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Research article

1-parameter formal deformations and abelian extensions of Lie color triple systems

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Abstract: The purpose of this paper is to discuss Lie color triple systems. The cohomology theory of Lie color triple systems is established, then 1-parameter formal deformations and abelian extensions of Lie color triple systems are studied using cohomology.

Keywords: Lie color triple system; cohomology; deformation; abelian extension

1. Introduction

The notion of Lie triple systems was introduced by Lister to obtain all simple Lie triple systems over an algebraically closed field [1]. The cohomology theory of Lie triple systems was investigated by Yamaguti [2]. In 2004, Kubo and Taniguchi gave that Yamaguti cohomology plays a crucial role in the theory of deformations of Lie triple systems [3]. Because of important applications in elementary particle theory and the theory of quantum mechanics, Lie triple systems have been discussed [4, 5]. Okubo reformulated the para-statistics as Lie supertriple systems and explained the relationship between Lie supertriple systems and ortho-symplectic supertriple systems [6]. Okubo and Kamiya introduced δ-Jordan Lie supertriple systems in [7]. Some examples and results of Lie supertriple systems were given in [6, 8, 9]. As a generalization of Lie triple systems and Lie supertriple systems, Cao studied split Lie color triple systems, and obtained the structure of split Lie color triple systems by the techniques of connections of roots in 2019 [10].

The purpose of this paper is to consider the cohomology theory and deformations of Lie color triple systems basing on some work in [2–4, 11–14]. The paper is organized as follows. Section 2 is devoted to some basic definitions and the cohomology theory of Lie color triple systems. Section 3 is dedicated to the 1-parameter formal deformation theory of Lie color triple systems. We show that the cohomology group defined in Section 2 is suitable for this 1-parameter formal deformation theory. In Section 4, we study abelian extensions of Lie color triple systems, and get that associated to any abelian extension, there is a representation and a 3-cocycle.

Throughout this paper, we assume that **F** denotes an arbitrary field.

2. Preliminaries

Definition 2.1. [15] *Let G be an abelian group. A bi-character on G is a map $\varepsilon : G \times G \to \mathbb{K} \setminus \{0\}$ satisfying*

$$\varepsilon(|x|,|y|)\varepsilon(|y|,|x|) = 1,$$
$$\varepsilon(|x|,|y|+|z|) = \varepsilon(|x|,|y|)\varepsilon(|x|,|z|),$$
$$\varepsilon(|x|+|z|,|y|) = \varepsilon(|x|,|y|)\varepsilon(|z|,|y|),$$

where $x, y, z \in G$. It is clear that

$$\varepsilon(|x|,|0|) = \varepsilon(|0|,|x|) = 1, \quad \varepsilon(|x|,|x|) = \pm 1, \quad \forall x \in G.$$

Definition 2.2. [1] *A Lie triple system $(T, [\cdot,\cdot,\cdot])$ consists of an* **F***-vector space T, a trilinear map $[\cdot,\cdot,\cdot] : T \times T \times T \to T$, such that for all $x, y, z, u, v \in T$,*

$$[x, x, z] = 0,$$

$$[x, y, z] + [y, z, x] + [z, x, y] = 0,$$

$$\begin{aligned}[u, v, [x, y, z]] =& [[u, v, x], y, z] + [x, [u, v, y], z] \\ &+ [x, y, [u, v, z]].\end{aligned}$$

Definition 2.3. [10] *A Lie color triple system $(T, [\cdot,\cdot,\cdot])$ consists of an* **F***-vector space T, a trilinear map $[\cdot,\cdot,\cdot] : T \times T \times T \to T$, such that for all $x, y, z, u, v \in T$,*

$$[x, y, z] = -\varepsilon(|x|,|y|)[y, x, z], \tag{2.1}$$

$$\varepsilon(|x|,|z|)[x, y, z] + \varepsilon(|y|,|x|)[y, z, x] + \varepsilon(|z|,|y|)[z, x, y] = 0, \tag{2.2}$$

$$\begin{aligned}[u, v, [x, y, z]] =& [[u, v, x], y, z] + \varepsilon(|x|,|u|+|v|)[x, [u, v, y], z] \\ &+ \varepsilon(|u|+|v|,|x|+|y|)[x, y, [u, v, z]].\end{aligned} \tag{2.3}$$

We generalize the notion of the representation of Lie triple systems to Lie color triple systems in the following.

Definition 2.4. *Let $(T, [\cdot,\cdot,\cdot])$ be a Lie color triple system, V an* **F***-vector space and $A \in \mathrm{End}(V)$. V is called a $(T, [\cdot,\cdot,\cdot])$-module with respect to A if there exists a bilinear map $\theta : T \times T \to \mathrm{End}(V)$, $(a, b) \mapsto \theta(a, b)$ such that for all $a, b, c, d \in T$,*

$$\theta(a, b) \circ A = A \circ \theta(a, b), \tag{2.4}$$

$$\begin{aligned}&\varepsilon(|a|+|b|,|c|+|d|)\theta(c, d)\theta(a, b) - \varepsilon(|a|,|b|)\varepsilon(|d|,|a|+|c|)\theta(b, d)\theta(a, c) \\ &-\theta(a, [b, c, d]) \circ A + \varepsilon(|a|,|b|+|c|)D(b, c)\theta(a, d) = 0,\end{aligned} \tag{2.5}$$

$$\varepsilon(|a|+|b|,|c|+|d|)\theta(c,d)D(a,b)-D(a,b)\theta(c,d)$$
$$+\theta([a,b,c],d)\circ A+\varepsilon(|c|,|a|+|b|)\theta(c,[a,b,d])\circ A=0, \tag{2.6}$$
$$\varepsilon(|a|+|b|,|c|+|d|)D(c,d)D(a,b)-D(a,b)D(c,d)$$
$$+D([a,b,c],d)\circ A+\varepsilon(|c|,|a|+|b|)D(c,[a,b,d])\circ A=0, \tag{2.7}$$

where $D(a,b)=\varepsilon(|a|,|b|)\theta(b,a)-\theta(a,b)$.

Then θ *is called the representation of* $(T,[\cdot,\cdot,\cdot])$ *on* V *with respect to* A. *In the case* $\theta=0$, V *is called the trivial* $(T,[\cdot,\cdot,\cdot])$*-module with respect to* A.

In particular, let $V=T$, and $\theta(x,y)(z)=\varepsilon(|z|,|x|+|y|)[z,x,y]$. Then $D(x,y)(z)=[x,y,z]$ and (2.4), (2.5), (2.6), (2.7) hold. In this case T is showed to be the adjoint $(T,[\cdot,\cdot,\cdot])$-module and θ is called the adjoint representation of $(T,[\cdot,\cdot,\cdot])$.

As follows, we give the semidirect product of a Lie color triple system.

Proposition 2.5. *Let* θ *be a representation of a Lie color triple system* $(T,[\cdot,\cdot,\cdot])$ *on* V *with respect to* A. *Assume the operation* $[\cdot,\cdot,\cdot]_V:(T\oplus V)\times(T\oplus V)\times(T\oplus V)\to T\oplus V$ *by*

$$[(x,a),(y,b),(z,c)]_V=([x,y,z],\varepsilon(|x|,|y|+|z|)\theta(y,z)(a)-\varepsilon(|y|,|z|)\theta(x,z)(b)+D(x,y)(c)),$$

then $T\oplus V$ *is a Lie color triple system.*

Proof. By $D(x,y)=\varepsilon(|x|,|y|)\theta(y,x)-\theta(x,y)$, we get

$$\begin{aligned}
&[(x,a),(y,b),(z,c)]_V\\
=&([x,y,z],\varepsilon(|x|,|y|+|z|)\theta(y,z)(a)-\varepsilon(|y|,|z|)\theta(x,z)(b)+D(x,y)(c))\\
=&-\varepsilon(|x|,|y|)([y,x,z],\varepsilon(|y|,|x|+|z|)\theta(x,z)(b)-\varepsilon(|x|,|z|)\theta(y,z)(a)-\varepsilon(|x|,|y|)D(x,y)(c))\\
=&-\varepsilon(|x|,|y|)([y,x,z],\varepsilon(|y|,|x|+|z|)\theta(x,z)(b)-\varepsilon(|x|,|z|)\theta(y,z)(a)+D(y,x)(c))\\
=&-\varepsilon(|x|,|y|)[(y,b),(x,a),(z,c)]_V,
\end{aligned}$$

and

$$\begin{aligned}
&\varepsilon(|x|,|z|)[(x,a),(y,b),(z,c)]_V+\varepsilon(|y|,|x|)[(y,b),(z,c),(x,a)]_V+\varepsilon(|z|,|y|)[(z,c),(x,a),(y,b)]_V\\
=&(\varepsilon(|x|,|z|)[x,y,z],\varepsilon(|x|,|y|)\theta(y,z)(a)-\varepsilon(|z|,|x|+|y|)\theta(x,z)(b)+\varepsilon(|x|,|z|)D(x,y)(c))\\
&+(\varepsilon(|y|,|x|)[y,z,x],\varepsilon(|y|,|z|)\theta(z,x)(b)-\varepsilon(|x|,|y|+|z|)\theta(y,x)(c)+\varepsilon(|y|,|x|)D(y,z)(a))\\
&+(\varepsilon(|z|,|y|)[z,x,y],\varepsilon(|z|,|x|)\theta(x,y)(c)-\varepsilon(|y|,|x|+|z|)\theta(z,y)(a)+\varepsilon(|z|,|y|)D(z,x)(b))\\
=&(0,\varepsilon(|x|,|y|)\theta(y,z)(a)-\varepsilon(|y|,|x|+|z|)\theta(z,y)(a)+\varepsilon(|y|,|x|)D(y,z)(a)\\
&+\varepsilon(|y|,|z|)\theta(z,x)(b)-\varepsilon(|z|,|x|+|y|)\theta(x,z)(b)+\varepsilon(|z|,|y|)D(z,x)(b)\\
&+\varepsilon(|z|,|x|)\theta(x,y)(c)-\varepsilon(|x|,|y|+|z|)\theta(y,x)(c)+\varepsilon(|x|,|z|)D(x,y)(c))\\
=&(0,0).
\end{aligned}$$

By (2.5), (2.6) and (2.7), it follows that

$$\begin{aligned}
&[[(x,a),(y,b),(u,c)]_V,(v,d),(w,e)]_V\\
=&[([x,y,u],\varepsilon(|x|,|y|+|u|)\theta(y,u)(a)-\varepsilon(|y|,|u|)\theta(x,u)(b)+D(x,y)(c)),
\end{aligned}$$

$$
\begin{aligned}
&(v,d),(w,e)]_V\\
=&([[x,y,u],v,w],\varepsilon(|x|+|y|+|u|,|v|+|w|)\theta(v,w)(\varepsilon(|x|,|y|+|u|)\theta(y,u)(a)\\
&-\varepsilon(|y|,|u|)\theta(x,u)(b)+D(x,y)(c))-\varepsilon(|v|,|w|)\theta([x,y,u],w)(d)\\
&+D([x,y,u],v)(e)),
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon(|u|,|x|+|y|)[(u,c),[(x,a),(y,b),(v,d)]_V,(w,e)]_V\\
=&\varepsilon(|u|,|x|+|y|)[(u,c),([x,y,v],\varepsilon(|x|,|y|+|v|)\theta(y,v)(a)-\varepsilon(|y|,|v|)\theta(x,v)(b)\\
&+D(x,y)(d)),(w,e)]_V\\
=&\varepsilon(|u|,|x|+|y|)([u,[x,y,v],w],\varepsilon(|u|,|x|+|y|+|v|+|w|)\theta([x,y,v],w)(c)\\
&-\varepsilon(|x|+|y|+|v|,|w|)\theta(u,w)(\varepsilon(|x|,|y|+|v|)\theta(y,v)(a)-\varepsilon(|y|,|v|)\theta(x,v)(b)\\
&+D(x,y)(d))+D(u,[x,y,v])(e)),
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon(|x|+|y|,|u|+|v|)[(u,c),(v,d),[(x,a),(y,b),(w,e)]_V]_V\\
=&\varepsilon(|x|+|y|,|u|+|v|)[(u,c),(v,d),([x,y,w],\varepsilon(|x|,|y|+|w|)\theta(y,w)(a)\\
&-\varepsilon(|y|,|w|)\theta(x,w)(b)+D(x,y)(e))]_V\\
=&\varepsilon(|x|+|y|,|u|+|v|)([u,v,[x,y,w]],\varepsilon(|u|,|v|+|x|+|y|+|w|)\theta(v,[x,y,w])(c)\\
&-\varepsilon(|v|,|x|+|y|+|w|)\theta(u,[x,y,w])(d)+D(u,v)(\varepsilon(|x|,|y|+|w|)\theta(y,w)(a)\\
&-\varepsilon(|y|,|w|)\theta(x,w)(b)+D(x,y)(e))),
\end{aligned}
$$

$$
\begin{aligned}
&[(x,a),(y,b),[(u,c),(v,d),(w,e)]_V]_V\\
=&[(x,a),(y,b),([u,v,w],\varepsilon(|u|,|v|+|w|)\theta(v,w)(c)-\varepsilon(|v|,|w|)\theta(u,w)(d)\\
&+D(u,v)(e))]_V\\
=&([x,y,[u,v,w]],\varepsilon(|x|,|y|+|u|+|v|+|w|)\theta(y,[u,v,w])(a)\\
&-\varepsilon(|y|,|u|+|v|+|w|)\theta(x,[u,v,w])(b)+D(x,y)(\varepsilon(|u|,|v|+|w|)\theta(v,w)(c)\\
&-\varepsilon(|v|,|w|)\theta(u,w)(d)+D(u,v)(e))).
\end{aligned}
$$

The calculation above shows that (2.1), (2.2) and (2.3) hold.
Thus, $(T\oplus V,[\cdot,\cdot,\cdot]_V)$ is a Lie color triple system. □

Let θ be a representation of $(T,[\cdot,\cdot,\cdot])$ on V with respect to A. If an n-linear map $f:\underbrace{T\times\cdots\times T}_{n\text{ times}}\to V$ satisfies

$$f(x_1,\cdots,x,y,x_n)=-\varepsilon(|x|,|y|)f(x_1,\cdots,y,x,x_n),$$

$$
\begin{aligned}
&\varepsilon(|x|,|z|)f(x_1,\cdots,x_{n-3},x,y,z)+\varepsilon(|y|,|x|)f(x_1,\cdots,x_{n-3},y,z,x)\\
&+\varepsilon(|z|,|y|)f(x_1,\cdots,x_{n-3},z,x,y)=0,
\end{aligned}
$$

then f is called an n-cochain on T. Denote by $C^n_A(T,V)$ the set of all n-cochains, $\forall n\geq 1$.

Definition 2.6. *For $n = 1, 2, 3, 4$, the coboundary operator $d^n : C^n_A(T, V) \to C^{n+2}_A(T, V)$ is defined as follows.*

- *If $f \in C^1(T, V)$, then*

$$\begin{aligned}&d^1 f(x_1, x_2, x_3)\\ =&\varepsilon(|f| + |x_1|, |x_2| + |x_3|)\theta(x_2, x_3) f(x_1) - \varepsilon(|x_2|, |x_3|)\varepsilon(|f|, |x_1| + |x_3|)\\ &\theta(x_1, x_3) f(x_2) + \varepsilon(|f|, |x_1| + |x_2|)D(x_1, x_2) f(x_3) - f([x_1, x_2, x_3]).\end{aligned}$$

- *If $f \in C^2(T, V)$, then*

$$\begin{aligned}&d^2 f(y, x_1, x_2, x_3)\\ =&\varepsilon(|f| + |y| + |x_1|, |x_2| + |x_3|)\theta(x_2, x_3) f(y, x_1) - \varepsilon(|x_2|, |x_3|)\varepsilon(|f| + |y_1|, |x_1| + |x_3|)\\ &\theta(x_1, x_3) f(y, x_2) + \varepsilon(|f| + |y|, |x_1| + |x_2|)D(x_1, x_2) f(y, x_3)\\ &- f(y, [x_1, x_2, x_3]).\end{aligned}$$

- *If $f \in C^3(T, V)$, then*

$$\begin{aligned}&d^3 f(x_1, x_2, x_3, x_4, x_5)\\ =&\varepsilon(|f| + |x_1| + |x_2| + |x_3|, |x_4| + |x_5|)\theta(x_4, x_5) f(x_1, x_2, x_3)\\ &- \varepsilon(|f| + |x_1| + |x_2|, |x_3| + |x_5|)\varepsilon(|x_4|, |x_5|)\theta(x_3, x_5) f(x_1, x_2, x_4)\\ &- \varepsilon(|f|, |x_1| + |x_2|)D(x_1, x_2) f(x_3, x_4, x_5)\\ &+ \varepsilon(|f| + |x_1| + |x_2|, |x_3| + |x_4|)D(x_3, x_4) f(x_1, x_2, x_5)\\ &+ f([x_1, x_2, x_3], x_4, x_5) + \varepsilon(|x_3|, |x_1| + |x_2|) f(x_3, [x_1, x_2, x_4], x_5)\\ &+ \varepsilon(|x_1| + |x_2|, |x_3| + |x_4|) f(x_3, x_4, [x_1, x_2, x_5]) - f(x_1, x_2, [x_3, x_4, x_5]).\end{aligned}$$

- *If $f \in C^4(T, V)$, then*

$$\begin{aligned}&d^4 f(y, x_1, x_2, x_3, x_4, x_5)\\ =&\varepsilon(|f| + |y| + |x_1| + |x_2| + |x_3|, |x_4| + |x_5|)\theta(x_4, x_5) f(y, x_1, x_2, x_3)\\ &- \varepsilon(|f| + |y| + |x_1| + |x_2|, |x_3| + |x_5|)\varepsilon(|x_4|, |x_5|)\theta(x_3, x_5) f(y, x_1, x_2, x_4)\\ &- \varepsilon(|f| + |y|, |x_1| + |x_2|)D(x_1, x_2) f(y, x_3, x_4, x_5)\\ &+ \varepsilon(|f| + |y| + |x_1| + |x_2|, |x_3| + |x_4|)D(x_3, x_4) f(y, x_1, x_2, x_5)\\ &+ f(y, [x_1, x_2, x_3], x_4, x_5)\\ &+ \varepsilon(|x_3|, |x_1| + |x_2|) f(y, x_3, [x_1, x_2, x_4], x_5)\\ &+ \varepsilon(|x_1| + |x_2|, |x_3| + |x_4|) f(y, x_3, x_4, [x_1, x_2, x_5])\\ &- f(y, x_1, x_2, [x_3, x_4, x_5]).\end{aligned}$$

Theorem 2.7. *The coboundary operator d^n defined above satisfies $d^{n+2}d^n = 0$, $n = 1, 2$.*

Proof. From the definition of the coboundary operator, it follows immediately that $d^3 d^1 = 0$ implies $d^4 d^2 = 0$. Then we only need to prove $d^3 d^1 = 0$. In fact, by (2.4)-(2.7), we get

$$d^3(d^1 f)(x_1, x_2, x_3, x_4, x_5)$$

$$
\begin{aligned}
=&\varepsilon(|f|+|x_1|+|x_2|+|x_3|,|x_4|+|x_5|)\theta(x_4,x_5)(d^1f)(x_1,x_2,x_3)\\
&-\varepsilon(|f|+|x_1|+|x_2|,|x_3|+|x_5|)\varepsilon(|x_4|,|x_5|)\theta(x_3,x_5)(d^1f)(x_1,x_2,x_4)\\
&-\varepsilon(|f|,|x_1|+|x_2|)D(x_1,x_2)(d^1f)(x_3,x_4,x_5)\\
&+\varepsilon(|f|+|x_1|+|x_2|,|x_3|+|x_4|)D(x_3,x_4)(d^1f)(x_1,x_2,x_5)\\
&+(d^1f)([x_1,x_2,x_3],x_4,x_5)+\varepsilon(|x_3|,|x_1|+|x_2|)(d^1f)(x_3,[x_1,x_2,x_4],x_5)\\
&+\varepsilon(|x_1|+|x_2|,|x_3|+|x_4|)(d^1f)(x_3,x_4,[x_1,x_2,x_5])\\
&-(d^1f)(x_1,x_2,[x_3,x_4,x_5])\\
=&\varepsilon(|f|+|x_1|+|x_2|+|x_3|,|x_4|+|x_5|)\theta(x_4,x_5)(\varepsilon(|f|+|x_1|,|x_2|+|x_3|)\theta(x_2,x_3)f(x_1)\\
&-\varepsilon(|x_2|,|x_3|)\varepsilon(|f|,|x_1|+|x_3|)\theta(x_1,x_3)f(x_2)+\varepsilon(|f|,|x_1|+|x_2|)D(x_1,x_2)f(x_3)-f([x_1,x_2,x_3]))\\
&-\varepsilon(|f|+|x_1|+|x_2|,|x_3|+|x_5|)\varepsilon(|x_4|,|x_5|)\theta(x_3,x_5)(\varepsilon(|f|+|x_1|,|x_2|+|x_4|)\theta(x_2,x_4)f(x_1)\\
&-\varepsilon(|x_2|,|x_4|)\varepsilon(|f|,|x_1|+|x_4|)\theta(x_1,x_4)f(x_2)+\varepsilon(|f|,|x_1|+|x_2|)D(x_1,x_2)f(x_4)-f([x_1,x_2,x_4]))\\
&-\varepsilon(|f|,|x_1|+|x_2|)D(x_1,x_2)(\varepsilon(|f|+|x_3|,|x_4|+|x_5|)\theta(x_4,x_5)f(x_3)\\
&-\varepsilon(|x_4|,|x_5|)\varepsilon(|f|,|x_3|+|x_5|)\theta(x_3,x_5)f(x_4)+\varepsilon(|f|,|x_3|+|x_4|)D(x_3,x_4)f(x_5)-f([x_3,x_4,x_5]))\\
&+\varepsilon(|f|+|x_1|+|x_2|,|x_3|+|x_4|)D(x_3,x_4)(\varepsilon(|f|+|x_1|,|x_2|+|x_5|)\theta(x_2,x_5)f(x_1)\\
&-\varepsilon(|x_2|,|x_5|)\varepsilon(|f|,|x_1|+|x_5|)\theta(x_1,x_5)f(x_2)+\varepsilon(|f|,|x_1|+|x_2|)D(x_1,x_2)f(x_5)-f([x_1,x_2,x_5]))\\
&+\varepsilon(|f|+|x_1|+|x_2|+|x_3|,|x_4|+|x_5|)(\theta(x_4,x_5)f([x_1,x_2,x_3])\\
&-\varepsilon(|x_4|,|x_5|)\varepsilon(|f|,|x_1|+|x_2|+|x_3|+|x_5|)\theta([x_1,x_2,x_3],x_5)f(x_4)\\
&+\varepsilon(|f|,|x_1|+|x_2|+|x_3|+|x_4|)D([x_1,x_2,x_3],x_4)f(x_5)-f([[x_1,x_2,x_3],x_4,x_5]))\\
&+\varepsilon(|x_3|,|x_1|+|x_2|)\varepsilon(|f|+|x_3|,|x_1|+|x_2|+|x_4|+|x_5|)(\theta([x_1,x_2,x_4],x_5)f(x_3)\\
&-\varepsilon(|f|+|x_1|+|x_2|,|x_3|+|x_5|)\varepsilon(x_4|,|x_5|)\theta(x_3,x_5)f([x_1,x_2,x_4])\\
&+\varepsilon(|x_3|,|x_1|+|x_2|)\varepsilon(|f|,|x_1|+|x_2|+|x_3|+|x_4|)D(x_3,[x_1,x_2,x_4])f(x_5)\\
&-\varepsilon(|x_3|,|x_1|+|x_2|)f([x_3,[x_1,x_2,x_4],x_5]))\\
&+\varepsilon(|x_1|+|x_2|,|x_3|+|x_4|)(\varepsilon(|f|+|x_3|,|x_1|+|x_2|+|x_4|+|x_5|)\theta(x_4,[x_1,x_2,x_5])f(x_3)\\
&-\varepsilon(|x_4|,|x_1|+|x_2|+|x_5|)\varepsilon(|f|,|x_1|+|x_2|+|x_3|+|x_5|)\theta(x_3,[x_1,x_2,x_5])f(x_4)\\
&+\varepsilon(|f|,|x_3|+|x_4|)D(x_3,x_4)f([x_1,x_2,x_5])-f([x_3,x_4,[x_1,x_2,x_5]]))\\
&-(\varepsilon(|f|+|x_1|,|x_2|+|x_3|+|x_4|+|x_5|)\theta(x_2,[x_3,x_4,x_5])f(x_1)\\
&-\varepsilon(|f|,|x_1|+|x_3|+|x_4|+|x_5|)\varepsilon(|x_2|,|x_3|+|x_4|+|x_5|)\theta(x_1,[x_3,x_4,x_5])f(x_2)\\
&+\varepsilon(|f|,|x_1|+|x_2|)D(x_1,x_2)f([x_3,x_4,x_5])-f([x_1,x_2,[x_3,x_4,x_5]]))\\
=&-f([[x_1,x_2,x_3],x_4,x_5])-\varepsilon(|x_3|,|x_1|+|x_2|)f([x_3,[x_1,x_2,x_4],x_5])\\
&-\varepsilon(|x_1|+|x_2|,|x_3|+|x_4|)f([x_3,x_4,[x_1,x_2,x_5]])+f([x_1,x_2,[x_3,x_4,x_5]])\\
&+\varepsilon(|f|+|x_1|+|x_2|+|x_3|,|x_4|+|x_5|)\varepsilon(|f|+|x_1|,|x_2|+|x_3|)\theta(x_4,x_5)\theta(x_2,x_3)f(x_1)\\
&-\varepsilon(|f|+|x_1|+|x_2|,|x_3|+|x_5|)\varepsilon(|x_4|,|x_5|)\varepsilon(|f|+|x_1|,|x_2|+|x_4|)\theta(x_3,x_5)\theta(x_2,x_4)f(x_1)\\
&+\varepsilon(|f|+|x_1|+|x_2|,|x_3|+|x_4|)\varepsilon(|f|+|x_1|,|x_2|+|x_5|)D(x_3,x_4)\theta(x_2,x_5)f(x_1)\\
&-\varepsilon(|f|+|x_1|,|x_2|+|x_3|+|x_4|+|x_5|)\theta(x_2,[x_3,x_4,x_5])f(x_1)\\
&-\varepsilon(|f|+|x_1|+|x_2|+|x_3|,|x_4|+|x_5|)\varepsilon(|x_2|,|x_3|)\varepsilon(|f|,|x_1|+|x_3|)\theta(x_4,x_5)\theta(x_1,x_3)f(x_2)\\
&+\varepsilon(|f|+|x_1|+|x_2|,|x_3|+|x_5|)\varepsilon(|x_4|,|x_2|+|x_5|)\varepsilon(|f|,|x_1|+|x_4|)\theta(x_3,x_5)\theta(x_1,x_4)f(x_2)
\end{aligned}
$$

$$
\begin{aligned}
&- \varepsilon(|f|+|x_1|+|x_2|,|x_3|+|x_4|)\varepsilon(|x_2|,|x_5|)\varepsilon(|f|,|x_1|+|x_5|)D(x_3,x_4)\theta(x_1,x_5)f(x_2)\\
&+ \varepsilon(|f|,|x_1|+|x_3|+|x_4|+|x_5|)\varepsilon(|x_2|,|x_3|+|x_4|+|x_5|)\theta(x_1,[x_3,x_4,x_5])f(x_2)\\
&+ \varepsilon(|f|+|x_1|+|x_2|+|x_3|,|x_4|+|x_5|)\varepsilon(|f|,|x_1|+|x_2|)\theta(x_4,x_5)D(x_1,x_2)f(x_3)\\
&- \varepsilon(|f|,|x_1|+|x_2|)\varepsilon(|f|+|x_3|,|x_4|+|x_5|)D(x_1,x_2)\theta(x_4,x_5)f(x_3)\\
&+ \varepsilon(|x_3|,|x_1|+|x_2|)\varepsilon(|f|+|x_3|,|x_1|+|x_2|+|x_4|+|x_5|)\theta([x_1,x_2,x_4],x_5)f(x_3)\\
&+ \varepsilon(|x_1|+|x_2|,|x_3|+|x_4|)\varepsilon(|f|+|x_3|,|x_1|+|x_2|+|x_4|+|x_5|)\theta(x_4,[x_1,x_2,x_5])f(x_3)\\
&- \varepsilon(|f|+|x_1|+|x_2|,|x_3|+|x_5|)\varepsilon(|x_4|,|x_5|)\varepsilon(|f|,|x_1|+|x_2|)\theta(x_3,x_5)D(x_1,x_2)f(x_4)\\
&+ \varepsilon(|f|,|x_1|+|x_2|)\varepsilon(|x_4|,|x_5|)\varepsilon(|f|,|x_3|+|x_5|)D(x_1,x_2)\theta(x_3,x_5)f(x_4)\\
&- \varepsilon(|x_4|,|x_5|)\varepsilon(|f|,|x_1|+|x_2|+|x_3|+|x_5|)\theta([x_1,x_2,x_3],x_5)f(x_4) - \varepsilon(|x_1|+|x_2|,|x_3|+|x_4|)\\
&\varepsilon(|x_4|,|x_1|+|x_2|+|x_5|)\varepsilon(|f|,|x_1|+|x_2|+|x_3|+|x_5|)\theta(x_3,[x_1,x_2,x_5])f(x_4)\\
&- \varepsilon(|f|,|x_1|+|x_2|+|x_3|+|x_4|)D(x_1,x_2)D(x_3,x_4)f(x_5)\\
&+ \varepsilon(|f|+|x_1|+|x_2|,|x_3|+|x_4|)\varepsilon(|f|,|x_1|+|x_2|)D(x_3,x_4)D(x_1,x_2)f(x_5)\\
&+ \varepsilon(|f|,|x_1|+|x_2|+|x_3|+|x_4|)D([x_1,x_2,x_3],x_4)f(x_5)\\
&+ \varepsilon(|x_3|,|x_1|+|x_2|)\varepsilon(|f|,|x_1|+|x_2|+|x_3|+|x_4|)D(x_3,[x_1,x_2,x_4])f(x_5)\\
&- \varepsilon(|f|+|x_1|+|x_2|+|x_3|,|x_4|+|x_5|)\theta(x_4,x_5)f([x_1,x_2,x_3])\\
&+ \varepsilon(|f|+|x_1|+|x_2|,|x_3|+|x_5|)\varepsilon(|x_4|,|x_5|)\theta(x_3,x_5)f([x_1,x_2,x_4])\\
&+ \varepsilon(|f|,|x_1|+|x_2|)D(x_1,x_2)f([x_3,x_4,x_5])\\
&- \varepsilon(|f|,|x_3|+|x_4|)\varepsilon(|x_1|+|x_2|,|x_3|+|x_4|)D(x_3,x_4)f([x_1,x_2,x_5])\\
&+ \varepsilon(|f|+|x_1|+|x_2|+|x_3|,|x_4|+|x_5|)\theta(x_4,x_5)f([x_1,x_2,x_3])\\
&- \varepsilon(|f|+|x_1|+|x_2|,|x_3|+|x_5|)\varepsilon(|x_4|,|x_5|)\theta(x_3,x_5)f([x_1,x_2,x_4])\\
&+ \varepsilon(|f|,|x_3|+|x_4|)\varepsilon(|x_1|+|x_2|,|x_3|+|x_4|)D(x_3,x_4)f([x_1,x_2,x_5])\\
&- \varepsilon(|f|,|x_1|+|x_2|)D(x_1,x_2)f([x_3,x_4,x_5])\\
=&0.
\end{aligned}
$$

Therefore, the proof is complete. □

For $n = 1,2,3,4$, the map $f \in C^n_A(T,V)$ is called an n-cocycle if $d^n f = 0$. We denote by $Z^n_A(T,V)$ the subspace spanned by n-cocycles and $B^n_A(T,V) = d^{n-2}C^{n-2}_A(T,V)$.

Since $d^{n+2}d^n = 0$, $B^n_A(T,V)$ is a subspace of $Z^n_A(T,V)$. Hence we can define a cohomology space $H^n_A(T,V)$ of $(T,[\cdot,\cdot,\cdot])$ as the factor space $Z^n_A(T,V)/B^n_A(T,V)$.

3. 1-parameter formal deformations of Lie color triple systems

Let $(T,[\cdot,\cdot,\cdot])$ be a Lie color triple system and $\mathbf{F}[[t]]$ be the ring of formal power series over $\mathbf{F}$. Assume that $T[[t]]$ is the set of formal power series over T. We extend an $\mathbf{F}$-trilinear map $f : T\times T\times T \to T$ to be an $\mathbf{F}[[t]]$-trilinear map $f : T[[t]]\times T[[t]]\times T[[t]] \to T[[t]]$ by

$$f\left(\sum_{i\geq 0} x_i t^i, \sum_{j\geq 0} y_j t^j, \sum_{k\geq 0} z_k t^k\right) = \sum_{i,j,k\geq 0} f(x_i,y_j,z_k)t^{i+j+k}.$$

Definition 3.1. *Let $(T, [\cdot,\cdot,\cdot])$ be a Lie color triple system over* $\mathbf{F}$*. A 1-parameter formal deformation of $(T, [\cdot,\cdot,\cdot])$ is a formal power series $m_t : T[[t]] \times T[[t]] \times T[[t]] \to T[[t]]$ of the form*

$$m_t(x, y, z) = \sum_{i \geq 0} m_i(x, y, z)t^i = m_0(x, y, z) + m_1(x, y, z)t + m_2(x, y, z)t^2 + \cdots ,$$

where each m_i is an $\mathbf{F}$*-trilinear map $m_i : T \times T \times T \to T$ (extended to be* $\mathbf{F}[[t]]$*-trilinear) and $m_0(x, y, z) = [x, y, z]$, such that the following equations hold*

$$m_t(x, y, z) = -\varepsilon(|x|, |y|)m_t(y, x, z), \tag{3.1}$$

$$\varepsilon(|x|, |z|)m_t(x, y, z) + \varepsilon(|y|, |x|)m_t(y, z, x) + \varepsilon(|z|, |y|)m_t(z, x, y) = 0, \tag{3.2}$$

$$\begin{aligned} m_t(u, v, m_t(x, y, z)) =& m_t(m_t(u, v, x), y, z) \\ &+ \varepsilon(|x|, |u| + |v|)m_t(x, m_t(u, v, y), z) \\ &+ \varepsilon(|u| + |v|, |x| + |y|)m_t(x, y, m_t(u, v, z)). \end{aligned} \tag{3.3}$$

Conditions (3.1)-(3.3) are called the deformation equations of a Lie color triple system.

Note that $T[[t]]$ is a module over $\mathbf{F}[[t]]$ and m_t defines the trilinear on $T[[t]]$ such that $T_t = (T[[t]], m_t)$ is a Lie color triple system. Now we discuss the deformation equations (3.1)-(3.3).

Conditions (3.1)-(3.2) are equivalent to the following equations

$$m_i(x, y, z) = -\varepsilon(|x|, |y|)m_i(y, x, z), \tag{3.4}$$

$$\varepsilon(|x|, |z|)m_i(x, y, z) + \varepsilon(|y|, |x|)m_i(y, z, x) + \varepsilon(|z|, |y|)m_i(z, x, y) = 0, \tag{3.5}$$

respectively, for $i = 0, 1, 2, \cdots$. The condition (3.3) can be showed as

$$\begin{aligned} &\sum_{i,j \geq 0} m_i(u, v, m_j(x, y, z)) \\ =& \sum_{i,j \geq 0} m_i(m_j(u, v, x), y, z) + \sum_{i,j \geq 0} \varepsilon(|x|, |u| + |v|)m_i(x, m_j(u, v, y), z) \\ &+ \sum_{i,j \geq 0} \varepsilon(|u| + |v|, |x| + |y|)m_i(x, y, m_j(u, v, z)). \end{aligned}$$

Then

$$\begin{aligned} \sum_{i+j=n} \Big(& m_i(m_j(u, v, x), y, z) + \varepsilon(|x|, |u| + |v|)m_i(x, m_j(u, v, y), z) \\ &+ \varepsilon(|u| + |v|, |x| + |y|)m_i(x, y, m_j(u, v, z)) \\ &- m_i(u, v, m_j(x, y, z)) \Big) = 0, \quad \forall n = 0, 1, 2 \cdots . \end{aligned}$$

Using two $\mathbf{F}$-trilinear maps $f, g : T \times T \times T \to T$ (extended to be $\mathbf{F}[[t]]$-trilinear), we assume a map $f \circ g : T[[t]] \times T[[t]] \times T[[t]] \times T[[t]] \times T[[t]] \to T[[t]]$ by

$$\begin{aligned} f \circ g(u, v, x, y, z) =& f(g(u, v, x), y, z) + \varepsilon(|x|, |u| + |v|)f(x, g(u, v, y), z) \\ &+ \varepsilon(|u| + |v|, |x| + |y|)f(x, y, g(u, v, z)) - f(u, v, g(x, y, z)). \end{aligned}$$

Then the deformation equation (3.3) can be given as

$$\sum_{i+j=n} m_i \circ m_j = 0.$$

For $n = 1$, $m_0 \circ m_1 + m_1 \circ m_0 = 0$.

For $n \geq 2$, $-(m_0 \circ m_n + m_n \circ_\alpha m_0) = m_1 \circ m_{n-1} + m_2 \circ m_{n-2} + \cdots + m_{n-1} \circ_\alpha m_1$.

Section 2 gets that T is the adjoint $(T, [\cdot, \cdot, \cdot])$-module by setting $\theta(x, y)(z) = \varepsilon(|z|, |x| + |y|)[z, x, y]$. In this case, by (3.4)-(3.5), we have $m_i \in C^3(T, T)$, and $m_i \circ m_j \in C^5(T, T)$. In general, if $f, g \in C^3(T, T)$, then $f \circ g \in C^5(T, T)$. Noticing that the definition of coboundary operator d^n, we obtain $d^3 m_n = m_0 \circ m_n + m_n \circ_\alpha m_0$, for $n = 0, 1, 2 \cdots$. Hence the deformation equation (3.3) can be rewritten as

$$d^3 m_1 = 0,$$
$$-d^3 m_n = m_1 \circ m_{n-1} + m_2 \circ m_{n-2} + \cdots + m_{n-1} \circ m_1.$$

Then m_1 is a 3-cocycle and called the **infinitesimal** of m_t.

Definition 3.2. *Let $(T, [\cdot, \cdot, \cdot])$ be a Lie color triple system. Assume that $m_t(x, y, z) = \sum_{i\geq 0} m_i(x, y, z)t^i$ and $m'_t(x, y, z) = \sum_{i\geq 0} m'_i(x, y, z)t^i$ are two 1-parameter formal deformations of $(T, [\cdot, \cdot, \cdot])$. They are called equivalent, denoted by $m_t \sim m'_t$, if there is a formal isomorphism of* $\mathbf{F}[[t]]$*-modules*

$$\phi_t(x) = \sum_{i\geq 0} \phi_i(x)t^i : (T[[t]], m_t) \longrightarrow (T[[t]], m'_t),$$

where each $\phi_i : T \to T$ is an $\mathbf{F}$*-linear map (extended to be* $\mathbf{F}[[t]]$*-linear) and $\phi_0 = \mathrm{id}_T$, satisfying*

$$\phi_t \circ m_t(x, y, z) = m'_t(\phi_t(x), \phi_t(y), \phi_t(z)).$$

When $m_1 = m_2 = \cdots = 0$, $m_t = m_0$ is said to be the null deformation. A 1-parameter formal deformation m_t is called trivial if $m_t \sim m_0$. A Lie color triple system $(T, [\cdot, \cdot, \cdot])$ is called analytically rigid, if every 1-parameter formal deformation m_t is trivial.

Theorem 3.3. *Let $m_t(x, y, z) = \sum_{i\geq 0} m_i(x, y, z)t^i$ and $m'_t(x, y, z) = \sum_{i\geq 0} m'_i(x, y, z)t^i$ be equivalent 1-parameter formal deformations of $(T, [\cdot, \cdot, \cdot])$. Then m_1 and m'_1 belong to the same cohomology class in $H^3(T, T)$.*

Proof. Assume that $\phi_t(x) = \sum_{i\geq 0} \phi_i(x)t^i$ is the formal $\mathbf{F}[[t]]$-module isomorphism and

$$\sum_{i\geq 0} \phi_i \left(\sum_{j\geq 0} m_j(x, y, z)t^j \right) t^i = \sum_{i\geq 0} m'_i \left(\sum_{k\geq 0} \phi_k(x)t^k, \sum_{l\geq 0} \phi_l(y)t^l, \sum_{m\geq 0} \phi_m(z)t^m \right) t^i.$$

We get that

$$\sum_{i+j=n} \phi_i(m_j(x, y, z))t^{i+j} = \sum_{i+k+l+m=n} m'_i(\phi_k(x), \phi_l(y), \phi_m(z))t^{i+k+l+m}.$$

In particular,

$$\sum_{i+j=1} \phi_i(m_j(x, y, z)) = \sum_{i+k+l+m=1} m'_i(\phi_k(x), \phi_l(y), \phi_m(z)),$$

that is,

$$\begin{aligned}&m_1(x,y,z)+\phi_1([x,y,z])\\=&[\phi_1(x),y,z]+[x,\phi_1(y),z]+[x,y,\phi_1(z)]+m_1'(x,y,z)\\=&\varepsilon(|x|,|y|+|z|)\theta(y,z)\phi_1(x)-\varepsilon(|y|,|z|)\theta(x,z)\phi_1(y)+D(x,y)\phi_1(z)+m_1'(x,y,z).\end{aligned}$$

Thus $m_1 - m_1' = d^1\phi_1 \in B^3(T,T)$. □

Theorem 3.4. *Let* $(T, [\cdot,\cdot,\cdot])$ *be a Lie color triple system such that* $H^3(T,T) = 0$. *Thus* $(T, [\cdot,\cdot,\cdot])$ *is analytically rigid.*

Proof. Let m_t be a 1-parameter formal deformation of $(T, [\cdot,\cdot,\cdot])$. Assume that $m_t = m_0 + \sum_{i\geq n} m_i t^i$. Then

$$d^3 m_n = m_1 \circ m_{n-1} + m_2 \circ m_{n-2} + \cdots + m_{n-1} \circ m_1 = 0,$$

that is, $m_n \in Z^3(T,T) = B^3(T,T)$. We obtain that there exists $f_n \in C^1(T,T)$ such that $m_n = d^1 f_n$.

Let ϕ_t be even, and $\phi_t = \mathrm{id}_T - f_n t^n : (T[[t]], m_t) \longrightarrow (T[[t]], m_t')$. Note that

$$\phi_t \circ \sum_{i\geq 0} f_n^i t^{in} = \sum_{i\geq 0} f_n^i t^{in} \circ \phi_t = \mathrm{id}_{T[[t]]}.$$

Thus ϕ_t is a linear isomorphism.

Now we consider $m_t'(x,y,z) = \phi_t^{-1} m_t(\phi_t(x), \phi_t(y), \phi_t(z))$. It is easy to give that m_t' is a 1-parameter formal deformation of $(T, [\cdot,\cdot,\cdot])$. In fact,

$$\begin{aligned}&m_t'(x,y,z)\\=&\phi_t^{-1} m_t(\phi_t(x),\phi_t(y),\phi_t(z))\\=&-\varepsilon(|x|,|y|)\phi_t^{-1} m_t(\phi_t(y),\phi_t(x),\phi_t(z))\\=&-\varepsilon(|x|,|y|)m_t'(y,x,z).\end{aligned}$$

$$\begin{aligned}&\varepsilon(|x|,|z|)m_t'(x,y,z)+\varepsilon(|y|,|x|)m_t'(y,z,x)+\varepsilon(|z|,|y|)m_t'(z,x,y)\\=&\varepsilon(|x|,|z|)\phi_t^{-1}m_t(\phi_t(x),\phi_t(y),\phi_t(z))+\varepsilon(|y|,|x|)\phi_t^{-1}m_t(\phi_t(y),\phi_t(z),\phi_t(x))\\&+\varepsilon(|z|,|y|)\phi_t^{-1}m_t(\phi_t(z),\phi_t(x),\phi_t(y))\\=&\phi_t^{-1}(\varepsilon(|x|,|z|)m_t(\phi_t(x),\phi_t(y),\phi_t(z))+\varepsilon(|y|,|x|)m_t(\phi_t(y),\phi_t(z),\phi_t(x))\\&+\varepsilon(|z|,|y|)m_t(\phi_t(z),\phi_t(x),\phi_t(y)))\\=&0.\end{aligned}$$

$$\begin{aligned}&m_t'(u,v,m_t'(x,y,z))\\=&m_t'(u,v,\phi_t^{-1}m_t(\phi_t(x),\phi_t(y),\phi_t(z)))\\=&\phi_t^{-1}m_t(\phi_t(u),\phi_t(v),m_t(\phi_t(x),\phi_t(y),\phi_t(z))).\end{aligned}$$

$$m_t'(m_t'(u,v,x),y,z)$$

$$
\begin{aligned}
&=m'_t(\phi_t^{-1}m_t(\phi_t(u),\phi_t(v),\phi_t(x)),y,z)\\
&=\phi_t^{-1}m_t(m_t(\phi_t(u),\phi_t(v),\phi_t(x)),\phi_t(y),\phi_t(z)).
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon(|x|,|u|+|v|)m'_t(x,m'_t(u,v,y),z)\\
=&\varepsilon(|x|,|u|+|v|)m'_t(x,\phi_t^{-1}m_t(\phi_t(u),\phi_t(v),\phi_t(y)),z)\\
=&\varepsilon(|x|,|u|+|v|)\phi_t^{-1}m_t(\phi_t(x),m_t(\phi_t(u),\phi_t(v),\phi_t(y)),\phi_t(z)).
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon(|u|+|v|,|x|+|y|)m'_t(x,y,m'_t(u,v,z))\\
=&\varepsilon(|u|+|v|,|x|+|y|)m'_t(x,y,\phi_t^{-1}m_t(\phi_t(u),\phi_t(v),\phi_t(z)))\\
=&\varepsilon(|u|+|v|,|x|+|y|)\phi_t^{-1}m_t(\phi_t(x),\phi_t(y),m_t(\phi_t(u),\phi_t(v),\phi_t(z))).
\end{aligned}
$$

The computations above show that equations (3.1)-(3.3) hold. Using the definition 3.2, we get $m_t \sim m'_t$. Assume that $m'_t = \sum_{i\geq 0} m'_i t^i$. Thus

$$
(\mathrm{id}_T - f_n t^n)\left(\sum_{i\geq 0} m'_i(x,y,z)t^i\right) = \left(m_0 + \sum_{i\geq n} m_i t^i\right)(x - f_n(x)t^n, y - f_n(y)t^n, z - f_n(z)t^n),
$$

i.e.,

$$
\begin{aligned}
&\sum_{i\geq 0} m'_i(x,y,z)t^i - \sum_{i\geq 0} f_n \circ m'_i(x,y,z)t^{i+n}\\
=&[x,y,z] - ([f_n(x),y,z] + [x,f_n(y),z] + [x,y,f_n(z)])t^n\\
&+ ([f_n(x),f_n(y),z] + [x,f_n(y),f_n(z)] + [f_n(x),y,f_n(z)])t^{2n} - [f_n(x),f_n(y),f_n(z)]t^{3n}\\
&+ \sum_{i\geq n} m_i(x,y,z)t^i - \sum_{i\geq n}(m_i(f_n(x),y,z) + m_i(x,f_n(y),z) + m_i(x,y,f_n(z)))t^{i+n}\\
&+ \sum_{i\geq n}(m_i(f_n(x),f_n(y),z) + m_i(x,f_n(y),f_n(z)) + m_i(f_n(x),y,f_n(z)))t^{i+2n}\\
&- \sum_{i\geq n} m_i(f_n(x),f_n(y),f_n(z))t^{i+3n}.
\end{aligned}
$$

Then we get $m'_1 = \cdots = m'_{n-1} = 0$ and

$$
\begin{aligned}
&m'_n(x,y,z) - f_n([x,y,z])\\
=&-([f_n(x),y,z] + [x,f_n(y),z] + [x,y,f_n(z)]) + m_n(x,y,z)\\
=&-\varepsilon(|x|,|y|+|z|)\theta(y,z)f_n(x) + \varepsilon(|y|,|z|)\theta(x,z)f_n(y) - D(x,y)f_n(z) + m_n(x,y,z)
\end{aligned}
$$

Hence $m'_n = m_n - d^1 f_n = 0$ and $m'_t = m_0 + \sum_{i\geq n+1} m'_i t^i$. Using induction, this procedure ends with $m_t \sim m_0$, that is, $(T,[\cdot,\cdot,\cdot])$ is analytically rigid. □

4. Abelian extensions of Lie color triple systems

In this section, we show that associated to any abelian extension, there is a representation and a 3-cocycle.

An ideal of a Lie color triple system T is a subspace I such that $[I, T, T] \subseteq I$. An ideal I of a Lie color triple system is called an abelian ideal if moreover $[T, I, I] = 0$. Notice that $[T, I, I] = 0$ implies that $[I, T, I] = 0$ and $[I, I, T] = 0$.

Definition 4.1. *Let* $(T, [\cdot,\cdot,\cdot]_T)$, $(V, [\cdot,\cdot,\cdot]_V)$, *and* $(\hat{T}, [\cdot,\cdot,\cdot]_{\hat{T}})$ *be Lie color triple systems and* $i : V \to \hat{T}$, $p : \hat{T} \to T$ *be homomorphisms. The following sequence of Lie color triple systems is a short exact sequence if* $\mathrm{Im}(i) = \mathrm{Ker}(p)$, $\mathrm{Ker}(i) = 0$ *and* $\mathrm{Im}(p) = T$,

$$0 \longrightarrow V \xrightarrow{i} \hat{T} \xrightarrow{p} T \longrightarrow 0. \tag{4.1}$$

In this case, we show $\hat{T}$ an extension of T by V, and denote it by $E_{\hat{T}}$. It is called an abelian extension if V is an abelian ideal of $\hat{T}$, i.e., $[u, v, \cdot]_{\hat{T}} = [u, \cdot, v]_{\hat{T}} = [\cdot, u, v]_{\hat{T}} = 0$, for all $u, v \in V$.

A section $\sigma : T \to \hat{T}$ of $p : \hat{T} \to T$ consists of the linear map $\sigma : T \to \hat{T}$ such that $p \circ \sigma = \mathrm{id}_T$.

Definition 4.2. *Two extensions of Lie color triple systems* $E_{\hat{T}} : 0 \longrightarrow V \xrightarrow{i} \hat{T} \xrightarrow{p} T \longrightarrow 0$ *and* $E_{\tilde{T}} : 0 \longrightarrow V \xrightarrow{j} \tilde{T} \xrightarrow{q} T \longrightarrow 0$ *are equivalent. If there exists a Lie color triple system homomorphism* $F : \hat{T} \to \tilde{T}$ *such that the following diagram commutes*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V & \xrightarrow{i} & \hat{T} & \xrightarrow{p} & T & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & V & \xrightarrow{j} & \tilde{T} & \xrightarrow{q} & T & \longrightarrow & 0
\end{array}$$

Let $\hat{T}$ be an abelian extension of T by V, and a linear mapping $\sigma : T \to \hat{T}$ be a section. Define maps $T \otimes T \to \mathrm{End}(V)$ by

$$D(x_1, x_2)(u) = [\sigma(x_1), \sigma(x_2), u]_{\hat{T}}, \tag{4.2}$$

$$\theta(x_1, x_2)(u) = [u, \sigma(x_1), \sigma(x_2)]_{\hat{T}} \tag{4.3}$$

Clearly, the following fact holds, i.e.,

$$D(x_1, x_2)(u) = \theta(x_2, x_1)(u) - \theta(x_1, x_2)(u),$$

for all $(x_1, x_2) \in T \otimes T, u \in V$.

Theorem 4.3. *Using the above notations,* (V, θ) *is a representation of* T *and does not depend on the choice of the section* σ. *Moreover, equivalent abelian extensions give the same representation.*

Proof. First, if we choose another section $\sigma' : T \to \hat{T}$, then

$$p(\sigma(x_i) - \sigma'(x_i)) = x_i - x_i = 0 \Rightarrow \sigma(x_i) - \sigma'(x_i) \in V \Rightarrow \sigma'(x_i) = \sigma(x_i) + u_i,$$

for some $u_i \in V$.

Note that $[u, v, \cdot]_{\hat{T}} = 0 = [u, \cdot, v]_{\hat{T}}$ for all $u, v \in V$, this yields that

$$[v, \sigma'(x_1), \sigma'(x_2)]_{\hat{T}} = [v, \sigma(x_1), \sigma(x_2)]_{\hat{T}}.$$

This shows that θ is independent on the choice of σ.

Second, we get that (V, θ) is a representation of T.

By the equality

$$\begin{aligned}&[u, \sigma(x_1), [\sigma(y_1), \sigma(y_2), \sigma(y_3)]_{\hat{T}}]_{\hat{T}}\\ =&[[u, \sigma(x_1), \sigma(y_1)]_{\hat{T}}, \sigma(y_2), \sigma(y_3)]_{\hat{T}} + \varepsilon(|y_1|, |u| + |x_1|)[\sigma(y_1), [u, \sigma(x_1), \sigma(y_2)]_{\hat{T}}, \sigma(y_3)]_{\hat{T}}\\ &+ \varepsilon(|u| + |x_1|, |y_1| + |y_2|)[\sigma(y_1), \sigma(y_2), [u, \sigma(x_1), \sigma(y_3)]_{\hat{T}}]_{\hat{T}},\end{aligned}$$

it follows that

$$\begin{aligned}&\varepsilon(|x_1| + |y_1|, |y_2| + |y_3|)\theta(y_2, y_3)\theta(x_1, y_1)(u) - \varepsilon(|x_1|, |y_1|)\varepsilon(|y_3|, |y_2| + |x_1|)\theta(y_1, y_3)\theta(x_1, y_2)(u)\\ &- \theta(x_1, [y_1, y_2, y_3])(u) + \varepsilon(|x_1|, |y_1| + |y_2|)D(y_1, y_2)\theta(x_1, y_3)(u) = 0.\end{aligned}$$

Thus, we obtain (2.5) holds.

Similarly, by the equality

$$\begin{aligned}&[\sigma(x_1), \sigma(x_2), [u, \sigma(y_1), \sigma(y_2)]_{\hat{T}}]_{\hat{T}}\\ =&[[\sigma(x_1), \sigma(x_2), u]_{\hat{T}}, \sigma(y_1), \sigma(y_2)]_{\hat{T}} + \varepsilon(|u|, |x_1| + |x_2|)[u, [\sigma(x_1), \sigma(x_2), \sigma(y_1)]_{\hat{T}}, \sigma(y_2)]_{\hat{T}}\\ &+ \varepsilon(|x_1| + |x_2|, |u| + |y_1|)[u, \sigma(y_1), [\sigma(x_1), \sigma(x_2), \sigma(y_2)]_{\hat{T}}]_{\hat{T}},\end{aligned}$$

we have

$$\begin{aligned}&\varepsilon(|x_1| + |x_2|, |y_1| + |y_2|)\theta(y_1, y_2)D(x_1, x_2)(u) - D(x_1, x_2)\theta(y_1, y_2)(u)\\ &+ \theta([x_1, x_2, y_1], y_2)(u) + \varepsilon(|y_1|, |x_1| + |x_2|)\theta(y_1, [x_1, x_2, y_2])(u) = 0,\end{aligned}$$

by the equality

$$\begin{aligned}&[\sigma(x_1), \sigma(x_2), [\sigma(y_1), \sigma(y_2), u]_{\hat{T}}]_{\hat{T}}\\ =&[[\sigma(x_1), \sigma(x_2), \sigma(y_1)]_{\hat{T}}, \sigma(y_2), u]_{\hat{T}} + \varepsilon(|y_1|, |x_1| + |x_2|)[\sigma(y_1), [\sigma(x_1), \sigma(x_2), \sigma(y_2)]_{\hat{T}}, u]_{\hat{T}}\\ &+ \varepsilon(|x_1| + |x_2|, |y_1| + |y_2|)[\sigma(y_1), \sigma(y_2), [\sigma(x_1), \sigma(x_2), u]_{\hat{T}}]_{\hat{T}},\end{aligned}$$

we get

$$\begin{aligned}&\varepsilon(|x_1| + |x_2|, |y_1| + |y_2|)D(y_1, y_2)D(x_1, x_2)(u) - D(x_1, x_2)D(y_1, y_2)(u)\\ &+ D([x_1, x_2, y_1], y_2)(u) + \varepsilon(|y_1|, |x_1| + |x_2|)D(y_1, [x_1, x_2, y_2])(u) = 0.\end{aligned}$$

Thus, we know that (2.6) and (2.7) hold. Therefore, it is proved that (V, θ) is a representation of T.

Third, we will show equivalent abelian extensions give the same θ.

Assume that $E_{\hat{T}}$ and $E_{\tilde{T}}$ are equivalent abelian extensions, and $F : \hat{T} \to \tilde{T}$ is the Lie color triple system homomorphism satisfying $F \circ i = j$, $q \circ F = p$. Choosing linear sections σ and σ' of p and q, we have $qF\sigma(x_i) = p\sigma(x_i) = x_i = q\sigma'(x_i)$, then $F\sigma(x_i) - \sigma'(x_i) \in \mathrm{Ker}(q) \cong V$. Moreover,

$$[u, \sigma(x_1), \sigma(x_2)]_{\hat{T}} = [u, F\sigma(x_1), F\sigma(x_2)]_{\tilde{T}} = [u, \sigma'(x_1), \sigma'(x_2)]_{\tilde{T}}.$$

The proof is complete. □

Let $\sigma : T \to \hat{T}$ be a section of the abelian extension. Assume the following map:

$$\omega(x_1, x_2, x_3) = [\sigma(x_1), \sigma(x_2), \sigma(x_3)]_{\hat{T}} - \sigma([x_1, x_2, x_3]_T), \tag{4.4}$$

for all $x_1, x_2, x_3 \in T$.

Theorem 4.4. *Let* $0 \longrightarrow V \longrightarrow \hat{T} \longrightarrow T \longrightarrow 0$ *be an abelian extension of* T *by* V. *Then* ω *defined by (4.4) is a* 3*-cocycle of* T *with coefficients in* V*, where the representation* θ *is given by (4.3).*

Proof. By the equality

$$\begin{aligned}
&[\sigma(x_1), \sigma(x_2), [\sigma(y_1), \sigma(y_2), \sigma(y_3)]_{\hat{T}}]_{\hat{T}} \\
=&[[\sigma(x_1), \sigma(x_2), \sigma(y_1)]_{\hat{T}}, \sigma(y_2), \sigma(y_3)]_{\hat{T}} + \varepsilon(|y_1|, |x_1| + |x_2|)[\sigma(y_1), [\sigma(x_1), \sigma(x_2), \sigma(y_2)]_{\hat{T}}, \sigma(y_3)]_{\hat{T}} \\
&+ \varepsilon(|x_1| + |x_2|, |y_1| + |y_2|)[\sigma(y_1), \sigma(y_2), [\sigma(x_1), \sigma(x_2), \sigma(y_3)]_{\hat{T}}]_{\hat{T}}.
\end{aligned}$$

The left hand side shows that

$$\begin{aligned}
&[\sigma(x_1), \sigma(x_2), [\sigma(y_1), \sigma(y_2), \sigma(y_3)]_{\hat{T}}]_{\hat{T}} \\
=&[\sigma(x_1), \sigma(x_2), \omega(y_1, y_2, y_3) + \sigma([y_1, y_2, y_3]_T)]_{\hat{T}} \\
=&D(x_1, x_2)\omega(y_1, y_2, y_3) + [\sigma(x_1), \sigma(x_2), \sigma([y_1, y_2, y_3]_T)]_{\hat{T}} \\
=&D(x_1, x_2)\omega(y_1, y_2, y_3) + \omega(x_1, x_2, [y_1, y_2, y_3]_T) + \sigma([x_1, x_2, [y_1, y_2, y_3]_T]_T).
\end{aligned}$$

Similarly, the right side is equal to

$$\begin{aligned}
&[[\sigma(x_1), \sigma(x_2), \sigma(y_1)]_{\hat{T}}, \sigma(y_2), \sigma(y_3)]_{\hat{T}} + \varepsilon(|y_1|, |x_1| + |x_2|)[\sigma(y_1), [\sigma(x_1), \sigma(x_2), \sigma(y_2)]_{\hat{T}}, \sigma(y_3)]_{\hat{T}} \\
&+ \varepsilon(|x_1| + |x_2|, |y_1| + |y_2|)[\sigma(y_1), \sigma(y_2), [\sigma(x_1), \sigma(x_2), \sigma(y_3)]_{\hat{T}}]_{\hat{T}} \\
&= [\omega(x_1, x_2, y_1) + \sigma([x_1, x_2, y_1]_T), \sigma(y_2), \sigma(y_3)]_{\hat{T}} \\
&+ \varepsilon(|y_1|, |x_1| + |x_2|)[\sigma(y_1), \omega(x_1, x_2, y_2) + \sigma([x_1, x_2, y_2]_T), \sigma(y_3)]_{\hat{T}} \\
&+ \varepsilon(|x_1| + |x_2|, |y_1| + |y_2|)[\sigma(y_1), \sigma(y_2), \omega(x_1, x_2, y_3) + \sigma([x_1, x_2, y_3]_T)]_{\hat{T}} \\
&= [\omega(x_1, x_2, y_1), \sigma(y_2), \sigma(y_3)]_{\hat{T}} + [\sigma([x_1, x_2, y_1]_T), \sigma(y_2), \sigma(y_3)]_{\hat{T}} \\
&+ \varepsilon(|y_1|, |x_1| + |x_2|)[\sigma(y_1), \omega(x_1, x_2, y_2), \sigma(y_3)]_{\hat{T}} + \varepsilon(|y_1|, |x_1| + |x_2|)[\sigma(y_1), \sigma([x_1, x_2, y_2]_T), \sigma(y_3)]_{\hat{T}} \\
&+ \varepsilon(|x_1| + |x_2|, |y_1| + |y_2|)[\sigma(y_1), \sigma(y_2), \omega(x_1, x_2, y_3)]_{\hat{T}} \\
&+ \varepsilon(|x_1| + |x_2|, |y_1| + |y_2|)[\sigma(y_1), \sigma(y_2), \sigma([x_1, x_2, y_3]_T)]_{\hat{T}} \\
&= \varepsilon(|x_1| + |x_2| + |y_1|, |y_2| + |y_3|)\theta(y_2, y_3)\omega(x_1, x_2, y_1) + \sigma([[x_1, x_2, y_1]_T, y_2, y_3]_T) \\
&+ \omega([[x_1, x_2, y_1]_T, y_2, y_3]_T) - \varepsilon(|y_1|, |x_1| + |x_2|)\varepsilon(|y_3|, |x_1| + |x_2| + |y_2|)\theta(y_1, y_3)\omega(x_1, x_2, y_2) \\
&+ \varepsilon(|y_1|, |x_1| + |x_2|)\omega(y_1, [x_1, x_2, y_2]_T, y_3) + \varepsilon(|y_1|, |x_1| + |x_2|)\sigma([y_1, [x_1, x_2, y_2]_T, y_3]_T) \\
&+ \varepsilon(|x_1| + |x_2|, |y_1| + |y_2|)D(y_1, y_2)\omega(x_1, x_2, y_3) + \varepsilon(|x_1| + |x_2|, |y_1| + |y_2|)\omega(y_1, y_2, [x_1, x_2, y_3]_T) \\
&+ \varepsilon(|x_1| + |x_2|, |y_1| + |y_2|)\sigma([y_1, y_2, [x_1, x_2, y_3]_T]_T).
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
&\omega([x_1, x_2, y_1]_T, y_2, y_3) + \varepsilon(|y_1|, |x_1| + |x_2|)\omega(y_1, [x_1, x_2, y_2]_T, y_3) \\
&+ \varepsilon(|x_1| + |x_2|, |y_1| + |y_2|)\omega(y_1, y_2, [x_1, x_2, y_3]_T)
\end{aligned}$$

$$
\begin{aligned}
&+ \varepsilon(|x_1| + |x_2| + |y_1|, |y_2| + |y_3|)\theta(y_2, y_3)\omega(x_1, x_2, y_1) \\
&- \varepsilon(|x_1| + |x_2|, |y_1| + |y_3|)\varepsilon(y_2, y_3)\theta(y_1, y_3)\omega(x_1, x_2, y_2) \\
&+ \varepsilon(|x_1| + |x_2|, |y_1| + |y_2|)D(y_1, y_2)\omega(x_1, x_2, y_3) \\
&- \omega(x_1, x_2, [y_1, y_2, y_3]_T) - D(x_1, x_2)\omega(y_1, y_2, y_3) \\
&= 0.
\end{aligned}
$$

Therefore, ω is a 3-cocycle. □

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Conflict of interest

The authors declare there is no conflicts of interest.

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