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Research article

A new method to construct model structures from left Frobenius pairs in extriangulated categories

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Abstract: Extriangulated categories were introduced by Nakaoka and Palu as a simultaneous generalization of exact categories and triangulated categories. In this paper, we first introduce the concept of left Frobenius pairs on an extriangulated category C, and then establish a bijective correspondence between left Frobenius pairs and certain cotorsion pairs in C. As an application, some new admissible model structures are established from left Frobenius pairs under certain conditions, which generalizes a result of Hu et al. (J. Algebra 551 (2020) 23–60).

Keywords: extriangulated category; left Frobenius pair; cotorsion pair; model structure

1. Introduction

The notion of extriangulated categories, whose extriangulated structures are given by $\mathbb{E}$-triangles with some axioms, was introduced by Nakaoka and Palu in [1] as a simultaneous generalization of exact categories and triangulated categories. They gave a bijective correspondence between Hovey twin cotorsion pairs and admissible model structures which unified the work of Hovey, Gillespie and Yang (see [2–4]). Exact categories and triangulated categories are extriangulated categories, while there exist some other examples of extriangulated categories which are neither exact nor triangulated, see [1, 5, 6].

Motivated by the ideas of projective covers and injective envelopes, Auslander and Buchweitz analyzed the framework in which the theory of maximal Cohen-Macaulay approximation can be developed. They systematically established their theory in abelian categories, which is known as Auslander-Buchweitz approximation theory. Up to now, Auslander-Buchweitz approximation theory has many important applications, see for example [7–10]. In particular, Becerril and coauthers [7] have revisited Auslander-Buchweitz approximation theory. From the notions of relative generators and cogenerators in approximation theory, they introduced the concept of left Frobenius pairs in an abelian category, established a bijective correspondence between left Frobenius pairs and relative cotorsion pairs, and showed how to construct an exact model structure from a strong left Frobenius pair, as a result of Hovey-Gillespie correspondence applied to two complete cotorsion pairs on an exact category (see [2, 3]).

The aim of this paper is to introduce the concept of left Frobenius pairs in an extriangulated category and give a method to construct more admissible model structures from strong left Frobenius pairs. For this purpose, we need to establish a bijective correspondence between left Frobenius pairs and cotorsion pairs in an extriangulated category under certain conditions.

The paper is organized as follows. In Section 2, we recall the definition of an extriangulated category and outline some basic properties that will be used later. In Section 3, we first introduce the concept of left Frobenius pairs (see Definition 3.4), and then study relative resolution dimension and thick subcategories with respect to a given left Frobenius pair. As a result, we give a bijective correspondence between left Frobenius pairs and cotorsion pairs in an extriangulated category under certain conditions (see Theorem 3.12). In Section 4, we give a method to construct the admissible model structure from a strong left Frobenius pair under certain conditions (see Theorem 4.4), which generalizes a main result of Hu et al. in [5]. This is based on the bijective correspondence established in Section 3.

2. Preliminaries

Throughout this paper, $\mathcal{C}$ denotes an additive category. By the term "*subcategory*" we always mean a full additive subcategory of an additive category closed under isomorphisms and direct summands. We denote by $\mathcal{C}(A, B)$ the set of morphisms from A to B in $\mathcal{C}$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two subcategories of $\mathcal{C}$, a morphism $f : X \to C$ in $\mathcal{C}$ is said to be an $\mathcal{X}$-precover of C if $X \in \mathcal{X}$ and $\mathcal{C}(X', f) : \mathcal{C}(X', X) \to \mathcal{C}(X', C)$ is surjective for all $X' \in \mathcal{X}$. If any $C \in \mathcal{Y}$ admits an $\mathcal{X}$-precover, then $\mathcal{X}$ is called a precovering class in $\mathcal{Y}$. By dualizing the definitions above, we get notions of an $\mathcal{X}$-preenvelope of C and a preenveloping class in $\mathcal{Y}$. For more details, we refer to [23].

Let us briefly recall some definitions and basic properties of extriangulated categories from [1]. We omit some details here, but the reader can find them in [1].

Assume that $\mathbb{E} : \mathcal{C}^{\rm op} \times \mathcal{C} \to \mathrm{Ab}$ is an additive bifunctor, where $\mathcal{C}$ is an additive category and Ab is the category of abelian groups. For any objects $A, C \in \mathcal{C}$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$-extension. Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. This $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$, if it makes the diagram in [1, Definition 2.9] commutative. A triplet $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is called an *extriangulated category* if it satisfies the following conditions.

1. $\mathbb{E} \colon \mathcal{C}^{\rm op} \times \mathcal{C} \to \mathrm{Ab}$ is an additive bifunctor.

2. $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.

3. $\mathbb{E}$ and $\mathfrak{s}$ satisfy certain axioms in [1, Definition 2.12].

In particular, we recall the following axioms which will be used later:
(ET4) Let $\delta \in \mathbb{E}(D, A)$ and $\delta' \in \mathbb{E}(F, B)$ be $\mathbb{E}$-extensions realized by

$$A \xrightarrow{f} B \xrightarrow{f'} D \text{ and } B \xrightarrow{g} C \xrightarrow{g'} F$$

respectively. Then there exists an object $E \in \mathcal{C}$, a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d} \\ A & \xrightarrow{h} & C & \xrightarrow{h'} & E \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle e} \\ & & F & = & F \end{array}$$

in $\mathcal{C}$, and an $\mathbb{E}$-extension $\delta'' \in \mathbb{E}(E, A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$, which satisfy the following compatibilities.

(i) $D \xrightarrow{d} E \xrightarrow{e} F$ realizes $\mathbb{E}(F, f')(\delta')$,

(ii) $\mathbb{E}(d, A)(\delta'') = \delta$,

(iii) $\mathbb{E}(E, f)(\delta'') = \mathbb{E}(e, B)(\delta')$.

(ET4)$^{\mathrm{op}}$ Dual of (ET4).

Remark 2.1. *Note that both exact categories and triangulated categories are extriangulated categories (see [1, Example 2.13]) and extension closed subcategories of extriangulated categories are again extriangulated (see [1, Remark 2.18]). Moreover, there exist extriangulated categories which are neither exact categories nor triangulated categories (see [1, Proposition 3.30], [6, Example 4.14] and [5, Remark 3.3]).*

Lemma 2.2. [1, Corollary 3.12] *Let $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and*

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

an $\mathbb{E}$-triangle. Then we have the following long exact sequences:

$$\mathcal{C}(C,-) \xrightarrow{\mathcal{C}(y,-)} \mathcal{C}(B,-) \xrightarrow{\mathcal{C}(x,-)} \mathcal{C}(A,-) \xrightarrow{\delta^\sharp} \mathbb{E}(C,-) \xrightarrow{\mathbb{E}(y,-)} \mathbb{E}(B,-) \xrightarrow{\mathbb{E}(x,-)} \mathbb{E}(A,-);$$

$$\mathcal{C}(-,A) \xrightarrow{\mathcal{C}(-,x)} \mathcal{C}(-,B) \xrightarrow{\mathcal{C}(-,y)} \mathcal{C}(-,C) \xrightarrow{\delta_\sharp} \mathbb{E}(-,A) \xrightarrow{\mathbb{E}(-,x)} \mathbb{E}(-,B) \xrightarrow{\mathbb{E}(-,y)} \mathbb{E}(-,C),$$

where natural transformations $\delta_\sharp$ and $\delta^\sharp$ are induced by $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$ via Yoneda's lemma.

Let $\mathcal{C}, \mathbb{E}$ be as above, we use the following notation:

• A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a *conflation* if it realizes some $\mathbb{E}$-extension $\delta \in \mathbb{E}(C, A)$. In this case, x is called an *inflation*, y is called a *deflation*, and we write it as

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}.$$

We usually do not write this "δ" if it is not used in the argument.

• Given an $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, we call A the *CoCone* of $y : B \to C$ and C the *Cone* of $x : A \to B$.

• An $\mathbb{E}$-triangle sequence in $\mathcal{C}$ [11] is displayed as a sequence

$$\cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots$$

over $\mathcal{C}$ such that for any n, there are $\mathbb{E}$-triangles $K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \overset{\delta^n}{\dashrightarrow}$ and the differential $d_n = g_{n-1}f_n$.

• An object $P \in \mathcal{C}$ is called projective if for any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and any morphism $c \in \mathcal{C}(P, C)$, there exists $b \in \mathcal{C}(P, B)$ satisfying $y \circ b = c$. Injective objects are defined dually. We denote the subcategory consisting of projective (resp., injective) objects in $\mathcal{C}$ by $Proj(\mathcal{C})$ (resp., $Inj(\mathcal{C})$).

• We say $\mathcal{C}$ has enough projectives (resp., enough injectives) if for any object $C \in \mathcal{C}$ (resp., $A \in \mathcal{C}$), there exists an $\mathbb{E}$-triangle $A \xrightarrow{x} P \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ (resp., $A \xrightarrow{x} I \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$) satisfying $P \in Proj(\mathcal{C})$ (resp., $I \in Inj(\mathcal{C})$).

Remark 2.3. *(1) If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an exact category, then the definitions of having enough projectives and having enough injectives agree with the usual definitions.*

(2) If $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is a triangulated category, then $Proj(\mathcal{C})$ and $Inj(\mathcal{C})$ consist of zero objects.

Definition 2.4. [1, Definition 4.2] Let $\mathcal{X}$, $\mathcal{Y}$ be two subcategories of $\mathcal{C}$. Define full subcategories $\mathrm{Cone}(\mathcal{X}, \mathcal{Y})$ and $\mathrm{CoCone}(\mathcal{X}, \mathcal{Y})$ of $\mathcal{C}$ as follows.

(1) C belongs to $\mathrm{Cone}(\mathcal{X}, \mathcal{Y})$ if and only if it admits a conflation $X \to Y \to C$ satisfying $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$;

(2) C belongs to $\mathrm{CoCone}(\mathcal{X}, \mathcal{Y})$ if and only if it admits a conflation $C \to X \to Y$ satisfying $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Suppose that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated categories with enough projectives and injectives. For a subcategory $\mathcal{B} \subseteq \mathcal{C}$, put $\Omega^0\mathcal{B} = \mathcal{B}$, and for $i > 0$, we define $\Omega^i\mathcal{B}$ inductively by

$$\Omega^i\mathcal{B} = \Omega(\Omega^{i-1}\mathcal{B}) = \mathrm{CoCone}(Proj(\mathcal{C}), \Omega^{i-1}\mathcal{B}).$$

We call $\Omega^i\mathcal{B}$ the i-th syzygy of $\mathcal{B}$ (see [12, Section 5]). Dually we define the i-th cosyzygy $\Sigma^i\mathcal{B}$ by $\Sigma^0\mathcal{B} = \mathcal{B}$ and $\Sigma^i\mathcal{B} = \mathrm{Cone}(\Sigma^{i-1}\mathcal{B}, Inj(\mathcal{C}))$ for $i > 0$.

Let X be any object in $\mathcal{C}$. It admits an $\mathbb{E}$-triangle

$$X \longrightarrow I^0 \longrightarrow \Sigma X \overset{\delta^X}{\dashrightarrow} \qquad (\text{resp., } \Omega X \longrightarrow P_0 \longrightarrow X \overset{\delta_X}{\dashrightarrow}),$$

where $I^0 \in Inj(\mathcal{C})$ (resp., $P_0 \in Proj(\mathcal{C})$). In [12] the authors defined higher extension groups in an extriangulated category having enough projectives and injectives as $\mathbb{E}^{i+1}(X, Y) \cong \mathbb{E}(X, \Sigma^i Y) \cong \mathbb{E}(\Omega^i X, Y)$ for $i \geqslant 0$, and they showed the following result:

Lemma 2.5. *[12, Proposition 5.2] Let $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle. For any object $X \in \mathcal{B}$, there are long exact sequences*

$$\cdots \longrightarrow \mathbb{E}^i(X, A) \xrightarrow{x_*} \mathbb{E}^i(X, B) \xrightarrow{y_*} \mathbb{E}^i(X, C) \longrightarrow \mathbb{E}^{i+1}(X, A) \xrightarrow{x_*} \mathbb{E}^{i+1}(X, B) \xrightarrow{y_*} \cdots (i \geqslant 1),$$

$$\cdots \longrightarrow \mathbb{E}^i(C, X) \xrightarrow{y^*} \mathbb{E}^i(B, X) \xrightarrow{x^*} \mathbb{E}^i(A, X) \longrightarrow \mathbb{E}^{i+1}(C, X) \xrightarrow{y^*} \mathbb{E}^{i+1}(B, X) \xrightarrow{x^*} \cdots (i \geqslant 1).$$

From now on to the end of the paper, we always suppose that $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated categories with enough projectives and injectives.

3. Frobenius pairs and cotorsion pairs

In this section, we introduce the concept of Frobenius pairs and show that it has very nice homological properties, which are necessary to construct cotorsion pairs from Frobenius pairs. At first, we need introduce the following definitions.

Definition 3.1. Let $\mathcal{X}$ be a subcategory of $\mathcal{C}$.

1. For any non-negative integer n, we denote by $\widecheck{\mathcal{X}_n}$ (resp., $\widehat{\mathcal{X}_n}$) the class of objects $C \in \mathcal{C}$ such that there exists an $\mathbb{E}$-triangle sequence

$$C \to X_0 \to \cdots \to X_{n-1} \to X_n \text{ (resp., } X_n \to X_{n-1} \to \cdots \to X_0 \to C)$$

with each $X_i \in \mathcal{X}$. Moreover, we set $\check{\mathcal{X}} = \bigcup_{n=0}^{\infty} \widecheck{\mathcal{X}_n}$, $\hat{\mathcal{X}} = \bigcup_{n=0}^{\infty} \widehat{\mathcal{X}_n}$.

2. For any $C \in \mathcal{C}$, the $\mathcal{X}$*-resolution dimension* of C is defined as

$$\operatorname{resdim}_{\mathcal{X}}(C) := \min\{n \in \mathbb{N} : C \in \widehat{\mathcal{X}_n}\}.$$

If $C \notin \widehat{\mathcal{X}_n}$ for any $n \in \mathbb{N}$, then $\operatorname{resdim}_{\mathcal{X}}(C) = \infty$.

For a subcategory $\mathcal{X}$ of $\mathcal{C}$, define $\mathcal{X}^{\perp} = \{Y \in \mathcal{C} | \mathbb{E}^i(X, Y) = 0 \text{ for all } i \geqslant 1, \text{ and all } X \in \mathcal{X}\}$. Similarly, we can define ${}^{\perp}\mathcal{X}$.

Definition 3.2. Let $\mathcal{X}$ and $\mathcal{W}$ be two subcategories of $\mathcal{C}$. We say that

(1) $\mathcal{W}$ is a *cogenerator* for $\mathcal{X}$, if $\mathcal{W} \subseteq \mathcal{X}$ and for each object $X \in \mathcal{X}$, there exists an $\mathbb{E}$-triangle $X \longrightarrow W \longrightarrow X' \overset{\delta}{\dashrightarrow}$ with $W \in \mathcal{W}$ and $X' \in \mathcal{X}$. The notion of a *generator* is defined dually.

(2) $\mathcal{W}$ is $\mathcal{X}$*-injective* if $\mathcal{W} \subseteq \mathcal{X}^{\perp}$. The notion of an $\mathcal{X}$*-projective* subcategory is defined dually.

(3) $\mathcal{W}$ is an $\mathcal{X}$*-injective cogenerator* for $\mathcal{X}$ if $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ and $\mathcal{W} \subseteq \mathcal{X}^{\perp}$. The notion of an $\mathcal{X}$*-projective generator* for $\mathcal{X}$ is defined dually.

(4) $\mathcal{X}$ is a *thick subcategory* if it is closed under direct summand and for any $\mathbb{E}$-triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

in $\mathcal{C}$ and two of A, B, C are in $\mathcal{C}$, then so is the third.

The following theorem unifies some results of [13] and [9]. It shows that any object in $\hat{\mathcal{X}}$ admits two $\mathbb{E}$-triangles: one giving rise to an $\mathcal{X}$-precover and the other to a $\widehat{\mathcal{W}}$-preenvelope.

Theorem 3.3. *Let $\mathcal{X}$ and $\mathcal{W}$ be two subcategories of $\mathcal{C}$. Suppose $\mathcal{X}$ is closed under extensions and $\mathcal{W}$ is a cogenerator for $\mathcal{X}$. Consider the following conditions:*

(1) C is in $\widehat{\mathcal{X}_n}$.

(2) There exists an $\mathbb{E}$-triangle $Y_C \longrightarrow X_C \xrightarrow{\varphi_C} C \overset{\delta}{\dashrightarrow}$ with $X_C \in \mathcal{X}$ and $Y_C \in \widehat{\mathcal{W}_{n-1}}$.

(3) There exists an $\mathbb{E}$-triangle $C \xrightarrow{\psi^C} Y^C \longrightarrow X^C \overset{\theta}{\dashrightarrow}$ with $X^C \in \mathcal{X}$ and $Y^C \in \widehat{\mathcal{W}_n}$.

Then, $(1) \Leftrightarrow (2) \Rightarrow (3)$. *If $\mathcal{X}$ is also closed under CoCone of deflations, then* $(3) \Rightarrow (2)$, *and hence all three conditions are equivalent. If $\mathcal{W}$ is $\mathcal{X}$-injective, then φ_C is an $\mathcal{X}$-precover of C and ψ^C is a $\widehat{\mathcal{W}}$-preenvelope of C.*

Proof. The proof is dual to that of [14, Proposition 3.6]. □

Definition 3.4. A pair $(\mathcal{X}, \mathcal{W})$ is called a left *Frobenius pair* in $\mathcal{C}$ if the following holds:

(1) $\mathcal{X}$ is closed under extensions and CoCone of deflations,

(2) $\mathcal{W}$ is an $\mathcal{X}$-injective cogenerator for $\mathcal{X}$.

If in addition $\mathcal{W}$ is also an $\mathcal{X}$-projective generator for $\mathcal{X}$, then we say $(\mathcal{X}, \mathcal{W})$ is a *strong left Frobenius pair*.

Example 3.5. (1) Assume that $\mathcal{C} = R$-Mod is the category of left R-modules for a ring R. A left R-module N is called *Gorenstein projective* [23, 24] if there is an exact sequence of projective left R-modules

$$\mathbf{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

with $N = \mathrm{Ker}(P_0 \to P^0)$ such that $\mathrm{Hom}_R(\mathbf{P}, Q)$ is exact for any projective left R-module Q. Let $\mathcal{GP}(R)$ be the full subcategory of R-Mod consisting of all Gorenstein projective modules and $\mathcal{P}(R)$ the subcategory of R-Mod consisting of all projective modules. Then $(\mathcal{GP}(R), \mathcal{P}(R))$ is a strong left Frobenius pair.

(2) Let $\mathcal{C}$ be a triangulated category with a proper class ξ of triangles. Asadollahi and Salarian [15] introduced and studied ξ-Gprojective and ξ-Ginjective objects, and developed a relative homological algebra in $\mathcal{C}$. Let $\mathcal{GP}(\xi)$ denotes the full subcategory of ξ-Gprojective objects and $\mathcal{P}(\xi)$ denotes the full subcategory of ξ-projective objects. Then $(\mathcal{GP}(\xi), \mathcal{P}(\xi))$ is a strong left Frobenius pair.

(3) Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{M}$ be a silting subcategory of $\mathcal{T}$ with $\mathcal{M} = add\mathcal{M}$, where $add\mathcal{M}$ is the smallest full subcategory of $\mathcal{T}$ which contains $\mathcal{M}$ and which is closed under taking isomorphisms, finite direct sums, and direct summands. Then $(\mathcal{T}_{\geqslant 0}, \mathcal{M})$ is a left Frobenius pair by [14, Corollary 3.7] and [16, Proposition 2.7], where $\mathcal{T}_{\geqslant 0} := \bigcup_{n\geqslant 0} \mathcal{M}[-n] * \cdots \mathcal{M}[-1] * \mathcal{M}$.

(4) In [17], the authors showed that if $(\mathcal{X}, \mathcal{Y})$ is a complete and hereditary cotorsion pair in an abelian category $\mathcal{A}$ and $\mathcal{Y}$ is closed under kernels of epimorphisms, then $(\mathcal{G}(\mathcal{X}) \bigcap \mathcal{Y}, \mathcal{X} \bigcap \mathcal{Y})$ is a strong left Frobenius pair, where $\mathcal{G}(\mathcal{X})$ is the class of objects M in $\mathcal{A}$ satisfying that there exists an exact sequence

$$\mathbf{X} = \cdots \to X_1 \to X_0 \to X^0 \to X^1 \to \cdots$$

with each term in $\mathcal{X}$ such that $M \cong \mathrm{Ker}(X^0 \to X^1)$ and $\mathrm{Hom}_{\mathcal{A}}(\mathbf{X}, Q)$ is exact for any object Q in $\mathcal{X} \bigcap \mathcal{Y}$.

Lemma 3.6. *Let $(\mathcal{X}, \mathcal{W})$ be a left Frobenius pair in $\mathcal{C}$. Given an $\mathbb{E}$-triangle $K \xrightarrow{x} X \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ with $X \in \mathcal{X}$, then $C \in \widehat{\mathcal{X}}$ if and only if $K \in \widehat{\mathcal{X}}$.*

Proof. The proof is dual to that of [14, Lemma 3.8]. □

Proposition 3.7. *Let $(\mathcal{X}, \mathcal{W})$ be a left Frobenius pair in $\mathcal{C}$. The following statements are equivalent for any $C \in \widehat{\mathcal{X}}$ and non-negative integer n.*

(1) $resdim_{\mathcal{X}}(C) \leqslant n$.

(2) If $U \to X_{n-1} \to \cdots \to X_0 \to C$ is an $\mathbb{E}$-triangle sequence with $X_i \in \mathcal{X}$ for $0 \leqslant i \leqslant n-1$, then $U \in \mathcal{X}$.

Proof. (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2). Let C be in $\widehat{\mathcal{X}}$. Then by Theorem 3.3, we have an $\mathbb{E}$-triangle sequence $W_n \to \cdots \to W_1 \to X \to C$ with $X \in \mathcal{X}$ and $W_i \in \mathcal{W}$ for $1 \leqslant i \leqslant n$. Since $\mathcal{W} \subseteq \mathcal{X}^{\perp}$, it is easy to see that $\widehat{\mathcal{W}} \subseteq \mathcal{X}^{\perp}$. Thus we have $\mathbb{E}^{n+i}(C, Y) \cong \mathbb{E}^i(W_n, Y) = 0$ for all $i \geqslant 1$ and $Y \in \widehat{\mathcal{W}}$. If $U \to X_{n-1} \to \cdots \to X_0 \to C$ is an $\mathbb{E}$-triangle sequence with $X_i \in \mathcal{X}$ for $0 \leqslant i \leqslant n-1$, then we have $\mathbb{E}^i(U, Y) \cong \mathbb{E}^{n+i}(C, Y) = 0$ for all $i \geqslant 1$ and $Y \in \widehat{\mathcal{W}}$. Note that $U \in \widehat{\mathcal{X}}$ by Lemma 3.6. Hence there exists an $\mathbb{E}$-triangle $Y_U \longrightarrow X_U \longrightarrow U \dashrightarrow$ with $X_U \in \mathcal{X}$ and $Y_U \in \widehat{\mathcal{W}}$ by Theorem 3.3. It follows that the above $\mathbb{E}$-triangle splits. Hence $U \in \mathcal{X}$. $\square$

If $\mathcal{X}$ is a subcategory of C, then we denote by Thick$(\mathcal{X})$ the smallest thick subcategory that contains $\mathcal{X}$. The following result shows that for a left Frobenius pair $(\mathcal{X}, \mathcal{W})$ in C, $\widehat{\mathcal{X}}$ is an extriangulated category. In particular, if C is a triangulated category, then $\widehat{\mathcal{X}}$ is the smallest triangulated subcategory of C containing $\mathcal{X}$ and is closed under direct summands and isomorphisms.

Proposition 3.8. *Let $(\mathcal{X}, \mathcal{W})$ be a left Frobenius pair in C. Then* Thick$(\mathcal{X}) = \widehat{\mathcal{X}}$.

Proof. For any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$, we need to check that if any two of A, B and C are in $\widehat{\mathcal{X}}$, then the third one is in $\widehat{\mathcal{X}}$. Since $\widehat{\mathcal{X}}$ is closed under extensions by the dual of [14, Corollary 3.7], it suffices to show that if $B \in \widehat{\mathcal{X}}$, then $A \in \widehat{\mathcal{X}}$ if and only if $C \in \widehat{\mathcal{X}}$. We first show that if A and B are in $\widehat{\mathcal{X}}$, then $C \in \widehat{\mathcal{X}}$. Since $B \in \widehat{\mathcal{X}}$, we have an $\mathbb{E}$-triangle $Y_B \longrightarrow X_B \longrightarrow B \dashrightarrow$ with $X_B \in \mathcal{X}, Y_B \in \widehat{\mathcal{W}}$. By $(ET4)^{op}$, we obtain a commutative diagram

$$\begin{array}{ccccc} Y_B & \longrightarrow & L & \longrightarrow & A \\ \| & & \downarrow & & \downarrow \\ Y_B & \longrightarrow & X_B & \longrightarrow & B \\ & & \downarrow & & \downarrow \\ & & C & = & C. \end{array}$$

It follows that $L \in \widehat{\mathcal{X}}$ as A and Y_B are in $\widehat{\mathcal{X}}$. Therefore $C \in \widehat{\mathcal{X}}$.

Suppose now B and C are in $\widehat{\mathcal{X}}$. It follows from Lemma 3.6 that $L \in \widehat{\mathcal{X}}$. Applying the just established result to the $\mathbb{E}$-triangle $Y_B \longrightarrow L \longrightarrow A \dashrightarrow$, one has that $A \in \widehat{\mathcal{X}}$.

Suppose $C_1 \oplus C_2 \in \widehat{\mathcal{X}}$. We proceed by induction on n =resdim$_{\mathcal{X}}(C_1 \oplus C_2)$. If $n = 0$, then C_1 and C_2 are in $\mathcal{X}$.

Suppose $n > 0$. There is an $\mathbb{E}$-triangle $K \xrightarrow{x} X \xrightarrow{\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]} C_1 \oplus C_2 \overset{\delta}{\dashrightarrow}$ with $X \in \mathcal{X}$ and resdim$_{\mathcal{X}}(K) = n - 1$. By $(ET4)^{op}$, we obtain the following commutative diagrams:

$$\begin{array}{ccccccc} K & \longrightarrow & L_2 & \longrightarrow & C_1 & \dashrightarrow & \\ \| & & \downarrow{\scriptstyle x_2} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} & & \\ K & \xrightarrow{x} & X & \xrightarrow{\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]} & C_1 \oplus C_2 & \overset{\delta}{\dashrightarrow} & \\ & & \downarrow{\scriptstyle y_2} & & \downarrow{\scriptstyle [0 \ \ 1]} & & \\ & & C_2 & = & C_2 & & \\ & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle 0} & & \end{array} \qquad \begin{array}{ccccccc} K & \longrightarrow & L_1 & \longrightarrow & C_2 & \dashrightarrow & \\ \| & & \downarrow{\scriptstyle x_1} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} & & \\ K & \xrightarrow{x} & X & \xrightarrow{\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]} & C_1 \oplus C_2 & \overset{\delta}{\dashrightarrow} & \\ & & \downarrow{\scriptstyle y_1} & & \downarrow{\scriptstyle [1 \ \ 0]} & & \\ & & C_1 & = & C_1 & & \\ & & \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle 0} & & \end{array}$$

Hence there is an $\mathbb{E}$-triangle

$$L_1 \oplus L_2 \xrightarrow{\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}} X \oplus X \xrightarrow{\begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}} C_1 \oplus C_2 \overset{\delta_1 \oplus \delta_2}{\dashrightarrow}.$$

By Lemma 3.6, $L_1 \oplus L_2 \in \widehat{\mathcal{X}}$, and Proposition 3.7 shows that $\mathrm{resdim}_{\mathcal{X}}(L_1 \oplus L_2) \leqslant n-1$. By the induction hypothesis, L_1 and L_2 are in $\widehat{\mathcal{X}}$. It follows that C_1 and C_2 are in $\widehat{\mathcal{X}}$. Hence $\widehat{\mathcal{X}}$ is closed under direct summands. Thus $\mathrm{Thick}(\mathcal{X}) = \widehat{\mathcal{X}}$. □

Definition 3.9. [1, Definition 4.1] Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{C}$ be a pair of full additive subcategories, closed under isomorphisms and direct summands. The pair $(\mathcal{U}, \mathcal{V})$ is called a *cotorsion pair* on $\mathcal{C}$ if it satisfies the following conditions:

(1) $\mathbb{E}(\mathcal{U}, \mathcal{V}) = 0$;

(2) For any $C \in \mathcal{C}$, there exists a conflation $V^C \to U^C \to C$ satisfying $U^C \in \mathcal{U}$ and $V^C \in \mathcal{V}$;

(3) For any $C \in \mathcal{C}$, there exists a conflation $C \to V_C \to U_C$ satisfying $U_C \in \mathcal{U}$ and $V_C \in \mathcal{V}$.

Lemma 3.10. *Let $\mathcal{X}$ and $\mathcal{W}$ be two subcategories of $\mathcal{C}$ such that $\mathcal{W}$ is $\mathcal{X}$-injecive. Then the following statements hold.*

(1) If $\mathcal{W}$ is a cogenerator for $\mathcal{X}$, then $\mathcal{W} = \mathcal{X} \bigcap \mathcal{X}^{\perp} = \mathcal{X} \bigcap \widehat{\mathcal{W}}$.

(2) If $\mathcal{W}$ is a cogenerator for $\mathcal{X}$, then $\widehat{\mathcal{W}} = \widehat{\mathcal{X}} \bigcap \mathcal{X}^{\perp}$.

Proof. The proof is dual to that of [14, Proposition 4.2]. □

The following result gives a method to construct cotorsion pairs on extriangulated categories.

Proposition 3.11. *Let $(\mathcal{X}, \mathcal{W})$ be a left Frobenius pair in $\mathcal{C}$. Then $(\mathcal{X}, \widehat{\mathcal{W}})$ is a cotorsion pair on the extriangulated category* $\mathrm{Thick}(\mathcal{X})$.

Proof. Note that $\mathrm{Thick}(\mathcal{X})$ is an extriangulated category by [1, Remark 2.18]. It suffices to show that $\widehat{\mathcal{W}}$ is closed under direct summands by Theorem 3.3. Note that $\widehat{\mathcal{W}} = \widehat{\mathcal{X}} \bigcap \mathcal{X}^{\perp}$ by Proposition 3.10. Since $\mathrm{Thick}(\mathcal{X})$=$\widehat{\mathcal{X}}$ is closed under direct summands by Proposition 3.8, so is $\widehat{\mathcal{W}}$. This completes the proof. □

Now we are in a position to state and prove the main result of this section.

Theorem 3.12. *Let $\mathcal{C}$ be an extriangulated category. The assignments*

$$(\mathcal{X}, \mathcal{W}) \mapsto (\mathcal{X}, \widehat{\mathcal{W}}) \quad \text{and} \quad (\mathcal{U}, \mathcal{V}) \mapsto (\mathcal{U}, \mathcal{U} \bigcap \mathcal{V})$$

give mutually inverse bijections between the following classes:

(1) Left Frobenius pairs $(\mathcal{X}, \mathcal{W})$ in $\mathcal{C}$.

(2) Cotorsion pairs $(\mathcal{U}, \mathcal{V})$ on the extriangulated category $\mathrm{Thick}(\mathcal{U})$ *with $\mathcal{V} \subseteq \mathcal{U}^{\perp}$.*

Proof. Let $(\mathcal{X}, \mathcal{W})$ be a left Frobenius pair. Then $(\mathcal{X}, \widehat{\mathcal{W}})$ is a cotorsion pair on the extriangulated category $\widehat{\mathcal{X}}$ by Proposition 3.11. Note that $\mathrm{Thick}(\mathcal{X}) = \widehat{\mathcal{X}}$ and $\widehat{\mathcal{W}} \subseteq \mathcal{X}^{\perp}$. Then $(\mathcal{X}, \widehat{\mathcal{W}})$ is a cotorsion pair on the extriangulated category $\mathrm{Thick}(\mathcal{X})$ with $\widehat{\mathcal{W}} \subseteq \mathcal{X}^{\perp}$.

Assume $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair on the extriangulated category $\mathrm{Thick}(\mathcal{U})$ with $\mathcal{V} \subseteq \mathcal{U}^{\perp}$. For $U \in \mathcal{U}$, we have an $\mathbb{E}$-triangle $U \xrightarrow{x} V \xrightarrow{y} U' \overset{\delta}{\dashrightarrow}$ with $V \in \mathcal{V}$ and $U' \in \mathcal{U}$. Thus $V \in \mathrm{Thick}(\mathcal{U})$ as $\mathrm{Thick}(\mathcal{U})$ is a thick subcategory. Since $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair on the extriangulated category Thick($\mathcal{U}$), it follows from [1, Remark 4.6] that $\mathcal{U}$ is closed under extensions in Thick($\mathcal{U}$). It implies that $V \in \mathcal{U}$. Thus $V \in \mathcal{U} \bigcap \mathcal{V}$. Note that $\mathcal{V} \subseteq \mathcal{U}^{\perp}$. It follows that $\mathcal{U} \bigcap \mathcal{V}$ is an $\mathcal{U}$-injective cogenerator. Let $Z \xrightarrow{a} U_1 \xrightarrow{b} U_2 \dashrightarrow$ be an $\mathbb{E}$-triangle with $U_1, U_2 \in \mathcal{U}$. Then we have an exact sequence $\mathbb{E}(U_1, V) \longrightarrow \mathbb{E}(Z, V) \longrightarrow \mathbb{E}^2(U_2, V)$ for any $V \in \mathcal{V}$. Since $\mathcal{V} \subseteq \mathcal{U}^{\perp}$, $\mathbb{E}(Z, V) = 0$. Note that $Z \in$ Thick($\mathcal{U}$) as Thick($\mathcal{U}$) is a thick subcategory. Thus there exists an $\mathbb{E}$-triangle $V^Z \longrightarrow U^Z \longrightarrow Z \dashrightarrow$ with $U^Z \in \mathcal{U}$ and $V^Z \in \mathcal{V}$ as $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair on the extriangulated category Thick($\mathcal{U}$). Therefore the above $\mathbb{E}$-triangle splits by $\mathbb{E}(Z, \mathcal{V}) = 0$. Hence $Z \in \mathcal{U}$. So $\mathcal{U}$ is closed under CoCone of deflations. Note that $\mathcal{U}$ is closed under extensions in Thick($\mathcal{U}$). It follows that $\mathcal{U}$ is closed under extensions in C. Thus $(\mathcal{U}, \mathcal{U} \bigcap \mathcal{V})$ is a left Frobenius pair in C.

Based on the above argument, it is enough to check that the compositions

$$(\mathcal{U}, \mathcal{V}) \mapsto (\mathcal{U}, \mathcal{U} \textstyle\bigcap \mathcal{V}) \mapsto (\mathcal{U}, \widehat{\mathcal{U} \bigcap \mathcal{V}}) \quad \text{and} \quad (\mathcal{X}, \mathcal{W}) \mapsto (\mathcal{X}, \widehat{\mathcal{W}}) \mapsto \mathcal{X}$$

are identities. Since $\mathcal{U} \bigcap \mathcal{V}$ is an $\mathcal{U}$-injective cogenerator for $\mathcal{U}$, $\widehat{\mathcal{U} \bigcap \mathcal{V}} = \hat{\mathcal{U}} \bigcap \mathcal{U}^{\perp} =$ Thick($\mathcal{U}$) $\bigcap \mathcal{U}^{\perp}$ where the first equality is due to Proposition 3.10 and the second equality is due to Proposition 3.8. It follows from [1, Remark 4.4] that Thick($\mathcal{U}$) $\bigcap \mathcal{U}^{\perp} = \mathcal{V}$. Thus $\widehat{\mathcal{U} \bigcap \mathcal{V}} = \mathcal{V}$. This completes the proof. □

As a consequence of Theorem 3.12 and Remark 2.3, we have the following result.

Corollary 3.13. *[7, Throrem 5.4] Let $\mathcal{A}$ be an abelian category with enough projectives and injectives. The assignments*

$$(\mathcal{X}, \mathcal{W}) \mapsto (\mathcal{X}, \widehat{\mathcal{W}}) \quad \textit{and} \quad (\mathcal{U}, \mathcal{V}) \mapsto (\mathcal{U}, \mathcal{U} \textstyle\bigcap \mathcal{V})$$

give mutually inverse bijections between the following classes:

(1) Left Frobenius pairs $(\mathcal{X}, \mathcal{W})$ in $\mathcal{A}$.

(2) Cotorsion pairs $(\mathcal{U}, \mathcal{V})$ on the exact category Thick($\mathcal{U}$) *with $\mathcal{V} \subseteq \mathcal{U}^{\perp}$.*

As an application, we have the following result in [10].

Corollary 3.14. *[10, Theorem 3.11] Let C be a triangulated category. The assignments*

$$(\mathcal{X}, \mathcal{W}) \mapsto (\mathcal{X}, \widehat{\mathcal{W}}) \quad \textit{and} \quad (\mathcal{U}, \mathcal{V}) \mapsto (\mathcal{U}, \mathcal{U} \textstyle\bigcap \mathcal{V})$$

give mutually inverse bijections between the following classes:

(1) Left Frobenius pairs $(\mathcal{X}, \mathcal{W})$ in C.

(2) Co-t-structures $(\mathcal{U}, \mathcal{V})$ on the triangulated category Thick($\mathcal{U}$).

Proof. Note that any triangulated category can be viewed as an extriangulated category, and its projective objects and injective objects consist of zero objects by Remark 2.3.

Let $(\mathcal{X}, \mathcal{W})$ be a left Frobenius pair. By Theorem 3.12, $(\mathcal{X}, \widehat{\mathcal{W}})$ is a cotorsion pair on the triangulated category Thick($\mathcal{X}$). Since $\mathcal{X}$ is closed under CoCone of deflations and extensions, it is easy to see that $\mathcal{X}[-1] \subseteq \mathcal{X}$. Hence $(\mathcal{X}, \widehat{\mathcal{W}})$ is a co-t-structure on the triangulated category Thick($\mathcal{X}$).

Assume $(\mathcal{U}, \mathcal{V})$ is a co-t-structure on the triangulated category Thick($\mathcal{U}$). It is easy to see that $(\mathcal{U}, \mathcal{V})$ is a cotorsion pair on Thick($\mathcal{U}$) with $\mathcal{V} \subseteq \mathcal{U}^{\perp}$. Hence the corollary follows from Theorem 3.12. □

Definition 3.15. [18, Definition 2.1] Let R and S be rings. An $(S\text{-}R)$-bimodule $C = {}_SC_R$ is semidualizing if:

(1) ${}_SC$ admits a degreewise finite S-projective resolution.

(2) C_R admits a degreewise finite R-projective resolution.

(3) The homothety map ${}_SS_S \xrightarrow{{}_S\gamma} \mathrm{Hom}_R(C,C)$ is an isomorphism.

(4) The homothety map ${}_RR_R \xrightarrow{\gamma_R} \mathrm{Hom}_S(C,C)$ is an isomorphism.

(5) $\mathrm{Ext}_S^{\geqslant 1}(C,C) = 0 = \mathrm{Ext}_R^{\geqslant 1}(C,C)$.

Definition 3.16. [18, Definition 3.1] A semidualizing bimodule $C = {}_SC_R$ is faithfully semidualizing if it satisfies the following conditions for all modules ${}_SN$ and M_R.

(1) If $\mathrm{Hom}_S(C,N) = 0$, then $N = 0$.

(2) If $\mathrm{Hom}_R(C,M) = 0$, then $M = 0$.

Definition 3.17. [18, Definition 4.1] The Bass class $\mathcal{B}_C(S)$ with respect to C consists of all S-modules N satisfying

(1) $\mathrm{Ext}_S^{\geqslant 1}(C,N) = 0 = \mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_S(C,N)) = 0$.

(2) The natural evaluation homomorphism $\nu_N : C \otimes_R \mathrm{Hom}_S(C,N) \to N$ is an isomorphism.

Remark 3.18. *Let $C = {}_SC_R$ be a faithfully semidualizing module. Then Bass class $\mathcal{B}_C(S)$ is an exact category by [18, Theorem 6.2] and $\mathcal{B}_C(S)$ has enough projectives and injectives by [20, Remark 3.13].*

By [18], the class of *C-projective* left S-modules, denoted by $\mathcal{P}_C(S)$ the collection of the left S-modules of the form $C \otimes_R P$ for some projective left R-module P. Recall from [20] that a left S-module M is called *C-Gorenstein projective* if there is an exact sequence of left S-modules

$$\mathbf{W} = \cdots \to W_1 \to W_0 \to W^0 \to W^1 \to \cdots$$

with each term in $\mathcal{P}_C(S)$ such that $N \cong \mathrm{Ker}(W_0 \to W^0)$ and both $\mathrm{Hom}_R(\mathbf{W}, Q)$ and $\mathrm{Hom}_R(Q, \mathbf{W})$ are exact for any object Q in $\mathcal{P}_C(S)$. It should be noted that C-Gorenstein projectives defined here are different from those defined in [19] when $S = R$ is a commutative Noetherian ring (see [20, Proposition 3.6]).

For convenience, we write G_C-$Proj(S)$ for the classes of C-Gorenstein projective left S-modules. By [20, Proposition 3.5], one has that $G_C\text{-}Proj(S) \subseteq \mathcal{B}_C(S)$. As a consequence of Theorem 3.12, we have the following result.

Corollary 3.19. *Let $C = {}_SC_R$ be a faithfully semidualizing module. Then*

(1) $(G_C\text{-}Proj(S), \mathcal{P}_C(S))$ is a strong left Frobenius pair in $\mathcal{B}_C(S)$.

(2) $(G_C\text{-}Proj(S), \widehat{\mathcal{P}_C(S)})$ is a cotorsion pair on $G_C\text{-}\widehat{Proj}(S)$.

Proof. Since $\mathcal{P}_C(S)$ is projectively resolving and $\mathcal{P}_C(S) \subseteq \mathcal{P}_C(S)^{\perp}$ by [18, Corollary 6.4] and [18, Theorem 6.4], G_C-$Proj(S)$ is closed under kernels of epimorphisms and direct summand by [21, Theorem 4.12] and [21, Proposition 4.11]. Hence $(G_C\text{-}Proj(S), \mathcal{P}_C(S))$ is a strong left Frobenius pair in $\mathcal{B}_C(S)$. (2) follows from Theorem 3.12. □

4. Admissible model structures associated with Frobenius pairs

In this section, we shall use our results in Section 3 to construct more admissible model structures in extriangulated categories. At first, we need to recall the following definition.

Definition 4.1. [1, Definition 5.1] Let $(\mathcal{S}, \mathcal{T})$ and $(\mathcal{U}, \mathcal{V})$ be cotorsion pairs on $\mathcal{C}$. Then $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ is called a *twin cotorsion pair* if it satisfies $\mathbb{E}(\mathcal{S}, \mathcal{V}) = 0$. Moreover, $\mathcal{P}$ is called a *Hovey twin cotorsion pair* if it satisfies $\mathrm{Cone}(\mathcal{V}, \mathcal{S}) = \mathrm{CoCone}(\mathcal{V}, \mathcal{S})$.

In [1] Nakaoka and Palu gave a correspondence between admissible model structures and Hovey twin cotorsion pairs on $\mathcal{C}$. Essentially, an admissible model structure on $\mathcal{C}$ is a Hovey twin cotorsion pair $\mathcal{P} = ((\mathcal{S}, \mathcal{T}), (\mathcal{U}, \mathcal{V}))$ on $\mathcal{C}$. For more details, we refer to [1, Section 5]. By a slight abuse of language we often refer to a Hovey twin cotorsion pair as an admissible model structure.

Lemma 4.2. *Let $(\mathcal{X}, \mathcal{W})$ be a strong left Frobenius pair in $\mathcal{C}$. Then $(\mathcal{W}, \widehat{\mathcal{X}})$ is a cotorsion pair on the extriangulated category* $\mathrm{Thick}(\mathcal{X})$.

Proof. Since $\mathcal{W} \subseteq {}^{\perp}\mathcal{X}$, one has $\mathbb{E}(\mathcal{W}, \widehat{\mathcal{X}}) = 0$. For any $C \in \widehat{\mathcal{X}}$, there exists an $\mathbb{E}$-triangle

$$Y_C \longrightarrow X_C \longrightarrow C \dashrightarrow$$

with $X_C \in \mathcal{X}$ and $Y_C \in \widehat{\mathcal{W}}$ by Theorem 3.3. Since $\mathcal{W}$ is a generator for $\mathcal{X}$, we have an $\mathbb{E}$-triangle $X \longrightarrow W \longrightarrow X_C \dashrightarrow$ with $W \in \mathcal{W}$ and $X \in \mathcal{X}$. By $(ET4)^{op}$, we obtain a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longrightarrow & Y_C \\ \| & & \downarrow & & \downarrow \\ X & \longrightarrow & W & \longrightarrow & X_C \\ & & \downarrow & & \downarrow \\ & & C & = & C. \end{array}$$

It follows that $Z \in \widehat{\mathcal{X}}$ as X and Y_C are in $\widehat{\mathcal{X}}$. Note that $\mathrm{Thick}(\mathcal{X}) = \widehat{\mathcal{X}}$. The second column and $\mathbb{E}(\mathcal{W}, \widehat{\mathcal{X}}) = 0$ show that $(\mathcal{W}, \widehat{\mathcal{X}})$ is a cotorsion pair on $\mathrm{Thick}(\mathcal{X})$. □

Proposition 4.3. *Let $(\mathcal{X}, \mathcal{W})$ be a strong left Frobenius pair in $\mathcal{C}$. Then $\mathcal{P} = ((\mathcal{W}, \widehat{\mathcal{X}}), (\mathcal{X}, \widehat{\mathcal{W}}))$ is an admissible model structure on the extriangulated category* $\mathrm{Thick}(\mathcal{X})$

Proof. By Theorem 3.12 and Lemma 4.2, we only need to check that $\mathrm{Cone}(\widehat{\mathcal{W}}, \mathcal{W}) = \mathrm{CoCone}(\widehat{\mathcal{W}}, \mathcal{W})$. It is obvious that $\mathrm{Cone}(\widehat{\mathcal{W}}, \mathcal{W}) = \widehat{\mathcal{W}} \subseteq \mathrm{CoCone}(\widehat{\mathcal{W}}, \mathcal{W})$. Let $C \in \mathrm{CoCone}(\widehat{\mathcal{W}}, \mathcal{W})$. Then we have an $\mathbb{E}$-triangle $C \longrightarrow Y \longrightarrow W \dashrightarrow$ with $Y \in \widehat{\mathcal{W}}$ and $W \in \mathcal{W}$. By Theorem 3.3, one has that $C \in \widehat{\mathcal{X}}$. Since $\mathbb{E}(\mathcal{W}, \widehat{\mathcal{X}}) = 0$, it follows that C is a direct summand of Y. Note that $\widehat{\mathcal{W}}$ is closed under direct summand by Proposition 3.11. Thus $C \in \widehat{\mathcal{W}}$. Hence the equality $\mathrm{Cone}(\widehat{\mathcal{W}}, \mathcal{W}) = \mathrm{CoCone}(\widehat{\mathcal{W}}, \mathcal{W})$ holds. □

Theorem 4.4. *Let $(\mathcal{X}, \mathcal{W})$ be a strong left Frobenius pair in $\mathcal{C}$. If n is a non-negative integer, then the following statements are equivalent:*

(1) $\widehat{\mathcal{X}_n} = \mathcal{C}$.

(2) $\mathcal{P} = ((\mathcal{W}, \mathcal{C}), (\mathcal{X}, \widehat{\mathcal{W}_n}))$ is an admissible model structure on $\mathcal{C}$.

Proof. (1) $\Rightarrow$ (2). If $\widehat{\mathcal{X}_n} = \mathcal{C}$, then $(\mathcal{W}, \mathcal{C})$ is a cotorsion pair on $\mathcal{C}$ by Lemma 4.2 and $(\mathcal{X}, \widehat{\mathcal{W}})$ is a cotorsion pair on $\mathcal{C}$ by Theorem 3.12. To prove (2), we only need to check that $\widehat{\mathcal{W}_n} = \widehat{\mathcal{W}}$. Note that $\widehat{\mathcal{W}_n} \subseteq \widehat{\mathcal{W}}$ is obvious. Let $C \in \widehat{\mathcal{W}}$. Then there is an $\mathbb{E}$-triangle $Y_C \longrightarrow X_C \longrightarrow C \dashrightarrow$ with $X_C \in \mathcal{X}$

and $Y_C \in \widehat{\mathcal{W}_{n-1}}$ by Theorem 3.3. Since $\widehat{\mathcal{W}}$ is closed under extensions, $X_C \in \mathcal{X} \bigcap \widehat{\mathcal{W}} = \mathcal{W}$. Hence $C \in \widehat{\mathcal{W}_n}$ implies $\widehat{\mathcal{W}_n} = \widehat{\mathcal{W}}$.

(2) $\Rightarrow$ (1). Since $(\mathcal{X}, \widehat{\mathcal{W}_n})$ is a cotorsion pair on C, one has that $C = \widehat{\mathcal{X}_n}$ by Theorem 3.3. □

As an application, we have the following result in [3].

Corollary 4.5. *[3, Theorem 8.6] Suppose R is a Gorenstein ring. Let $\mathcal{GP}(R)$ be the subcategory of* R-Mod *consisting of Gorenstein projective modules and $\mathcal{P}(R)$ the subcategory of* R-Mod *consisting of projective modules. Then $((\mathcal{P}(R), R\text{-Mod}), (\mathcal{GP}(R), \widehat{\mathcal{P}(R)}))$ is an admissible model structure on* R-Mod.

Proof. It follows from Example 3.5 and Proposition 4.3. □

Let n be a non-negative integer. In the following, we denote by $G_C\text{-}Proj(S)_{\leqslant n}$ (resp., $\mathcal{P}_C(S)_{\leqslant n}$) the class of modules with C-Gorenstein projective (resp., C-projective) dimension at most n

Corollary 4.6. *Let $C = {}_SC_R$ be a faithfully semidualizing module. Then the following statements are equivalent:*

(1) $G_C\text{-}Proj(S)_{\leqslant n} = \mathcal{B}_C(S)$.

(2) $\mathcal{P} = ((\mathcal{P}_C(S), \mathcal{B}_C(S)), (G_C\text{-}Proj(S), \mathcal{P}_C(S)_{\leqslant n})$ is an admissible model structure on $\mathcal{B}_C(S)$.

Proof. It follows from Corollary 3.19 and Theorem 4.4. □

Let $(C, \mathbb{E}, \mathfrak{s})$ be an extriangulated category and ξ a proper class of $\mathbb{E}$-triangles. By [5], an object $P \in C$ is called *ξ-projective* if for any $\mathbb{E}$-triangle

$$A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$$

in ξ, the induced sequence of abelian groups $0 \longrightarrow C(P, A) \longrightarrow C(P, B) \longrightarrow C(P, C) \longrightarrow 0$ is exact. We denote $\mathcal{P}(\xi)$ the class of ξ-projective objects of C. Recall from [5] that an object $M \in C$ is called *ξ-$\mathcal{G}$projective* if there exists a diagram

$$\mathbf{P} : \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \longrightarrow \cdots$$

in C satisfying that: (1) P_n is ξ-projective for each integer n; (2) there is a $C(-, \mathcal{P}(\xi))$-exact $\mathbb{E}$-triangle $K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \overset{\delta_n}{\dashrightarrow}$ in ξ and $d_n = g_{n-1}f_n$ for each integer n such that $M \cong K_n$ for some $n \in \mathbb{Z}$. We denote by $\mathcal{GP}(\xi)$ the class of ξ-$\mathcal{G}$projective objects in C. Specializing Theorem 4.4 to the case $\mathcal{X} = \mathcal{GP}(\xi)$, we have the following result in [5].

Corollary 4.7. *[5, Theorem 5.9] Let $(C, \mathbb{E}, \mathfrak{s})$ be an extriangulated category satisfying Condition* (WIC) (*see [1, Condition 5.8]*). *Assume that ξ is a proper class in C. Set $\mathbb{E}_\xi := \mathbb{E}|_\xi$, that is,*

$$\mathbb{E}_\xi(C, A) = \{\delta \in \mathbb{E}(C, A) \mid \delta \text{ is realized as an } \mathbb{E}\text{-triangle } A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow} \text{ in } \xi\}$$

for any $A, C \in C$, and $\mathfrak{s}_\xi := \mathfrak{s}|_{\mathbb{E}_\xi}$. If n is a non-negative integer, then the following conditions are equivalent:

(1) $\sup\{\xi\text{-}\mathcal{G}\mathrm{pd}A \mid A \in C\} \leqslant n$.

(2) $\mathcal{P} = ((\mathcal{P}(\xi), C), (\mathcal{GP}(\xi), \mathcal{P}^{\leqslant n}(\xi)))$ *is an admissible model structure on $(C, \mathbb{E}_\xi, \mathfrak{s}_\xi)$, where $\mathcal{P}^{\leqslant n}(\xi) = \{A \in C \mid \xi\text{-}\mathrm{pd}A \leqslant n\}$.*

Proof. It is easy to check that $(\mathcal{GP}(\xi), \mathcal{P}(\xi))$ is a strong left Frobenius pair in $(C, \mathbb{E}_\xi, \mathfrak{s}_\xi)$. Thus the corollary follows from Theorem 4.4. □

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. H. Nakaoka, Y. Palu, Extriangulated categories, Hovey twin cotorsion pairs and model structures, *Cah. Topol. Géom. Différ. Catég.*, **60** (2019), 117–193.
2. J. Gillespie, Model structures on exact categories, *J. Pure Appl. Algebra*, **215** (2011), 2892–2902. https://doi.org/10.1016/j.jpaa.2011.04.010
3. M. Hovey, Cotorsion pairs, model category structures, and representation theory, *Math. Z.*, **241** (2002), 553–592. https://doi.org/10.1007/s00209-002-0431-9
4. X. Y. Yang, Model structures on triangulated categories, *Glasg. Math. J.*, **57** (2015), 263–284. https://doi.org/10.1017/S0017089514000299
5. J. S. Hu, D. D. Zhang, P. Y. Zhou, Proper classes and Gorensteinness in extriangulated categories, *J. Algebra*, **551** (2020), 23–60. https://doi.org/10.1016/j.jalgebra.2019.12.028
6. P. Y. Zhou, B. Zhu, Triangulated quotient categories revisited, *J. Algebra*, **502** (2018), 196–232. https://doi.org/10.1016/j.jalgebra.2018.01.031
7. V. Becerril, O. Mendoza, M. A. Pérez, V. Santiago, Frobenius pairs in abelian categories: Correspondences with cotorsions pairs, exact model categories, and Auslander-Buchweitz contexts, *J. Homotopy Relat. Struct.*, **14** (2019), 1–50.
8. Z. X. Di, Z. K. Liu, J. P. Wang, J. Q. Wei, An Auslander-Buchweitz approximation approach to (pre)silting subcategories in triangulated categories, *J. Algebra*, **525** (2019), 42–63. https://doi.org/10.1016/j.jalgebra.2019.01.021
9. O. Mendoza Hernández, E. Sáenz, V. Santiago Vargas, M. Souto Salorio, Auslander-Buchweitz approximation theory for triangulated categories, *Appl. Categ. Structures*, **21** (2013a), 119–139. https://doi.org/10.1007/s10485-011-9261-4
10. O. Mendoza Hernández, E. Sáenz, V. Santiago Vargas, M. Souto Salorio, Auslander-Buchweitz context and co-t-structures, *Appl. Categ. Structures*, **21** (2013b), 417–440. https://doi.org/10.1007/s10485-011-9271-2
11. B. Zhu, X. Zhuang, Tilting subcategories in extriangulated categories, *Front. Math. China*, **15** (2020), 225–253. https://doi.org/10.1007/s11464-020-0811-7
12. Y. Liu, H. Nakaoka, Hearts of twin cotorsion pairs on extriangulated categories, *J. Algebra*, **528** (2019), 96–149. https://doi.org/10.1016/j.jalgebra.2019.03.005

13. M. Auslander, R. O. Buchweitz, The homological theory of maximal Cohen-Macaulay approximations, *Mem. Soc. Math. France*, **38** (1989), 5–37. https://doi.org/10.24033/msmf.339

14. Y. J. Ma, N. Q. Ding, Y. F. Zhang, J. S. Hu, A new characterization of silting subcategories in the stable category of a Frobenius extriangulated category, *arXiv preprint*, arXiv: 2012. 03779v2.

15. J. Asadollahi, S. Salarian, Gorenstein objects in triangulated categories, *J. Algebra*, **281** (2004), 264–286. https://doi.org/10.1016/j.jalgebra.2004.07.027

16. O. Iyama, D. Yang, Silting reduction and Calabi-Yau reduction of triangulated categories, *Trans. Amer. Math. Soc.*, **370** (2018), 7861–7898. https://doi.org/10.1090/tran/7213

17. W. J. Chen, Z. K. Liu, X. Y. Yang, A new method to construct model structures from a cotorsion pair, *Comm. Algebra*, **47** (2017), 4420–4431.

18. H. Holm, D. White, Foxby equivalence over associative rings, *J. Math. Kyoto Univ.*, **47** (2007), 781–808. https://doi.org/10.1215/kjm/1250692289

19. H. Holm, P. Jørgensen, *Semi-dualizing modules and related Gorenstein homological dimensions*, *J. Pure Appl. Algebra*, **205** (2006), 423–445. https://doi.org/10.1016/j.jpaa.2005.07.010

20. Y. X. Geng, N. Q. Ding, $\mathcal{W}$-Gorenstein modules, *J. Algebra*, **325** (2011), 132–146.

21. S. Sather-Wagstaff, T. Sharif, D. White, Stability of Gorenstein categories, *J. Lond. Math. Soc.*, **77** (2008), 481–502. https://doi.org/10.1112/jlms/jdm124

22. T. Bühler, Exact categories, *Expo. Math.*, **28** (2010), 1–69. https://doi.org/10.1016/j.exmath.2009.04.004

23. E. E. Enochs, O. M. G. Jenda, Relative Homological Algebra, Walter de Gruyter, Berlin, New York, 2000. https://doi.org/10.1515/9783110803662

24. H. Holm, Gorenstein homological dimensions, *J. Pure Appl. Algebra*, **189** (2004), 167–193. https://doi.org/10.1016/j.jpaa.2003.11.007