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Research article

Flows with ergodic pseudo orbit tracing property

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Abstract: In the manuscript, we deal with a type of pseudo orbit tracing property and hyperbolicity about a vector field (or a divergence free vector field). We prove that a vector field (or a divergence free vector field) of a smooth closed manifold M has the robustly ergodic pseudo orbit tracing property then it does not contain any singularities and it is Anosov. Additionally, there is a dense and open set $\mathcal{R}$ in the set of C^1 a vector field (or a divergence free vector field) of a smooth closed manifold M such that given a vector field (or a divergence free vector field) has the ergodic pseudo orbit tracing property then it does not contain singularities and it is Anosov.

Keywords: pseudo orbit tracing property; ergodic pseudo orbit tracing property; chain transitive; transitive; star condition; hyperbolic

1. Introduction

In smooth dynamical systems, various pseudo trajectory(orbit) tracing properties have been studied to investigate hyperbolic systems (see [1–21]). About the ergodic pseudo orbit tracing property (see [22]), Barzanouni and Honary [23] and Lee [24–26] studied that a C^1 diffeomorphism f of a smooth closed manifold M has a hyperbolic structure if f has the robustly ergodic pseudo orbit tracing property. The results are a version of discrete smooth dynamic system. In general, many results of discrete smooth dynamic systems can be extended for continuous smooth dynamic systems. However, the results of discrete dynamic systems can not directly lead to the results of successive smooth dynamic systems (see [27,28]). For instance, Sakai [19] proved that a diffeomorphism f has the robustly pseudo orbit tracing property if and only if it is structural stable. But, for continuous dynamical systems, it is still open problem if the system having sinuglar points (see [6]). In the paper, we consider an extended version, that is, continuous dynamic systems, of the result for the ergodic pseudo orbit tracing property of a C^1 diffeomorphism f of a smooth closed manifold M.

2. Basic notions and results

2.1. Vector fields

Assume that $M^n = M$ is smooth closed $n(\geq 3)$ dimensional manifold with Riemannian metric $d(\cdot,\cdot)$. The flow $\varphi : \mathbb{R}\times M \to M$ satisfies the followings: (i) $\varphi_0(x) = x$, $\forall x \in M$, and (ii) $\varphi_s(\varphi_t(x)) = \varphi_{s+t}(x)$ $\forall x \in M$ and all $s,t \in \mathbb{R}$. Denote by $\mathfrak{X}(M)$ the set of C^1-vector fields on M. For any $x \in M$, the $Orb(x,\varphi) = \{\varphi_t(x) : t \in \mathbb{R}\}$ is called the *orbit* of φ through x. Let **Rep** $= \{h : \mathbb{R} \to \mathbb{R} : h$ is an oriented homeomorphism with $h(0) = 0\}$. For any $\epsilon > 0$, we define $\mathbf{Rep}(\epsilon)$ as follows:

$$\mathbf{Rep}(\epsilon) = \Big\{h \in \mathbf{Rep} : \Big|\frac{h(t)-h(s)}{t-s} - 1\Big| < \epsilon (t \neq s)\Big\}.$$

Let $\xi = \{(x_i,t_i) : t_i \geq \tau, i \in \mathbb{Z}\}$, and $\Gamma_\delta(\xi) = \{i \in \mathbb{Z} : d(\varphi_{t_i}(x_i), x_{i+1}) \geq \delta\}$. Then a sequence of points $\{(x_i,t_i) : t_i \geq \tau, i \in \mathbb{Z}\}$ is called (δ,τ)-*ergodic pseudo orbit* of φ if for $i \in \mathbb{Z}$, $t_i \geq \tau$,

$$\lim_{n\to\infty}\frac{\#(\Gamma_\delta(\xi)\cap\{0,\ldots,n-1\})}{n} = 0, \text{ and}$$

$$\lim_{n\to\infty}\frac{\#(\Gamma_\delta(\xi)\cap\{0,-1,\ldots,-n+1\})}{-n} = 0,$$

where $\#(A)$ is the number of the elements of A. Let $s_0 = 0, s_n = \sum_{i=0}^{n-1} t_i$, and $s_n = -\sum_{i=-n}^{-1} t_i$, $n = 1,2,\ldots$. Then we define

$$\Gamma_\epsilon(\xi,x)^+ = \{i \in \mathbb{N}\cup\{0\} : \int_{s_i}^{s_{i+1}} d(\varphi_{h(t)}(x), \varphi_{t-s_i}(x_i))dt \geq \epsilon\}, \text{ and}$$

$$\Gamma_\epsilon(\xi,x)^- = \{i \in \mathbb{N} : \int_{s_{-i}}^{s_{-i+1}} d(\varphi_{h(t)}(x), \varphi_{t-s_{-i}}(x_i))dt \geq \epsilon\}.$$

We set $\Gamma_\epsilon(\xi,x) = \Gamma_\epsilon(\xi,x)^+ \cup \Gamma_\epsilon(\xi,x)^-$.

Definition 2.1. *A vector field φ has the ergodic pseudo orbit tracing property if for any positive $\epsilon > 0$, we can find positive $\delta > 0$ such that for any (δ,τ)-ergodic pseudo orbit $\xi = \{(x_i,t_i) : t_i \geq \tau, i \in \mathbb{Z}\}$ there are a point $z \in M$ and $h \in$ **Rep**(ϵ) for which*

$$\lim_{|n|\to\infty}\frac{\#(\Gamma_\epsilon(\xi,z)\cap\{0,1,\ldots,n-1\})}{n} = 0.$$

Denote by ESP the set of all vector fields possessing the ergodic pseudo orbit tracing property and by $int(ESP)$ the set of all C^1 interior of the set of all vector fields possessing the ergodic pseudo orbit tracing property. A closed φ_t-invariant set $\Lambda \subset M$ is called *hyperbolic* for φ_t if

(a) Λ have a continuous $D\varphi$-invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus F \oplus E^u_\Lambda$,
(b) $\|D\varphi_t|_{E^s_x}\| \leq Ce^{-\lambda t}$, $\forall x \in \Lambda$ and every $t \geq 0$, and
(c) $m(D\varphi_t|_{E^u_x}) \geq Ce^{\lambda t}$, $\forall x \in \Lambda$ and every $t \geq 0$,

where $m(T) = \inf_{\|v\|=1}\|T(v)\|$ is the minimum norm of a linear operator T, and F is generated by $X(x)$. We say that $\varphi \in \mathfrak{X}(M)$ is *Anosov* if M is hyperbolic for φ. Let $Sing(\varphi) = \{x \in M : \varphi(x) = 0\}$ and $P(\varphi) = \{x \in M :$ there is $T > 0$ such that $\varphi_T(x) = x\}$. Denote by $Crit(\varphi) = Sing(\varphi) \cup P(\varphi)$.

Theorem A Let $\varphi \in \mathfrak{X}(M)$. we have the followings:

(a) If $\varphi \in int(ESP)$ then $Sing(\varphi) = \emptyset$ and φ is Anosov.
(b) There is a dense and open set $\mathcal{R}$ in $\mathfrak{X}(M)$ such that given $\varphi \in \mathcal{R}$, if $\varphi \in ESP$ then $Sing(\varphi) = \emptyset$ and φ is Anosov.

2.2. Divergence free vector fields

Let $M^n = M$ be a smooth closed $n(\geq 3)$-dimensional Riemannian manifold endowed with a volume form μ(Lebesgue measure). A vector field φ is *divergence-free* if its divergence is zero. Notice that, by Liouville formula, a flow φ_t is volume-preserving means that it is the corresponding divergence-free vector field φ. Let $\mathfrak{X}_\mu(M)$ denote the space of C^1 divergence-free vector fields on M and we consider the usual C^1 Whitney topology on this space. Lee proved in reference [8] that a volume preserving diffeomorphism f of a smooth compact manifold M has the robustly the ergodic pseudo orbit tracing property is equivalent to Anosov, and there is a dense and open set $\mathcal{M}$ in the set of all volume preserving diffeomorphisms of a smooth compact manifold M such that given $f \in \mathcal{M}$, f has the the ergodic pseudo orbit tracing property is equivalent to Anosov. It is a discrete version of volume preserving dynamical systems.

Denote by ESP_μ the set of all divergence-free vector fields possessing the ergodic pseudo orbit tracing property and by $int(ESP)_\mu$ the set of all C^1 interior of the set of all divergence-free vector fields possessing the ergodic pseudo orbit tracing property.

According the result and the previous results, we have the following.

Theorem B Let $\varphi \in \mathfrak{X}_\mu(M)$. We have the followings:

(a) If $\varphi \in int(ESP)_\mu$ then it is Anosov.
(b) There is a dense and open set $\mathcal{M}$ in $\mathfrak{X}_\mu(M)$ such that given $\varphi \in \mathcal{M}$, if $\varphi \in ESP_\mu$ then it is Anosov.

3. Proof of Theorem A

3.1. Proof of item(a) of Theorem A

Let M be as before, and let $\varphi \in \mathfrak{X}_\mu(M)$. Given $\delta > 0$ and $\tau > 0$, a sequence of points $\{(x_i, t_i) : t_i \geq \tau, i \in \mathbb{Z}\}$ is called (δ, τ)-pseudo orbit of φ if $t_i \geq \tau$ and $d(\varphi_{t_i}(x_i), x_{i+1}) < \delta\ \forall i \in \mathbb{Z}$. For $x, y \in M$, a finite (δ, τ)-pseudo orbit $\{(x_i, t_i) : t_i \geq \tau, i = 0, 1, \ldots, n\}$ of φ is said to be a (δ, τ)-chain of φ from x to y with length $n + 1$ if $x_0 = x$ and $x_n = y$. A $\varphi \in \mathfrak{X}(M)$ is said to be *chain transitive* if M is a chain transitive set. Let U, V be non-empty open subsets of M. A vector field φ is *transitive* if $\varphi_T(U) \cap V \neq \emptyset$, for some $T > 0$. Obviously, if φ is transitive then it is chain transitive (see [31, Proposition 3.3.2]). A closed φ_t-invariant set $\Lambda \subset M$ is *attracting* if Λ equals $\bigcap_{t\geq 0} \varphi_t(U)$ for some neighborhood U satisfying $\overline{\varphi_t(U)} \subset U$, $\forall t > 0$. An attractor of φ is a transitive attracting set of φ and a repeller is an attractor for $-\varphi$. A closed φ_t-invariant set $\Lambda \subset M$ is a *proper attractor* or *repeller* if $\phi \neq \Lambda \neq M$. In [18, Proposition 3], a vector field φ is chain transitive in an isolated set Λ if and only if Λ has no proper attractor for φ. Here Λ is *isolated* if there exists an open set U of Λ which is called *isolated block* for which $\bigcap_{t\in\mathbb{R}} \varphi_t(U) = \Lambda$.

A vector field φ has the *pseudo orbit tracing property* if for any positive $\eta > 0$, one can find $\delta > 0$ such that any $(\delta, 1)$-pseudo orbit $\{(x_i, t_i) : t_i \geq T, i \in \mathbb{Z}\}$, there are a point $y \in M$ and $h \in \mathbf{Rep}(\epsilon)$ having the following property;

$$d(\varphi_{h(t)}(y), \varphi_{t-s_i}(x_i)) < \eta,$$

$\forall i \in \mathbb{Z}$ and $s_i \le t < s_{i+1}$, where $s_0 = 0$, $s_i = t_0 + t_1 + \ldots + t_i$ for $i > 0$ and $s_i = -(t_{-1} + t_{-2} + \ldots + t_{-i})$ for $i > 0$.

Lemma 3.1. *If a vector field $\varphi \in ESP$ then φ has the finite pseudo orbit tracing property.*

Proof. Firstly, we show that φ is chain transitive. To prove, it is enough to show that Λ is a proper attractor. Since Λ is compact, one can take a positive $\eta >$ for which $\Lambda \subset B(\eta, \Lambda)$. Set $\epsilon = \eta/4$ and $U = B(\eta, \Lambda)$. We take two points $a \in \Lambda$ and $b \in M \setminus U$. Let $\delta > 0$ be the number of the definition of the ergodic pseudo orbit tracing property. Then we construct a $(\delta, 1)$-ergodic pseudo orbit of φ as follows: for $i \in \mathbb{Z}$ and $t_i = 1$, (i) $\varphi_i(a) = x_i$ for $i \le 0$, and (ii) $\varphi_i(b) = x_i$ for $i > 0$. Clearly, $\zeta = \{(x_i, t_i) : t_i = 1, i \in \mathbb{Z}\}$ is $(\delta, 1)$-ergodic pseudo orbit of φ. Since $\varphi \in ESP$, there are a point $z \in M$ and $h \in \mathbf{Rep}(\epsilon)$ having the following property;

$$\lim_{|n|\to\infty} \frac{\#(\Gamma_\epsilon(\zeta, z) \cap \{0, 1, \ldots, n-1\})}{n} = 0.$$

Then we can find $\tau > 0$ such that $\varphi_{h(-\tau)}(z) \in U$. Since Λ is an attractor, $\varphi_t(\varphi_{h(-\tau)}(z)) \in U$, $\forall t > 0$. Set $\varphi_{h(-\tau)}(z) = z'$. Then we know that

$$d(\varphi_t(z'), \varphi_t(b)) > \eta$$

$\forall t > 0$. Since $d(\varphi_t(z'), \varphi_t(b)) > \eta$ $\forall t > 0$ and by ergodic pseudo orbit ζ, we have

$$\int_i^{i+1} d(\varphi_{h(t)}(z'), \varphi_{t-i}(x_i))dt > \epsilon,$$

$\forall i \in \mathbb{N} \cup \{0\}$. Thus one can see that

$$\lim_{|n|\to\infty} \frac{\#(\Gamma_\epsilon(\zeta, z) \cap \{0, 1, \ldots, n-1\})}{n} \ne 0.$$

This is a contradiction.

Now, we prove that φ has the finite pseudo orbit tracing property.

For any $n \in \mathbb{N}$, the finite sequence $\xi_n = \{(x_i^n, t_i) : t_i \ge 1, 0 \le i \le n\}$ is a $(1/n, 1)$-pseudo orbit of φ. Since $\varphi \in ESP$, as the above, for any $n \in \mathbb{N}$, there is a $(1/n, 1)$-chain $\zeta_n = \{(y_i^n, t_i) : t_i \ge 1, 0 \le i \le n\}$ such that $\xi_n \zeta_n \xi_{n+1}$ is a $1/n$-pseudo orbit. Clearly, the sequence

$$\tau = \{\xi_1 \zeta_1 \xi_2 \zeta_2 \cdots\} = \{x_0^1, x_1^1, \ldots, x_n^1, y_0^1, y_0^2, \ldots, y_n^1, \ldots,\}$$

is a $(1/n, 1)$-ergodic pseudo orbit of φ. We denote $\tau = \{(w_i, t_i) : t_i \ge 1, i \ge 0\}$. Since $\varphi \in ESP$, there are a point $z \in M$ and $h \in \mathbf{Rep}(\epsilon)$ having the following property;

$$\lim_{n\to\infty} \frac{\#(\Gamma_\epsilon(\tau, z) \cap \{0, 1, \ldots, n-1\})}{n} = 0.$$

Then there are $t' \in \mathbb{R}$ and $w_j \in \tau$ for which $d(\varphi_{t'}(z), w_j) < \epsilon$. Thus we can find a finite $(1/n, 1)$-pseudo orbit $\xi_n \subset \tau$ such that ξ_n is ϵ-pseudo orbit traced by the point $\varphi_{t'}(z)$. Thus φ has the finite pseudo orbit tracing property which is a contradiction. □

Notice that Thomas proved in [21] that if a flow φ_t has no singular points then φ_t has the finite pseudo orbit tracing property with respect to $h \in \mathbf{Rep}$ if and only if φ_t has the pseudo orbit tracing property with respect to $h \in \mathbf{Rep}$. However, if a flow φ_t has a singular point then it is not true (see [33]). On the other hand, the flow φ_t has the finite pseudo orbit tracing property if and only if φ_t has the pseudo orbit tracing property, for some $h \in \mathbf{Rep}(\epsilon)$.

Remark 3.2. *If vector field $\varphi \in ESP$ then by Lemma 3.1, it has the pseudo orbit tracing property. Then we can easily show that it is transitive.*

Indeed, since $\varphi \in ESP$, φ is chain transitive and by Lemma 3.1, φ has the pseudo orbit tracing property. Let U, V be given non-empty open sets. We take $x \in U$ and $y \in V$ and choose a positive ϵ small enough such that $B_\epsilon(x) \subset U$ and $B_\epsilon(y) \subset V$. Let $\delta(\epsilon) > 0$ be the number of the pseudo orbit tracing property of φ. Then there is a finite $(\delta, 1)$-pseudo orbit $\{(x_i, t_i) : t_i \geq 1, 0 \leq i \leq n\}$ such that $x_0 = x$ and $x_n = y$. Since φ has the pseudo orbit tracing property, there are a point $z \in M$ and $h \in \mathbf{Rep}(\epsilon)$ for which $d(\varphi_{h(t)}(z), x_{t-s_i})) < \epsilon$ for all $0 \leq i \leq n$ and $s_0 = 0, s_i = t_0 + t_1 + \cdots + t_n$. Then we know $\varphi_{h(t)}(z) \in V$. Put $h(t) = T$. Then $\varphi_T(U) \cap V \neq \emptyset$ which means that φ is transitive. □

Let $\gamma \in P(\varphi)$ be hyperbolic, and let $p \in \gamma$ be such that $\varphi_{\pi(p)}(p) = p$. The stable manifold $V^s(\gamma)$ of γ and the unstable manifold $V^u(\gamma)$ of γ are defined as follows: $V^s(\gamma) = \{y \in M : d(\varphi_t(y), \varphi_t(\gamma)) \to 0 \,\text{as}\, t \to \infty\}$, and $V^u(\gamma) = \{y \in M : d(\varphi_t(y), \varphi_t(\gamma)) \to 0 \,\text{as}\, t \to -\infty\}$. For any small $\eta > 0$, the local stable manifold $V^s_{\eta(p)}(p)$ of p and the local unstable manifold $V^u_{\eta(p)}(p)$ of p are defined by $V^s_{\eta(p)}(p) = \{y \in M : d(\varphi_t(y), \varphi_t(p)) < \eta(p), \text{if}\ t \geq 0\}$, and $V^u_{\eta(p)}(p) = \{y \in M : d(\varphi_t(y), \varphi_t(p)) < \eta(p), \text{if}\ t \leq 0\}$. Let $\sigma \in Sing(\varphi)$ be a hyperbolic. Then such as the hyperbolic periodic orbits, the stable/unstable manifold, and local stable/unstable manifold are defined for $\sigma \in Sing(\varphi)$.

Lemma 3.3. *Let $\gamma, \sigma \in Crit(\varphi)$ be hyperbolic. If a vector field $\varphi \in ESP$ then $V^s(\sigma) \cap V^u(\gamma) \neq \emptyset$ and $V^s(\gamma) \cap V^u(\sigma) \neq \emptyset$.*

Proof. To prove, we consider that $\gamma, \sigma \in P(\varphi)$ are hyperbolic(other case is similar). Since $\varphi \in ESP$, by Remark 3.2, it is transitive. Thus one can take a point $x \in M$ for which $\overline{Orb(x)} = M$. Let $\epsilon(\sigma) > 0$ and $\epsilon(\gamma) > 0$ be as before with respect to σ and γ. Set $\eta = \min\{\epsilon(\sigma), \epsilon(\gamma)\}$. Since, by Lemma 3.1, φ has the pseudo orbit tracing property, we let $0 < \delta = \delta(\eta) < \eta$ be the number of the pseudo orbit tracing property of φ. For a finite $(\delta, 1)$-pseudo orbit $\{(x_i, t_i) : t_i \geq 1, i = 0, \ldots, n\}$. Let $t_i = 1 (i = 0, \ldots, n)$. Since φ is transitive, there are positive t and s such that

$$d(\varphi_t(x), \sigma) < \delta \ \text{ and } d(\varphi_s(x), \gamma) < \delta.$$

Then there are natural number $k, l \in \mathbb{N}$ such that $k \leq t < k + 1$ and $l \leq s < l + 1$, and so $d(\varphi_k(x), \sigma) < \delta$ and $d(\varphi_l(x), \gamma) < \delta$.

Take $p \in \sigma$ and $q \in \gamma$ such that $d(\varphi_1(p), \varphi_k(x)) < \delta$ and $d(\varphi_l(x), q) < \delta$. We may assume that $l > k$. Then we have a finite $(\delta, 1)$-pseudo orbit of φ such that

$$\{p, \varphi_k(x), \varphi_{k+1}(x), \ldots, \varphi_{l-1}(x), q\}.$$

Assume that $l = k + j$ for some $j \in \mathbb{R}$. Then we construct a $(\delta, 1)$-pseudo orbit $\{(x_i, t_i) : t_i = 1, i \in \mathbb{Z}\} = \{(x_i, 1) : i \in \mathbb{Z}\}$ as follows: (i) $x_i = \varphi_i(p)$ for $i \leq 0$, (ii) $x_{i+1} = \varphi_{k+i}(x)$ for $0 \leq i < j$, and (iii) $x_i = \varphi_{l+i}(q)$ for $i \geq 1$. Then

$$\{(x_i, t_i) : t_i = 1, i \in \mathbb{Z}\}$$

$$= \{\ldots, \varphi_{-1}(p), x_0(= p), \varphi_k(x), \varphi_{k+1}(x), \ldots, \varphi_{k+j-1}(x), q, \ldots, \}$$
$$= \{\ldots, x_{-1}, x_0(= p), x_1, \ldots, x_j, x_{j+1}(= q), \ldots\}$$

is a $(\delta, 1)$-pseudo orbit of φ. Since φ has the pseudo orbit tracing property, there exist a point $z \in M$ and $h \in \mathbf{Rep}(\epsilon)$ such that

$$d(\varphi_{h(t)}(z), \varphi_{t-s_i}(x_i)) < \eta \ \forall i \in \mathbb{Z},$$

where $s_0 = 0$, $s_i = t_0 + t_1 + \ldots + t_i$ for $i > 0$ and $s_{-i} = t_{-1} + t_{-2} + \ldots + t_{-i}$ for $i > 0$. Then $d(z, p) < \eta$ and

$$d(\varphi_{h(t)}(z), \varphi_{t-s_{-i}}(p)) = d(\varphi_{h(t)}(z), \varphi_{t+i}(p)) = d(\varphi_{h(t)}(z), \varphi_t(\varphi_i(p))) < \eta$$

for $i \geq 0$ and $s_{-i} \leq t < s_{-i+1}$. Thus if $t \to -\infty$ then $z \in V^{uu}_\eta(p) \subset V^u(\sigma)$.

Similarly, we obtain $z \in V^s(\gamma)$. Indeed, for $s_{j+1} \leq t < s_{j+2}$ we know that

$$d(\varphi_{h(t)}(z), \varphi_{t-s_{j+1}}(x_{j+1})) = d(\varphi_{h(t)}(z), \varphi_{t-s_{j+1}}(q)) < \eta.$$

Let $\varphi_{h(t)}(z) = z'$. Then by Lemma 3.1, $d(\varphi_s(z'), \varphi_s(\varphi_{t-s_{j+1}}(q))) < \eta$ for $s \to \infty$. One can see that $z' \in V^{uu}_\eta(q) \subset V^u(\gamma)$. Thus $Orb(\varphi, z) \subset V^u(\sigma) \cap V^u(\gamma)$ and so $V^u(\sigma) \cap V^s(\gamma) \neq \emptyset$. □

A set $\Lambda \subset M$ is *robustly transitive* if Λ is closed φ_t-invariant, and there exist a neighborhood $\mathcal{U}(\varphi)$ of φ and a neighborhood U of Λ such that for any $\phi \in \mathcal{U}(\varphi)$, $\Lambda_\phi(U) = \bigcap_{t\in\mathbb{R}} \phi_t$ is transitive. A vector field φ is robustly transitive if $\Lambda = M$.

Remark 3.4. *Let $\varphi \in \mathfrak{X}(M) \in int(ESP)$. Since $\varphi \in ESP$, by Remark 3.2, φ is transitive. Thus if $\varphi \in int(ESP)$ then φ is robustly transitive. According to Vivier's result [35, Theorem 1], φ has no singularity. Thus if $\varphi \in int(ESP)$ then $Sing(\varphi) = \emptyset$.*

For a hyperbolic $\gamma \in Crit(\varphi)$, we denote index$(\gamma) = \dim V^s(\gamma)$. Note that if $\gamma \in Crit(\varphi)$ is hyperbolic then there are a neighborhood $\mathcal{U}(\varphi)$ of φ and a neighborhood U of γ such that for any $\phi \in \mathcal{U}(\varphi)$, ϕ has a critical hyperbolic orbit $\gamma_\phi \in U$ and index(γ) = index(γ_ϕ), where γ_ϕ is called the *continuation* of γ. We say that a vector field φ is *Kupka-Smale* if every $p \in Crit(\varphi)$ is hyperbolic and their stable and unstable manifolds intersect transversally. It is well known that the Kupka-Smale vector fields form a dense and open set in $\mathfrak{X}(M)$ (see [29]). Denote by $\mathcal{KS}$ the set of all Kupka-Smale vector fields.

Proposition 3.5. *Let $\varphi \in int(ESP)$. Then the index of all hyperbolic $\gamma \in P(\varphi)$ is constant.*

Proof. Let $\mathcal{U}(\varphi) \subset \mathfrak{X}(M)$ be a neighborhood of φ. Since $\varphi \in ESP$, by Theorem 3.2 φ is transitive, and so, φ does not admit sink and source. As in the proof of Arbieto, Senos and Sodero [1], assume that there exist two hyperbolic $\gamma, \tau \in P(\varphi)$ such that index$(\gamma) \neq$ index(τ). Then we can use Kupka-Smale vector fields and so, we will derive a contradiction. Indeed, assume index$(\gamma) = i$ and index$(\tau) = j$. If $j < i$, then we have

$$\dim V^s(\gamma) + \dim V^u(\tau) \leq \dim M.$$

Since $\varphi \in int(ESP)$, we can take $\phi \in \mathcal{KS} \cap \mathcal{U}(\varphi)$ such that $\phi \in ESP$. Then there exist two hyperbolic $\gamma_\phi, \tau_\phi \in P(\phi)$ such that index(γ) = index(γ_ϕ) and index(τ) = index(τ_ϕ). Thus we know

$$\dim V^s(\gamma_\phi) + \dim V^u(\tau_\phi) \leq \dim M.$$

We consider the case $\dim V^s(\gamma_\phi) + \dim V^u(\tau_\phi) < \dim M$. Then we know $V^s(\gamma_\phi) \cap V^u(\tau_\phi) = \emptyset$.

We consider other case $\dim V^s(\gamma_\phi) + \dim V^u(\tau_\phi) = \dim M$. Since $\phi \in ESP$, by Lemma 3.3, we may assume that $x \in V^s(\gamma_\phi) \cap V^u(\tau_\phi)$. Then as in the proof of Arbieto, Senos and Sodero [1], we have

$$\dim(T_x(V^s(\gamma_\phi)) + T_x(V^u(\tau_\phi))) < \dim V^s(\gamma_\phi) + \dim V^u(\tau_\phi) = \dim M.$$

This means that $V^s(\gamma_\phi)$ is not transverse to $V^u(\tau_\phi)$. Since $\phi \in \mathcal{KS}$, we know $V^s(\gamma) \cap V^u(\tau) = \emptyset$. This is a contradiction.

Finally, we consider $j > i$. Then as the above arguments, we can get a contradiction. □

Lemma 3.6. [1, Theorem 4.3.] *Let $\varphi \in int(ESP)$. If a periodic orbit γ is not hyperbolic then there is ϕ C^1-close to φ for which ϕ has two hyperbolic periodic orbits γ_1, γ_2 with different indices.*

A vector field φ is *star* if there exists a neighborhood $\mathcal{U}(\varphi) \subset \mathfrak{X}(M)$ such that for any $\phi \in \mathcal{U}(\varphi)$, every $\gamma \in Crit(\phi)$ is hyperbolic. Denote by $\mathcal{G}(M)$ the set of all star vector fields and $\mathcal{G}^*(M)$ the set of all non-singular star vector fields. Note that if $\varphi \in \mathcal{G}^*(M)$ then φ is Axiom A without cycles (see [27]) and so φ is Anosov if transitive $\varphi \in \mathcal{G}^*(M)$.

Proposition 3.7. *If a vector field $\varphi \in int(ESP)$, then $\varphi \in \mathcal{G}^*(M)$.*

Proof. Since $\varphi \in int(ESP)$, by Remark 3.4 we have $Sing(\varphi) = \emptyset$. Assume that $\varphi \notin \mathcal{G}^*(M)$. Then there is ϕ C^1-close to φ such that ϕ has a non-hyperbolic periodic orbit γ. By Lemma 3.6, there is ϕ_1 C^1 close to ϕ (also, C^1 close to φ) such that ϕ_1 has two hyperbolic periodic orbits γ_1, γ_2 with $\text{index}(\gamma_1) \neq \text{index}(\gamma_2)$. Since $\varphi \in int(ESP)$, by Proposition 3.5, we have $\text{index}(\gamma_1) = \text{index}(\gamma_2)$. This is a contradiction. □

End of the proof of Item (a). Since $\varphi \in int(ESP)$, by Remark 3.4, φ is robustly transitive and $Sing(\varphi) = \emptyset$. By Proposition 3.5, for any hyperbolic $\gamma, \eta \in P(\varphi)$, we know $\text{index}(\gamma) = \text{index}(\eta)$. By Proposition 3.7, $\varphi \in \mathcal{G}^*(M)$. According to Remark 3.2, φ is transitive Anosov. □

3.2. Proof of item(b) of Theorem A

Lemma 3.8. [1, Lemma 3.4] *Let $\varphi \in \mathcal{KS}$ and let $\gamma, \tau \in Crit(\varphi)$. If $\dim V^s(\gamma) + \dim V^u(\tau) \leq \dim M$ then $V^s(\gamma) \cap V^u(\tau) = \emptyset$.*

Lemma 3.9. *There is an dense and open set $\mathcal{G}_1 \subset \mathfrak{X}(M)$ such that given $\varphi \in \mathcal{G}_1$, if $\varphi \in ESP$ then $Sing(\varphi) = \emptyset$ and the index of all $\gamma \in P(\varphi)$ is constant.*

Proof. We first show that $Sing(\varphi) = \emptyset$. Let $\varphi \in \mathcal{G}_1 = \mathcal{KS}$, $\varphi \in ESP$ and let $\gamma \in P(\varphi)$ be hyperbolic with $\text{index}(\gamma) = j$. Assume that there exists a hyperbolic $\sigma \in Sing(\varphi)$ such that $\text{index}(\sigma) = i$. We consider two cases: (i) $j < i$, and (ii) $j > i$. However, the cases (i) and (ii) have

$$\dim V^s(\gamma) + \dim V^u(\sigma) \leq \dim M \text{ and } \dim V^u(\gamma) + \dim V^s(\sigma) \leq \dim M.$$

By Lemma 3.8, this is a contradiction. Thus if a vector field $\varphi \in \mathcal{G}_1$ and $\varphi \in ESP$ then $Sing(\varphi) = \emptyset$.

Finally, we show that $\text{index}(\gamma) = \text{index}(\eta)$, for any hyperbolic $\gamma, \eta \in P(\varphi)$. Let $\gamma, \eta \in P(\varphi)$ be hyperbolic. Assume that $\text{index}(\gamma) \neq \text{index}(\eta)$. Then we have $\dim V^s(\gamma) + \dim V^u(\tau) \leq \dim M$. This is a contradiction by Lemma 3.8. □

For any positive $\delta > 0$, a point $p \in \gamma \in P(\varphi)$ is *δ-hyperbolic* if the derivative of the Poincaré map of φ has an eigenvalue λ of p such that $(1 - \delta) < |\lambda| < (1 + \delta)$.

Lemma 3.10. [1, Lemma 5.1, Lemma 5.3] *There is a dense and open set $\mathcal{G}_2 \subset \mathfrak{X}(M)$ such that given $\varphi \in \mathcal{G}_2$,*

(a) if any neighborhood $\mathcal{U}(\varphi)$ of φ one can take $\phi \in \mathcal{U}(\varphi)$ for which ϕ has two hyperbolic periodic orbits γ_1, τ_1 with different indices then φ has two hyperbolic periodic orbits γ, τ with different indices.

(b) for any positive $\delta > 0$, if any neighborhood $\mathcal{U}(\varphi)$ of φ one can take $\phi \in \mathcal{U}(\varphi)$ for which g has a δ-hyperbolic periodic orbit γ_1 then φ has a 2δ-hyperbolic periodic orbit γ.

Note that if $p \in \gamma \in P(\varphi)$ is a δ- hyperbolic then by [34, Lemma 1.3], one can take ϕ C^1-close to φ for which $D\phi_{\pi(p)}(p)$ has 1 as an eigenvalue, where $\pi(p)$ is the period of p.

Lemma 3.11. *There is a dense and open set $\mathcal{G}_3 \subset \mathfrak{X}(M)$ such that given $\varphi \in \mathcal{G}_3$, if $\varphi \in ESP$ then one can find a positive δ such that every $\gamma \in P(\varphi)$ is not δ-hyperbolic.*

Proof. Let $\varphi \in \mathcal{G}_3 = \mathcal{G}_2 \cap \mathcal{G}_1$ and $\varphi \in ESP$. To derive a contradiction, we may assume that for any $\delta > 0$, one can take $\gamma \in P(\varphi)$ for which γ is a δ-hyperbolic. Then by [34, Lemma 1.3] and Lemma 3.6, one can take ϕ C^1-close to φ such that ϕ has two hyperbolic $\eta_1, \tau_1 \in P(\phi)$ with different indices. By Lemma 3.10(a), φ has two hyperbolic $\eta, \tau \in P(\varphi)$ with different indices. This is a contradiction by Lemma 3.9. □

Proposition 3.12. *For $\varphi \in \mathcal{G}_3$, if φ has ESP then $\varphi \in \mathcal{G}(M)$.*

Proof. Let $\varphi \in \mathcal{G}_3$ and $\varphi \in ESP$. By Lemma 3.9, $Sing(\varphi) = \emptyset$. To prove, it is enough to show that $\varphi \in \mathcal{G}^*(M)$, that is, for any ϕ C^1 close to φ, every $\gamma \in P(\phi)$ is hyperbolic. Assume that $\varphi \notin \mathcal{G}^*(M)$. Then one can take ϕ C^1 close to φ such that ϕ has a periodic orbit γ which is not hyperbolic. As [34, Lemma 1.3], one can take ϕ_1 C^1 close to ϕ(also C^1 close to φ) for which ϕ_1 has a $\delta/2$-hyperbolic $\gamma_{\phi_1} \in P(\phi_1)$. Since $\varphi \in \mathcal{G}_2$, φ has a δ-hyperbolic $\gamma' \in P(\varphi)$. This is a contradiction by Lemma 3.11. □

End of the proof of Item (b). Let $\varphi \in \mathcal{G}_3$ and $\varphi \in ESP$. Since $\varphi \in ESP$, by Remark 3.2, it is enough to show that $\varphi \in \mathcal{G}^*(M)$. By Lemma 3.9 and Proposition 3.12, every $\gamma \in P(\varphi)$ is hyperbolic, and so, $\varphi \in \mathcal{G}^*(M)$. Thus φ is transitive Anosov. □

4. Proof of Theorem B

Remark 4.1. *Let $\varphi \in \mathfrak{X}_\mu(M)$. By Zuppa's Theorem* [30], *we can find ϕ C^1-close to φ such that $\phi \in \mathfrak{X}^\infty_\mu(M)$, $\phi_{\pi(p)}(p) = p$ and $P^{\pi(p)}_\phi(p)$ has an eigenvalue λ with $|\lambda| = 1$.*

A $\varphi \in \mathfrak{X}_\mu(M)$ is a *divergence-free star* if there exists a neighborhood $\mathcal{U}(\varphi)$ of φ in $\mathfrak{X}_\mu(M)$ such that every point in $Crit(\phi)$ is hyperbolic, for any $\phi \in \mathcal{U}(\varphi) \subset \mathfrak{X}_\mu(M)$. The set of divergence-free star vector fields is denoted by $\mathcal{G}_\mu(M)$. Then we get the following.

Theorem 4.2. [32, Theorem 1] *If $\varphi \in \mathcal{G}_\mu(M)$ then $Sing(\varphi) = \emptyset$ and φ is Anosov.*

Proof of Theorem B. To prove Theorem B, it is enough to show that $\varphi \in \mathcal{G}_\mu(M)$. At first, we assume that $\varphi \in int(ESP)_\mu$. As in the proof of Theorem A, we can easily show that $\varphi \in \mathcal{G}_\mu(M)$. Thus φ is transitive Anosov. Finally, there is a dense and open set $\mathcal{M}$ in $\mathfrak{X}_\mu(M)$ such that given $\varphi \in \mathcal{M}$, we assume that $\varphi \in ESP_\mu$. As in the proof of Theorem A, we can show that $\varphi \in \mathcal{G}_\mu(M)$. Thus φ is transitive Anosov. □

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Conflict of interest

The authors declare there is no conflicts of interest.

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