every $\xi_0 = (-A_{11}^{-1}B_1 y, x_2, y)^\top \in W^\perp$, we have

$$\begin{aligned} \langle \mathcal{A}(0)\xi_0, \xi_0\rangle &= \left\langle \begin{bmatrix} A_{11} & 0 & B_1 \\ 0 & 0 & B_2 \\ B_1^* & B_2^* & C \end{bmatrix} \begin{bmatrix} -A_{11}^{-1}B_1 y \\ x_2 \\ y \end{bmatrix}, \begin{bmatrix} -A_{11}^{-1}B_1 y \\ x_2 \\ y \end{bmatrix} \right\rangle \\ &= \langle -B_1 y, -A_{11}^{-1}B_1 y\rangle + \langle B_1 y, -A_{11}^{-1}B_1 y\rangle + \langle B_2 y, x_2\rangle \\ &\quad + \langle -B_1^* A_{11}^{-1}B_1 y, y\rangle + \langle B_2^* x_2, y\rangle + \langle Cy, y\rangle \\ &= \langle (C - B_1^* A_{11}^{-1}B_1)y, y\rangle. \end{aligned}$$

Therefore, we have $m^-(\mathcal{A}(0)|_{W^\perp}) = m^-((C - B_1^* A_{11}^{-1}B_1)|_{\ker B_2})$. This concludes the proof. □

Lemma 3.11. *We let* $\mathcal{A}(s) = \begin{bmatrix} A & sB \\ sB^* & sC \end{bmatrix}$, *for* $s \in [0, 1]$ *and we assume that* A *is invertible. Then we have*

$$\operatorname{sf}(\mathcal{A}(s), s \in [0, 1]) = -m^-(C - B^* A^{-1} B).$$

Proof. By a similar discussion as provided in the proof of Eq (3.9), we get

$$\operatorname{sf}(\mathcal{A}(s), s \in [0, 1]) = \operatorname{sf}(\mathcal{B}(s), s \in [0, 1]), \text{ where } \mathcal{B}(s) = \begin{bmatrix} A & 0 \\ 0 & s[C - sB^* A^{-1} B] \end{bmatrix}.$$

Since A is invertible, we infer that

$$\operatorname{sf}(\mathcal{A}(s), s \in [0, 1]) = \operatorname{sf}(s(C - sB^* A^{-1} B), s \in [0, 1]).$$

We observe that the operator $s(C - sB^*A^{-1}B)$ is defined on a finite dimensional space V, and since in this case the spectral flow is equal to the Morse index at the starting point minus the Morse index at the end point, we get that

$$\mathrm{sf}(s(C - sB^*A^{-1}B), s \in [0,1]) = -m^-(C - B^*A^{-1}B).$$

□

By using the previous results, we are now ready to compute the difference between two spectral indices defined in Definition 3.3. By taking into account Eq (3.6), we denote by $\mathcal{A}(s)$ the realization of I_s meaning the bounded linear operator representing the bilinear form I_s w.r.t. the $H^1_A \times \mathbb{R}$ scalar product; so, we get

$$I_s[(u,b),(v,d)] = \left\langle \mathcal{A}(s)\begin{bmatrix} u \\ b \end{bmatrix}, \begin{bmatrix} v \\ d \end{bmatrix} \right\rangle.$$

Similarly, we define the bounded linear operators $A(s), B, C(s)$ representing w.r.t. the $H^1_A \times \mathbb{R}$ scalar product the three terms defining I_s. So, we get

$$\begin{aligned} &\langle A(s)u, v\rangle \\ &\quad = \int_0^1 \left\{ \left\langle \frac{1}{T} P(t)u'(t), v'(t) \right\rangle + \langle Q(t)u(t), v'(t)\rangle + \left\langle Q^{\mathsf{T}}(t)u'(t), v(t)\right\rangle + \langle TR(t)u(t), v(t)\rangle \right\} dt \\ &\qquad\qquad + s\int_0^1 \left\{ \left\langle \frac{1}{T}P(t)u(t), v(t)\right\rangle \right\} dt \\ &\left\langle B\begin{bmatrix} u \\ b \end{bmatrix}, \begin{bmatrix} v \\ d \end{bmatrix} \right\rangle = \int_0^1 \left\{ -\frac{1}{T^2}\langle P(t)x'(t), v'(t)\rangle \cdot b + \left\langle L(t) - \frac{1}{T}Q^{\mathsf{T}}(t)x'(t), v(t)\right\rangle \cdot b \right\} dt \\ &\langle C(s)\, b, d\rangle = (s+1)\int_0^1 \frac{1}{T^3}\kappa(t) bd\, dt. \end{aligned}$$

In matrix form the operator $\mathcal{A}(s)$ can be written as

$$\mathcal{A}(s) = \begin{bmatrix} A(s) & B \\ B^* & C(s) \end{bmatrix}.$$

Let now consider the homotopy pointwise defined by

$$\mathcal{A}(s,\epsilon) = \begin{bmatrix} A(s) & (1-\epsilon)B \\ (1-\epsilon)B^* & (1-\epsilon)C(s) \end{bmatrix} \qquad \text{for } \epsilon \in [0,1].$$

Recall the discussions below Remark 3.7 we proved that the index form $I_{s,T}$ is non-degenerate for s_0 large enough. Therefore, $A(s_0)$ is invertible. The next result provides a more strinking propery about the spectrum of $A(s_0)$.

Lemma 3.12. *There exists $\delta > 0$ such that $\sigma(A(s_0)) \cap [-\delta, \delta] = \emptyset$. In particular, the operator $A^{-1}(s_0)$ is bounded.*

Proof. Arguing by contradiction, we assume that for every $\delta > 0$ there exists $\lambda_\delta \in [-\delta, \delta]$ and $u_\delta \in H^1_A([0,1], \mathbb{R}^n)$ such that $A(s_0)u_\delta = \lambda_\delta u_\delta$. Take $v_\delta = Pu_\delta$, then we have

$$I_{s_0,T}(u_\delta, v_\delta) = \langle A(s_0)u_\delta, v_\delta\rangle = \lambda_\delta\langle u_\delta, v_\delta\rangle \leq \lambda_\delta \int_0^1 \|P^{-1}\|\|Pu_\delta\|^2 dt. \tag{3.13}$$

By taking into account Eq (3.8), we have

$$I_{s_0,T}(u_\delta, v_\delta) \geq \int_0^1 \left[\frac{1}{T^2}\|Pu'_\delta\|^2 - C_1\|Pu'_\delta\|\|Pu_\delta\| + (\frac{s_0}{T^2} - C_2)\|Pu_\delta\|^2\right] dt. \tag{3.14}$$

Inequalities provided at Eq (3.13) and Eq (3.14) contradict each other for s_0 sufficiently large and δ (consequently λ_δ) sufficiently small. This concludes the proof. □

By the homotopy invariance property of the spectral flow, we get that

$$\begin{aligned}\mathrm{sf}(\mathcal{A}(s,0), s \in [0, s_0]) = \mathrm{sf}(\mathcal{A}(0,\epsilon), \epsilon \in [0,1]) + \mathrm{sf}(\mathcal{A}(s,1), s \in [0, s_0])\\ + \mathrm{sf}(\mathcal{A}(s_0, 1-\epsilon), \epsilon \in [0,1]).\end{aligned} \tag{3.15}$$

Let us now compute the spectral flow $\mathrm{sf}(\mathcal{A}(s_0, 1-\epsilon), \epsilon \in [0,1])$. By using Lemma 3.12, we infer that $B^*A^{-1}(s_0)B$ is a bounded operator (on a one-dimensional space). So, we get

$$m^-(C(s_0) - B^*A^{-1}(s_0)B) = \begin{cases}1 & \text{if} \quad \kappa < 0\\ 0 & \text{if} \quad \kappa > 0.\end{cases}$$

By Lemma 3.11, we have

$$\mathrm{sf}(\mathcal{A}(s_0, 1-\epsilon), \epsilon \in [0,1]) = m^-(C(s_0) - B^*A^{-1}(s_0)B) = \begin{cases}1 & \text{if} \quad \kappa < 0\\ 0 & \text{if} \quad \kappa > 0.\end{cases} \tag{3.16}$$

Let us now compute the spectral flow $\mathrm{sf}(\mathcal{A}(0,\epsilon), \epsilon \in [0,1])$. By using Lemma 3.10 we have

$$\mathrm{sf}(\mathcal{A}(0,\epsilon), \epsilon \in [0,1]) = m^-(\mathcal{A}(0,0)|_{W^\perp}) + \dim(W \cap W^\perp) - \dim(W \cap \ker\mathcal{A}(0,0))$$

where $W = H^1_A([0,1], \mathbb{R}^n)$ and $V = \mathbb{R}$.

The next step is to provide an explicit description of

$$m^-(\mathcal{A}(0,0)|_{W^\perp}) + \dim(W \cap W^\perp) - \dim(W \cap \ker\mathcal{A}(0,0)).$$

The basic idea comes from [9, Section 2.1].

Let (x, T) be a non-null critical point of $\mathbb{E}_h$ with orbit cylinder (x_{h+s}, T_{h+s}). Then for every $(\xi, b) \in \mathcal{H}(x) \times \mathbb{R}$ we have

$$d\mathbb{E}_{h+s}(x_{h+s}, T_{h+s})[(\xi, b)] \equiv 0. \tag{3.17}$$

By differentiating w.r.t. s both sides of Eq (3.17), we get

$$d^2\mathbb{E}_h(x,T)[(\xi_h, T'(h)), (\xi, b)] + \frac{\partial}{\partial s}\Big|_{s=0} d\mathbb{E}_{h+s}(x,T)[(\xi, b)] = 0, \tag{3.18}$$

where $\xi_h(t) = \frac{\partial}{\partial s}|_{s=0} x_{h+s}(t), T'(h) = \frac{d}{ds}|_{s=0} T_{h+s}$. Let now choose a variation $\{(x_{h,r}, T_{h,r}), r \in (-\epsilon, \epsilon)\}$ such that $(x_{h,0}, T_{h,0}) = (x, T)$ and $\frac{\partial}{\partial r}|_{r=0}(x_{h,r}, T_{h,r}) = (\xi, b)$, then we have

$$\begin{aligned}\frac{\partial}{\partial s}\Big|_{s=0} d\mathbb{E}_{h+s}(x,T)[(\xi,b)] &= \frac{\partial}{\partial s}\Big|_{s=0}\frac{\partial}{\partial r}\Big|_{r=0}\mathbb{E}_{h+s}(x,T)[(x_{h,r},T_{h,r})]\\ &= \frac{\partial}{\partial r}\Big|_{r=0}\frac{\partial}{\partial s}\Big|_{s=0}\mathbb{E}_{h+s}(x,T)[(x_{h,r},T_{h,r})]\\ &= \frac{\partial}{\partial r}\Big|_{r=0} T_{h,r} = b.\end{aligned}$$

By taking into account Eq (3.18) we have

$$d^2\mathbb{E}_h(x,T)[(\xi_h, T'(h)), (\xi, b)] = -b \tag{3.19}$$

for every $(\xi, b) \in \mathcal{H}(x) \times \mathbb{R}$. Taking $b = 0$ and $(\xi, b) = (\xi_h, T'(h))$ respectively, we have

$$d^2\mathbb{E}_h(x,T)[(\xi_h, T'(h)), (\xi, 0)] = 0, \quad d^2\mathbb{E}_h(x,T)[(\xi_h, T'(h)), (\xi_h, T'(h))] = -T'(h). \tag{3.20}$$

Let us identify $\mathcal{H}(x)$ with $\mathcal{H}(x) \times \{0\}$ and we denote the Hessian of $\mathbb{E}_h$ by $\nabla^2\mathbb{E}_h(x,T)$. So, $\ker d^2\mathbb{E}_h(x,T) = \ker \nabla^2\mathbb{E}_h(x,T)$. We now set

$$\mathcal{H}^\perp(x) = \{(\xi, b) \in \mathcal{H}(x) \times \mathbb{R} \mid d^2\mathbb{E}_h(x,T)[(\xi,b),(\eta,0)] = 0,\ \forall(\eta,0) \in \mathcal{H}(x)\}.$$

The following result holds.

Lemma 3.13. *Under above notations, we get*

$$\ker d^2\mathbb{E}_h(x,T) \subset \mathcal{H}(x), \quad \textit{and} \quad \mathcal{H}^\perp(x) = \ker d^2\mathbb{E}_h(x,T) \oplus \mathbb{R}(\xi_h, T'(h)).$$

Proof. We argue by contradiction. If $\mathcal{H}(x) + \ker d^2\mathbb{E}_h(x,T) = \mathcal{H}(x) \times \mathbb{R}$, then

$$\mathcal{H}^\perp(x) = (\mathcal{H}(x) + \ker d^2\mathbb{E}_h(x,T))^\perp = (\mathcal{H}(x) \times \mathbb{R})^\perp = \ker d^2\mathbb{E}_h(x,T). \tag{3.21}$$

By taking into account Eq (3.20) we get $(\xi_h, T'(h)) \in \mathcal{H}^\perp(x)$ and by using Eq (3.19) we know $(\xi_h, T'(h)) \notin \ker d^2\mathbb{E}_h(x,T)$ which contradicts Eq (3.21).
So, $\mathcal{H}(x) + \ker d^2\mathbb{E}_h(x,T) \neq \mathcal{H}(x) \times \mathbb{R}$. Since $\dim((\mathcal{H}(x) \times \mathbb{R})/\mathcal{H}(x)) = 1$, then we finally get $\ker d^2\mathbb{E}_h(x,T) \subset \mathcal{H}(x)$.

Now observe that $\ker d^2\mathbb{E}_h(x,T) \oplus \mathbb{R}(\xi_h, T'(h)) \subset \mathcal{H}^\perp(x)$ and $\dim(\mathcal{H}^\perp(x)/\ker d^2\mathbb{E}_h(x,T)) \leq 1$. In particular $\mathcal{H}^\perp(x) = \ker d^2\mathbb{E}_h(x,T) \oplus \mathbb{R}(\xi_h, T'(h))$. This concludes the proof. □

By invoking Lemma 3.13, we get

$$\mathcal{H}^\perp(x) = \ker d^2\mathbb{E}_h(x,T) \oplus \mathbb{R}(\xi_h, T'(h)) \supset \mathcal{H}(x) \cap \mathcal{H}^\perp(x) \supset \ker d^2\mathbb{E}_h(x,T).$$

If $T'(h) \neq 0$, the $\mathbb{R}(\xi_h, T'(h)) \subsetneq \mathcal{H}(x)$. Thus, we have $\mathcal{H}(x) \cap \mathcal{H}^\perp(x) = \ker d^2\mathbb{E}_h(x,T)$.

At the same time, by Lemma 3.13 we have $d^2\mathbb{E}_h(x,T)|_{\mathcal{H}^\perp(x)} = d^2\mathbb{E}_h(x,T)|_{\mathbb{R}(\xi_h, T'(h))}$. Then by Eq (3.20), we have

$$m^-(d^2\mathbb{E}_h(x,T)|_{\mathcal{H}^\perp(x)}) = \begin{cases} 1 & \text{if} \quad T'(h) > 0 \\ 0 & \text{if} \quad T'(h) < 0. \end{cases}$$

Hence, we have

$$m^-(d^2\mathbb{E}_h(x,T)|_{\mathcal{H}^\perp(x)}) + \dim(\mathcal{H}(x)\cap\mathcal{H}^\perp(x)) - \dim(\mathcal{H}(x)\cap\ker d^2\mathbb{E}_h(x,T)) = \begin{cases} 1 & \text{if} \quad T'(h) > 0 \\ 0 & \text{if} \quad T'(h) < 0. \end{cases}$$

If $T'(h) = 0$, then by Eq (3.20), we get that $(\xi_h, T'(h)) \in \mathcal{H}(x)\cap\mathcal{H}^\perp(x)$. Therefore, we have

$$\mathcal{H}(x)\cap\mathcal{H}^\perp(x) = \ker d^2\mathbb{E}_h(x,T) \oplus \mathbb{R}(\xi_h, 0) = \mathcal{H}^\perp(x).$$

As a result, we have $\dim(\mathcal{H}(x)\cap\mathcal{H}^\perp(x)) - \dim(\mathcal{H}(x)\cap\ker d^2\mathbb{E}_h(x,T)) = 1$, and

$$d^2\mathbb{E}_h(x,T)|_{\mathcal{H}^\perp(x)} = d^2\mathbb{E}_h(x,T)|_{\mathcal{H}^\perp(x)\cap\mathcal{H}(x)} = 0.$$

So, we get that if $T'(h) = 0$ there holds

$$m^-(d^2\mathbb{E}_h(x,T)|_{\mathcal{H}^\perp(x)}) + \dim(\mathcal{H}(x)\cap\mathcal{H}^\perp(x)) - \dim(\mathcal{H}(x)\cap\ker d^2\mathbb{E}_h(x,T)) = 1.$$

Summing up, we have

$$m^-(d^2\mathbb{E}_h(x,T)|_{\mathcal{H}^\perp(x)}) + \dim(\mathcal{H}(x)\cap\mathcal{H}^\perp(x)) - \dim(\mathcal{H}(x)\cap\ker d^2\mathbb{E}_h(x,T)) = \begin{cases} 1 & \text{if} \quad T'(h) \geq 0 \\ 0 & \text{if} \quad T'(h) < 0. \end{cases}$$

In conclusion, we get

$$\begin{aligned} \mathrm{sf}(\mathcal{A}(0,\epsilon), \epsilon\in[0,1]) &= m^-(\mathcal{A}(0,0)|_{W^\perp}) + \dim(W\cap W^\perp) - \dim(W\cap\ker\mathcal{A}(0,0)) \\ &= m^-(d^2\mathbb{E}_h(x,T)|_{\mathcal{H}^\perp(x)}) + \dim(\mathcal{H}(x)\cap\mathcal{H}^\perp(x)) \\ &- \dim(\mathcal{H}(x)\cap\ker d^2\mathbb{E}_h(x,T)) \\ &= \begin{cases} 1 & \text{if} \quad T'(h) \geq 0 \\ 0 & \text{if} \quad T'(h) < 0 \end{cases}, \end{aligned} \tag{3.22}$$

where $W = H^1_A([0,1],\mathbb{R}^n)$. By summarizing all the previous results, the following theorem holds.

Theorem 3.14. *Under above notations the following equalities hold:*

$$\begin{aligned} \iota_{\mathrm{spec}}(x,T) - \iota^T_{\mathrm{spec}}(x) &= \mathrm{sf}\,(\mathcal{Q}_s, s\in[0,s_0]) - \mathrm{sf}\,(\mathcal{Q}_{s,T}, s\in[0,s_0]) \\ &= \mathrm{sf}(\mathcal{A}(s,0), s\in[0,s_0]) - \mathrm{sf}(\mathcal{A}(s,1), s\in[0,s_0]) \\ &= \mathrm{sf}(\mathcal{A}(0,\epsilon), \epsilon\in[0,1]) + \mathrm{sf}(\mathcal{A}(s_0, 1-\epsilon), \epsilon\in[0,1]) \\ &= \begin{cases} 2 & \textit{if} \quad \kappa<0,\ T'(h)\geq 0 \\ 1 & \textit{if} \quad \kappa<0,\ T'(h)<0 \quad \textit{or} \quad \kappa>0,\ T'(h)\geq 0 \\ 0 & \textit{if} \quad \kappa>0,\ T'(h)<0 \end{cases} \end{aligned} \tag{3.23}$$

Proof. The proof readily follows by invoking Eqs (3.15)-(3.16) and Eq (3.22). This concludes the proof. □

Remark 3.15. We observe that the main role of orbit cylinder is to ensure the existence of vector $(\xi_h, T'(h))$ in Eq (3.19). A natural problem is to find out some more general conditions to insure the existence of a vector in $\mathcal{H}^{\perp}(x)$ but not in $\ker d^2\mathbb{E}_h(x,T)$. The bifurcation theory of Hamiltonian system could be the right direction for answering this question.

By using Proposition 3.5, the following result holds.

Corollary 3.16. *If L is $\mathscr{C}^2$-strictly convex on TM, then the difference between two Morse indices is given by*

$$\iota_{\mathrm{Mor}}(x,T) - \iota^{T}_{\mathrm{Mor}}(x) = \begin{cases} 1 & \text{if} \quad T'(h) \geq 0 \\ 0 & \text{if} \quad T'(h) < 0 \end{cases}$$

Proof. Since L is $\mathscr{C}^2$-strictly convex, then $\kappa = \langle Px', x'\rangle > 0$. By Proposition 3.5 and Eq (3.23), we conclude the proof. □

Remark 3.17. Here we point out that if (x,T) is a minimizer of system (2.4) when L is $\mathscr{C}^2$-strictly convex, then we must have $\iota_{\mathrm{Mor}}(x,T) = 0$. By Corollary 3.16 there must hold $T'(h) < 0$. Or vice versa, if $T'(h) \geq 0$, then (x,T) cannot be a minimizer.

4. Linear instability and proof of the main result

In this section we recall some well-known results about the fixed period problem. We refer the interested reader to [20] for the complete proofs.

4.1. Hamiltonian system and geometrical index

It is well-known that under the assumption (N1) the Legendre transform

$$\mathscr{L}_L : TM \to T^*M, \qquad (q,v) \mapsto \left(q, DL(q,v)\big|_{T^v_{(q,v)}TM}\right)$$

is a local smooth diffeomorphism. The Fenchel transform of L is the autonomous Hamiltonian on T^*M

$$H(q,p) = \max_{v \in T_qM} \big(p[v] - L(q,v)\big) = p[v(q,p)] - L(q, v(q,p)),$$

for every $(q,p) \in T^*M$, where the map v is a component of the fiber-preserving diffeomorphism

$$\mathscr{L}_L^{-1} : T^*M \to TM, \qquad (q,p) \mapsto (q, v(q,p))$$

the inverse of $\mathscr{L}_L$.

By the above Legendre transform, the Euler-Lagrange Equation (2.5) is changed into the following Hamiltonian equation:

$$z_x'(t) = J\nabla H(z_x(t)).$$

By trivializing the pull-back bundle $x^*(TM)$ over TM through the frame $\mathbf{E}$ defined in Subsection 3, Eq (2.6) is changed into

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{dt}}\big(\tfrac{1}{T}P(t)u'(t) + Q(t)u(t)\big) + Q^{\mathsf{T}}(t)u'(t) + TR(t)u(t) = 0, & t \in (0,1) \\ u(0) = Au(1), \qquad u'(0) = Au'(1). & \end{cases}$$

By setting $y(t) = \frac{1}{T}P(t)u'(t) + Q(t)u(t)$ and $z(t) = (y(t), u(t))^\mathsf{T}$ we finally get

$$\begin{cases} z'(t) = JB(t)z(t), & t \in [0,1] \\ z(0) = A_d z(1) \end{cases} \quad \text{where}$$

$$B(t) := \begin{bmatrix} TP^{-1}(t) & -TP^{-1}(t)Q(t) \\ -TQ(t)P^{-1}(t) & TQ^\mathsf{T}(t)P^{-1}(t)Q(t) - TR(t) \end{bmatrix} \quad (4.1)$$

and A_d has been defined in Eq (3.4).

In the standard symplectic space $(\mathbb{R}^{2n}, \omega)$, we denote by J the standard symplectic matrix defined by $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$. Thus the symplectic form ω can be represented with respect to the Euclidean product $\langle \cdot, \cdot \rangle$ by J as follows $\omega(z_1, z_2) = \langle Jz_1, z_2 \rangle$ for every $z_1, z_2 \in \mathbb{R}^{2n}$.

Given $M \in \mathrm{Sp}(2n, \mathbb{R})$, we denote by $\mathrm{Gr}(M) = \{(x, Mx) | x \in \mathbb{R}^{2n}\}$ its graph and we recall that $\mathrm{Gr}(M)$ is a Lagrangian subspace of the symplectic space $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\omega \times \omega)$.

Definition 4.1. Let x be a 1-periodic solution of Eq (2.5), z_x be the solution of corresponding Hamiltonian equation and let us consider the path

$$\gamma_\Phi : [0,1] \to \mathrm{Sp}(2n, \mathbb{R}) \quad \text{given by} \quad \gamma_\Phi(t) := A_d[\Phi^E(t)]^{-1} D\phi_H^t(z_x(0))\Phi^E(0).$$

We define the *geometrical index of x* as follows

$$\iota_{\mathrm{geo}}(x) := \iota_{\mathrm{CLM}}(\Delta, \mathrm{Gr}(\gamma_\Phi(t)), t \in [0,1])$$

where the (RHS) in Eq (4.3) denotes the ι_{CLM} Maslov index between the Lagrangian path $t \mapsto \mathrm{Gr}(\gamma_\Phi(t))$ and the Lagrangian path $\Delta := \mathrm{Gr}(I)$ defined at Appendix A and references therein.

Let x be a 1-periodic solution of Eq (2.5) and z_x be the solution of corresponding Hamiltonian equation, we can define the *linearized Poincaré map of z_x* as follows.

$\mathfrak{P}_{z_x} : T_{x(0)}M \oplus T^*_{x(0)}M \to T_{x(0)}M \oplus T^*_{x(0)}M$ is given by

$$\mathfrak{P}_{z_x}(\alpha_0, \delta_0) := \bar{A}_d\Big(\zeta(T), \frac{1}{T}\bar{P}(T)\nabla_t\zeta(T) + \bar{Q}(T)\zeta(T)\Big)^\mathsf{T}$$

$$\text{for} \quad \bar{A}_d := \begin{bmatrix} \bar{A} & 0 \\ 0 & \bar{A} \end{bmatrix} \quad (4.2)$$

where ζ is the unique vector field along x such that $\zeta(0) = \alpha_0$ and $\frac{1}{T}\bar{P}(0)\nabla_t\zeta(0) + \bar{Q}(0)\zeta(0) = \delta_0$. Fixed points of $\mathfrak{P}_{z_x}$ correspond to periodic vector fields along z_x.

By pulling back the linearized Poincaré map defined in Eq (4.2) through the unitary trivialization $\Phi^{\mathbf{E}}$ of $z_x(TT^*M)$ over $[0,1]$ we get the map

$$P^{\mathbf{E}} : \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n \text{ defined by } P^{\mathbf{E}}(y_0, u_0) = A_d\left(\frac{1}{T}Pu'(T) + Qu(T), u(T)\right)^\mathsf{T}$$

where $z(t) = (y(t), u(t))$ is the unique solution of the Hamiltonian system given in Eq (4.1) such that $z(0) = (y_0, u_0)$.

Denoting by $t \mapsto \psi(t)$ the fundamental solution of (linear) Hamiltonian system given in Eq (4.1), then we get the geometrical index given in Definition 4.1 reduces to

$$\iota_{\text{geo}}(x) := \iota_{\text{CLM}}(\Delta, \text{Gr}(A_d\psi(t)), t \in [0,1]). \tag{4.3}$$

Moreover, the linearized Poincaré map can be given by the symplectic matrix $A_d\psi(1)$.

By choosing a suitable coordinates and trivialization we can split $A_d\psi(t)$ into following form:

$$A_d\psi(t) = \begin{bmatrix} 1 & 0 & 0 \\ -tT'(h) & 1 & 0 \\ 0 & 0 & P_x(t) \end{bmatrix},$$

where $P_x(t)$ is a path of $2(n-1) \times 2(n-1)$ symplectic matrices. It is referred to [9, Page 104-105].

Definition 4.2. Under above notations, z_x is termed *spectrally stable* if the spectrum $\sigma(P_x(1)) \subset \mathbb{U}$ where $\mathbb{U} \subset \mathbb{C}$ denotes the unit circle of the complex plane. If $P_x(1)$ is also semi-simple, then z_x is termed *linearly stable*.

Denote $\gamma_1(t) = \left\{ \begin{bmatrix} 1 & 0 \\ -tT'(h) & 1 \end{bmatrix}, t \in [0,1] \right\}$ and $\gamma_2(t) = \{P_x(t) \mid t \in [0,1]\}$, then by invoking Eq (A.2), then we get

$$\begin{aligned} \iota_{\text{geo}}(x) &= \iota_{\text{CLM}}(\Delta, \text{Gr}(\gamma_1(t)), t \in [0,1]) + \iota_{\text{CLM}}(\Delta, \text{Gr}(\gamma_2(t)), t \in [0,1]) \\ &= \iota_{\text{CLM}}(\Delta, \text{Gr}(\gamma_2(t)), t \in [0,1]) + \begin{cases} 1, & \text{if } \; T'(h) < 0 \\ 0, & \text{if } \; T'(h) \geq 0. \end{cases} \end{aligned} \tag{4.4}$$

The, next result is well-known and relates the parity of the ι_{CLM}-index to the linear instability of the periodic orbit.

Lemma 4.3. *The following implication holds*

$$\iota_{\text{CLM}}(\Delta, \text{Gr}(\gamma_2(t)), t \in [0,1]) \textit{ is odd } \Rightarrow x \textit{ is linearly unstable}$$

Proof. It is referred to the proof of [20, Lemma 3.15].

□

In [20, Equation 4.25] we give the precise relationship between $\iota_{\text{geo}}(x)$ and $\iota^T_{\text{spec}}(x)$:

$$\iota_{\text{geo}}(x) = \iota^T_{\text{spec}}(x) + \dim\ker(A - I).$$

Remark 4.4. We conclude this section, by observing that even if not explicitly stated, all arguments provided above, also work when the periodic orbit is transversally degenerate.

4.2. Proof of Main Theorem

Proof of Theorem 1.4. We prove only the (contrapositive of) the first statement in Theorem 1.4, being the others completely analogous. Thus, we aim to prove that

$$\text{if } x \text{ is L-positive, orientation preserving and linearly stable} \quad \Rightarrow \quad \iota_{\text{spec}}(x,T) + n \text{ is odd.}$$

First of all, we have

$$\begin{aligned} n + \iota_{\text{spec}}(x,T) &= n + (\iota_{\text{spec}}(x,T) - \iota^T_{\text{spec}}(x)) + \iota^T_{\text{spec}}(x) \\ &= (n - \dim\ker(A-I)) + (\iota_{\text{spec}}(x,T) - \iota^T_{\text{spec}}(x)) + (\iota^T_{\text{spec}}(x) + \dim\ker(A-I)) \\ &= (n - \dim\ker(A-I)) + (\iota_{\text{spec}}(x,T) - \iota^T_{\text{spec}}(x)) + \iota_{\text{geo}}(x) \\ &= (n - \dim\ker(A-I)) + (\iota_{\text{spec}}(x,T) - \iota^T_{\text{spec}}(x)) + (\iota_{\text{geo}}(x) - \iota_{\text{CLM}}(\Delta, \text{Gr}(\gamma_2(t)))) \\ &\qquad + \iota_{\text{CLM}}(\Delta, \text{Gr}(\gamma_2(t))). \end{aligned} \tag{4.5}$$

Being x orientation preserving (by assumption), then $\det A = 1$ and being A also orthogonal, then we get that

- n even $\Rightarrow \dim\ker(A-I)$ even
- n odd $\Rightarrow \dim\ker(A-I)$ odd.

So, in both cases we have $n - \dim\ker(A-I)$ is even. Since x is L-Positive, then $\kappa(t) > 0$. By Equations (3.23) and (4.4), we have

- $T'(h) \geq 0 \Rightarrow \iota_{\text{spec}}(x,T) - \iota^T_{\text{spec}}(x)$ odd and $\iota_{\text{geo}}(x) - \iota_{\text{CLM}}(\Delta, \text{Gr}(\gamma_2(t)))$ even
- $T'(h) < 0 \Rightarrow \iota_{\text{spec}}(x,T) - \iota^T_{\text{spec}}(x)$ even and $\iota_{\text{geo}}(x) - \iota_{\text{CLM}}(\Delta, \text{Gr}(\gamma_2(t)))$ odd .

So, in both cases we have $(\iota_{\text{spec}}(x,T) - \iota^T_{\text{spec}}(x)) + (\iota_{\text{geo}}(x) - \iota_{\text{CLM}}(\Delta, \text{Gr}(\gamma_2(t))))$ is odd. If x is linear stable, then by taking into account Lemma 4.3, we get that $\iota_{\text{CLM}}(\Delta, \text{Gr}(\gamma_2(t)))$ is even. Then, by Eq (4.5), we finally get that $n + \iota_{\text{spec}}(x,T)$ is odd. This concludes the proof. □

5. A classical example

In this section, we will give a simple example where $T'(h) = 0$ inspired by [9, Section 5].

Let (r,θ) denote the polar coordinate on $\mathbb{R}^2$. Suppose $D := \{(r,\theta) \mid 0 < r < 4\}$ and $f(r) = -\frac{1}{2}(r^3 - 4r^2 + 3r)$, then define $L : TD \to \mathbb{R}$ by

$$L(r,\theta,\dot r,\dot\theta) = \frac{1}{2}(\dot r^2 + r^2\dot\theta^2) - f(r)\dot\theta.$$

The energy function $E : TD \to \mathbb{R}$ is given by $E(r,\theta,\dot r,\dot\theta) = \frac{1}{2}(\dot r^2 + r^2\dot\theta^2)$.

By a direct computation we have $\gamma(t) := (r(t),\theta(t))$ satisfies the Euler-Lagrange equations if and only if

$$\ddot r = \dot\theta(r\dot\theta - f'(r)); \quad r^2\dot\theta - f(r) = \text{constant}. \tag{5.1}$$

From now on we only consider the circular orbit. Suppose

$$r_k(t) = \rho(k), \theta_k(t) = a(k)t,$$

where $\rho(k)$ and $a(k)$ are both positive constants. If $\gamma(t)$ is an orbit, since the second equation of (5.1) is automatically satisfied, then we only requir

$$0 = \ddot r = a(k)(\rho(k)a(k) - f'(\rho(k))) \Leftrightarrow \rho(k)a(k) = f'(\rho(k)). \tag{5.2}$$

The energy $E = \frac{1}{2}(\rho^2(k)a^2(k)) = k$, then we have

$$\rho(k)a(k) = \sqrt{2k}. \tag{5.3}$$

We note that $f'(r) = (3r^2 - 8r + 3)/2$, then by Equations (5.2)-(5.3), we have $\sqrt{2k} = -(3\rho^2(k) - 8\rho(k) + 3)/2$, namely,

$$\rho^2(k) - \frac{8}{3}\rho(k) + 1 + \frac{2}{3}\sqrt{2k} = 0.$$

By solving the above equation, then we get $\rho(k) = \frac{4}{3} - \sqrt{\frac{7}{9} - \frac{2}{3}\sqrt{2k}}$.

Consider $k \in (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon)$, where ϵ is a sufficiently small positive number. For $k = \frac{1}{2}$, it is easy to calculate that

$$\rho(1/2) = \rho'(1/2) = 1.$$

By using Eq (5.3), then we have $T(k) = \dfrac{2\pi}{a(k)} = \dfrac{2\pi\rho(k)}{\sqrt{2k}}$, then

$$T'(k) = 2\pi\frac{2k\rho'(k) - \rho(k)}{2k\sqrt{2k}}.$$

Therefore, $T'(1/2) = 2\pi(\rho'(1/2) - \rho(1/2)) = 0$. By Corollary 3.16, then we get

$$\iota_{\mathrm{Mor}}(x, T) - \iota^T_{\mathrm{Mor}}(x) = 1.$$

A. A symplectic excursion on the Maslov index

The purpose of this section is to provide the symplectic preliminaries used in the paper. In Subsection A.1, we give the definition of the Maslov index. Then we compute the Maslov index of a special symplectic path. Our basic references are [13, 14, 21, 22].

A.1. A quick recap on the ι_{CLM}-index

Given a $2n$-dimensional (real) symplectic space (V, ω), a *Lagrangian subspace* of V is an n-dimensional subspace $L \subset V$ such that $L = L^\omega$ where L^ω denotes the *symplectic orthogonal*, i.e., the orthogonal with respect to the symplectic structure. We denote by $\Lambda = \Lambda(V, \omega)$ the *Lagrangian Grassmannian of* (V, ω), namely the set of all Lagrangian subspaces of (V, ω)

$$\Lambda(V, \omega) := \{\, L \subset V \mid L = L^\omega \,\}.$$

It is well-known that $\Lambda(V, \omega)$ is a manifold. For each $L_0 \in \Lambda$, let

$$\Lambda^k(L_0) := \left\{ L \in \Lambda(V, \omega) \,\middle|\, \dim(L \cap L_0) = k \right\} \qquad k = 0, \ldots, n.$$

Each $\Lambda^k(L_0)$ is a real compact, connected submanifold of codimension $k(k+1)/2$. The topological closure of $\Lambda^1(L_0)$ is the *Maslov cycle* that can be also described as follows

$$\Sigma(L_0) := \bigcup_{k=1}^{n} \Lambda^k(L_0)$$

The top-stratum $\Lambda^1(L_0)$ is co-oriented meaning that it has a transverse orientation. To be more precise, for each $L \in \Lambda^1(L_0)$, the path of Lagrangian subspaces $(-\delta, \delta) \mapsto e^{tJ}L$ cross $\Lambda^1(L_0)$ transversally, and as t increases the path points to the transverse direction. Thus the Maslov cycle is two-sidedly embedded in $\Lambda(V, \omega)$. Based on the topological properties of the Lagrangian Grassmannian manifold, it is possible to define a fixed endpoints homotopy invariant called *Maslov index*.

Definition A.1. Let $L_0 \in \Lambda(V, \omega)$ and let $\ell : [0, 1] \to \Lambda(V, \omega)$ be a continuous path. We define the *Maslov index* ι_{CLM} as follows:

$$\iota_{\text{CLM}}(L_0, \ell(t); t \in [a, b]) := \left[e^{-\varepsilon J}\, \ell(t) : \Sigma(L_0)\right]$$

where the right hand-side denotes the intersection number and $0 < \varepsilon << 1$.

For further reference and properties of the Maslov index we refer the interested reader to [21] and references therein.

For the special symplectic path $\gamma(t) = \begin{bmatrix} M_{11}(t) & 0 \\ M_{21}(t) & M_{22}(t) \end{bmatrix}, t \in [0, T]$, there is a very useful formula to compute its Maslov index [23, Theorem 2.2]. Here we only give the simplified version. Let V be a subspace of $\mathbb{C}^{2n}$, define

$$V^I = \{x \in \mathbb{C}^{2n} \mid \omega(x, y) = 0\ \forall y \in V\}, W_I(V) = \{(x, u, y, v)^\top \in \mathbb{C}^{4n} \mid (x, y)^\top \in V^J, (u, v)^\top \in V\}.$$

Then there holds

$$\begin{aligned} &\mu^{CLM}(W_I(V), Gr(\gamma(t))) \\ &= m^+(M_{1,1}(T)^* M_{2,1}(T)|_{S(T)}) - m^+(M_{1,1}(0)^* M_{2,1}(0)|_{S(0)}) + \dim S(0) - \dim S(T), \end{aligned} \tag{A.1}$$

where m^+ denotes the Morse positive index and $S(t) = \{x \in \mathbb{C}^n \mid (x, M_{1,1}x)^\top \in V^I\}$. Please note that in our situation we take K, R in [23, Theorem 2.2] as I and V respectively.

Take $V = \{(x, x)^\top \mid x \in \mathbb{R}\}$, then $V^I = V$ and $\Delta := Gr(I) = W_I(V)$. Let $\gamma(t) = \left\{\begin{bmatrix} 1 & 0 \\ tT_0 & 1 \end{bmatrix}, t \in [0, 1]\right\}$ where T_0 is a given real constant, then we have

$$\iota_{\text{CLM}}(\Delta, \gamma(t); t \in [0, 1]) = \begin{cases} 1, & \text{if} \quad T_0 > 0; \\ 0, & \text{if} \quad T_0 \leq 0. \end{cases} \tag{A.2}$$

In fact, note that in this case we have $S(0) = S(T) = \mathbb{R}$ and $M_{21}(0) = 0, M_{21}(T) = T_0$, then it is just the consequence of (A.1).

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Conflict of interest

The authors declare there is no conflicts of interest.

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