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Research article

Blow-up properties of solutions to a class of p-Kirchhoff evolution equations

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Abstract: This paper is devoted to an initial-boundary value problem for a class of p-Kirchhoff type parabolic equations. Firstly, we consider this problem with a general nonlocal coefficient $M(\|\nabla u\|_p^p)$ and a general nonlinearity $k(t)f(u)$. A new finite time blow-up criterion is established, also, the upper and lower bounds for the blow-up time are derived. Secondly, we deal with the case that $M(\|\nabla u\|_p^p) = a + b\|\nabla u\|_p^p$, $k(t) \equiv 1$ and $f(u) = |u|^{q-1}u$, which was considered by Li and Han [Math. Model. Anal. 2019; 24: 195-217] only for $q > 2p-1$. The threshold results for the existence of global and finite time blow-up solutions to this problem are obtained for the case $1 < q \le 2p-1$, which, together with the results given by Li and Han, shows that $q = 2p-1$ is critical for the existence of finite time blow-up solutions to this problem. These results partially generalize and extend some recent ones in previous literature.

Keywords: p-Kirchhoff equation; general nonlinearity; blow-up; critical exponent

1. Introduction

In this paper, we consider the following initial-boundary value problem for a p-Kirchhoff type parabolic equation with a general nonlocal coefficient and a general nonlinearity

$$\begin{cases} u_t - M(\int_\Omega |\nabla u|^p \mathrm{d}x)\Delta_p u = k(t)f(u), & x \in \Omega, 0 < t < T, \\ u(x,t) = 0, & x \in \partial\Omega, 0 < t < T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the standard p-Laplace operator with $p > 2$, $\Omega \subset \mathbb{R}^n (n \ge 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $T \in (0, +\infty]$ is the maximal existence time of the solution $u(x,t)$ and $u_0 \in W_0^{1,p}(\Omega)$. The nonlocal coefficient $M(t)$, the nonlinearity $f(s)$ and the time-dependent function $k(t)$ are supposed to satisfy the following assumptions:

(H1) $M(t) \in C[0,\infty)$ and $M(t) \geq m_0$ for some $m_0 > 0$ and for all $t \geq 0$. Moreover, there exists a constant $\sigma \in (0,1)$ such that

$$\overline{M}(t) \geq \sigma t M(t), \ \ \forall\, t \in \mathbb{R}^+, \tag{1.2}$$

where $\overline{M}(t) = \int_0^t M(s)\mathrm{d}s$;

(H2) $sf(s) \geq 0, \ \forall\, s \in \mathbb{R}$;

(H3) $f(s) \in C^1(\mathbb{R})$, and there exists a constant $\alpha > \frac{p}{\sigma} - 1 (> 1)$ such that

$$s[sf'(s) - \alpha f(s)] \geq 0, \ \forall\, s \in \mathbb{R};$$

(H4) There exist a positive integer l and constants $a_i > 0 (1 \leq i \leq l)$ such that

$$|f(s)| \leq \sum_{i=1}^{l} a_i |s|^{p_i}, \ \forall\, s \in \mathbb{R},$$

where $1 < p_1 < \ldots < p_l < p^* - 1$, p^* is the Sobolev conjugate of p, i.e., $p^* = +\infty$ for $n \leq p$ and $p^* = \frac{np}{n-p}$ for $n > p$;

(H5) $k(t) \in C^1[0,\infty)$, $k(0) > 0$ and $k'(t) \geq 0$ for all $t \in [0,\infty)$.

After the pioneer work of Lions [1], where a functional analysis method was proposed, the well-posedness of solutions to Kirchhoff-type (elliptic or evolution) equations have drawn more and more attention. Many authors focused on the following Kirchhoff type elliptic equation

$$- M(\int_{\Omega} |\nabla u|^{\alpha} \mathrm{d}x)(\Delta_p u) = f(x,u). \tag{1.3}$$

When $\alpha = 1, p = 2, M(s) = \theta + s, \theta \in \mathbb{R}$, Chen et al. [2] derived a sequence of positive isolated singular solutions such that the nonlocal coefficient is positive, by applying the Schauder fixed point theorem. When $\alpha = p = 2$, $M(s) = a + bs, a, b > 0$, He et al. [3] obtained the existence of at least one or two positive solutions by employing the monotonicity trick and established a nonexistence criterion by using Pohožaev identity. When $\alpha = p > 1$ and $f(x,u)$ satisfies critical growth condition, Hamydy et al. [4] proved that (1.3) admits at least one nontrivial solution via the variational method.

Next we review some works related to evolution equation (1.1). When $M(s) \equiv 1$, (1.1) is called the p-Laplace type parabolic problem. Local existence, uniqueness and regularity of weak solutions to this kind of problems have been studied extensively. Interested readers may refer to, for example, [5,6] and the references therein for such results. In particular, Tsutsumi [7] considered the following nonlinear p-Laplace problem

$$\begin{cases} u_t - \Delta_p u = u^{1+\alpha}, & x \in \Omega, 0 < t < T, \\ u(x,t) = 0, & x \in \partial\Omega, 0 < t < T, \\ u(x,0) = u_0(x). & x \in \Omega \end{cases} \tag{1.4}$$

He showed that $p = 2 + \alpha$ is the critical blow-up exponent, i.e., when $p > 2 + \alpha$, problem (1.4) has a unique nonnegative global solution for any nonnegative initial values, and when $p < 2 + \alpha$, problem (1.4) has a global solution for sufficiently small (nonnegative) initial data $u_0(x)$, while the solution blows up in finite time if $u_0(x)$ is large enough. The blow-up result for $p < 2 + \alpha$ was also given in [8].

When $M(s)$ is not a constant function, (1.1) is usually classified as p-Kirchhoff type parabolic problems, which evolved from the following Kirchhoff type hyperbolic equation

$$\epsilon u_{tt}^{\epsilon} + u_t^{\epsilon} - M\left(\int_\Omega |\nabla u^{\epsilon}|^p \mathrm{d}x\right)\Delta_p u^{\epsilon} = f(x,t,u^{\epsilon}). \tag{1.5}$$

(1.5) is an extension of the classical D'Alembert wave equation for free oscillations of elastic strings (see [9]). In [10], Lin et al. discussed the Kirchhoff type hyperbolic problem involving the fractional Laplacian

$$\begin{cases} u_{tt} + [u]_s^{2(\theta-1)}(-\Delta)^s u = f(u), & x \in \Omega, 0 < t < T, \\ u(x,t) = 0, & x \in \partial\Omega, 0 < t < T, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega \end{cases} \tag{1.6}$$

where $[u]_s$ is the Gagliardo seminorm of u. They establish some sufficient conditions under which the solutions to problem (1.6) blow up in finite time at arbitrary positive initial energy level. For more recent works on Kirchhoff type hyperbolic problems, for instance, we refer the readers to [11, 12] and the references therein. By formally taking $\epsilon = 0$, (1.5) changes into the following p-Kirchhoff type parabolic equation

$$u_t - M\left(\int_\Omega |\nabla u|^p \mathrm{d}x\right)\Delta_p u = f(x,t,u). \tag{1.7}$$

As a mathematical model, problem (1.7) can be used to describe the motion of a nonstationary fluid or gas in a nonhomogeneous and anisotropic medium [13, 14]. When $0 < m \le M(s) < M_0$ for all $s > 0$ and $f(x,t,u) \equiv f(x)$, the existence, uniqueness and the asymptotic behavior of the solution to (1.7) were well studied for both $p = 2$ and $p > 1$ in [13, 15].

Our motivation to study problem (1.1) is that it can model the density of population (for example of bacteria) affected by spreading [16]. The diffusion coefficient $M(\int_\Omega |\nabla u|^p \mathrm{d}x)$ depends on a nonlocal quantity which related to the total population in Ω. The so-called nonlocal quantity represents the average value of the measurement. The nonlinearity $k(t)f(u)$ represents the change of the population over time. In particular, when $M(s) = a + bs$, $a, b > 0$, $k(t) \equiv 1$ and $f(s) = |s|^{q-1}s$, problem (1.1) turns into the following form

$$\begin{cases} u_t - (a + b\int_\Omega |\nabla u|^p \mathrm{d}x)\Delta_p u = |u|^{q-1}u, & x \in \Omega, 0 < t < T, \\ u(x,t) = 0, & x \in \partial\Omega, 0 < t < T, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.8}$$

Li et al. [17] comprehensively investigated the global well-posedness and finite time blow-up of solutions to problem (1.8) for the case $p > 1$ and $2p - 1 < q \le p^* - 1$. By using the modified potential well method (see [18, 19] for example) and some variational tricks, the authors gave the sufficient conditions for the existence of global and finite time blow-up solutions to problem (1.8) for subcritical, critical and supercritical initial energy. However, it should be pointed out that only the case $q > 2p - 1$ was considered in [17], but whether or not problem (1.8) admits blow-up solutions for $1 < q \le 2p - 1$ is still unknown.

Our consideration of problems (1.1) and (1.8) is mainly motivated by [7, 17], and the main purpose of this paper is twofold. The first one is that we aim to establish a new blow-up criterion for more general p-Kirchhoff problem, i.e., for problem (1.1) with a general nonlocal coefficient M and a general nonlinearity f. Here the general nonlocal coefficient covers the case $M(s) = a + bs$ which was considered in [3, 17]. Moreover, a prototype of f is a combination of some power type nonlinearities, in particular, it includes the case $f(u) = u^p$ which was studied in [2, 7, 17]. Finite time blow-up of solutions to problem (1.1) with negative initial energy will be proved, by applying the first order differential inequality method. Moreover, by using Levine's concavity argument, we also show that the solutions to problem (1.1) blow up in finite time for positive initial energy. In particular, we have revealed the influence of the constraint relationship between the general nonlocal coefficient and the general nonlinear source term (assumption (H1) and (H3)) on the blow-up properties of solutions to problem (1.1). Besides, the upper and lower bounds for the blow-up time of both cases are also derived. It is noteworthy that we consider a class of quite general p-Kirchhoff equations which include the equation considered in [17]. Moreover, as long as the nonlocal coefficient and the nonlinear term satisfy the assumptions (H1)–(H4), our results will hold. Thus, our blow-up results are more general than those obtained in [17].

The second one is that we would like to reveal what will happen to problem (1.8) when $1 < q \le 2p - 1$ and $q < p^* - 1$. We shall show that $q = 2p - 1$ is in some sense critical for problem (1.8) to admit finite time blow-up solutions. To be more precise, we will show that all the solutions to problem (1.8) exist globally when $1 < q < \min\{2p - 1, p^* - 1\}$ or when $q = 2p - 1 < p^* - 1$ and $b > 0$ is suitably small. When $q = 2p - 1 < p^* - 1$ and $b > 0$ is suitably large, problem (1.8) admits both global and finite time blow-up solutions, depending on the initial data. Therefore, by combining these results with that in [17], it is clear that the power $2p - 1$ is critical for the existence of finite time blow-up solutions to problem (1.8). In addition, by comparing the blow-up results for p-Laplace problem (1.4) with $\alpha = q - 1$ in [7] (the power $p - 1$ is the critical blow-up exponent) and p-Kirchhoff problem (1.8), we can show the effect of the nonlocal coefficient $b \int_\Omega |\nabla u|^p \mathrm{d}x$ on finite time blow-up solutions (see Remark 4.2).

The rest of the paper is organized as follows. In Section 2, we present some definitions and notations as preliminaries. Blow-up results for problem (1.1) will be stated and proved in Section 3. In Section 4, we investigate the global well-posedness and finite time blow-up properties of solutions to problem (1.8) with $1 < q \le 2p - 1$ and $q < p^* - 1$.

2. Preliminaries

In this section, we present some notations, definitions and necessary lemmas, which will be used in the sequel. As in [17], we denote by $\|u\|_r$ the $L^r(\Omega)$ norm of a Lebesgue function $u \in L^r(\Omega)$ for $r \ge 1$ and by $(\cdot, \cdot)$ the inner product in $L^2(\Omega)$. By $W_0^{1,p}(\Omega)$ we denote the Sobolev space such that both u and $|\nabla u|$ belong to $L^p(\Omega)$ for any $u \in W_0^{1,p}(\Omega)$, and equip it with the norm $\|u\|_{W_0^{1,p}(\Omega)} = \|\nabla u\|_p$. This norm is equivalent to the full $W_0^{1,p}(\Omega)$ norm due to Poincaré's inequality.

We always associate problem (1.1) with the time-dependent energy functional $J(u; t)$ and Nehari's functional $I(u;t)$, which are defined, respectively, by

$$J(u;t) = \frac{1}{p}\overline{M}\left(\|\nabla u\|_p^p\right) - k(t)\int_\Omega F(u)\mathrm{d}x, \qquad u \in H_0^1(\Omega),\ t \geq 0, \tag{2.1}$$

$$I(u;t) = M\left(\|\nabla u\|_p^p\right)\|\nabla u\|_p^p - k(t)\int_\Omega uf(u)\mathrm{d}x, \qquad u \in H_0^1(\Omega),\ t \geq 0, \tag{2.2}$$

where $F(u) = \int_0^u f(s)\mathrm{d}s$. Since $f(u)$ satisfies (H3) and (H4), both functionals are well defined and continuous on $W_0^{1,p}(\Omega)$ for each $t \geq 0$.

In this paper, we consider weak solutions to problem (1.1) in the following sense.

Definition 2.1. *(see [17]) A function $u = u(x,t) \in L^\infty(0,T; W_0^{1,p}(\Omega))$ with $u_t \in L^2(0,T; L^2(\Omega))$ is called a weak solution to problem* (1.1) *on $\Omega \times [0,T)$, if $u(x,0) = u_0 \in W_0^{1,p}(\Omega)$ and $u(x,t)$ satisfies*

$$(u_t, \phi) + M(\|\nabla u\|_p^p)(|\nabla u|^{p-2}\nabla u, \nabla\phi) = (k(t)f(u), \phi), \quad a.e.\ t \in (0,T), \tag{2.3}$$

for any $\phi \in W_0^{1,p}(\Omega)$. We say that $u(x,t)$ blows up at a finite time T provided that

$$\lim_{t\to T}\|u(\cdot,t)\|_2^2 = +\infty. \tag{2.4}$$

If (2.4) *does not occur for any finite T, we say that $u(x,t)$ is global.*

Local existence of weak solutions to problem (1.1) can be obtained by combining the standard Galerkin's method with standard limit process and the details are omitted. If no confusion arises, we simply write $u(t)$ to denote $u(x,t)$ sometimes.

Lemma 2.1. *Let $J(u;t)$ and $I(u;t)$ be given in* (2.1) *and* (2.2)*, respectively, and let $u = u(t)$ be a weak solution to problem* (1.1). *Then the following statements hold:*

$$\int_0^t \|u_\tau\|_2^2\mathrm{d}\tau + \int_0^t k'(\tau)\int_\Omega F(u)\mathrm{d}x\mathrm{d}\tau + J(u(t);t) = J(u_0;0), \quad a.e.\ t \in (0,T), \tag{2.5}$$

$$\frac{d}{dt}\left(\frac{1}{2}\|u(t)\|_2^2\right) = -I(u(t);t), \quad a.e.\ t \in (0,T). \tag{2.6}$$

Proof. Taking u_t and u as a test function in (2.3), respectively, one obtains (2.5) and (2.6). The proof is complete. □

From (2.5) and the assumptions (H2) and (H5), it is easily seen that $J(u(t);t)$ is nonincreasing with respect to t on $[0,T)$.

Since problem (1.8) is a special form of (1.1), Definition 2.1 and Lemma 2.1 are also applicable to problem (1.8). The energy functional associated with (1.8) is defined by

$$J(u) = \frac{a}{p}\|\nabla u\|_p^p + \frac{b}{2p}\|\nabla u\|_p^{2p} - k(t)\int_\Omega F(u)\mathrm{d}x, \quad u \in W_0^{1,p}(\Omega). \tag{2.7}$$

For any $\delta > 0$, we define the modified Nehari's functional and Nehari's manifold as follows

$$I_\delta(u) = \delta(a\|\nabla u\|_p^p + b\|\nabla u\|_p^{2p}) - \|u\|_{q+1}^{q+1}, \tag{2.8}$$

$$\mathcal{N}_\delta = \{u \in W_0^{1,p}(\Omega) \mid I_\delta(u) = 0,\ \|\nabla u\|_p \neq 0\}.$$

Both $J(u)$ and $I_\delta(u)$ are well defined and continuous on $W_0^{1,p}(\Omega)$ since $q < p^* - 1$. We define the modified potential wells and their corresponding sets, respectively, by

$$W_\delta = \{u \in W_0^{1,p}(\Omega) \mid I_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\},$$
$$V_\delta = \{u \in W_0^{1,p}(\Omega) \mid I_\delta(u) < 0, J(u) < d(\delta)\},$$

where $d(\delta)$, the depth of the potential well W_δ, is characterized by

$$d(\delta) = \inf_{u \in \mathcal{N}_\delta} J(u).$$

When $\delta = 1$, I_δ, $\mathcal{N}_\delta$, W_δ, V_δ and $d(\delta)$ will be simply written, respectively, as I, $\mathcal{N}$, W, V and d.

Moreover, for any $r \in (1, p^*]$, we will use $S_r > 0$ to denote the optimal embedding constant from $W_0^{1,p}(\Omega)$ to $L^r(\Omega)$, i.e.,

$$\frac{1}{S_r} = \inf_{0 \neq u \in W_0^{1,p}(\Omega)} \frac{\|\nabla u\|_p}{\|u\|_r}, \quad u \in W_0^{1,p}(\Omega). \tag{2.9}$$

When $r = 2p$, $S_r = S_{2p}$ will be simply written as S.

3. Finite time blow-up for problem (1.1).

We begin this section with two lemmas. The first one is a concavity lemma due to Levine [20], which will be needed to estimate an upper bound for the blow-up time of $u(x,t)$ when the initial energy $J(u_0; 0)$ is nonnegative. The second one is a special form of Gagliardo-Nirenberg's interpolation inequality, with the help of which, a lower bound for the blow-up time T can be derived.

Lemma 3.1. *(see [20, 21]) Suppose that a positive, twice-differentiable function $\psi(t)$ satisfies the inequality*

$$\psi''(t)\psi(t) - (1+\theta)(\psi'(t))^2 \geq 0,$$

where $\theta > 0$. If $\psi(0) > 0$ and $\psi'(0) > 0$, then $\psi(t) \to \infty$ as

$$t \to t_* \leq t^* = \frac{\psi(0)}{\theta\psi'(0)}.$$

Lemma 3.2. *(see [22]) Let $1 < p_l < p^* - 1$. Then, for any $u \in W_0^{1,p}(\Omega)$, it holds that*

$$\|u\|_{p_l+1} \leq C_{p_l+1}\|\nabla u\|_p^\gamma \|u\|_2^{1-\gamma},$$

where $\gamma = \frac{np(p_l-1)}{(p_l+1)(np+2p-2n)} \in (0,1)$ and $C_{p_l+1} > 0$ is a constant depending only on p_l and n.

Next we show that the solutions to problem (1.1) blow up in finite time with negative initial energy or positive initial energy that is bounded from the above by $C_0(\|u_0\|_2^2 - 1)$ for some $C_0 > 0$. Moreover, we derive an upper bound for the blow-up time of both cases.

Theorem 3.1. *Any weak solution $u(x,t)$ to problem (1.1) blows up at some finite time T provided one of the following statements holds:*

(i) $J(u_0; 0) < 0$;

(ii) $0 \le J(u_0; 0) < \frac{m_0}{S_2^p}\left(\frac{\sigma}{p} - \frac{1}{\alpha+1}\right)(\|u_0\|_2^2 - 1) \equiv C_0(\|u_0\|_2^2 - 1)$, where $C_0 > 0$ by (H1) and (H3).

Moreover, an upper bound for T has the following form:

(a)When (i) holds, $T \le \frac{\|u_0\|_2^2}{(1-\alpha^2)J(u_0;0)}$;

(b)When (ii) holds, $T \le \frac{4\alpha\|u_0\|_2^2}{(\alpha-1)^2(\alpha+1)[C_0(\|u_0\|_2^2-1)-J(u_0;0)]}$.

Proof. (i) To deal with Case (i), we use the first order differential inequality method which is picked from [23]. Set

$$L(t) = \frac{1}{2}\|u(t)\|_2^2, \quad H(t) = -J(u(t); t).$$

Clearly, $L(0) > 0, H(0) > 0$. Recalling (2.5), (H2) and (H5), we obtain

$$H'(t) = -\frac{d}{dt}J(u(t); t) = \int_\Omega u_t^2 \mathrm{d}x + k'(t)\int_\Omega F(u)\mathrm{d}x \ge 0,$$

which implies that $H(t) \ge H(0) > 0$ for any $t \in [0, T)$. On the other hand, it follows from the assumption (H3) that (see [24])

$$uf(u) \ge (\alpha + 1)F(u), \quad \alpha > \frac{p}{\sigma} - 1. \tag{3.1}$$

Taking (1.2), (2.6) and (3.1) into account, we have, for any $t \in [0, T)$

$$\begin{aligned} L'(t) = -I(u(t); t) &\ge -\frac{1}{\sigma}\overline{M}(\|\nabla u\|_p^p) - (\alpha + 1)J(u(t); t) + \frac{\alpha + 1}{p}\overline{M}(\|\nabla u\|_p^p) \\ &\ge (\alpha + 1)H(t). \end{aligned} \tag{3.2}$$

In accordance with $k'(t) \ge 0$ on $[0, +\infty)$ and the assumption (H2), by using Cauchy-Schwarz inequality, we get that

$$\begin{aligned} L(t)H'(t) &= \frac{1}{2}\int_\Omega u^2\mathrm{d}x\int_\Omega u_t^2\mathrm{d}x + \frac{1}{2}\|u\|_2^2 k'(t)\int_\Omega F(u)\mathrm{d}x \\ &\ge \frac{1}{2}\left(\int_\Omega uu_t\mathrm{d}x\right)^2 = \frac{1}{2}\left(L'(t)\right)^2 \ge \frac{\alpha + 1}{2}L'(t)H(t). \end{aligned} \tag{3.3}$$

Following from (3.3), through direct computations we have

$$\left(H(t)L^{-\frac{\alpha+1}{2}}(t)\right)' = \left(L(t)H'(t) - \frac{\alpha + 1}{2}L'(t)H(t)\right)L^{-\frac{\alpha+3}{2}}(t) \ge 0.$$

Thus,

$$
\begin{aligned}
0 < \xi \triangleq H(0)L^{-\frac{\alpha+1}{2}}(0) \leq & H(t)L^{-\frac{\alpha+1}{2}}(t) \\
& \leq \frac{1}{\alpha+1}L'(t)L^{-\frac{\alpha+1}{2}}(t) = \frac{2}{1-\alpha^2}\left(L^{\frac{1-\alpha}{2}}(t)\right)'.
\end{aligned}
\tag{3.4}
$$

Integrating (3.4) over $[0,t]$ for any $t \in (0,T)$, we obtain that

$$
0 \leq L^{-\frac{\alpha-1}{2}}(t) \leq L^{-\frac{\alpha-1}{2}}(0) - \frac{\alpha^2-1}{2}\xi t, \quad t \in (0,T). \tag{3.5}
$$

Since $\alpha > 1$, the right hand side of inequality (3.5) tends to $-\infty$ as $t \to +\infty$, which is a contradiction. Hence, $T < +\infty$. Moreover, we can derive from (3.5) that

$$
T \leq \frac{2}{(\alpha^2-1)\xi}L^{-\frac{\alpha-1}{2}}(0) = \frac{\|u_0\|_2^2}{(1-\alpha^2)J(u_0;0)}.
$$

(ii) First, by virtue of (H1) and (3.1), we have

$$
\begin{aligned}
I(u_0;0) &\leq M(\|\nabla u_0\|_p^p)\|\nabla u_0\|_p^p + (\alpha+1)J(u_0;0) - \frac{(\alpha+1)\sigma}{p}M(\|\nabla u_0\|_p^p)\|\nabla u_0\|_p^p \\
&\leq (\alpha+1)\left[J(u_0;0) - \left(\frac{\sigma}{p} - \frac{1}{\alpha+1}\right)m_0\|\nabla u_0\|_p^p\right].
\end{aligned}
\tag{3.6}
$$

Since $p > 2 > \frac{2n}{n+2}$, by combining (2.9) with the basic inequality $z \leq z^\beta + 1$ for any $z > 0$ and $\beta \geq 1$, we have

$$
S_2^p\|\nabla u\|_p^p \geq \|u\|_2^2 - 1, \quad \forall t \in [0,T), \tag{3.7}
$$

which together with the assumption (ii) implies that $I(u_0;0) < 0$. We claim that $I(u(t);t) < 0$ for all $t \in [0,T)$. Otherwise, there would exist a $t_0 \in (0,T)$ such that $I(u(t);t) < 0$ for all $t \in [0,t_0)$ and $I(u(t_0);t_0) = 0$. From (2.6) we know that $\|u\|_2^2$ is strictly increasing and continuous on $[0,t_0)$, which guarantees that

$$
0 \leq J(u_0;0) < C_0(\|u_0\|_2^2 - 1) < C_0(\|u(t_0)\|_2^2 - 1). \tag{3.8}
$$

On the other hand, by the monotonicity of $J(u(t);t)$, (H1) and (3.7), we obtain

$$
\begin{aligned}
J(u_0;0) &\geq J(u(t_0);t_0) \\
&\geq \frac{1}{p}\overline{M}(\|\nabla u(t_0)\|_p^p) + \frac{1}{\alpha+1}I(u(t_0);t_0) - \frac{1}{\alpha+1}M(\|\nabla u(t_0)\|_p^p)\|\nabla u(t_0)\|_p^p \\
&\geq \left(\frac{\sigma}{p} - \frac{1}{\alpha+1}\right)\frac{m_0}{S_2^p}(\|u(t_0)\|_2^2 - 1) \\
&= C_0(\|u(t_0)\|_2^2 - 1),
\end{aligned}
$$

which contradicts with (3.8). Consequently, $I(u(t);t) < 0$ for all $t \in [0,T)$ as claimed, and $\|u\|_2^2$ is strictly increasing on $[0,T)$.

For any $T^* \in (0, T), \beta > 0$ and $\eta > 0$, we define

$$F(t) = \int_0^t \|u(\tau)\|_2^2 \mathrm{d}\tau + (T^* - t)\|u_0\|_2^2 + \beta(t+\eta)^2, \quad t \in [0, T^*]. \tag{3.9}$$

Taking the first and second derivatives of the function $F(t)$, we have

$$\begin{aligned} F'(t) &= \|u(t)\|_2^2 - \|u_0\|_2^2 + 2\beta(t+\eta) = \int_0^t \mathrm{d}\|u(\tau)\|_2^2 + 2\beta(t+\eta) \\ &= 2\int_0^t (u, u_\tau)\mathrm{d}\tau + 2\beta(t+\eta), \end{aligned} \tag{3.10}$$

$$\begin{aligned} F''(t) =& -2I(u;t) + 2\beta \\ \geq& -2M(\|\nabla u\|_p^p)\|\nabla u\|_p^p - 2(\alpha+1)J(u(t);t) + \frac{2(\alpha+1)\sigma}{p}\overline{M}(\|\nabla u\|_p^p) + 2\beta \\ \geq& \left[\frac{2(\alpha+1)\sigma}{p} - 2\right] M(\|\nabla u\|_p^p)\|\nabla u\|_p^p - 2(\alpha+1)J(u_0;0) \\ &+ 2(\alpha+1)\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + 2\beta. \end{aligned} \tag{3.11}$$

For $t \in [0, T^*]$, set

$$Q(t) = \left(\int_0^t \|u(\tau)\|_2^2 \mathrm{d}\tau + \beta(t+\eta)^2\right)\left(\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \beta\right) - \left(\int_0^t (u, u_\tau)\mathrm{d}\tau + \beta(t+\eta)\right)^2.$$

According to Cauchy-Schwarz inequality and Hölder's inequality, we can derive that $Q(t)$ is nonnegative on $[0, T^*]$. Hence, combining (3.9)–(3.11) with (3.7) and the monotonicity of $\|u(t)\|_2^2$, we have

$$\begin{aligned} & F''(t)F(t) - \frac{\alpha+1}{2}(F'(t))^2 \\ =& F''(t)F(t) - 2(\alpha+1)\left(\int_0^t (u, u_\tau)\mathrm{d}\tau + \beta(t+\sigma)\right)^2 \\ =& F''(t)F(t) + 2(\alpha+1)\left[Q(t) - \left(F(t) - (T^*-t)\|u_0\|_2^2\right)\left(\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \beta\right)\right] \\ \geq& F''(t)F(t) - 2(\alpha+1)F(t)\left(\int_0^t \|u_\tau\|_2^2 \mathrm{d}\tau + \beta\right) \\ \geq& F(t)\left\{\left[\frac{2(\alpha+1)\sigma}{p} - 2\right] M(\|\nabla u\|_p^p)\|\nabla u\|_p^p - 2(\alpha+1)J(u_0;0) - 2\alpha\beta\right\} \\ \geq& F(t)\left\{\left[\frac{2(\alpha+1)\sigma}{p} - 2\right] \frac{m_0}{S_2^p}(\|u\|_2^2 - 1) - 2(\alpha+1)J(u_0;0) - 2\alpha\beta\right\} \\ \geq& F(t)\left\{\left[\frac{2(\alpha+1)\sigma}{p} - 2\right] \frac{m_0}{S_2^p}(\|u_0\|_2^2 - 1) - 2(\alpha+1)J(u_0;0) - 2\alpha\beta\right\} \\ =& 2(\alpha+1)F(t)\left[C_0(\|u_0\|_2^2 - 1) - J(u_0;0) - \frac{\alpha\beta}{\alpha+1}\right] \geq 0, \end{aligned} \tag{3.12}$$

for any $t \in [0, T^*]$ and $\beta \in \left(0, \frac{\alpha+1}{\alpha}(C_0(\|u_0\|_2^2 - 1) - J(u_0; 0))\right]$. Therefore, it follows from the fact that $\alpha > 1$ and Lemma 3.1 that

$$T^* \leq \frac{2F(0)}{(\alpha-1)F'(0)} = \frac{2(T^*\|u_0\|_2^2 + \beta\eta^2)}{2(\alpha-1)\beta\eta},$$

which implies that

$$T^*\left(1 - \frac{\|u_0\|_2^2}{(\alpha-1)\beta\eta}\right) \leq \frac{\eta}{\alpha-1},$$

for all $\beta \in \left(0, \frac{\alpha+1}{\alpha}(C_0(\|u_0\|_2^2 - 1) - J(u_0; 0))\right]$ and $\eta > 0$.

To estimate the upper bound for T^*, we fix a $\beta_0 \in \left(0, \frac{\alpha+1}{\alpha}(C_0(\|u_0\|_2^2 - 1) - J(u_0; 0))\right]$. Thus, for all $\eta \in (\frac{\|u_0\|_2^2}{(\alpha-1)\beta_0}, +\infty)$, we have

$$T^* \leq \frac{\beta_0\eta^2}{(\alpha-1)\beta_0\eta - \|u_0\|_2^2}. \tag{3.13}$$

Minimizing the right hand side term in (3.13) for $\eta \in (\frac{\|u_0\|_2^2}{(\alpha-1)\beta_0}, +\infty)$, one sees

$$T^* \leq \frac{4\|u_0\|_2^2}{(\alpha-1)^2\beta_0}, \quad \beta_0 \in \left(0, \frac{\alpha+1}{\alpha}(C_0(\|u_0\|_2^2 - 1) - J(u_0; 0))\right]. \tag{3.14}$$

Minimizing the right hand side term in (3.14) with respect to $\beta_0 \in \left(0, \frac{\alpha+1}{\alpha}(C_0(\|u_0\|_2^2 - 1) - J(u_0; 0))\right]$, we can obtain

$$T^* \leq \frac{4\alpha\|u_0\|_2^2}{(\alpha-1)^2(\alpha+1)[C_0(\|u_0\|_2^2 - 1) - J(u_0; 0)]}.$$

By the arbitrariness of $T^* < T$, it follows that

$$T \leq \frac{4\alpha\|u_0\|_2^2}{(\alpha-1)^2(\alpha+1)[C_0(\|u_0\|_2^2 - 1) - J(u_0; 0)]}.$$

The proof is complete. □

At the end of this section, we give an estimation of the blow-up time of solutions to problem (1.1) from below. Since the negativity of $I(u(t); t)$ is preserved in both case (i) and (ii) in Theorem 3.1, a lower bound for the blow-up time will be deduced in a uniform way.

Theorem 3.2. *Let all the assumptions in Theorem 3.1 hold, and assume that* $1 < p_l < p + \frac{2p}{n} - 1$. *Then the maximal existence time of problem* (1.1) *satisfies* $T \geq \frac{\|u_0\|_2^{2(1-h)}}{2(h-1)\hat{C}}$, *where* $h, \hat{C}$ *are positive constants that will be determined in the proof.*

Proof. First we claim that $I(u(t);t) < 0$ for all $t \in [0,T)$ when either assumption (i) or (ii) in Theorem 3.1 holds. In fact, when assumption (ii) holds, the negativity of $I(u(t);t)$ on $[0,T)$ has been proved above. Now we prove the other case. Due to Lemma 2.1, we have

$$J(u(t);t) \le J(u_0;0) < 0, \quad t \in [0,T).$$

In addition, by (H1) and (H3), we can obtain

$$I(u;t) \le (\alpha+1)J(u;t) - \left(\frac{(\alpha+1)\sigma}{p} - 1\right) M(\|\nabla u\|_p^p)\|\nabla u\|_p^p < 0, \quad t \in [0,T).$$

Here we still denote $L(t) = \frac{1}{2}\|u(t)\|_2^2$. By assumptions (H1) and (H4), we know that

$$\begin{aligned} L'(t) = -I(u;t) &= - M(\|\nabla u\|_p^p)\|\nabla u\|_p^p + k(t)\int_\Omega uf(u)\mathrm{d}x \\ &\le k(t)\sum_{i=1}^{l}\int_\Omega a_i|u|^{p_i+1}\mathrm{d}x, \quad t \in [0,T). \end{aligned} \tag{3.15}$$

Since $1 < p_1 < ... < p_l$, we obtain, by applying Young's inequality, that

$$\int_\Omega a_i|u|^{p_i+1}\mathrm{d}x \le \frac{p_i+1}{p_l+1}\|u\|_{p_l+1}^{p_l+1} + \frac{(a_i)^{\frac{p_l+1}{p_l-p_i}}(p_l-p_i)}{p_l+1}|\Omega|, \quad t \in [0,T). \tag{3.16}$$

Recalling that $I(u(t);t) < 0$ for any $t \in [0,T)$, we have

$$\frac{d}{dt}\|u\|_2^2 = -2I(u;t) > 0, \quad t \in [0,T), \tag{3.17}$$

which, together with Hölder's inequality, implies that

$$\|u_0\|_2^{p_l+1} \le \|u(t)\|_2^{p_l+1} \le |\Omega|^{\frac{p_l-1}{2}}\|u(t)\|_{p_l+1}^{p_l+1}, \quad t \in [0,T), \tag{3.18}$$

or equivalently,

$$\frac{|\Omega|^{\frac{p_l-1}{2}}\|u(t)\|_{p_l+1}^{p_l+1}}{\|u_0\|_2^{p_l+1}} \ge 1, \quad t \in [0,T).$$

Substituting (3.16) and (3.18) into (3.15) we arrive at

$$\begin{aligned} L'(t) &\le k(t)\sum_{i=1}^{l-1}\frac{(p_l-p_i)a_i^{\frac{p_l+1}{p_l-p_i}}}{p_l+1} + k(t)\left(\sum_{i=1}^{l-1}\frac{p_i+1}{p_l+1} + a_l\right)\|u\|_{p_l+1}^{p_l+1} \\ &\le \chi k(t)\|u\|_{p_l+1}^{p_l+1}, \quad t \in [0,T), \end{aligned} \tag{3.19}$$

where $\chi = \frac{|\Omega|^{\frac{p_l+1}{2}}}{\|u_0\|_2^{p_l+1}}\sum_{i=1}^{l-1}\frac{(p_l-p_i)a_i^{\frac{p_l+1}{p_l-p_i}}}{p_l+1} + \sum_{i=1}^{l-1}\frac{p_i+1}{p_l+1} + a_l > 0.$
In view of (H1) and the negativity of $I(u(t);t)$, we obtain

$$m_0\|\nabla u\|_p^p \le M(\|\nabla u\|_p^p)\|\nabla u\|_p^p < k(t)\int_\Omega uf(u)\mathrm{d}x \le \chi k(t)\|u\|_{p_l+1}^{p_l+1}. \tag{3.20}$$

Following from Lemma 3.2 and (3.20), we have

$$\begin{aligned}\|u\|_{p_l+1}^{p_l+1} &\le C_{p_l+1}\|\nabla u\|_p^{(p_l+1)\gamma}\|u\|_2^{(p_l+1)(1-\gamma)}\\ &\le C_{p_l+1}\left(\frac{\chi k(t)\|u\|_{p_l+1}^{p_l+1}}{m_0}\right)^{\frac{(p_l+1)\gamma}{p}}\|u\|_2^{(p_l+1)(1-\gamma)}\\ &\le \widetilde{C}(\|u\|_{p_l+1}^{p_l+1})^{\frac{(p_l+1)\gamma}{p}}(\|u\|_2^2)^{\frac{(p_l+1)(1-\gamma)}{2}}, \quad t\in[0,T),\end{aligned}$$

where $\widetilde{C} = C_{p_l+1}\left(\frac{\chi\kappa}{m_0}\right)^{\frac{(p_l+1)\gamma}{p}}$, $\kappa = \max\limits_{t\in[0,T)} k(t)$. This shows

$$(\|u\|_{p_l+1}^{p_l+1})^{1-\frac{(p_l+1)\gamma}{p}} \le \widetilde{C}(\|u\|_2^2)^{\frac{(p_l+1)(1-\gamma)}{2}}, \quad t\in[0,T). \tag{3.21}$$

Since $1 < p_l < p + \frac{2p}{n} - 1$, it is clear that $1 - \frac{(p_l+1)\gamma}{p} > 0$ and $h = \frac{(p_l+1)(1-\gamma)/2}{[p-(p_l+1)\gamma]/p} > 1$. According to (3.19) and (3.21), we know

$$L'(t) \le \hat{C}2^h L^h(t), \quad t\in[0,T), \tag{3.22}$$

where $\hat{C} = \chi\kappa\widetilde{C}$. Indeed, it is obvious that $L(0) > 0$ when either assumption (i) or (ii) in Theorem 3.1 holds. From (3.17), then $L(t) > 0$. Dividing both sides of (3.22) by $L^\gamma(t)$ and integrating the resulting inequality over $[0,t)$, we can deduce

$$\frac{1}{1-h}\left[L^{1-h}(t) - L^{1-h}(0)\right] \le 2^h\hat{C}t, \quad t\in(0,T).$$

Since $\lim\limits_{t\to T} L(t) = +\infty$, we obtain, by letting $t\to T$ in the above inequality, that

$$T \ge \frac{L^{1-h}(0)}{2^h(h-1)\hat{C}} = \frac{\|u_0\|_2^{2(1-h)}}{2(h-1)\hat{C}}.$$

The proof is complete. □

4. Critical exponent for problem (1.8).

In this section, we shall deal with problem (1.8) for the case $1 < q \le 2p-1$ and $q \le p^*-1$. The first result tells that all the weak solutions to problem (1.8) exist globally when $1 < q < 2p-1$ or when $q = 2p-1$ and $b \ge S^{2p}$.

Theorem 4.1. *Any weak solution $u(x,t)$ to problem* (1.8) *exists globally provided one of the following assumptions holds:*

(i) $1 < q < \min\{2p-1, p^*-1\}$*;*

(ii) $q = 2p-1 < p^*-1$ *and* $b \ge S^{2p}$.

Moreover, the decay rate is estimated as follows:

(a) When (i) holds, $\|u(\cdot,t)\|_2^2 \le \frac{\widetilde{C^*}S_2^p}{a} + e^{-\frac{2a}{S_2^p}t}(\|u_0\|_2^2 - \frac{\widetilde{C^*}S_2^p}{a})$,

where $\widetilde{C^*} = \frac{2p-q-1}{2p}\left(\frac{S_{q+1}^{q+1}}{b^{\frac{q+1}{2p}}}\right)^{\frac{2p}{2p-q-1}} + \frac{a}{S_2^p}$.

(b) When (ii) holds, $\|u(\cdot,t)\|_2^2 \le 1 + e^{-\frac{2a}{S_2^p}t}(\|u_0\|_2^2 - 1)$.

Proof. (i) Taking $\phi = u$ in (2.3), recalling $W_0^{1,p}(\Omega) \hookrightarrow L^{q+1}(\Omega)$ and applying Young's inequality, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + a\|\nabla u\|_p^p + b\|\nabla u\|_p^{2p} = \|u\|_{q+1}^{q+1} \le S_{q+1}^{q+1}\|\nabla u\|_p^{q+1} \le \frac{(q+1)b}{2p}\|\nabla u\|_p^{2p} + C^*, \tag{4.1}$$

which together with (3.7) implies that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \frac{a}{S_2^p}\|u\|_2^2 \le \widetilde{C^*}, \tag{4.2}$$

where $C^* = \frac{2p-q-1}{2p}\left(\frac{S_{q+1}^{q+1}}{b^{\frac{q+1}{2p}}}\right)^{\frac{2p}{2p-q-1}}$, $\widetilde{C^*} = C^* + \frac{a}{S_2^p}$. Direct computations guarantee that

$$\|u(\cdot,t)\|_2^2 \le \frac{\widetilde{C^*}S_2^p}{a} + e^{-\frac{2a}{S_2^p}t}\left(\|u_0\|_2^2 - \frac{\widetilde{C^*}S_2^p}{a}\right).$$

which means that $u(x,t)$ is a global weak solution to problem (1.8).

(ii) In accordance with (4.1) and $b \ge S^{2p}$, it is clear that

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \frac{a}{S_2^p}\|u\|_2^2 \le \frac{a}{S_2^p}.$$

By solving the above ordinary differential inequality, we obtain

$$\|u(\cdot,t)\|_2^2 \le 1 + e^{-\frac{2a}{S_2^p}t}(\|u_0\|_2^2 - 1).$$

The proof is complete. □

To show that problem (1.8) admits both global and finite time blow-up solutions for the case $q = 2p-1$, the following lemma, which asserts that $\mathcal{N}$ is non-empty when b is suitably small, is crucial.

Lemma 4.1. *Assume $q = 2p-1 < p^*-1$ and $b < S^{2p}$. Then $\mathcal{N} \neq \emptyset$.*

Proof. Since Ω is a bounded smooth domain in $\mathbb{R}^n$ and $q = 2p-1 < p^*-1$, it is well known that the constant defined in (2.9) can be attained, i.e., there exists a $\overline{u} \in W_0^{1,p}(\Omega)\backslash\{0\}$ such that $\|\overline{u}\|_{2p} = S\|\nabla\overline{u}\|_p$.

Set $\widetilde{\lambda} = \left[\frac{a}{(S^{2p}-b)\|\nabla\overline{u}\|_p^p}\right]^{1/p} > 0$. A direct computation shows that

$$\begin{aligned}
I(\widetilde{\lambda}\overline{u}) &= a\widetilde{\lambda}^p\|\nabla\overline{u}\|_p^p + b\widetilde{\lambda}^{2p}\|\nabla\overline{u}\|_p^{2p} - \widetilde{\lambda}^{2p}\|\overline{u}\|_{2p}^{2p} \\
&= a\widetilde{\lambda}^p\|\nabla\overline{u}\|_p^p + b\widetilde{\lambda}^{2p}\|\nabla\overline{u}\|_p^{2p} - \widetilde{\lambda}^{2p}S^{2p}\|\nabla\overline{u}\|_p^{2p} \\
&= \widetilde{\lambda}^p\|\nabla\overline{u}\|_p^p\left[a - (S^{2p}-b)\widetilde{\lambda}^p\|\nabla\overline{u}\|_p^p\right] \\
&= 0,
\end{aligned}$$

which means that $\widetilde{\lambda}\overline{u} \in \mathcal{N}$. Therefore, $\mathcal{N} \neq \emptyset$, as required. □

Once $\mathcal{N}$ is shown to be non-empty, the following three lemmas can be deduced by similar arguments to that in [17] for the case $2p-1 < q < p^*-1$, and their proofs are therefore omitted.

Lemma 4.2. *Let $q = 2p - 1 < p^* - 1$ and $b < S^{2p}$. Then the depth d of the potential well W is positive.*

Lemma 4.3. *Let $q = 2p-1 < p^*-1$, $b < S^{2p}$, $u \in W_0^{1,p}(\Omega)$ and $r(\delta) = (\frac{a\delta}{S^{2p}})^{\frac{1}{p}}$. We have*
(i) If $0 \le \|\nabla u\|_p \le r(\delta)$, then $I_\delta(u) \ge 0$.
(ii) If $I_\delta(u) < 0$, then $\|\nabla u\|_p > r(\delta)$.
(iii) If $I_\delta(u) = 0$, then $\|\nabla u\|_p = 0$ or $\|\nabla u\|_p \ge r(\delta)$.

Lemma 4.4. *Let $q = 2p-1 < p^*-1$ and $b < S^{2p}$. Assume that $u(x,t)$ is a weak solution to problem (1.8) with $0 < J(u_0) < d$ and T is the maximal existence time. Let $\delta_1 < 1 < \delta_2$ be the two roots of the equation $d(\delta) = J(u_0)$.*
(i) If $I(u_0) > 0$, then $u(x,t) \in W_\delta$ for $\delta_1 < 1 < \delta_2$ and $0 < t < T$.
(ii) If $I(u_0) < 0$, then $u(x,t) \in V_\delta$ for $\delta_1 < 1 < \delta_2$ and $0 < t < T$.

Next, we show that when $q = 2p - 1 < p^* - 1$ and $b < S^{2p}$, problem (1.8) may admit both global and finite time blow-up solutions, depending on the initial data. The proof of the existence of global solutions (Theorem 4.2) is similar to that of Theorem 1 in [17], and hence is omitted here.

Theorem 4.2. *Assume that $q = 2p - 1 < p^* - 1$, $b < S^{2p}$ and $u_0 \in W_0^{1,p}(\Omega)$. If $J(u_0) \le d$ and $I(u_0) > 0$, then problem (1.8) admits a global weak solution $u \in L^\infty(0,\infty; W_0^{1,p}(\Omega))$ with $u_t \in L^2(0,\infty; L^2(\Omega))$. If $J(u_0) < d$, then $u(t) \in W$ for $0 \le t < \infty$ and $\|u\|_2^2 \le [\|u_0\|_2^{2-2p} + A_*(p-1)t]^{-1/(p-1)}$. If $J(u_0) = d$, then $u(t) \in \overline{W} = W \cup \partial W$ for $0 \le t < \infty$ and there exists a $t_0 > 0$ such that $\|u\|_2^2 \le [\|u(t_0)\|_2^{2-2p} + A_*(p-1)(t-t_0)]^{-1/(p-1)}$. Here $A_* = 2b(1-\delta_1)/S_2^{2p} > 0$ and $\delta_1 \in (0,1)$ is given in Lemma 4.4.*

Theorem 4.3. *Assume $q = 2p - 1 < p^* - 1$, $b < S^{2p}$ and let $u(x,t)$ be a weak solution to problem (1.8) with $u_0 \in W_0^{1,p}(\Omega)$. If $J(u_0) \le d$ and $I(u_0) < 0$, then $u(x,t)$ blows up at some finite time T.*

Proof. Assume on the contrary that $u(x,t)$ exists globally, then $G(t) = \int_0^t \|u\|_2^2 d\tau$ is well defined for all $t \ge 0$. Taking derivative successively to obtain

$$G'(t) = \|u\|_2^2, \tag{4.3}$$

and

$$G''(t) = 2(u_t, u) = -2\left(a\|\nabla u\|_p^p + b\|\nabla u\|_p^{2p} - \|u\|_{2p}^{2p}\right) = -2I(u). \tag{4.4}$$

Direct computations show that

$$J(u) = \frac{a}{2p}\|\nabla u\|_p^p + \frac{1}{2p}I(u). \tag{4.5}$$

By Lemma 2.1, (4.4) and (4.5), we deduce

$$G''(t) = 2a\|\nabla u\|_p^p - 4pJ(u) \ge \frac{2a}{S_2^p}(G'(t)-1) + 4p\int_0^t \|u_\tau\|_2^2 d\tau - 4pJ(u_0).$$

Noticing that

$$(G'(t))^2 = 4\left(\int_0^t \int_\Omega u_\tau u dx d\tau\right)^2 + 2\|u_0\|_2^2 G'(t) - \|u_0\|_2^4.$$

Applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}G''(t)G(t)-p(G'(t))^2 \geq & p\|u_0\|_2^4+4p\int_0^t\|u_\tau\|_2^2\mathrm{d}\tau\int_0^t\|u\|_2^2\mathrm{d}t-\left(4pJ(u_0)+\frac{2a}{S_2^p}\right)G(t)\\ &+\frac{2a}{S_2^p}G'(t)G(t)-4p\left(\int_0^t\int_\Omega u_\tau u\mathrm{d}x\mathrm{d}\tau\right)^2-2p\|u_0\|_2^2G'(t)\\ \geq & \frac{2a}{S_2^p}G'(t)G(t)-(4pJ(u_0)+\frac{2a}{S_2^p})G(t)-2p\|u_0\|_2^2G'(t).\end{aligned} \tag{4.6}$$

In order to complete the proof, we consider the following three cases.

Case I: $0<J(u_0)<d$. Lemma 4.5 implies that $u(t)\in V_\delta$ for $t\geq 0$ and $\delta_1<\delta<\delta_2$. Here $\delta_1<1<\delta_2$ are the two roots of $d(\delta)=J(u_0)$. Thus $I_{\delta_2}(u)\leq 0$ and $\|\nabla u\|_p\geq r(\delta_2)$ for $t\geq 0$. Then from (4.4) one sees that for $t\geq 0$

$$\begin{aligned}G''(t)&=-2I(u)\\ &=2a(\delta_2-1)\|\nabla u\|_p^p+2b(\delta_2-1)\|\nabla u\|_p^{2p}-2I_{\delta_2}(u)\\ &\geq 2a(\delta_2-1)r^p(\delta_2).\end{aligned} \tag{4.7}$$

This make sure that

$$G'(t)\geq 2a(\delta_2-1)r^p(\delta_2)t,\quad G(t)\geq a(\delta_2-1)r^p(\delta_2)t^2.$$

Hence there exists a $t^*>0$ such that for $t\geq t^*$, we get

$$\frac{a}{S_2^p}G(t)>2p\|u_0\|_2^2,$$

$$\frac{a}{S_2^p}G'(t)>4pJ(u_0)+\frac{2a}{S_2^p},$$

which together with (4.6) show that for sufficiently large $t\geq t^*$, we have

$$G''(t)G(t)-p(G'(t))^2>0. \tag{4.8}$$

Case II: $J(u_0)\leq 0$. Firstly, if $J(u_0)<0$ or $J(u_0)=0$ with $\|\nabla u_0\|_p\neq 0$, then every solution $u(x,t)$ to problem (1.8) belongs to V_δ for any $0<\delta<1$ and $0\leq t<T$, where $T>0$ is the maximum existence time. In fact, by

$$\frac{\|\nabla u\|_p^p}{2p}\{a(2-\delta)+b(1-\delta)\|\nabla u\|_p^p\}+\frac{I_\delta(u)}{2p}=J(u)\leq J(u_0),$$

it can be seen that if $J(u_0)<0$, then $J(u(x,t))<0<d(\delta)$ and $I(u(x,t))<0$ for $0\leq t<T$, which means that $u(x,t)\in V_\delta$. Next, if $J(u_0)=0$ and $\|\nabla u_0\|_p\neq 0$, then $J(u(x,t))\leq 0$ for $0\leq t<T$, which implies that there exists a constant $c>0$ such that $\|\nabla u(\cdot,t)\|_p\geq c$. From (4.8), we can also obtain that $I(u(x,t))<0$ and $J(u(x,t))<0<d(\delta)$, i.e., $u(x,t)\in V_\delta$. Therefore, by replacing δ_2 in case I with δ, after similar calculation, we can get that (4.8) still holds for sufficiently large t.

Case III: $J(u_0)=d$, $I(u_0)<0$. Since $J(u)$ and $I(u)$ are continuous with respect to t, there exists a $t^\circ>0$ such that $J(u(t))>0$, $I(u(t))<0$ for $0<t\leq t^\circ$. From $(u,u_t)=-I(u)$, we arrive at $u_t\neq 0$ for $0<t\leq t^\circ$. Moreover,

$$J(u(t^\circ))=d-\int_0^{t^\circ}\|u_\tau\|_2^2\mathrm{d}\tau=d_1<d.$$

Choosing $t = t^\circ$ as the initial time, recalling Lemma 4.4 (ii), we know $u(x,t) \in V_\delta$ for $\delta_1 < \delta < \delta_2$ and $t > t^\circ$, where $\delta_1 < 1 < \delta_2$ are the two roots of $d(\delta) = d_1$. Thus, $I_\delta(u) < 0$ and $\|\nabla u\|_p > r(\delta)$ for any $\delta_1 < \delta < \delta_2$ and $t > t^\circ$. Furthermore, $I_{\delta_2}(u) \leq 0$ and $\|\nabla u\|_p > r(\delta_2)$ for $t > t^\circ$. Consequently, the estimate (4.7) still holds. In addition, for $t > t^\circ$, we have

$$G'(t) \geq 2a(\delta_2 - 1)r^p(\delta_2)(t - t^\circ), \quad G(t) \geq a(\delta_2 - 1)r^p(\delta_2)(t - t^\circ)^2,$$

which together with (4.6) implies that (4.8) is still valid for sufficiently large $t > t^\circ$.

The remainder of the proof follows from the standard concavity arguments as those in [18–20] and the details are therefore omitted. The proof is complete. □

Remark 4.1. *By combining the results in Section 4 and that in [17] we can see that* $q = 2p - 1$ *is in some sense the critical exponent for the existence of global or finite time blow-up solutions to problem* (1.8). *More precisely, when* $q < 2p - 1$, *all the weak solutions to problem* (1.8) *exist globally. When* $q = 2p - 1$, *all the weak solutions to problem* (1.8) *exist globally provided that* $b > 0$ *is suitably large, while there are both global and finite time blow-up solutions (depending on the initial data) provided that* $b > 0$ *is suitably small. When* $q > 2p - 1$, *problem* (1.8) *also admits both global and finite time blow-up solutions for different initial data.*

Remark 4.2. *Although the blow-up conditions for p-Kirchhoff problem* (1.8) *and the nonlinear p-Laplace problem* (1.4) *with* $\alpha = q - 1$ *are similar, from the initial energy point of view (see [8, 25] for example). It is worth pointing out that the critical blow-up exponents for these two problems are quite different. As we know, the critical blow-up exponent for problem* (1.4) *is* $q = p - 1$ *in [7]. However, we can see from Remark 4.1 that the critical blow-up exponent for p-Kirchhoff problem* (1.8) *is* $q = 2p - 1$. *This is obviously caused by the nonlocal term* $b \int_\Omega |\nabla u|^p \mathrm{d}x \Delta_p u$.

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Conflict of interest

The authors declare there is no conflicts of interest.

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