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Research article

Multiplicity of symmetric brake orbits of asymptotically linear symmetric reversible Hamiltonian systems

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Abstract: In this paper, we give the relation between a relative Morse index for two continuous symmetric matrices paths in $\mathbf{R}^{2n}$ satisfying condition (BS1) and the Maslov-type indices under symmetric brake orbit boundary value of these two symmetric matrices paths. As application, we obtain a multiple existence of symmetric brake orbit solutions of asymptotically linear symmetric reversible Hamiltonian systems.

Keywords: symmetric brake orbits; reversible; Hamiltonian systems; Maslov-type index; asymptotic linear

1. Introduction

In this paper, let $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and $N = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ with I being the identity matrix on $\mathbf{R}^n$. Denote by $\mathcal{L}_s(\mathbf{R}^{2n})$ the space of all symmetric matrices in $\mathbf{R}^{2n}$ and $\mathrm{Sp}(2n)$ the symplectic group of $2n \times 2n$ matrices.

We call $B \in C(S^1, \mathcal{L}_s(\mathbf{R}^{2n}))$ satisfies **condition (BS1)** if $B(-t) = NB(t)N$ for all $t \in \mathbf{R}$ and B is $\frac{1}{2}$-periodic, where $S^1 = \mathbf{R}/\mathbf{Z}$.

Let $H \in C^1(\mathbf{R} \times \mathbf{R}^{2n}, \mathbf{R})$ and H' denote the gradient of H with respect to the last $2n$ variables. We assume H satisfies the following conditions:

(H1)$H'(t,x) = B_0(t)x + o(|x|)$ as $|x| \to \ 0$ uniformly in t,

(H2)$H'(t,x) = B_\infty x + o(|x|)$ as $|x| \to \ \infty$ uniformly in t,

(H3) $H(-t, Nx) = H(t,x) = H(t,-x) = H(t+\frac{1}{2}, -x)$, $\forall (t,x) \in (\mathbf{R} \times \mathbf{R}^{2n})$.

In [1] of 2008, the second author of this paper considered the multiplicity of 1-periodic brake orbits of asymptotically linear symmetric reversible Hamiltonian systems. Since condition (H3) holds, it is natural to consider 1-periodic solution of the asymptotically linear Hamiltonian systems

$$\dot{x} = JH'(t, x), \tag{1.1}$$

$$x(t+1) = x(t), \qquad x(\frac{1}{2}+t) = Nx(\frac{1}{2}-t), \qquad x(\frac{1}{2}+t) = -x(t), \quad \forall t \in \mathbf{R}. \tag{1.2}$$

We call the above 1-periodic solutions *symmetric brake orbits*. If $H(-t, Nx) = H(t, x)$ for all $(t, x) \in (\mathbf{R} \times \mathbf{R}^{2n})$, we say H is *reversible*.

Note that conditions (H1)-(H3) yield that both B_0 and B_∞ belong to $C(S^1, \mathcal{L}_s(\mathbf{R}^{2n}))$ and satisfy the above (BS1) condition.

In 1948, Seifert firstly studied brake orbits in Hamiltonian system in [2]. For the existences and multiple existence results and more details on brake orbits one can refer the paper [3, 4] and the references therein. In [1] the second author of this paper obtained multiple existence of brake orbits of the asymptotically linear Hamiltonian systems (1.1)-(1.2) under certain conditions. In this paper we will study multiplicity of symmetric brake orbits of Hamiltonian systems (1.1)-(1.2).

In [5], the difference between B_0 and B_∞, i.e., the different behaviors of H at zero and infinity plays an important role in the study of 1-periodic solutions of (1.1). For this reason we also define the relative Morse index to measure the "true" difference between B_0 and B_∞ under symmetric brake orbit boundary value. We shall study the relation between the relative Morse index and the Maslov-type index $i^{L_0}_{\sqrt{-1}}$ and $\nu^{L_0}_{\sqrt{-1}}$ defined below. As application, we obtain a multiplicity of symmetric brake orbits (1.1)-(1.2).

For any symplectic path γ in Sp($2n$), the Maslov-type index for symmetric brake orbits boundary values of γ is defined in [6] to be a pair of integers $(i_{L_0}(\gamma), \nu_{L_0}(\gamma)) \in \mathbf{Z} \times \{0, 1, 2, ..., n\}$.

For any continuous path B in $\mathcal{L}_s(\mathbf{R}^{2n})$ satisfying condition (BS1), as in [6], we define

$$(i^{L_0}_{\sqrt{-1}}(B), \nu^{L_0}_{\sqrt{-1}}(B)) = (i^{L_0}_{\sqrt{-1}}(\gamma_B, [0, \frac{1}{4}]), \nu^{L_0}_{\sqrt{-1}}(\gamma_B(\frac{1}{4}))), \tag{1.3}$$

where the symplectic path γ_B is the fundamental solution of the following linear Hamiltonian system

$$\frac{d}{dt}\gamma_B(t) = JB(t)\gamma_B(t) \qquad \text{and} \qquad \gamma_B(0) = I_{2n}. \tag{1.4}$$

We will briefly introduce such Malov-type index theory in Section 2.

In order to consider the multiplicity of symmetric brake orbits, define

$$\tilde{E} = \{x \in W^{1/2,2}(S^1, \mathbf{R}^{2n}) \mid x(-t) = Nx(t), \quad x(t+\frac{1}{2}) = -x(t) \qquad a.e.\ t \in \mathbf{R}\}. \tag{1.5}$$

Equip $\tilde{E}$ with the usual $W^{1/2,2}$ norm. Then $\tilde{E}$ is a Hilbert space with the associated inner product $\langle , \rangle$. We define two self adjoint liner operators $\tilde{A}$, $\tilde{B}$ from $\tilde{E}$ to $\tilde{E}$ by

$$\langle \tilde{A}x, y\rangle = \int_0^1 (-J\dot{x}, y)dt,$$

$$\langle \tilde{B}x, y\rangle = \int_0^1 (B(t)x, y)dt, \qquad \forall x, y \in \tilde{E}, \tag{1.6}$$

where B is a continuous path in $\mathcal{L}_s(\mathbf{R}^{2n})$ satisfying condition (BS1).

As in [5], we denote by $M^+(\cdot)$, $M^-(\cdot)$, and $M^0(\cdot)$ the positive definite, negative definite, and null subspaces of the self adjoint linear operator defining it respectively.

Definition 1.1. Let B_1 and B_2 be continuous paths in $\mathcal{L}_s(\mathbf{R}^{2n})$ satisfying condition (BS1). We define the relative Morse index of $\tilde{B}_1$ and $\tilde{B}_2$ by

$$\begin{aligned} I(\tilde{B}_1, \tilde{B}_2) &= \dim(M^+(\tilde{A} - \tilde{B}_1) \cap M^-(\tilde{A} - \tilde{B}_2)) \\ &\quad - \dim((M^-(\tilde{A} - \tilde{B}_1) \oplus M^0(\tilde{A} - \tilde{B}_1)) \bigcap (M^+(\tilde{A} - \tilde{B}_2) \oplus M^0\tilde{A} - \tilde{B}_2)), \end{aligned} \tag{1.7}$$

where we also denote by $\tilde{B}_1$ and $\tilde{B}_2$ the operators defined by (1.6) respectively. We call $I(\tilde{B}_1, \tilde{B}_2)$ the relative Morse index, following [5]. By Theorem 1.2 below this relative More index is well defined. Note that such definition of relative Morse index is different from those defined in [7], [8], and [9] etc, which is difference of Morse index or spectral flow or definition from Garlerkin approximation, the definition if $\dim\ker(A - B_1)$.

Using the iteration theory of Maslov-type index theory and result in [1], we obtain the relation between the Maslov-type index $(i^{L_0}_{\sqrt{-1}}, \nu^{L_0}_{\sqrt{-1}})$ and the relative Morse index as the following

Theorem 1.1. *Let B_1 and B_2 be be continuous paths in $\mathcal{L}_s(\mathbf{R}^{2n})$ satisfying condition (BS1). We have*

$$I(\tilde{B}_1, \tilde{B}_2) = i^{L_0}_{\sqrt{-1}}(B_2) - i^{L_0}_{\sqrt{-1}}(B_1) - \nu^{L_0}_{\sqrt{-1}}(B_1). \tag{1.8}$$

As application, we obtain the main result of this paper.

Theorem 1.2. *Suppose that H satisfies (H1), (H2), (H3), and $\nu^{L_0}_{\sqrt{-1}}(B_0) = \nu^{L_0}_{\sqrt{-1}}(B_\infty) = 0$. Then (1.1)-(1.2) has at least $|i^{L_0}_{\sqrt{-1}}(B_1) - i^{L_0}_{\sqrt{-1}}(B_\infty)|$ pairs of nontrivial 1-periodic symmetric brake orbits.*

Organization: In Section 2, we will briefly introduce the Maslov-type index and its iteration theory for symplectic path under brake orbit boundary value. Based on this index theory we give he proof of Theorem 1.1. As application, in Section 3, we give he proof of Theorem 1.2.

Throughout this paper, let **N**, **Z**, **R**, **C** and **U** denote the set of natural integers, integers, rational numbers, real numbers, complex numbers and the unit circle in **C**, respectively.

2. Maslov-type index theory and Relative Morse index

In this section we will prove Theorem 1.1. We first simply recall the Maslov-type index and its iteration theory for brake orbits.

As we know, in 1984, Conley and Zehnder in their celebrated paper [10] introduced an index theory for the non-degenerate symplectic paths in the real symplectic matrix group $\mathrm{Sp}(2n)$ for $n \geq 2$. Since then, there are tremendous works about this kind of index theory developed or generalized in various directions. In 2006, combined with the Maslov index formulated in [11], Long, Zhang and Zhu developed an index theory called μ-index in [12] and obtained an result on the existence of multiple brake orbits. The difference of the Maslove-type μ-index in [12] and the Maslov-type L-index in that paper is constant n (half of the dimension). In [13], Liu and Zhang established the Bott-type iteration formulas and some precise iteration formula of the L-index theory and proved the multiplicity of brake orbits on every C^2 compact convex symmetric hypersurface in $\mathbf{R}^{2n}$.

Set

$$\mathcal{P}_T(2n) = \{\gamma \in C([0, T], \mathrm{Sp}(2n)) \mid \gamma(0) = I_{2n}\},$$

where we omit T from the notation of $\mathcal{P}_T$ if $[0,T]$ is replaced by $[0,+\infty)$. Let J be the standard almost complex in $(\mathbf{R}^{2n},\omega_0)$ and J is a compatible with ω_0, i.e.,

$$\omega_0(x,y)=Jx\cdot y,\quad \omega_0(Jx,Jy)=\omega_0(x,y)\quad\text{and}\quad \omega_0(x,Jx)>0\quad\text{for}\quad x\neq 0.$$

A n-dimensional subspace $\Lambda\subseteq\mathbf{R}^{2n}$ is called a Lagrangian subspace if $\omega_0(x,y)=0$, for any $x,y\in\Lambda$. Let $F=\mathbf{R}^{2n}\bigoplus\mathbf{R}^{2n}$ be equipped with symplectic form $(-\omega_0)\bigoplus\omega_0$. Then $\mathcal{J}=(-J)\bigoplus J$ is an almost complex structure on F and $\mathcal{J}$ is compatible with $(-\omega_0)\bigoplus\omega_0$. Denote by $\mathcal{L}ag(F)$ the set of Lagrangian subspaces of F. Then for any $M\in\mathrm{Sp}(2n)$, its graph

$$Gr(M)=\left\{\left(\begin{array}{c}x\\ Mx\end{array}\right)\Big| x\in\mathbf{R}^{2n}\right\}\in\mathcal{L}ag(F).$$

Denote by $L_0=\{0\}\times\mathbf{R}^n$ and $L_1=\mathbf{R}^n\times\{0\}$ the two fixed Lagrangian subspaces of $\mathbf{R}^{2n}$ and let

$$\begin{aligned} V_0 &= L_0\times L_0,\quad V_1=L_1\times L_1, \\ Gr(M)|_{V_j} &= \left\{\left(\begin{array}{c}x\\ My\end{array}\right)\Big| x,y\in L_j\right\}. \end{aligned} \tag{2.1}$$

Then both $V_j, Gr(M)|_{V_j}\in\mathcal{L}ag(F)$ for $M\in\mathrm{Sp}(2n)$ and $j=0,1$.

Denote by $\mu_F^{CLM}(V,W,[a,b])$ the Maslov-type index for (ordered) pair of paths of Lagrangian subspaces (V,W) in F on $[a,b]$, which is defined by Cappel, Lee and Miller in [11].

Definition 2.1. (cf. [6,12,13]) For $\gamma\in\mathcal{P}_\tau(2n)$, define

$$i_\omega^{L_j}(\gamma) = \begin{cases} \mu_F^{CLM}(Gr(e^{\theta J})|_{V_j}, Gr(\gamma(t)), t\in[0,\tau]), & \omega=e^{\sqrt{-1}\theta}\in\mathbf{U}\setminus\{1\}, \\ \mu_F^{CLM}(V_j, Gr(\gamma(t)), t\in[0,\tau])-n, & \omega=1, \end{cases} \tag{2.2}$$

$$\nu_\omega^{L_j}(\gamma) = \nu_\omega^{L_j}(\gamma(\tau))=\dim_{\mathbf{C}}(\gamma(\tau)L_j\cap e^{\theta J}L_j),\qquad \omega=e^{\sqrt{-1}\theta}\in\mathbf{U}. \tag{2.3}$$

For $j=0,1$, we define $(i_{L_j}(\gamma),\nu_{L_j}(\gamma))=(i_\omega^{L_j}(\gamma),\nu_\omega^{L_j}(\gamma))$ if $\omega=1$. Note that, for any continuous path $\Psi\in\mathcal{P}_\tau$, the following Maslov-type indices of Ψ is defined by (cf [1,12])

$$\begin{aligned} \mu_1(\Psi,[a,b]) &= \mu_F^{CLM}(V_0,\mathrm{Gr}(\Psi),[0,\tau]), \\ \nu(\Psi,[0,\tau]) &= \dim\Psi(\tau)L_0\cap L_0. \end{aligned} \tag{2.4}$$

When there is no confusion we will omit the intervals in the above definitions. Hence we have

$$i_{L_0}(\gamma)=\mu_1(\gamma)-n,\qquad \nu_{L_0}(\gamma)=\nu_1(\gamma), \tag{2.5}$$

For $B\in C(S_T,\mathcal{L}_s(\mathbf{R}^{2n}))$, the fundamental solution γ_B of the linear Hamiltonian system

$$\begin{cases} \dot\gamma(t)=JB(t)\gamma(t), \\ \gamma(0)=I_{2n}. \end{cases} \tag{2.6}$$

satisfies $\gamma_B\in\mathcal{P}_T(2n)$, and is called the **associated symplectic path** of B. For $\omega\in\mathbf{U}$, we define the Maslov-type indices of B via the restriction $\gamma_B|_{[0,T/2]}\in\mathcal{P}_{T/2}(2n)$:

$$\left(i_\omega^{L_j}\left(B,\frac{T}{2}\right),\nu_\omega^{L_j}\left(B,\frac{T}{2}\right)\right):=\left(i_\omega^{L_j}\left(\gamma_B|_{[0,T/2]}\right),\nu_\omega^{L_j}\left(\gamma_B\left(\frac{T}{2}\right)\right)\right).$$

In 1956, Bott in [14] established the famous iteration formula of the Morse index for closed geodesics on Riemannian manifolds. For convex Hamiltonian systems, Ekeland developed the similar Bott-type iteration index formulas for the Ekeland index theory (cf. [15] of 1990). In 1999 (cf. [16]), Long established the Bott-type iteration formulas for the Maslov-type index theory. Motivated by the above results, in [13] of Liu and Zhang in 2014, the following Bott-type iteration formulas for the L_0-index was established.

Definition 2.2. (cf. [13]) Given an $\tau > 0$, a positive integer k and a path $\gamma \in \mathcal{P}_\tau(2n)$, the k-th iteration γ^k of γ in brake orbit boundary sense is defined by $\tilde{\gamma}|_{[0,k\tau]}$ with

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t-2j\tau)(\gamma(2\tau))^j, & t \in [2j\tau, (2j+1)\tau], \ j \in \mathbf{N} \cup \{0\}, \\ N\gamma((2j+2)\tau - t)N(\gamma(2\tau))^{j+1}, & t \in [(2j+1)\tau, (2j+2)\tau], \ j \in \mathbf{N} \cup \{0\}, \end{cases}$$

where $\gamma(2\tau) := N\gamma(\tau)^{-1}N\gamma(\tau)$.

Theorem 2.1. (cf. [13] of Liu and Zhang in 2014) Suppose $\gamma \in \mathcal{P}_\tau(2n)$, for the iteration symplectic paths γ^k, when k is odd, there hold

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + \sum_{i=1}^{\frac{k-1}{2}} i_{\omega_k^{2i}}(\gamma^2), \ \nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^1) + \sum_{i=1}^{\frac{k-1}{2}} \nu_{\omega_k^{2i}}(\gamma^2);$$

when k is even, there hold

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + i^{L_0}_{\sqrt{-1}}(\gamma^1) + \sum_{i=1}^{\frac{k}{2}-1} i_{\omega_k^{2i}}(\gamma^2), \ \nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^1) + \nu^{L_0}_{\sqrt{-1}}(\gamma^1) + \sum_{i=1}^{\frac{k}{2}-1} \nu_{\omega_k^{2i}}(\gamma^2),$$

where $\omega_k = e^{\pi\sqrt{-1}/k}$, and (i_ω, ν_ω) is the ω-index pair defined by Long(cf. [16]).

Proof of Theorem 1.1. For any $B \in \mathcal{L}_s(\mathbf{R}^{2n})$ satisfying condition (BS1), as in [1], we define

$$E = \{x \in W^{1/2,2}(S^1, \mathbf{R}^{2n}) \mid x(-t) = Nx(t), \quad a.e.\ t \in \mathbf{R}\}. \tag{2.7}$$

Equip E with the usual $W^{1/2,2}$ norm. Then E is a Hilbert space with the associated inner product $\langle , \rangle$. We define two self adjoint liner operators A, B from E to E by

$$\langle Ax, y\rangle = \int_0^1 (-J\dot{x}, y)dt,$$
$$\langle Bx, y\rangle = \int_0^1 (B(t)x, y)dt, \qquad \forall x, y \in E. \tag{2.8}$$

We also define

$$\hat{E} = \{x \in W^{1/2,2}(S^1, \mathbf{R}^{2n}) \mid x(-t) = Nx(t), \quad x(t+\frac{1}{2}) = x(t), \quad a.e.\ t \in \mathbf{R}\}. \tag{2.9}$$

Equip $\hat{E}$ with the usual $W^{1/2,2}$ norm. We define two self adjoint liner operators $\hat{A}$, $\hat{B}$ from $\hat{E}$ to $\hat{E}$ by

$$\langle \hat{A}x, y\rangle = \int_0^1 (-J\dot{x}, y)dt,$$

$$\langle \hat{B}x, y\rangle = \int_0^1 (B(t)x, y)dt, \qquad \forall x, y \in \hat{E}. \tag{2.10}$$

Then $\tilde{E}$ and $\hat{E}$ are both subspaces of E and A invariant, we have both the A orthogonal and B orthogonal decomposition

$$E = \tilde{E} \oplus \hat{E}, \quad \tilde{A} = A|_{\tilde{E}}, \quad \hat{A} = A|_{\hat{E}}.$$

Since B satisfies condition (BS1), one can verify the following orthogonal decomposition

$$B = \tilde{B} \oplus \hat{B}, \quad \tilde{B} = B|_{\tilde{E}}, \quad \hat{B} = B|_{\hat{E}}.$$

Then we have the orthogonal decomposition

$$M^*(A - B) = M^*(\tilde{A} - \tilde{B}) \oplus M^*(\hat{A} - \hat{B}), \text{ for } * = \pm, 0,$$

where $M^*(A - B) \subset E$, $M^*(\tilde{A} - \tilde{B}) \subset \tilde{E}$, $M^*(\hat{A} - \hat{B}) \subset \hat{E}$. So by the definitions of $I(B_1, B_2)$, $I(\tilde{B}_1, \tilde{B}_2)$, $I(\hat{B}_1, \hat{B}_2)$ we have

$$I(B_1, B_2) = I(\tilde{B}_1, \tilde{B}_2) + I(\hat{B}_1, \hat{B}_2), \tag{2.11}$$

where $I(B_1, B_2)$ and $I(\hat{B}_1, \hat{B}_2)$ are defined similarly as (1.3).

Since B satisfies condition (BS1), one has $B \in C(S^{1/2}, \mathcal{L}_s(\mathbf{R}^{2n}))$. Thus

$$\langle \hat{A}x, y\rangle = 2\int_0^{\frac{1}{2}} (-J\dot{x}, y)dt,$$

$$\langle \hat{B}x, y\rangle = 2\int_0^{\frac{1}{2}} (B(t)x, y)dt, \qquad \forall x, y \in \hat{E}.$$

So by Theorem 1.2 of [1] and (2.8) we have

$$I(\hat{B}_1, \hat{B}_2) = i_{L_0}(\gamma_{B_2}(t), [0, \frac{1}{4}]) - i_{L_0}(\gamma_{B_1}(t), [0, \frac{1}{4}]) - \nu_1(\gamma_{B_1}(\frac{1}{4})) \tag{2.12}$$

Also by Theorem 1.2 of [1] and (2.8) we have

$$I(B_1, B_2) = i_{L_0}(\gamma_{B_2}(t), [0, \frac{1}{2}]) - i_{L_0}(\gamma_{B_1}(t), [0, \frac{1}{2}]) - \nu_1(\gamma_{B_1}(\frac{1}{2})) \tag{2.13}$$

Since both B_1 and B_2 satisfy condition (BS1), for $j = 1, 2$, by Theorem 2.3 we have

$$i_{L_0}(\gamma_{B_j}(t), [0, \frac{1}{2}]) = i_{L_0}(\gamma_{B_j}(t), [0, \frac{1}{4}]) + i^{L_0}_{\sqrt{-1}}(\gamma_{B_j}(t), [0, \frac{1}{4}]), \tag{2.14}$$

$$\nu_{L_0}(\gamma_{B_j}(t), [0, \frac{1}{2}]) = \nu_{L_0}(\gamma_{B_j}(t), [0, \frac{1}{4}]) + \nu^{L_0}_{\sqrt{-1}}(\gamma_{B_j}(t), [0, \frac{1}{4}]). \tag{2.15}$$

By (2.11)–(2.15) one has

$$I(\tilde{B}_1, \tilde{B}_2) = i^{L_0}_{\sqrt{-1}}(\gamma_{B_2}(t), [0, \frac{1}{4}]) - i^{L_0}_{\sqrt{-1}}(\gamma_{B_2}(t), [0, \frac{1}{4}]) - \nu^{L_0}_{\sqrt{-1}}(\gamma_{B_2}(t), [0, \frac{1}{4}]). \tag{2.16}$$

Thus Theorem 1.1 holds by the definitions of $(i^{L_0}_{\sqrt{-1}}(\gamma_B), \nu^{L_0}_{\sqrt{-1}}(\gamma_B))$ in (1.3). ∎

3. Proof of Theorem 1.2.

In this section we prove Theorem 1,2.

We study the 1-periodic brake orbit solution of Hamiltonian system (1.1)-(1.2)

$$\begin{aligned}&\dot{x} = JH'(t,x),\\ &x(t+1) = x(t), \qquad x(\frac{1}{2}+t) = Nx(\frac{1}{2}-t).\end{aligned}$$

It is well know that x is a solution of (1.1)-(1.2) if and only if it is a critical point of the functional f defined on $\tilde{E}$ as follows

$$f(x) = \frac{1}{2}\langle \tilde{A}x, x\rangle + \tilde{\Phi}(x), \qquad x \in \tilde{E}, \tag{3.1}$$

where $\tilde{E}$ is defined by (1.5), $\tilde{A}$ is defined in (1.6), $\tilde{\Phi}(x) = \int_0^1 -H(t,x)dt$. It is easy to check that $\tilde{\Phi}'(x)$ is compact.

In [17], Benci proved the following important abstract theorem:

Theorem 3.1. Let $f \in C^1(E, \mathbf{R})$ have the form (3.1) and satisfy

(f1) Every sequence $\{u_j\}$ such that $f(u_j) \to c < \tilde{\Phi}(0)$ and $\|f'(u_j)\| \to 0$ as $j \to +\infty$ is bounded.

(f2) $\tilde{\Phi}(u) = \tilde{\Phi}(-u)$, $u \in \tilde{E}$.

(f3) There are two closed subspaces of $\tilde{E}$, E^+ and E^-, and a constant $\rho > 0$ such that

(a) $f(u) > 0$ for $u \in E^+$, where $c_0 < c_\infty < \tilde{\Phi}(0)$ be two constants.

(b) $f(u) < c_\infty < \tilde{\Phi}(0)$ for $u \in E^- \cap S_\rho$, ($S_\rho = \{u \in E| \, \|u\| = \rho\}$).

Then the number of pairs of nontrivial critical points of f is greater than or equal to $\dim(E^+ \cap E^-) - \mathrm{cod}(E^- + E^+)$. More over, the corresponding critical values belong to to $[c_0, c_\infty]$.

Proof of Theorem 1.2. We take the method in [1,5] to prove this theorem.

We set $\tilde{E}^+ = M^+(\tilde{A} - \tilde{B}_\infty)$ and $\tilde{E}^- = M^-(\tilde{A} - B_0)$. By Definition 1.1 and Theorem 1.2, we have

$$\begin{aligned} & \dim(\tilde{E}^+ \cap \tilde{E}^-) - \mathrm{cod}(\tilde{E}^- + \tilde{E}^+)\\ = \; & \dim(M^+(\tilde{A} - \tilde{B}_\infty) \cap M^0(\tilde{A} - \tilde{B}_0))\\ & - \dim((M^-(\tilde{A} - \tilde{B}_\infty) \oplus M^-(\tilde{A} - \tilde{B}_\infty)) \cap (M^+(\tilde{A} - \tilde{B}_0) \oplus M^0(\tilde{A} - \tilde{B}_0)))\\ = \; & I(\tilde{B}_\infty, \tilde{B}_0)\\ = \; & i^{L_0}_{\sqrt{-1}}(\tilde{B}_0) - i^{L_0}_{\sqrt{-1}}(\tilde{B}_\infty). \end{aligned} \tag{3.2}$$

Here $\tilde{B}_0$ and $\tilde{B}_\infty$ are compact operators from $\tilde{E}$ to $\tilde{E}$ defined by (1.6). Since 0 is an isolated eigenvalue of $\tilde{A}$ with n-dimensional eigenspace $\tilde{E}_0$, by (4-4′) of [17], there exist two real numbers $\alpha < 0$ and $\beta > 0$ such that

$$\langle \tilde{A} - \tilde{B}_0 u, u\rangle \le \alpha \|u\|^2, \qquad \forall u \in \tilde{E}^-, \tag{3.3}$$

$$\langle \tilde{A} - \tilde{B}_\infty u, u\rangle \ge \beta \|u\|^2, \qquad \forall u \in \tilde{E}^+. \tag{3.4}$$

Define

$$V_\infty(t,x) = H(t,x) - \frac{1}{2}\langle \tilde{B}_\infty(t)x, x\rangle, \qquad V_0(t,x) = H(t,x) - \frac{1}{2}\langle \tilde{B}_0(t)x, x\rangle, \tag{3.5}$$

and let $g_\infty(x) = \int_0^1 V_\infty(t,x)dt$ and $g_0(x) = \int_0^1 V_0(t,x)dt$, then we have

$$f(x) = \frac{1}{2}\langle(\tilde{A} - \tilde{B}_\infty)x, x\rangle - g_\infty(x), \qquad \forall x \in \tilde{E}, \tag{3.6}$$

$$f(x) = \frac{1}{2}\langle(\tilde{A} - \tilde{B}_0)x, x\rangle - g_0(x), \qquad \forall x \in \tilde{E}. \tag{3.7}$$

By (H1)-(H2) and the same arguments in the proof of Lemma 5.5 of [17], we get

$$\lim_{\|x\|\to+\infty} \frac{\|g'_\infty(x)\|}{\|x\|} = 0, \tag{3.8}$$

$$\lim_{\|x\|\to+0} \frac{\|g'_0(x)\|}{\|x\|} = 0. \tag{3.9}$$

So by definition of g_0 and (3.9), we have

$$g_0(u) = -\tilde{\Phi}(0) + o(\|u\|^2), \qquad \text{for } \|u\| \to 0. \tag{3.10}$$

By (3.3) and (3.10) we have

$$f(u) \le \alpha\|u\|^2 + \tilde{\Phi}(0) + o(\|u\|^2), \qquad \text{for } u \in E^- \text{ and } \|u\| \to 0. \tag{3.11}$$

Since $\alpha < 0$, there exist a constant $\rho > 0$ and $\gamma_1 < 0$ such that

$$f(u) < \gamma_1 + \tilde{\Phi}(0), \quad \forall u \in E^- \cap S_\rho. \tag{3.12}$$

Setting $c_\infty = \frac{\gamma_1}{2} + \tilde{\Phi}(0)$, (f3)(b) of Theorem 3.1 is satisfied.

By (H2) for there exist $M > 0$ such that

$$|V_\infty(t,x)| \le \frac{\beta}{2}|x|^2 + M|x|, \qquad \forall x \in \mathbf{R}^{2n}. \tag{3.13}$$

Thus

$$\begin{aligned} |g_\infty(u)| &= |\int_0^1 V_\infty(t,u)dt| \\ &\le \int_0^1 |V_\infty(t,u)|dt \\ &\le \int_0^1 \frac{\beta}{2}|u|^2 + M|u| \\ &\le \frac{\beta}{2}\|u\|^2 + M\|u\|. \end{aligned} \tag{3.14}$$

Then by (3.4) and (3.14), for every $u \in \tilde{E}^+$, we get

$$\begin{aligned} f(u) &= \frac{1}{2}\langle(\tilde{A} - \tilde{B}_\infty)u, u\rangle + g_\infty(u) \\ &\ge \beta\|u\|^2 - |g_\infty(u)| \end{aligned}$$

$$\geq \quad \frac{\beta}{2}\|u\|^2 - M\|u\|. \tag{3.15}$$

This implies that f is bounded from below on $\tilde{E}^+$ and we can set

$$c_0 = \inf_{u\in E^+} f(u) - w \text{ with } w > 0 \text{ such that } c_0 < c_\infty.$$

Thus (f3)(a) of Theorem 3.1 is satisfied.

Since $\nu_1(\tilde{B}_\infty) = 0$, $M^0(\tilde{A} - \tilde{B}_\infty) = 0$. Now we prove that (f1) is satisfied. other wise we can suppose $\|u_j\| \to +\infty$ as $j \to +\infty$, then by (3.6) and (3.8) we have

$$0 = \lim_{j\to+\infty} f'(u_j) = \lim_{j\to+\infty}((\tilde{A} - \tilde{B}_\infty)u_j + g'_\infty(u_j)) = \lim_{j\to+\infty}(\tilde{A} - \tilde{B}_\infty)u_j. \tag{3.16}$$

But by (4-4′) of [17] there exists a real number $\alpha' > 0$ such that

$$\|(\tilde{A} - \tilde{B}_\infty)u\| \geq \alpha'\|u\|, \qquad \forall u \in E. \tag{3.17}$$

Hence by (3.17) we have

$$\lim_{j\to+\infty} \|(\tilde{A} - \tilde{B}_\infty)u_j\| = +\infty, \tag{3.18}$$

which contradicts (3.16). This proves (f1) in Theorem 3.1.

(H3) implies (f2) of Theorem 3.1 holds. Hence by Theorem 3.1, (1.1)-(1.2) has at least $i^{L_0}_{\sqrt{-1}}(\tilde{B}_0) - i^{L_0}_{\sqrt{-1}}(\tilde{B}_\infty)$ pairs of nontrivial solutions whenever $i^{L_0}_{\sqrt{-1}}(\tilde{B}_0) - i^{L_0}_{\sqrt{-1}}(\tilde{B}_\infty) > 0$. If $i^{L_0}_{\sqrt{-1}}(\tilde{B}_0) - i^{L_0}_{\sqrt{-1}}(\tilde{B}_\infty) > 0$, we replace f by $-f$ and let $E^+ = M^-(\tilde{A} - \tilde{B}_\infty)$ and $E^- = M^+(\tilde{A} - \tilde{B}_0)$. By almost the same proof we can show that (f1)-(f3) of Theorem 3.1 hold. And by Theorems 1.2 and 3.1 (1.1)-(1.2) has at least $i^{L_0}_{\sqrt{-1}}(\tilde{B}_\infty) - i^{L_0}_{\sqrt{-1}}(\tilde{B}_0)$ pairs of nontrivial brake orbit solution. The proof of Theorem 1.3 is completed. ∎

Similarly to Theorems 1.4 and 1.5 of [5] or [1]), we have

Remark 3.1. If $\nu^{L_0}_{\sqrt{-1}}(\tilde{B}_\infty) > 0$, we can prove (f1) of Theorem 3.1 under other additional conditions while we can prove (f2) and (f3) are satisfied under (H1)-(H3) by the same proof of Theorem 1.3.

Suppose the following condition:

(H4) $V'_\infty(t, x)$ is bounded and $V(t, x) \to +\infty$ as $|x| \to +\infty$, uniformly in t.

By the proof of Theorem 5.2 of [17] and Theorem 4.1 of [18] (f1) holds.

Suppose the following conditions:

(H5) There is $r > 0$ and $p \in (1, 2)$ such that

$$pV_\infty(t, x) \geq (z, V'_\infty(t, x)) > 0 \qquad \text{for} \quad |z| \geq r,\ t \in \mathbf{R}.$$

(H6) $\overline{\lim}_{|x|\to\infty}|x|^{-1}|V'_\infty(t, x)| \leq c < \frac{1}{2}$.

(H7) There are constant $a_1 > 0$ and $a_2 > 0$ such that $V_\infty(t, x) \geq a|z|^p - a_2$.

By the proof of Theorem 4.11 of [18] (f1) holds.

Then under either additional condition (H4) or (H5)-(H7), (1.1)-(1.2) has at least $i^{L_0}_{\sqrt{-1}}(\tilde{B}_0) - i^{L_0}_{\sqrt{-1}}(\tilde{B}_\infty) - \nu^{L_0}_{\sqrt{-1}}(\tilde{B}_\infty)$ pairs of nontrivial solutions whenever $i^{L_0}_{\sqrt{-1}}(\tilde{B}_0) - i^{L_0}_{\sqrt{-1}}(\tilde{B}_\infty) - \nu^{L_0}_{\sqrt{-1}}(\tilde{B}_\infty) > 0$.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. D. Zhang, Relative Morse index and multiple brake orbits of asymptotically linear Hamiltonian systems in the presence of symmetries, *J. Differ. Equ.*, **245** (2008), 925–938. https://doi.org/10.1016/j.jde.2008.04.020
2. H. Seifert, Periodische Bewegungen mechanischer Systeme, *Math. Z.*, **51** (1948), 197–216. https://doi.org/10.1007/BF01291002
3. R. Giambò, F. Giannoni, P. Piccione, Multiple brake orbits in m-dimensional disks, *Calc. Var. Partial Differ. Equ.*, **54** (2015), 2553–2580.
4. C. Liu, D. Zhang, Seifert conjecture in the even convex case, *Comm. Pure Appl. Math.*, **67** (2014), 1563–1604. https://doi.org/10.1002/cpa.21525
5. G. Fei, Relative morse index and its Application to Hamiltonian systems in the presence of Symmetries, *J. Differ. Equ.*, **122** (1995), 302–315. https://doi.org/10.1006/jdeq.1995.1150
6. D. Zhang, Minimal period problems for brake orbits of nonlinear autonomous reversible semipositive Hamiltonian systems, *Discrete. Contin. Dyn. Sys.*, **5** (2015), 2227–2272. https://doi.org/10.3934/dcds.2015.35.2227
7. K. C. Chang, J. Q. Liu, M. J. Liu, Nontrivial periodic resonance Hamiltonian systems, In *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **14** (1997), no. 1, 103–117. Elsevier Masson.
8. Y. Dong, Maslov type index theory for linear Hamiltonian systems with Bolza boundary value conditions and multiple solutions for nonlinear Hamiltonian systems, *Pacific J. Math.*, **221** (2005), 253–280. https://doi.org/10.2140/pjm.2005.221.253
9. C. Zhu, Y. Long, Maslov-type index theory for symplectic paths and spectral flow. I, *Chinese Ann. Math. Ser. B*, **20** (1999), 413–424. https://doi.org/10.1142/S0252959999000485
10. C. Conley, E. Zehnder, Morse type index theory for flows and periodic solutions for Hamiltonian equation, *Comm. Pure Appl. Math.*, **45** (1984), 1303–1360. https://doi.org/10.1002/cpa.3160370204
11. S. E. Cappell, R. Lee, E. Y. Miller, On the Maslov-type index, *Comm. Pure Appl. Math.*, **47** (1994), 121–186.
12. Y. Long, D. Zhang, C. Zhu, Multiple brake orbits in bounded convex symmetric domains, *Adv. Math.*, **203** (2006), 568–635. https://doi.org/10.1016/j.aim.2005.05.005

13. C. Liu, D. Zhang, Iteration theory of L-index and multiplicity of brake orbits, *J. Differ. Equ.*, **257** (2014), 1194–1245. https://doi.org/10.1016/j.jde.2014.05.006

14. R. Bott, On the iteration of closed geodesics and the Sturm intersection theory, *Comm. Pure Appl. Math.*, **9** (1956), 171–206. https://doi.org/10.1002/cpa.3160090204

15. I. Ekeland, *Convexity Methods in Hamiltonian Mechanics*. Spring-Verlag, Berlin, 1990.

16. Y. Long, *Index Theory for Symplectic Paths with Applications*. Birkhäuser, Basel, 2002.

17. V. Benci, On critical point theory for indefinite functionals in the presences of symmetries, *Trans. Am. Math. Soc.*, **274** (1982), 533–572. https://doi.org/10.1090/S0002-9947-1982-0675067-X

18. V. Benci, P. Rabinowitz, Critical point theorems for indefinite functionals, *Invent. Math.*, **52** (1979), 241–273. https://doi.org/10.1007/BF01389883