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ERA, 30(7): 2428–2445.
DOI: 10.3934/era.2022124
Received: 28 August 2021
Revised: 17 January 2022
Accepted: 20 January 2022
Published: 29 April 2022

Research article

On the Drazin inverse of anti-triangular block matrices

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Abstract: Our aim is to present new expressions for the Drazin inverse of anti-triangular block matrices under some circumstances. Applying the established new formulae for anti-triangular block matrices, we derive explicit representations for the Drazin inverse of a 2×2 complex block matrix under corresponding assumptions. We extend several well known results in the literature in this way.

Keywords: Drazin inverse; block matrix; anti-triangular block matrix; index

1. Introduction

The Drazin inverse is a very useful tool in various fields of applied mathematics such as Markov chains, control theory, iterative methods in numerical linear algebra, singular differential and difference equations [1–3]. There are many researches about finding expressions of the Drazin inverse of a block matrix under certain conditions [4, 5], but it is still an open problem proposed by Campbell and Meyer [6] in 1979.

It is well known that the *Drazin inverse* of a square complex matrix A is the unique matrix A^d for which the following equations hold

$$AA^d = A^dA, \quad A^dAA^d = A^d, \quad A^k = A^{k+1}A^d,$$

where k is the index of A (i.e., the smallest non-negative integer such that $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$) and denoted by $\mathrm{ind}(A)$. Recall that $A^e = AA^d$, and $A^\pi = I - A^e$ is the spectral idempotent of A corresponding to $\{0\}$. Because $A^0 = I$, for the identity matrix I of the proper size, and $(A^d)^n = (A^n)^d$ for any non-negative integer n, we adopt the conventions that $A^{dn} = A^{nd} = (A^d)^n$. Some interesting results related to the Drazin inverse can be found in [7–11].

Under adequate restrictions, various representations for the Drazin inverse of a 2×2 complex block matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.1}$$

are proved and we list some of them:

1. in [12], $BC = 0, BD = 0$ and $DC = 0$;
2. in [13], $BC = 0, BDC = 0$ and $BD^2 = 0$;
3. in [14], $BC = 0, DC = 0$ (or $BD = 0$) and D is nilpotent;
4. in [15], $A = 0$ and $D = 0$;
5. in [16], $ABC = 0, DC = 0$ and $BD = 0$ (or BC is nilpotent, or D is nilpotent);
6. in [17], $ABC = 0, CBC = 0$ and $BD = 0$;
7. in [18], $ABC = 0$ and $BD = 0$ (or $DC = 0$).

Let us recall that the solutions to singular systems of differential equations are determined by the formula for the Drazin inverse of an anti-triangular block matrix. We consider the following anti-triangular block matrices:

$$\bar{N} = \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \tag{1.2}$$

and

$$N = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{1.3}$$

Several interesting investigations related to the Drazin inverse of the anti-triangular block matrix N partitioned as in the form (1.3) can be seen in [19–25]. Under assumptions $AB = 0$ and $ABC = 0$, the formulae for the Drazin inverse of the anti-triangular block matrix N given by (1.3), were respectively proved in [26]. Also, note that the Drazin inverse of $\tilde{N} = \begin{bmatrix} A & I \\ B & 0 \end{bmatrix}$ was studied in [27–29].

The first goal of this paper is to present a new formula for the Drazin inverse of $\bar{N}$ in the case that $A^3B = 0$, $BAB = 0$ and $BA^2B = 0$. Applying this new formula and the splitting of N in terms of $\bar{N}$, we get the expression for N^d which recovers some earlier results from [26]. Then, using the obtained formula for the Drazin inverse of N, we obtain explicit expressions for the Drazin inverse of M under corresponding assumptions and extend several results in the literature in this manner.

The symbol $\mathbb{C}^{m\times n}$ presents the set of all $m \times n$ complex matrices and all matrices are proper sizes over $\mathbb{C}^{m\times n}$ in this paper. If the lower limit of a sum is greater than its upper limit, we define the sum to be 0, i.e., for example, the sum $\sum_{n=0}^{-1} * = 0$. Notice that $[x]$ stands for the truncates integer of x.

2. Key lemma

Some auxiliary results concerning the Drazin inverse, which will be often used, are given in this section.

Firstly, the so-called Cline's formula is stated.

Lemma 2.1. [30] (Cline's Formula) *For $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$, $(BA)^d = B[(AB)^{2d}]A$.*

We cite one important result about the Drazin inverse of anti-triangular block matrices as follows.

Lemma 2.2. [31, 32] *Let* $M = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ *and* $N = \begin{bmatrix} D & 0 \\ B & A \end{bmatrix} \in \mathbb{C}^{n\times n}$*, where* A *and* D *are square matrices such that* $r = \mathrm{ind}(A)$ *and* $s = \mathrm{ind}(D)$*. Then*

$$M^d = \begin{bmatrix} A^d & X \\ 0 & D^d \end{bmatrix} \quad and \quad N^d = \begin{bmatrix} D^d & 0 \\ X & A^d \end{bmatrix},$$

where

$$X = \sum_{i=0}^{s-1} A^{(i+2)d} B D^i D^\pi + A^\pi \sum_{i=0}^{r-1} A^i B D^{(i+2)d} - A^d B D^d.$$

The following expressions for the Drazin inverse of the sum of two matrices proved in [33], are very useful.

Lemma 2.3. [33, Theorem 2.2] *Let* $QPQ = 0$ *and* $P^2Q = 0$*, where* $P, Q \in \mathbb{C}^{n\times n}$*,* $\mathrm{ind}(P) = r$ *and* $\mathrm{ind}(Q) = s$*. Then*

$$\begin{aligned} (P+Q)^d &= Q^\pi \sum_{i=0}^{s-1} Q^i P^{(i+1)d} + \sum_{i=0}^{r-1} Q^{(i+1)d} P^i P^\pi + P \sum_{i=0}^{r-1} Q^{(i+2)d} P^i P^\pi + PQ^\pi \sum_{i=0}^{s-2} Q^{i+1} P^{(i+3)d} \\ &- PQ^d P^d - PQQ^d P^{2d}. \end{aligned} \tag{2.1}$$

Lemma 2.4. [33, Theorem 2.1] *Let* $PQP = 0$ *and* $PQ^2 = 0$*, where* $P, Q \in \mathbb{C}^{n\times n}$*,* $\mathrm{ind}(P) = r$ *and* $\mathrm{ind}(Q) = s$*. Then*

$$\begin{aligned} (P+Q)^d &= Q^\pi \sum_{i=0}^{s-1} Q^i P^{(i+1)d} + \sum_{i=0}^{r-1} Q^{(i+1)d} P^i P^\pi + Q^\pi \sum_{i=0}^{s-1} Q^i P^{(i+2)d} Q + \sum_{i=0}^{r-2} Q^{(i+3)d} P^{i+1} P^\pi Q \\ &- Q^d P^d Q - Q^{2d} P P^d Q. \end{aligned} \tag{2.2}$$

3. Main results

The aim of this section is to derive the representations for the Drazin inverse of N expressed by (1.3) under new conditions in the literature. To establish the representations of N^d, we obtain our first main result considering the Drazin inverse of $\bar{N}$, which is given as in (1.2), in the case that $A^3B = 0$, $BAB = 0$ and $BA^2B = 0$.

Theorem 3.1. *Let* $\bar{N}$ *be a matrix of the form* (1.2)*, where* A *and* B *are square matrices of the same size such that* $\mathrm{ind}(A) = r$ *and* $\mathrm{ind}(B) = s$*. If*

$$A^3B = 0, \quad BAB = 0 \quad and \quad BA^2B = 0,$$

then

$$\bar{N}^d = \begin{bmatrix} E_1 & A^2B^d + BB^d \\ E_3 & AB^d \end{bmatrix},$$

where

$$\begin{aligned}
E_1 &= AB^{\pi}\sum_{i=0}^{s-1}B^iA^{(2i+2)d} + A^2B^{\pi}\sum_{i=0}^{s-1}B^iA^{(2i+3)d} + B^{\pi}\sum_{i=0}^{s-1}B^iA^{(2i+1)d}\\
&+ A^2\sum_{i=1}^{[\frac{r}{2}]}B^{(i+1)d}A^{2i-1}A^{\pi} + A\sum_{i=0}^{[\frac{r}{2}]}B^{(i+1)d}A^{2i}A^{\pi} + \sum_{i=1}^{[\frac{r}{2}]}B^{id}A^{2i-1}A^{\pi}\\
&- A^2B^dA^d - 2A^d,\\
E_3 &= B^{\pi}\sum_{i=0}^{s-1}B^iA^{(2i+2)d} + AB^{\pi}\sum_{i=0}^{s-1}B^iA^{(2i+3)d} + A^2B^{\pi}\sum_{i=0}^{s-1}B^iA^{(2i+4)d}\\
&+ \sum_{i=0}^{[\frac{r}{2}]}B^{(i+1)d}A^{2i}A^{\pi} + A\sum_{i=1}^{[\frac{r}{2}]}B^{(i+1)d}A^{2i-1}A^{\pi} + A^2\sum_{i=0}^{[\frac{r}{2}]}B^{(i+2)d}A^{2i}A^{\pi}\\
&- AB^dA^d - A^2B^dA^{2d} - 2A^{2d}.
\end{aligned}$$

Proof. The next splitting of $\bar{N}^2$ will be used:

$$\bar{N}^2 = \begin{bmatrix} A^2+B & AB\\ A & B\end{bmatrix} = \begin{bmatrix} A^2 & 0\\ A & 0\end{bmatrix} + \begin{bmatrix} B & AB\\ 0 & B\end{bmatrix} := P + Q.$$

By Lemma 2.2, we get

$$P^d = \begin{bmatrix} A^{2d} & 0\\ A^{3d} & 0\end{bmatrix} \quad \text{and} \quad Q^d = \begin{bmatrix} B^d & AB^d\\ 0 & B^d\end{bmatrix}.$$

Then we verify that

$$P^{\pi} = \begin{bmatrix} A^{\pi} & 0\\ -A^d & I\end{bmatrix} \quad \text{and} \quad Q^{\pi} = \begin{bmatrix} B^{\pi} & -AB^e\\ 0 & B^{\pi}\end{bmatrix}.$$

For any $n \geq 1$, we observe that

$$P^n = \begin{bmatrix} A^{2n} & 0\\ A^{2n-1} & 0\end{bmatrix}, \quad Q^n = \begin{bmatrix} B^n & AB^n\\ 0 & B^n\end{bmatrix},$$

and

$$P^{nd} = \begin{bmatrix} A^{(2n)d} & 0\\ A^{(2n+1)d} & 0\end{bmatrix}, \quad Q^{nd} = \begin{bmatrix} B^{nd} & AB^{nd}\\ 0 & B^{nd}\end{bmatrix}.$$

Since $\mathrm{ind}(A) = r$ and $\mathrm{ind}(B) = s$, it is clearly that r and s are respectively the least nonnegative integers as follows

$$A^rA^{\pi} = 0, \quad B^sB^{\pi} = 0.$$

One can observe that

$$r-2 \leq 2[\frac{r}{2}]-1 \leq r-1, \quad r-1 \leq 2[\frac{r}{2}] \leq r \;\; \textit{and} \;\; r \leq 2[\frac{r}{2}]+1 \leq r+1$$

for any nonnegative integer r. Notice that, for $i \geq 1$,

$$P^iP^\pi = \begin{bmatrix} A^{2i} & 0 \\ A^{2i-1} & 0 \end{bmatrix} \begin{bmatrix} A^\pi & 0 \\ -A^d & I \end{bmatrix} = \begin{bmatrix} A^{2i}A^\pi & 0 \\ A^{2i-1}A^\pi & 0 \end{bmatrix},$$

and

$$Q^iQ^\pi = \begin{bmatrix} B^i & AB^i \\ 0 & B^i \end{bmatrix} \begin{bmatrix} B^\pi & -AB^e \\ 0 & B^\pi \end{bmatrix} = \begin{bmatrix} B^iB^\pi & AB^iB^\pi \\ 0 & B^iB^\pi \end{bmatrix}.$$

So, we conclude that $\mathrm{ind}(P) = [\frac{r}{2}] + 1$ and $\mathrm{ind}(Q) = s$. Because the assumptions $P^2Q = 0$ and $QPQ = 0$ of Lemma 2.3 are satisfied, we calculate the following terms as in (2.1):

$$Q^\pi \sum_{i=0}^{s-1} Q^iP^{(i+1)d} = \begin{bmatrix} B^\pi \sum_{i=0}^{s-1} B^iA^{(2i+2)d} + A \sum_{i=1}^{s-1} B^iA^{(2i+3)d} - AB^e \sum_{i=0}^{s-1} B^iA^{(2i+3)d} & 0 \\ B^\pi \sum_{i=0}^{s-1} B^iA^{(2i+3)d} & 0 \end{bmatrix},$$

$$\sum_{i=0}^{[\frac{r}{2}]} Q^{(i+1)d}P^iP^\pi = \begin{bmatrix} \sum_{i=0}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i}A^\pi + A \sum_{i=1}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i-1}A^\pi - AB^dA^d & AB^d \\ \sum_{i=1}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i-1}A^\pi - B^dA^d & B^d \end{bmatrix},$$

$$P \sum_{i=0}^{[\frac{r}{2}]} Q^{(i+2)d}P^iP^\pi = \begin{bmatrix} A^2 \sum_{i=0}^{[\frac{r}{2}]} B^{(i+2)d}A^{2i}A^\pi & 0 \\ A \sum_{i=0}^{[\frac{r}{2}]} B^{(i+2)d}A^{2i}A^\pi + A^2 \sum_{i=1}^{[\frac{r}{2}]} B^{(i+2)d}A^{2i-1}A^\pi - A^2B^{2d}A^d & A^2B^{2d} \end{bmatrix},$$

$$PQ^\pi \sum_{i=0}^{s-2} Q^{i+1}P^{(i+3)d} = \begin{bmatrix} A^2B^\pi \sum_{i=0}^{s-2} B^{i+1}A^{(2i+6)d} & 0 \\ AB^\pi \sum_{i=0}^{s-2} B^{i+1}A^{(2i+6)d} + A^2B^\pi \sum_{i=0}^{s-2} B^{i+1}A^{(2i+7)d} & 0 \end{bmatrix},$$

$$PQ^dP^d = \begin{bmatrix} A^2B^dA^{2d} & 0 \\ AB^dA^{2d} + A^2B^dA^{3d} & 0 \end{bmatrix}$$

and

$$PQQ^dP^{2d} = \begin{bmatrix} A^2BB^dA^{4d} & 0 \\ ABB^dA^{4d} + A^2BB^dA^{5d} & 0 \end{bmatrix}.$$

Substituting the above expressions into (2.1), we get

$$\bar{N}^{2d} = (P+Q)^d = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

where

$$\alpha = B^\pi \sum_{i=0}^{s-1} B^iA^{(2i+2)d} + A \sum_{i=1}^{s-1} B^iA^{(2i+3)d} - AB^e \sum_{i=0}^{s-1} B^iA^{(2i+3)d} + A^2B^\pi \sum_{i=0}^{s-2} B^{i+1}A^{(2i+6)d}$$

$$
\begin{aligned}
&+ \quad A\sum_{i=1}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i-1}A^{\pi} + A^2\sum_{i=0}^{[\frac{r}{2}]} B^{(i+2)d}A^{2i}A^{\pi} + \sum_{i=0}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i}A^{\pi} \\
&- \quad A^2B^dA^{2d} - A^2BB^dA^{4d} - AB^dA^d, \\
\beta \quad &= \quad AB^d, \\
\gamma \quad &= \quad B^{\pi}\sum_{i=0}^{s-1} B^iA^{(2i+3)d} + AB^{\pi}\sum_{i=0}^{s-2} B^{i+1}A^{(2i+6)d} + A^2B^{\pi}\sum_{i=0}^{s-2} B^{i+1}A^{(2i+7)d} \\
&+ \quad \sum_{i=1}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i-1}A^{\pi} + A\sum_{i=0}^{[\frac{r}{2}]} B^{(i+2)d}A^{2i}A^{\pi} + A^2\sum_{i=1}^{[\frac{r}{2}]} B^{(i+2)d}A^{2i-1}A^{\pi} \\
&- \quad AB^dA^{2d} - A^2B^dA^{3d} - ABB^dA^{4d} - A^2BB^dA^{5d} - A^2B^{2d}A^d - B^dA^d, \\
\delta \quad &= \quad B^d + A^2B^{2d}.
\end{aligned}
$$

Computing $\bar{N}^d = \bar{N}\bar{N}^{2d}$, we obtain

$$
\bar{N}^d = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix},
$$

where

$$
\begin{aligned}
E_1 \quad &= \quad AB^{\pi}\sum_{i=0}^{s-1} B^iA^{(2i+2)d} + A^2\sum_{i=1}^{s-1} B^iA^{(2i+3)d} - A^2B^e\sum_{i=0}^{s-1} B^iA^{(2i+3)d} + B^{\pi}\sum_{i=0}^{s-1} B^{i+1}A^{(2i+3)d} \\
&+ \quad A^2\sum_{i=1}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i-1}A^{\pi} + \sum_{i=1}^{[\frac{r}{2}]} B^{id}A^{2i-1}A^{\pi} + A\sum_{i=0}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i}A^{\pi} - A^2B^dA^d - B^eA^d, \\
E_2 \quad &= \quad A^2B^d + BB^d, \\
E_3 \quad &= \quad B^{\pi}\sum_{i=0}^{s-1} B^iA^{(2i+2)d} + A\sum_{i=1}^{s-1} B^iA^{(2i+3)d} - AB^e\sum_{i=0}^{s-1} B^iA^{(2i+3)d} + A^2B^{\pi}\sum_{i=0}^{s-2} B^{i+1}A^{(2i+6)d} \\
&+ \quad A\sum_{i=1}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i-1}A^{\pi} + A^2\sum_{i=0}^{[\frac{r}{2}]} B^{(i+2)d}A^{2i}A^{\pi} + \sum_{i=0}^{[\frac{r}{2}]} B^{(i+1)d}A^{2i}A^{\pi} \\
&- \quad A^2B^dA^{2d} - A^2BB^dA^{4d} - AB^dA^d, \\
E_4 \quad &= \quad AB^d.
\end{aligned}
$$

We finish this proof by modulating appropriately the upper and lower limits of the corresponding sums. □

By direct calculations, we can obtain the following corollary using Theorem 3.1.

Corollary 3.2. *Let $\bar{N}$ be a matrix of the form* (1.2)*, where A and B are square matrices of the same size such that* $\operatorname{ind}(A) = r$ *and* $\operatorname{ind}(BC) = s$*. If $A^2B = 0$ and $BAB = 0$, then*

$$
\bar{N}^d = \begin{bmatrix} E_1 & BB^d \\ E_3 & AB^d \end{bmatrix},
$$

where

$$\begin{aligned} E_1 &= AB^\pi \sum_{i=0}^{s-1} B^i A^{(2i+2)d} + B^\pi \sum_{i=0}^{s-1} B^i A^{(2i+1)d} + A \sum_{i=0}^{[\frac{r}{2}]} B^{(i+1)d} A^{2i} A^\pi + \sum_{i=1}^{[\frac{r}{2}]} B^{id} A^{2i-1} A^\pi - A^d, \\ E_3 &= B^\pi \sum_{i=0}^{s-1} B^i A^{(2i+2)d} + AB^\pi \sum_{i=0}^{s-1} B^i A^{(2i+3)d} + \sum_{i=0}^{[\frac{r}{2}]} B^{(i+1)d} A^{2i} A^\pi + A \sum_{i=1}^{[\frac{r}{2}]} B^{(i+1)d} A^{2i-1} A^\pi \\ &- AB^d A^d - A^{2d}. \end{aligned}$$

A natural motivation is from the Drazin inverse of $\bar{N}$ to give a new expression for the Drazin inverse of N.

Theorem 3.3. *Let N be a matrix of the form* (1.3)*, where A and BC are square matrices of the same size such that* $\mathrm{ind}(A) = r$ *and* $\mathrm{ind}(BC) = s$*. If*

$$A^3BC = 0, \quad BCABC = 0 \quad \text{and} \quad BCA^2BC = 0,$$

then

$$N^d = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}, \tag{3.1}$$

where

$$\begin{aligned} F_1 &= A(BC)^\pi \sum_{i=0}^{s-1} (BC)^i A^{(2i+2)d} + A^2(BC)^\pi \sum_{i=0}^{s-1} (BC)^i A^{(2i+3)d} + (BC)^\pi \sum_{i=0}^{s-1} (BC)^i A^{(2i+1)d} \\ &+ A \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i} A^\pi + A^2 \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+2)d} A^{2i+1} A^\pi + \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i+1} A^\pi \\ &- A^2(BC)^d A^d - 2A^d, \\ F_2 &= A(BC)^\pi \sum_{i=0}^{s-1} (BC)^i A^{(2i+3)d} B + (BC)^\pi \sum_{i=0}^{s-1} (BC)^i A^{(2i+2)d} B + A^2(BC)^\pi \sum_{i=0}^{s-1} (BC)^i A^{(2i+4)d} B \\ &+ A \sum_{i=1}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i-1} A^\pi B + A^2 \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+2)d} A^{2i} A^\pi B + \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i} A^\pi B \\ &- 2A^{2d}B - A^2(BC)^d A^{2d} B - A(BC)^d A^d B, \\ F_3 &= CA(BC)^\pi \sum_{i=0}^{s-1} (BC)^i A^{(2i+3)d} + CA^2 \sum_{i=0}^{s-1} (BC)^i (BC)^\pi A^{(2i+4)d} + C(BC)^\pi \sum_{i=0}^{s-1} (BC)^i A^{(2i+2)d} \\ &+ C \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i} A^\pi + CA^2 \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+2)d} A^{2i} A^\pi + CA \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+2)d} A^{2i+1} A^\pi \\ &- 2CA^{2d} - CA(BC)^d A^d - CA^2(BC)^d A^{2d}, \\ F_4 &= CA(BC)^\pi \sum_{i=0}^{s-1} (BC)^i A^{(2i+4)d} B + CA^2 \sum_{i=0}^{s-1} (BC)^i (BC)^\pi A^{(2i+5)d} B + C(BC)^\pi \sum_{i=0}^{s-1} (BC)^i A^{(2i+3)d} B \end{aligned}$$

$$+ \quad C\sum_{i=1}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i-1}A^{\pi}B + CA^2\sum_{i=1}^{[\frac{r}{2}]}(BC)^{(i+2)d}A^{2i-1}A^{\pi}B + CA\sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+2)d}A^{2i}A^{\pi}B$$
$$- \quad 2CA^{3d}B - C(BC)^dA^dB - CA(BC)^dA^{2d}B - CA^2(BC)^{2d}A^dB - CA^2(BC)^dA^{3d}B.$$

Proof. We denote by P and Q, respectively, the left matrix and the right matrix of the right-hand side of the next splitting of N:

$$N = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}.$$

Thus,

$$QP = \begin{bmatrix} A & BC \\ I & 0 \end{bmatrix},$$

and, applying Theorem 3.1, we have

$$(QP)^d = \begin{bmatrix} \lambda & \mu \\ \nu & \xi \end{bmatrix},$$

where $\mathrm{ind}(A) = r$, $\mathrm{ind}(BC) = s$,

$$\begin{aligned}
\lambda &= A(BC)^{\pi}\sum_{i=0}^{s-1}(BC)^iA^{(2i+2)d} + A^2(BC)^{\pi}\sum_{i=0}^{s-1}(BC)^iA^{(2i+3)d} + (BC)^{\pi}\sum_{i=0}^{s-1}(BC)^iA^{(2i+1)d} \\
&+ A^2\sum_{i=1}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i-1}A^{\pi} + A\sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i}A^{\pi} + \sum_{i=1}^{[\frac{r}{2}]}(BC)^{id}A^{2i-1}A^{\pi} \\
&- A^2(BC)^dA^d - 2A^d, \\
\mu &= A^2(BC)^d + BC(BC)^d, \\
\nu &= (BC)^{\pi}\sum_{i=0}^{s-1}(BC)^iA^{(2i+2)d} + A(BC)^{\pi}\sum_{i=0}^{s-1}(BC)^iA^{(2i+3)d} + A^2(BC)^{\pi}\sum_{i=0}^{s-1}(BC)^iA^{(2i+4)d} \\
&+ \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i}A^{\pi} + A\sum_{i=1}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i-1}A^{\pi} + A^2\sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+2)d}A^{2i}A^{\pi} \\
&- A(BC)^dA^d - A^2(BC)^dA^{2d} - 2A^{2d}, \\
\xi &= A(BC)^d.
\end{aligned}$$

According to Lemma 2.1, notice that

$$N^d = P(QP)^{2d}Q = \begin{bmatrix} \lambda^2A + \mu\nu A + \lambda\mu + \mu\xi & \lambda^2B + \mu\nu B \\ C\nu\lambda A + C\xi\nu A + C\nu\mu + C\xi^2 & C\nu\lambda B + C\xi\nu B \end{bmatrix}. \tag{3.2}$$

By direct computations, we obtain $\xi^2 = 0$, $\mu\xi = 0$,

$$\lambda^2 = A(BC)^{\pi}\sum_{i=0}^{s-1}(BC)^iA^{(2i+3)d} + A^2\sum_{i=1}^{s-1}(BC)^iA^{(2i+4)d} - A^2(BC)^e\sum_{i=0}^{s-1}(BC)^iA^{(2i+4)d}$$

$$
\begin{aligned}
&+ && \sum_{i=0}^{s-1}(BC)^{i+1}(BC)^{\pi}A^{(2i+4)d} + A\sum_{i=1}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i-1}A^{\pi} \\
&- && A(BC)^dA^d - A^2(BC)^dA^{2d} - (BC)^eA^{2d}, \\
\mu\nu &= && A^2\sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+2)d}A^{2i}A^{\pi} + \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i}A^{\pi}, \\
\lambda\mu &= && A(BC)^d, \\
\nu\lambda &= && A(BC)^{\pi}\sum_{i=0}^{s-1}(BC)^iA^{(2i+4)d} + A^2\sum_{i=0}^{s-1}(BC)^i(BC)^{\pi}A^{(2i+5)d} + (BC)^{\pi}\sum_{i=0}^{s-1}(BC)^iA^{(2i+3)d} \\
&+ && \sum_{i=1}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i-1}A^{\pi} + A^2\sum_{i=1}^{[\frac{r}{2}]}(BC)^{(i+2)d}A^{2i-1}A^{\pi} \\
&- && 2A^{3d} - (BC)^dA^d - A(BC)^dA^{2d} - A^2(BC)^{2d}A^d - A^2(BC)^dA^{3d}, \\
\nu\mu &= && (BC)^d + A^2(BC)^{2d}, \\
\xi\nu &= && A\sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+2)d}A^{2i}A^{\pi}.
\end{aligned}
$$

The proof is finished by substituting the above expressions into (3.2). □

Several particular consequences of our main result are investigated now. We combine Theorem 3.3 and routine computations to obtain the following expressions for the Drazin inverse of N as in (1.3).

Corollary 3.4. *Let N be a matrix of the form* (1.3)*, where A and BC are square matrices of the same size such that* $\operatorname{ind}(A) = r$ *and* $\operatorname{ind}(BC) = s$*. If $A^2BC = 0$ and $CABC = 0$, then*

$$
N^d = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix},
$$

where

$$
\begin{aligned}
F_1 &= && A(BC)^{\pi}\sum_{i=0}^{s-1}(BC)^iA^{(2i+2)d} + (BC)^{\pi}\sum_{i=0}^{s-1}(BC)^iA^{(2i+1)d} \\
&+ && A\sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i}A^{\pi} + \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i+1}A^{\pi} - A^d, \\
F_2 &= && A(BC)^{\pi}\sum_{i=0}^{s-1}(BC)^iA^{(2i+3)d}B + (BC)^{\pi}\sum_{i=0}^{s-1}(BC)^iA^{(2i+2)d}B \\
&+ && A\sum_{i=1}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i-1}A^{\pi}B + \sum_{i=0}^{[\frac{r}{2}]}(BC)^{(i+1)d}A^{2i}A^{\pi}B \\
&- && A(BC)^dA^dB - A^{2d}B,
\end{aligned}
$$

$$F_3 = C(BC)^{\pi} \sum_{i=0}^{s-1} (BC)^i A^{(2i+2)d} + C \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i} A^{\pi},$$

$$F_4 = C(BC)^{\pi} \sum_{i=0}^{s-1} (BC)^i A^{(2i+3)d} B + C \sum_{i=1}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i-1} A^{\pi} B - C(BC)^d A^d B.$$

The following corollary presents a special case of Corollary 3.4.

Corollary 3.5. *Let N be a matrix of the form* (1.3)*, where A and BC are square matrices of the same size such that* $\text{ind}(A) = r$ *and* $\text{ind}(BC) = s$*. If $A^2B = 0$ and $CAB = 0$, then*

$$N^d = \begin{bmatrix} F_1 & (BC)^d B \\ F_3 & 0 \end{bmatrix},$$

where

$$\begin{aligned} F_1 &= A(BC)^{\pi} \sum_{i=0}^{s-1} (BC)^i A^{(2i+2)d} + (BC)^{\pi} \sum_{i=0}^{s-1} (BC)^i A^{(2i+1)d} \\ &+ A \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i} A^{\pi} + \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i+1} A^{\pi} - A^d, \\ F_3 &= C(BC)^{\pi} \sum_{i=0}^{s-1} (BC)^i A^{(2i+2)d} + C \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i} A^{\pi}. \end{aligned}$$

We can easily check that Corollary 3.4 extend both [26, Theorem 3.1] and [26, Theorem 3.3] as follows.

Corollary 3.6. *Let N be a matrix of the form* (1.3)*, where A and BC are square matrices of the same size such that* $\text{ind}(A) = r$ *and* $\text{ind}(BC) = s$*.*

(i) [26, Theorem 3.3] *If $ABC = 0$, then*

$$N^d = \begin{bmatrix} XA & XB \\ CX & C[XA^d + (BC)^d(XA - A^d)]B \end{bmatrix};$$

(ii) [26, Theorem 3.1] *If $AB = 0$, then*

$$N^d = \begin{bmatrix} XA & (BC)^d B \\ CX & 0 \end{bmatrix},$$

where

$$X = (BC)^{\pi} \sum_{i=0}^{s-1} (BC)^i A^{(2i+2)d} + \sum_{i=0}^{[\frac{r}{2}]} (BC)^{(i+1)d} A^{2i} A^{\pi}. \quad (3.3)$$

4. Applications to express M^d

Under new conditions, applying the formulae for the Drazin inverse of anti-triangular block matrices proved in Section 3, we present several representations for the Drazin inverse of a 2×2 block matrix M and generalize a series of results, most of whom are from the references.

Theorem 4.1. *Let M be a matrix of the form* (1.1) *and N be a matrix of the form* (1.3)*, where A, D and BC are square matrices such that A and BC are of the same size. If*

$$A^3BC = 0, \quad BCABC = 0, \quad BCA^2BC = 0, \quad BDC = 0 \quad \text{and} \quad BD^2 = 0,$$

then

$$\begin{aligned} M^d &= \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i N^{(i+1)d} \\ &+ \sum_{i=0}^{r-1} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} N^i \begin{bmatrix} (BC)^\pi - F_1A - A^2(BC)^d & -F_1B \\ -F_3A - CA(BC)^d & I - F_3B \end{bmatrix} \\ &+ \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i N^{(i+2)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \\ &+ \sum_{i=0}^{r-2} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+3)d} \end{bmatrix} N^{i+1} \begin{bmatrix} 0 & -F_1BD \\ 0 & (I - F_3B)D \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D^dF_4D + D^{2d}CF_2D \end{bmatrix}, \end{aligned}$$

where N^d and F_n, $n = \overline{1,4}$, are represented as in (3.1)*,* $\mathrm{ind}(N) = r$ *and* $\mathrm{ind}(D) = s$*.*

Proof. We can write $M = N + Q$, where

$$N = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}.$$

Hence,

$$Q^d = \begin{bmatrix} 0 & 0 \\ 0 & D^d \end{bmatrix} \quad \text{and} \quad Q^\pi = \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix}.$$

Using Theorem 3.3, N^d is represented as in (3.1) and so

$$N^\pi = \begin{bmatrix} I - F_1A - F_2C & -F_1B \\ -F_3A - F_4C & I - F_3B \end{bmatrix}.$$

From the equalities

$$\begin{aligned} F_2C &= A^2(BC)^d + (BC)^dBC, \\ F_4C &= CA(BC)^d, \end{aligned}$$

we have

$$N^\pi = \begin{bmatrix} I - F_1A - A^2(BC)^d - (BC)^dBC & -F_1B \\ -F_3A - CA(BC)^d & I - F_3B \end{bmatrix}.$$

Because $NQN = 0$ and $NQ^2 = 0$, the rest is clear by Lemma 2.4. ◻

In order to illustrate the width of Theorem 4.1, we only need to list the results generalized through whose following corollary.

Corollary 4.2. *Let M be a matrix of the form* (1.1) *and N be a matrix of the form* (1.3)*, where A, D and BC are square matrices such that A and BC are of the same size. If*

$$A^2BC = 0, \quad CABC = 0, \quad BDC = 0 \quad \textit{and} \quad BD^2 = 0,$$

then

$$\begin{aligned} M^d &= \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i N^{(i+1)d} \\ &+ \sum_{i=0}^{r-1} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} N^i \begin{bmatrix} (BC)^\pi - F_1A & -F_1B \\ -F_3A & I - F_3B \end{bmatrix} \\ &+ \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i N^{(i+2)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \\ &+ \sum_{i=0}^{r-2} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+3)d} \end{bmatrix} N^{i+1} \begin{bmatrix} 0 & -F_1BD \\ 0 & (I - F_3B)D \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D^dF_4D + D^{2d}CF_2D \end{bmatrix}, \end{aligned}$$

where N^d and F_n, $n = \overline{1,4}$, are represented as in Corollary 3.4, $\mathrm{ind}(N) = r$ *and* $\mathrm{ind}(D) = s$.

We can verify that Corollary 4.2 generalizes and unifies the following conditions about the expression for M^d:

1. $BC = 0$, $BD = 0$ and $DC = 0$ (see [12, Theorem 5.3]);
2. $BC = 0$, $BD = 0$ and D is nilpotent (see [14, Corollary 2.3]);
3. $ABC = 0, CBC = 0$ and $BD = 0$ (see [17, Corollary 3.3]);
4. $ABC = 0$ and $BD = 0$ (see [18, Theorem 2.3]).

In addition, we utilize Corollary 4.2 to obtain the following expression for M^d as in [13, Corollary 2.3].

Corollary 4.3. [13, Corollary 2.3] *Let M be a matrix of the form* (1.1)*, where A, D and BC are square matrices such that A and BC are of the same size. If*

$$BC = 0, \quad BDC = 0 \quad \textit{and} \quad BD^2 = 0,$$

then

$$\begin{aligned} M^d &= \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} A^d & A^{2d}B \\ CA^{2d} & CA^{3d}B \end{bmatrix}^{i+1} + \sum_{i=0}^{r-1} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} N^i \begin{bmatrix} A^\pi & -A^dB \\ -CA^d & I - CA^{2d}B \end{bmatrix} \\ &+ \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} A^d & A^{2d}B \\ CA^{2d} & CA^{3d}B \end{bmatrix}^{i+2} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \\ &+ \sum_{i=0}^{r-2} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+3)d} \end{bmatrix} N^{i+1} \begin{bmatrix} 0 & -A^dBD \\ 0 & (I - CA^{2d}B)D \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D^dCA^{3d}BD + D^{2d}CA^{2d}BD \end{bmatrix}, \end{aligned}$$

where $\mathrm{ind}(N) = r$ *and* $\mathrm{ind}(D) = s$.

Utilizing Corollary 4.2, we also obtain the following expression for M^d.

Corollary 4.4. *Let M be a matrix of the form* (1.1)*, where A, D and BC are square matrices such that A and BC are of the same size. If*

$$A^2B = 0, \quad CAB = 0, \quad BDC = 0 \quad \textit{and} \quad BD^2 = 0,$$

then

$$\begin{aligned} M^d &= \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i N^{(i+1)d} \\ &+ \sum_{i=0}^{r-1} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+1)d} \end{bmatrix} N^i \begin{bmatrix} (BC)^\pi - F_1A & -F_1B \\ -F_3A & I - F_3B \end{bmatrix} \\ &+ \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} \sum_{i=0}^{s-1} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i N^{(i+2)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \\ &+ \sum_{i=0}^{r-2} \begin{bmatrix} 0 & 0 \\ 0 & D^{(i+3)d} \end{bmatrix} N^{i+1} \begin{bmatrix} 0 & -F_1BD \\ 0 & (I - F_3B)D \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D^{2d}C(BC)^dBD \end{bmatrix}, \end{aligned}$$

where F_1, F_3 and N^d are given as in Corollary 3.5, $\operatorname{ind}(N) = r$ *and* $\operatorname{ind}(D) = s$*.*

Utilizing Corollary 4.2, we obtain the expression for M^d as in [16, Theorem 1].

Corollary 4.5. [16, Theorem 1] *Let M be a matrix of the form* (1.1)*, where A, D and BC are square matrices such that A and BC are of the same size. If*

$$ABC = 0, \quad BD = 0 \quad \textit{and} \quad DC = 0,$$

then

$$M^d = \begin{bmatrix} XA & XB \\ CX & D^d + C[XA^d + (BC)^d(XA - A^d)]B \end{bmatrix},$$

where X is represented by (3.3)*.*

We note that Corollary 4.5 generalizes the next formula proved in [15, Theorem 2.1].

Corollary 4.6. [15, Theorem 2.1] *Let M be a matrix of the form* (1.1)*, where A, D and BC are square matrices such that A and BC are of the same size. If*

$$A = 0 \quad \textit{and} \quad D = 0,$$

then

$$M^d = \begin{bmatrix} 0 & (BC)^dB \\ C(BC)^d & 0 \end{bmatrix}.$$

The following formula, which is a dual version of Theorem 4.1, can be proved similarly.

Theorem 4.7. *Let M be a matrix of the form* (1.1) *and N be a matrix of the form* (1.3)*, where A, D and BC are square matrices such that A and BC are of the same size. If*

$$A^3BC = 0, \quad BCABC = 0, \quad BCA^2BC = 0, \quad DCA = 0 \quad and \quad DCB = 0,$$

then

$$\begin{aligned} M^d &= \begin{bmatrix} (BC)^\pi - F_1A - A^2(BC)^d & -F_1B \\ -F_3A - CA(BC)^d & I - F_3B \end{bmatrix} \sum_{i=0}^{r-1} N^i \begin{bmatrix} 0 & 0 \\ D^{(i+2)d}C & D^{(i+1)d} \end{bmatrix} \\ &+ \sum_{i=0}^{s-1} N^{(i+1)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} + \sum_{i=0}^{s-2} N^{(i+3)d} \begin{bmatrix} 0 & 0 \\ D^{i+1}D^\pi C & 0 \end{bmatrix} \\ &- \begin{bmatrix} F_2D^dC & 0 \\ F_4D^dC & 0 \end{bmatrix} - N^{2d} \begin{bmatrix} 0 & 0 \\ DD^dC & 0 \end{bmatrix}, \end{aligned}$$

where N^d and F_n, $n = \overline{1,4}$, are given by (3.1)*,* $\text{ind}(N) = r$ *and* $\text{ind}(D) = s$.

Proof. Notice that $QNQ = 0$ and $QN^2 = 0$, where Q and N are given as in the proof of Theorem 4.1. As in the proof of Theorem 4.1, this proof can be finished by using Theorem 3.3 and Lemma 2.4. □

Applying Theorem 4.7, we prove the next formula for M^d.

Corollary 4.8. *Let M be a matrix of the form* (1.1) *and N be a matrix of the form* (1.3)*, where A, D and BC are square matrices such that A and BC are of the same size. If*

$$A^2BC = 0, \quad CABC = 0, \quad DCA = 0 \quad and \quad DCB = 0,$$

then

$$\begin{aligned} M^d &= \begin{bmatrix} -F_1A + (BC)^\pi & -F_1B \\ -F_3A & I - F_3B \end{bmatrix} \sum_{i=0}^{r-1} N^i \begin{bmatrix} 0 & 0 \\ D^{(i+2)d}C & D^{(i+1)d} \end{bmatrix} \\ &+ \sum_{i=0}^{s-1} N^{(i+1)d} \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix}^i \begin{bmatrix} I & 0 \\ 0 & D^\pi \end{bmatrix} + \sum_{i=0}^{s-2} N^{(i+3)d} \begin{bmatrix} 0 & 0 \\ D^{i+1}D^\pi C & 0 \end{bmatrix} \\ &- \begin{bmatrix} F_2D^dC & 0 \\ F_4D^dC & 0 \end{bmatrix} - N^{2d} \begin{bmatrix} 0 & 0 \\ DD^dC & 0 \end{bmatrix}, \end{aligned}$$

where N^d and F_n, $n = \overline{1,4}$, are given as in Corollary 3.4, $\text{ind}(N) = r$ *and* $\text{ind}(D) = s$.

It is worth mentioning that Corollary 4.8 recovers the formulae for the Drazin inverse of M under the following assumptions:

1. $ABC = 0$ and $DC = 0$ (see [18, Theorem 2.2]);
2. $ABC = 0$, $DC = 0$ and BC is nilpotent (or D is nilpotent) (see [16, Theorem 2, Theorem 3]);
3. $BC = 0$, $DC = 0$ and D is nilpotent (see [14, Lemma 2.2]).

5. An example

In this section, we give an example to illustrate our results. Precisely, we present matrices N and M whose blocks are 4×4 complex matrices A, B, C and D which do not satisfy the conditions of [26, Theorem 3.1 and Theorem 3.3], [12, Theorem 5.3], [14, Corollary 2.3] and [18, Theorem 2.3], but the assumptions of Theorem 3.3 (or Corollary 3.4) and Theorem 4.1 (or Corollary 4.4) are met, which allows us to find N^d and M^d.

Example 5.1. Let M be a matrix of the form (1.1) and N be a matrix of the form (1.3), where A, B, C and D are 4×4 complex matrices given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ e & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & g & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & t & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d & d & d & 1 \end{bmatrix},$$

where $0 \notin \{a, b, c, d, e, f, g, u, v, t\}$. Notice that

$$BC = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ fu & 0 & 0 & 0 \\ 0 & gv & 0 & 0 \end{bmatrix} \neq 0,$$

$$AB = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ be & 0 & 0 & 0 \\ 0 & cf & 0 & 0 \end{bmatrix} \neq 0 \quad \text{and} \quad ABC = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ cfu & 0 & 0 & 0 \end{bmatrix} \neq 0.$$

The conclusions above imply that [26, Theorem 3.1 and Theorem 3.3], [12, Theorem 5.3], [14, Corollary 2.3] and [18, Theorem 2.3] can not be applied. We note that the equalities $A^2BC = 0$, $CABC = 0$ and $BD = 0$ are satisfied, and utilize Theorem 3.3 (or Corollary 3.4) to obtain

$$N^d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & e & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & f & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^d = 0.$$

Since $D = D^2 = D^{\#}$, we apply Theorem 4.1 (or Corollary 4.2) to get

$$M^d = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & e & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & f & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & d & d & d & 1 \end{bmatrix}^d$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d(u+va)+t(ba+fu) & dv+tb & t & 0 & d+(dv+tb)e & d+tf & d & 1 \end{bmatrix}.$$

Acknowledgments

The first author (D. Zhang) is supported by the National Natural Science Foundation of China (NSFC) (No. 11901079), China Postdoctoral Science Foundation (No. 2021M700751) and the Scientific and Technological Research Program Foundation of Jilin Province (No. JJKH20190690KJ; No. 20200401085GX; No. JJKH20220091KJ). The second author (D. Mosić) is supported by the Ministry of Education, Science and Technological Development, Republic of Serbia (No. 174007/451-03-9/2021-14/200124) and the bilateral project between Serbia and Slovenia (Generalized inverses, operator equations and applications, No. 337-00-21/2020-09/32).

Conflict of interest

The authors declare there is no conflicts of interest.

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