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Research article

Combinatorial structure and sumsets associated with Beatty sequences generated by powers of the golden ratio

Prapanpong Pongsriiam*

Department of Mathematics, Faculty of Science, Silpakorn University, Nakhon Pathom 73000, Thailand

* **Correspondence:** Email: prapanpong@gmail.com, pongsriiam_p@silpakorn.edu.

Abstract: Let α be the golden ratio, $m \in \mathbb{N}$, and $B(\alpha^m)$ the Beatty sequence (or Beatty set) generated by α^m. In this article, we give some combinatorial structures of $B(\alpha^m)$ and use them in the study of associated sumsets. In particular, we obtain, for each $m \in \mathbb{N}$, a positive integer $h = h(m)$ such that the h-fold sumset $hB(\alpha^m)$ is a cofinite subset of $\mathbb{N}$. In addition, we explicitly give the integer $N = N(m)$ such that $hB(\alpha^m)$ contains every integer that is larger than or equal to N, and show that this choice of N is best possible when m is small. We also propose some possible research problems. This paper extends the previous results on sumsets associated with upper and lower Wythoff sequences.

Keywords: Beatty sequence; sumset; Wythoff sequence; Fibonacci number; golden ratio; fractional part

1. Introduction

Let A and B be nonempty subsets of $\mathbb{Z}$, $h \in \mathbb{N}$, and $x \in \mathbb{Z}$. The sumset $A + B$, the h-fold sumset hA, the translation $x + A$, and the dilation $x * A$ are defined by

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\},\ hA = \{a_1 + a_2 + \cdots + a_h \mid a_i \in A \text{ for all } i\},$$
$$x + A = A + x = \{a + x \mid a \in A\},\ \text{and } x * A = A * x = \{ax \mid a \in A\}.$$

The central problem in additive number theory is to determine whether a given subset A of $\mathbb{N}$ (or $\mathbb{N}_0$) is an additive basis or an asymptotic additive basis of finite order, and if it is, then it is desirable to explicitly obtain positive integers h and N such that the h-fold sumset hA contains all positive integers, or every positive integer larger than N. For additional details and references on sumsets and additive number theory, we refer the reader to the books by Freiman [1], Halberstam and Roth [2], Nathanson [3], Tao and Vu [4], and Vaughan [5].

Let α be the golden ratio, $B(\alpha)$ and $B(\alpha^2)$ the lower and upper Wythoff sequences, respectively, and in general, let $B(x)$ be the Beatty sequence (or Beatty set) generated by x. Previously, Pongsriiam and his coauthors [6] obtained various results on sumsets associated with $B(\alpha)$ and $B(\alpha^2)$ and showed that $2B(\alpha)$ and $3B(\alpha^2)$ are cofinite while $B(\alpha)$ and $2B(\alpha^2)$ are not. Using automata theory and a computer program called Walnut, Shallit [7, 8] gave a different proof of this result and also calculated the Frobenius numbers for some classical automatic sequences. Dekking [9] introduced a different technique to compute the sumsets $2B(\alpha)$ and $3B(\alpha^2)$ by using combinatorics on words and the theory of two dimensional substitution; see also [10]. Napp Phunphayap, Pongsriiam, and Shallit [11] studied sumsets associated with $B(x)$ where x is any irrational number in the interval $(1, 3)$. For more information on related research, we refer the reader to Fraenkel [12–14], Kawsumarng et al. [15], Kimberling [16, 17], Pitman [18], Zhou [19], and in the online encyclopedia OEIS [20].

One of our motivations comes from the recent results on palindromes as an additive basis: Banks [21] showed that every positive integer can be written as the sum of at most 49 palindromes in base 10; Cilleruelo, Luca, and Baxter [22] improved it by showing that if $b \geq 5$ is fixed, then every positive integer is the sum of at most three b-adic palindromes; Rajasekaran, Shallit, and Smith [23] completed the study by proving that the theorem of Cilleruelo, Luca, and Baxter [22] also holds when $b \in \{3, 4\}$, and if $b = 2$, then we need four summands to write every positive integer as a sum of b-adic palindromes. Comparing these complete results on palindromes [21–23] and those satisfactory but incomplete answers on the classical bases such as primes or powers of nonnegative integers in Goldbach's or Waring's problems, we are led to an idea of studying a new arithmetic or combinatorial sequence as an additive basis like they did for palindromes in [21–23].

Clearly, arithmetic progressions are bad for sumsets because the calculation is too easy and many of them cannot be a basis. For instance, the set of all positive integers that are congruent to a modulo m where m and $\gcd(a, m)$ are larger than 1 is not a basis. Being a generalization of arithmetic progressions, Beatty sequences $B(x)$ are nearly periodic but not really periodic when their generator x is an irrational number. So it is interesting to replace arithmetic progressions by Beatty sequences and see what happen.

It seems that there are some connections between the sumsets of $B(x)$ and the members of a particular linear recurrence relation whose characteristic polynomial has x as one of its root; see, for example, in Theorems 3.5, 3.16, 3.17, Remark 3.19, and open questions in the article by Kawsumarng et al. [15]. In particular, if $m \geq 3$ is an integer and $h = h(m)$ is the smallest positive integer such that $hB(\alpha^m)$ is cofinite, then it seems that there are infinitely many Fibonacci numbers that do not belong to $(h-1)B(\alpha^m)$. While many combinatorial properties of lower and upper Wythoff sequences have been extensively studied, there are only a few arithmetic results concerning sumsets associated with Beatty sequences, a generalization of Wythoff sequences. These motivate us to investigate more on this problem.

In this article, we continue the investigation on $B(\alpha^m)$ for $m \geq 3$. In particular, we provide some combinatorial structures of $B(\alpha^m)$ in Theorems 3.4 and 3.5 and use them to calculate, for each $m \geq 3$, a positive integer $h = h(m)$ such that $hB(\alpha^m)$ is cofinite. In addition, we explicitly give in Theorems 4.3 and 4.4 a positive integer $N = N(m)$, which is best possible when m is small, such that $hB(\alpha^m)$ contains every integer that is larger than or equal to N. For example, we obtain that $12B(\alpha^5)$ contains every integer larger than or equal to 2684 and that 2684 is the smallest integer having this property. By some numerical evidence as shown in Remark 2, Corollary 1, and Question 1, we believe that our choices of the integers h and N are actually the smallest possible for all $m \in \mathbb{N}$. Nevertheless, the proof seems very long and difficult, and so we postpone this for future research.

We organize this article as follows. In Section 2, we recall some definitions and useful results for the reader's convenience. In Section 3, we give some combinatorial structures of $B(\alpha^m)$, and then in Section 4, we use them to obtain the desired results on sumsets.

2. Preliminaries and lemmas

We first introduce the notation which will be used throughout this article as follows: $\alpha = (1+\sqrt{5})/2$ is the golden ratio, $\beta = (1-\sqrt{5})/2$, and if $x \in \mathbb{R}$, then $\lfloor x \rfloor$ is the largest integer less than or equal to x, $\{x\} = x - \lfloor x \rfloor$, $\lceil x \rceil$ is the smallest integer larger than or equal to x, and with a little abuse of notation

$$B(x) = \{\lfloor nx \rfloor \mid n \in \mathbb{N}\} = (\lfloor nx \rfloor)_{n \geq 1},$$

where we consider $B(x)$ as a sequence $(\lfloor nx \rfloor)_{n\geq 1}$ when we show its combinatorial structure, and we treat $B(x)$ as a set when we give a result on sumsets. In addition, for $n \geq 0$, we write F_n and L_n to denote the nth Fibonacci and Lucas numbers, which are defined by the same recursive pattern $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$ but with different initial values $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, and $L_1 = 1$. We can also extend them to negative indices by the formula $F_{-n} = (-1)^{n+1} F_n$ and $L_{-n} = (-1)^n L_n$. Furthermore, if P is a mathematical statement, then the Iverson notation $[P]$ is defined by

$$[P] = \begin{cases} 1, & \text{if } P \text{ holds;} \\ 0, & \text{otherwise.} \end{cases}$$

We often use the following fact: $-1 < \beta < 0$, $(|\beta^n|)_{n\geq 1}$ is strictly decreasing, if $a_1 > a_2 > \cdots > a_r$ are even positive integers, then $0 < \beta^{a_1} < \beta^{a_2} < \cdots < \beta^{a_r}$, and if $b_1 > b_2 > \cdots > b_r$ are odd positive integers, then $0 > \beta^{b_1} > \beta^{b_2} > \cdots > \beta^{b_r}$. In addition, α and β are roots of the equation $x^2 - x - 1 = 0$. So, for instance, $\alpha\beta = -1$, $\beta^2 = \beta + 1$, and $\beta^2 + \beta^4 = 4\beta + 3$. Finally, we remark that we apply Lemmas 2.1 to 2.3 throughout this article sometimes without reference.

Lemma 2.1. *For $n \in \mathbb{Z}$ and $x, y \in \mathbb{R}$, the following statements hold.*

(i) $\lfloor n + x \rfloor = n + \lfloor x \rfloor$ *and* $\lceil n + x \rceil = n + \lceil x \rceil$.
(ii) $\{n + x\} = \{x\}$.
(iii) $0 \leq \{x\} < 1$ *and if x is not an integer, then* $\{-x\} = 1 - \{x\}$.
(iv) $\lfloor x + y \rfloor = \begin{cases} \lfloor x \rfloor + \lfloor y \rfloor, & \text{if } \{x\} + \{y\} < 1; \\ \lfloor x \rfloor + \lfloor y \rfloor + 1, & \text{if } \{x\} + \{y\} \geq 1. \end{cases}$
(v) $\lfloor (n+1)x \rfloor - \lfloor nx \rfloor = \lfloor x \rfloor$ *or* $\lfloor x \rfloor + 1$.

Proof. These are well known and can be proved easily. For more details, see for instance in [24, Chapter 3].

Lemma 2.2. *The following statements hold for all nonnegative integers m and n.*

(i) (Binet's formula) $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ *and* $L_n = \alpha^n + \beta^n$.
(ii) $\alpha^n = \alpha F_n + F_{n-1}$ *and* $\beta^n = \beta F_n + F_{n-1}$.

(iii) $L_n = F_{n-1} + F_{n+1}$.

Proof. These are well known, not difficult to prove, and can be found on pages 78–80 in [25].

By Lemma 2.5 of [6], we know the formulas for $\lfloor F_n\alpha \rfloor$, $\left\lfloor F_n\alpha^2 \right\rfloor$, $\{F_n\alpha\}$, and $\{F_n\alpha^2\}$. In this article, we extend those to the following lemma.

Lemma 2.3. *Let k and m be positive integers. Then the following statements hold.*

(i) *If* $k \geq m$, *then* $\left\lfloor \beta^k F_m \right\rfloor = -[k \equiv 1 \pmod 2]$.
(ii) *If* $k < m$, *then* $\left\lfloor \beta^k F_m \right\rfloor = (-1)^k F_{m-k} - [m \equiv 1 \pmod 2]$.
(iii) *If* $k \geq m$, *then* $\lfloor F_k\alpha^m \rfloor = F_{k+m} - [k \equiv 0 \pmod 2]$.
(iv) *If* $k < m$, *then* $\lfloor F_k\alpha^m \rfloor = F_{k+m} + (-1)^{k+1} F_{m-k} - [m \equiv 0 \pmod 2]$.
(v) *If* $k \geq m$, *then* $\{F_k\alpha^m\} = \left\{-\beta^k F_m\right\} = [k \equiv 0 \pmod 2] - \beta^k F_m$.
(vi) *If* $k \geq m$, *then* $\left\{\beta^k F_m\right\} = [k \equiv 1 \pmod 2] + \beta^k F_m$.

For (vii) *and* (viii), *let* $k < m$ *and* $g(m,k) = \beta^{m-k}((-1)^k - \beta^{2k})/\sqrt{5}$. *Then*

(vii) $\{F_k\alpha^m\} = \left\{-\beta^k F_m\right\} = \{-g(m,k)\} = -g(m,k) + [m \equiv 0 \pmod 2]$,
(viii) $\left\{\beta^k F_m\right\} = \{g(m,k)\} = g(m,k) + [m \equiv 1 \pmod 2]$.

Proof. By Binet's formula and the fact that $\alpha\beta = -1$,we see that

$$F_k\alpha^m = \frac{\alpha^{k+m} - \beta^{k+m}}{\alpha - \beta} + \frac{\beta^{k+m} - \beta^k\alpha^m}{\alpha - \beta} = F_{k+m} - \beta^k F_m, \text{ and} \tag{2.1}$$

$$|\beta^m F_m| = \left|\frac{(-1)^m - \beta^{2m}}{\alpha - \beta}\right| \leq \frac{1 + \beta^{2m}}{\sqrt{5}} \leq \frac{1 + \beta^2}{\sqrt{5}} < 1. \tag{2.2}$$

Case 1 $k \geq m$. Then $|\beta^k F_m| \leq |\beta^m F_m| < 1$. So if k is even, then $0 < \beta^k F_m < 1$; if k is odd, then $-1 < \beta^k F_m < 0$. Therefore $\left\lfloor \beta^k F_m \right\rfloor = -[k \equiv 1 \pmod 2]$, $\left\lfloor -\beta^k F_m \right\rfloor = -[k \equiv 0 \pmod 2]$, and (2.1) implies that

$$\lfloor F_k\alpha^m \rfloor = F_{k+m} + \left\lfloor -\beta^k F_m \right\rfloor = F_{k+m} - [k \equiv 0 \pmod 2].$$

Therefore (i) and (iii) are proved.
Case 2 $k < m$. Let $\ell = m - k$ and $g(m,k) = \beta^{m-k}((-1)^k - \beta^{2k})/\sqrt{5}$. Then $m = k + \ell$ and

$$\begin{aligned}\beta^k F_m = \beta^k\left(\frac{\alpha^{k+\ell} - \beta^{k+\ell}}{\alpha - \beta}\right) &= \frac{(-1)^k\left(\alpha^\ell - \beta^\ell\right)}{\alpha - \beta} + \frac{(-1)^k\beta^\ell - \beta^{2k+\ell}}{\alpha - \beta} \\ &= (-1)^k F_\ell + \frac{\beta^\ell((-1)^k - \beta^{2k})}{\alpha - \beta} = (-1)^k F_\ell + g(m,k). \end{aligned} \tag{2.3}$$

For convenience, let $A = g(m,k)$ throughout the remaining proof. Then

$$|A| \leq \frac{|\beta^\ell|(1 + \beta^{2k})}{\sqrt{5}} \leq \frac{|\beta|(1 + \beta^2)}{\sqrt{5}} < 1.$$

Therefore, if k and ℓ are even, then $A = \beta^{\ell}(1-\beta^{2k})/\sqrt{5} \in (0,1)$; if k is even and ℓ is odd, then $A \in (-1,0)$; if k and ℓ are odd, then $A = -\beta^{\ell}(1+\beta^{2k})/\sqrt{5} \in (0,1)$; if k is odd and ℓ is even, then $A \in (-1,0)$. These imply that

$$\lfloor A \rfloor = -[\ell \equiv k+1 \pmod 2] \quad \text{and} \quad \lfloor -A \rfloor = -[\ell \equiv k \pmod 2]. \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\left\lfloor \beta^k F_m \right\rfloor = (-1)^k F_\ell + \lfloor A \rfloor = (-1)^k F_\ell - [\ell \equiv k+1 \pmod 2], \tag{2.5}$$

$$\left\lfloor -\beta^k F_m \right\rfloor = (-1)^{k+1} F_\ell + \lfloor -A \rfloor = (-1)^{k+1} F_\ell - [\ell \equiv k \pmod 2]. \tag{2.6}$$

Since $\ell = m-k$, we see that (2.5) implies (ii). In addition, we obtain (iv) from (2.1) and (2.6) as

$$\lfloor F_k \alpha^m \rfloor = F_{k+m} + (-1)^{k+1} F_{m-k} - [m \equiv 0 \pmod 2].$$

Next, we use some of the above calculation to prove (v) to (viii). We obtain from (2.1) that $\{F_k\alpha^m\} = \{-\beta^k F_m\}$, and the analysis of $\beta^k F_m$ is already done. Suppose $k \geq m$. Then $|\beta^k F_m| < 1$. So if k is even, then $\{\beta^k F_m\} = \beta^k F_m$ and $\{-\beta^k F_m\} = 1-\beta^k F_m$; if k is odd, then $\{\beta^k F_m\} = 1+\beta^k F_m$ and $\{-\beta^k F_m\} = -\beta^k F_m$. These imply (v) and (vi). Next, let $k < m$. From (2.3), we know that $\{\beta^k F_m\} = \{A\}$, $\{-\beta^k F_m\} = \{-A\}$, and the analysis of A is already done. We have

$$\{A\} = A - \lfloor A \rfloor = A + [\ell \equiv k+1 \pmod 2] = A + [m \equiv 1 \pmod 2],$$

$$\{-A\} = -A - \lfloor -A \rfloor = -A + [\ell \equiv k \pmod 2] = -A + [m \equiv 0 \pmod 2].$$

These imply (vii) and (viii). So the proof is complete.

Lemma 2.4. *Let m and n be positive integers. Then the following statements hold.*

(i) $\lfloor \alpha^m \rfloor = L_m - [m \equiv 0 \pmod 2]$.
(ii) $\{\alpha^m\} = -\beta^m + [m \equiv 0 \pmod 2]$.
(iii) *If m is odd and $n < \alpha^m$, then $n\{\alpha^m\} = \{n\alpha^m\}$ and $\lfloor n\{\alpha^m\} \rfloor = 0$.*
(iv) *If m is even and $n < \alpha^m$, then $\{n\alpha^m\} = 1 - n\beta^m$ and $\lfloor n\{\alpha^m\} \rfloor \leq \lfloor \alpha^m \rfloor - 1$.*
(v) *If m is odd, then $\lfloor \alpha^m \rfloor \{\alpha^m\} = 1 - \beta^{2m}$, $\alpha^m\{\alpha^m\} = 1$, and $\lceil \alpha^m \rceil \{\alpha^m\} = 1 - \beta^{2m} - \beta^m \in (1,2)$.*
(vi) *If m is even and $n \in \{\lfloor \alpha^m \rfloor, \lceil \alpha^m \rceil\}$, then $\lfloor n\{\alpha^m\} \rfloor = \lfloor \alpha^m \rfloor - 1$.*
(vii) *If m is even, then $\lceil \alpha^m \rceil \beta^m = 1 + \beta^{2m}$ and $\lfloor \alpha^m \rfloor \beta^m = 1 - \beta^m + \beta^{2m}$.*

Proof. For (i), we obtain by Lemmas 2.1 and 2.2 that

$$\lfloor \alpha^m \rfloor = \lfloor L_m - \beta^m \rfloor = L_m + \lfloor -\beta^m \rfloor = L_m - [m \equiv 0 \pmod 2].$$

Substituting $L_m = \alpha^m + \beta^m = \lfloor \alpha^m \rfloor + \{\alpha^m\} + \beta^m$ in (i) leads to (ii). For (iii), suppose m is odd and $n < \alpha^m$. Then $\{n\alpha^m\} = \{n\lfloor \alpha^m \rfloor + n\{\alpha^m\}\} = \{n\{\alpha^m\}\}$. In addition, $0 < n\{\alpha^m\} = -n\beta^m < -\alpha^m\beta^m = 1$, and so $\{n\{\alpha^m\}\} = n\{\alpha^m\}$ and $\lfloor n\{\alpha^m\} \rfloor = 0$. This proves (iii). For (iv), suppose m is even and $n < \alpha^m$. Then similar to (iii), we have

$$\{n\alpha^m\} = \{n\{\alpha^m\}\} = \{n(1-\beta^m)\} = \{-n\beta^m\} = 1 - \{n\beta^m\}. \tag{2.7}$$

Since $0 < n\beta^m < \alpha^m\beta^m = 1$, we obtain $\{n\alpha^m\} = 1 - n\beta^m$. In addition, $n\{\alpha^m\} < \alpha^m(1-\beta^m) = \alpha^m - 1$, which implies $\lfloor n\{\alpha^m\}\rfloor \le \lfloor\alpha^m\rfloor - 1$. For (v), suppose m is odd. Then $\lfloor\alpha^m\rfloor\{\alpha^m\} = L_m\{\alpha^m\} = (\alpha^m + \beta^m)(-\beta^m) = 1 - \beta^{2m}$, $\alpha^m\{\alpha^m\} = -\alpha^m\beta^m = 1$, and

$$\lceil\alpha^m\rceil\{\alpha^m\} = \lfloor\alpha^m\rfloor\{\alpha^m\} + \{\alpha^m\} = 1 - \beta^{2m} - \beta^m.$$

In addition, we have $0 > \beta^m + \beta^{2m} > \beta^m \ge \beta > -1$. Therefore $1 - \beta^{2m} - \beta^m$ lies in the interval $(1, 2)$, as required. For (vi), suppose m is even and $n = \lfloor\alpha^m\rfloor$ or $\lceil\alpha^m\rceil$. If $n = \lfloor\alpha^m\rfloor$, then

$$n\{\alpha^m\} = (1 - \beta^m)\lfloor\alpha^m\rfloor = \lfloor\alpha^m\rfloor - \beta^m\lfloor\alpha^m\rfloor \ge \lfloor\alpha^m\rfloor - \beta^m\alpha^m = \lfloor\alpha^m\rfloor - 1,$$

which implies $\lfloor n\{\alpha^m\}\rfloor = \lfloor\alpha^m\rfloor - 1$, by (iv). Next, suppose $n = \lceil\alpha^m\rceil$. Then by (i) and (ii), we obtain $n = L_m = \alpha^m + \beta^m$, $\{\alpha^m\} = 1 - \beta^m$, and so $n\{\alpha^m\} = L_m - 1 - \beta^{2m}$. Therefore $\lfloor n\{\alpha^m\}\rfloor = L_m - 2 = \lfloor\alpha^m\rfloor - 1$. This proves (vi). For (vii), we have $(\lfloor\alpha^m\rfloor + 1)\beta^m = L_m\beta^m = (\alpha^m + \beta^m)\beta^m = 1 + \beta^{2m}$, which implies the first part of (vii). Subtracting both sides of the above equation by β^m, we obtain the second part. This completes the proof.

3. Combinatorial structure of $B(\alpha^m)$

Let $A = (a_n)_{n\ge1}$ be a sequence of real numbers. We say that C is a segment of A if C is a finite sequence of consecutive terms of A, that is, $C = (a_k, a_{k+1}, \ldots, a_{k+m})$ for some $k, m \in \mathbb{N}$. In this case, the length of C is $m + 1$, and if $a_k = a_{k+1} = \cdots = a_{k+m}$, then we call C a constant segment. We often refer to the following segments:

$$S = (a_1, a_2, \ldots, a_{\lfloor\alpha^m\rfloor-1}, \lceil\alpha^m\rceil), \tag{3.1}$$
$$S_0 = (a_1, a_2, \ldots, a_{\lfloor\alpha^m\rfloor}, \lceil\alpha^m\rceil), \tag{3.2}$$
$$T = (b_1, b_2, \ldots, b_{\lfloor\alpha^m\rfloor}, \lfloor\alpha^m\rfloor), \tag{3.3}$$
$$T_0 = (b_1, b_2, \ldots, b_{\lfloor\alpha^m\rfloor-1}, \lfloor\alpha^m\rfloor), \tag{3.4}$$

where $a_i = \lfloor\alpha^m\rfloor$ and $b_i = \lceil\alpha^m\rceil$ for all i. In addition, we define Diff(A) to be the sequence of the difference between consecutive elements of A, that is,

$$\text{Diff}(A) = (a_{n+1} - a_n)_{n\ge1}.$$

In particular, $\text{Diff}(B(x)) = (\lfloor(n + 1)x\rfloor - \lfloor nx\rfloor)_{n\ge1}$. Our purpose in this section is to give a structure of $\text{Diff}(B(\alpha^m))$ in terms of its segments. We begin with the following lemma.

Lemma 3.1. *Let $m \ge 3$ and $n \ge 1$ be integers. Then the following statements hold.*

(i) *If m is odd and $\lfloor(n + 1)\alpha^m\rfloor - \lfloor n\alpha^m\rfloor = \lceil\alpha^m\rceil$, then*

$$\lfloor(n + k)\alpha^m\rfloor - \lfloor(n + k - 1)\alpha^m\rfloor = \lfloor\alpha^m\rfloor \quad \textit{for each } k = 2, 3, \ldots, \lfloor\alpha^m\rfloor. \tag{3.5}$$

(ii) *If m is even and $\lfloor(n + 1)\alpha^m\rfloor - \lfloor n\alpha^m\rfloor = \lfloor\alpha^m\rfloor$, then*

$$\lfloor(n + k)\alpha^m\rfloor - \lfloor(n + k - 1)\alpha^m\rfloor = \lceil\alpha^m\rceil \quad \textit{for each } k = 2, 3, \ldots, \lfloor\alpha^m\rfloor.$$

Proof. For convenience, let $\ell = \lfloor \alpha^m \rfloor$ and $u = \lceil \alpha^m \rceil$. By Lemma 2.1, the difference between consecutive terms of $B(\alpha^m)$ is either ℓ or u. For (i), suppose m is odd and $\lfloor (n+1)\alpha^m \rfloor - \lfloor n\alpha^m \rfloor = u$ but (3.5) does not hold. Then

$$\begin{aligned} \lfloor (n+\ell)\alpha^m \rfloor - \lfloor n\alpha^m \rfloor &= \sum_{k=1}^{\ell} \left(\lfloor (n+k)\alpha^m \rfloor - \lfloor (n+k-1)\alpha^m \rfloor \right) \\ &\geq 2u + (\ell - 2)\,\ell. \end{aligned} \tag{3.6}$$

On the other hand, we obtain by Lemma 2.1 that

$$\lfloor (n+\ell)\alpha^m \rfloor - \lfloor n\alpha^m \rfloor \leq \lfloor \ell\alpha^m \rfloor + 1. \tag{3.7}$$

Writing $\alpha^m = (\ell - 2) + 2 + \{\alpha^m\}$ and applying Lemmas 2.1 and 2.4, we see that the right-hand side of (3.7) is

$$(\ell-2)\,\ell + 2\ell + \lfloor \ell\{\alpha^m\} \rfloor + 1 = (\ell-2)\,\ell + 2\ell + 1 < (\ell-2)\,\ell + 2u,$$

which contradicts (3.6). Hence (i) holds. Similarly, suppose (ii) does not hold. Since m is even, we obtain by Lemma 2.4 that $\lfloor \ell\{\alpha^m\} \rfloor = \ell - 1$. Similar to (3.6) and (3.7), we obtain

$$\begin{aligned} \ell^2 + \lfloor \ell\{\alpha^m\} \rfloor = \left\lfloor \ell^2 + \ell\{\alpha^m\} \right\rfloor = \lfloor \ell\alpha^m \rfloor &\leq \lfloor (n+\ell)\alpha^m \rfloor - \lfloor n\alpha^m \rfloor \\ &= \sum_{k=1}^{\ell} \left(\lfloor (n+k)\alpha^m \rfloor - \lfloor (n+k-1)\alpha^m \rfloor \right) \\ &\leq 2\ell + (\ell-2)u = \ell^2 + \ell - 2, \end{aligned}$$

which implies $\lfloor \ell\{\alpha^m\} \rfloor \leq \ell - 2$, a contradiction. Hence the proof is complete.

Lemma 3.2. *Let $m \geq 3$ be an integer. Then the following statements hold.*

(i) *If m is odd, then the list of the first $\lfloor \alpha^m \rfloor$ elements of* $\mathrm{Diff}(B(\alpha^m))$ *is the segment S given in* (3.1).
(ii) *If m is even, then the list of the first $\lfloor \alpha^m \rfloor$ elements of* $\mathrm{Diff}(B(\alpha^m))$ *is the segment T_0 given in* (3.4).

Proof. For convenience, let $\ell = \lfloor \alpha^m \rfloor$. We first consider the case that m is odd. To prove (i), it is enough to show that $\lfloor (n+1)\alpha^m \rfloor - \lfloor n\alpha^m \rfloor = \lfloor \alpha^m \rfloor$ for each $n = 1, 2, \ldots, \ell - 1$ and that $\lfloor (\ell+1)\alpha^m \rfloor - \lfloor \ell\alpha^m \rfloor = \lceil \alpha^m \rceil$. So suppose that $1 \leq n \leq \ell - 1$. Then $n + 1 < \alpha^m$ and we obtain by Lemma 2.4 that

$$\{n\alpha^m\} + \{\alpha^m\} = n\{\alpha^m\} + \{\alpha^m\} = (n+1)\{\alpha^m\} = \{(n+1)\alpha^m\} < 1.$$

By Lemma 2.1, we obtain $\lfloor (n+1)\alpha^m \rfloor - \lfloor n\alpha^m \rfloor = \lfloor \alpha^m \rfloor$. Next, suppose $n = \ell$. Again, by Lemma 2.4, we have $\{n\alpha^m\} = n\{\alpha^m\}$ and so

$$\{n\alpha^m\} + \{\alpha^m\} = (n+1)\{\alpha^m\} \geq \alpha^m\{\alpha^m\} = 1.$$

Thus $\lfloor (n+1)\alpha^m \rfloor - \lfloor n\alpha^m \rfloor = \lceil \alpha^m \rceil$. This proves (i). For (ii), assume that m is even. Similar to (i), if $1 \leq n \leq \ell - 1$, then we obtain by Lemma 2.4 that

$$\{n\alpha^m\} + \{\alpha^m\} = 1 - n\beta^m + 1 - \beta^m = 2 - \beta^m(n+1) \geq 2 - \beta^m\ell > 2 - \beta^m\alpha^m = 1,$$

and thus $\lfloor (n+1)\alpha^m \rfloor - \lfloor n\alpha^m \rfloor = \lceil \alpha^m \rceil$. Next, if $n = \ell$, then

$$\{n\alpha^m\} + \{\alpha^m\} = 1 - n\beta^m + 1 - \beta^m = 2 - \beta^m(n+1) < 2 - \beta^m\alpha^m = 1,$$

implying $\lfloor (n+1)\alpha^m \rfloor - \lfloor n\alpha^m \rfloor = \lfloor \alpha^m \rfloor$, as desired. This completes the proof.

Lemma 3.3. *Let $m \geq 3$ be an integer. Then the following statements hold.*

(i) *If m is odd and $\mathrm{Diff}(B(\alpha^m))$ contains the constant segment $(\lfloor\alpha^m\rfloor, \lfloor\alpha^m\rfloor, \ldots, \lfloor\alpha^m\rfloor)$ of length k, then $k \leq \lfloor\alpha^m\rfloor$.*

(ii) *If m is even and $\mathrm{Diff}(B(\alpha^m))$ contains the constant segment $(\lceil\alpha^m\rceil, \lceil\alpha^m\rceil, \ldots, \lceil\alpha^m\rceil)$ of length k, then $k \leq \lfloor\alpha^m\rfloor$.*

Remark 1. By Theorem 3.4 to be proved later, we see that the inequality $k \leq \lfloor\alpha^m\rfloor$ in Lemma 3.3 is sharp in the sense that there exists such a constant segment of length $\lfloor\alpha^m\rfloor$ in $\mathrm{Diff}(B(\alpha^m))$.

Proof of Lemma 3.3. For (i), suppose for a contradiction that m is odd but the sequence $\mathrm{Diff}(B(\alpha^m))$ contains a constant segment $(\lfloor\alpha^m\rfloor, \lfloor\alpha^m\rfloor, \ldots, \lfloor\alpha^m\rfloor)$ of length k with $k > \lfloor\alpha^m\rfloor$. By considering a shorter segment (if necessary), we can choose $k = \lceil\alpha^m\rceil$. This implies that $B(\alpha^m)$ contains a finite arithmetic progression

$$a, a+d, a+2d, \ldots, a+kd,$$

where $a = \lfloor n\alpha^m\rfloor$ for some $n \in \mathbb{N}$, $d = \lfloor\alpha^m\rfloor$, $a + d = \lfloor(n+1)\alpha^m\rfloor$, ..., $a + kd = \lfloor(n+k)\alpha^m\rfloor$. Therefore $kd = a + kd - a$, which is equal to

$$\lfloor(n+k)\alpha^m\rfloor - \lfloor n\alpha^m\rfloor \geq \lfloor k\alpha^m\rfloor = k\lfloor\alpha^m\rfloor + \lfloor k\{\alpha^m\}\rfloor = kd + \lfloor\lceil\alpha^m\rceil\{\alpha^m\}\rfloor = kd + 1,$$

where the last equality is obtained by using Lemma 2.4. This is a contradiction. So (i) is proved. The proof of (ii) is similar, so we omit some details. If (ii) is not true, we would obtain an arithmetic progression $a, a+d, \ldots, a+kd$, where $k = \lceil\alpha^m\rceil$, $a = \lfloor n\alpha^m\rfloor$ for some $n \in \mathbb{N}$, $d = \lceil\alpha^m\rceil$, and $a + kd = \lfloor(n+k)\alpha^m\rfloor$, and therefore

$$\begin{aligned} kd &= \lfloor(n+k)\alpha^m\rfloor - \lfloor n\alpha^m\rfloor \leq \lfloor k\alpha^m\rfloor + 1 = k\lfloor\alpha^m\rfloor + \lfloor k\{\alpha^m\}\rfloor + 1 \\ &= k\lfloor\alpha^m\rfloor + \lfloor\alpha^m\rfloor = k(d-1) + d - 1 = kd - 1 < kd, \end{aligned}$$

which is a contradiction. So (ii) is verified and the proof is complete.

Before proceeding further, we need to define a concept that is similar to a concatenation of words as in combinatorics. Suppose $A = (a_{k+1}, a_{k+2}, \ldots, a_{k+m})$ and $B = (b_{\ell+1}, b_{\ell+2}, \ldots, b_{\ell+r})$ are finite sequences. Then we define the concatenation of A and B, and the n copies of A by

$$A \cdot B = (a_{k+1}, a_{k+2}, \ldots, a_{k+m}, b_{\ell+1}, b_{\ell+2}, \ldots, b_{\ell+r}), \tag{3.8}$$

$$A^{(1)} = A, \text{ and } A^{(n)} = A^{(n-1)} \cdot A \quad \text{for each positive integer } n \geq 2. \tag{3.9}$$

For example, $(1,2,3)^{(2)} = (1,2,3,1,2,3)$, and $((-1)^n)_{1\leq n\leq 10} = (-1,1)^{(5)}$. We now give a structure of $\mathrm{Diff}(B(\alpha^m))$ in terms the segments given in (3.1) to (3.4). We first deal with the case that m is odd.

Theorem 3.4. *Let $m \geq 3$ be an odd integer. Then the first $\lfloor\alpha^m\rfloor^2 + \lceil\alpha^m\rceil$ elements of $\mathrm{Diff}(B(\alpha^m))$ can be written as*

$$\mathrm{Diff}(B(\alpha^m)) = (S, S, S, \ldots, S, S_0, \ldots), \tag{3.10}$$

where S and S_0 are the segments given in (3.1) and (3.2), and there are exactly $\lfloor\alpha^m\rfloor$ of S appearing before S_0 in (3.10).

Proof. By Lemma 3.2, the first $\lfloor\alpha^m\rfloor$ elements of $\mathrm{Diff}(B(\alpha^m))$ is indeed the segment S. To prove (3.10), it is enough to show that if we write $\mathrm{Diff}(B(\alpha^m))$ as

$$\mathrm{Diff}(B(\alpha^m)) = (\underbrace{S, S, S, \ldots, S}_{b \text{ copies of } S}, \ldots), \tag{3.11}$$

then S follows $S^{(b)}$ if $1 \le b < \lfloor\alpha^m\rfloor$, and S_0 follows $S^{(b)}$ if $b = \lfloor\alpha^m\rfloor$, where $S^{(b)}$ is the b copies of S as defined in (3.9) and written in (3.11). So suppose that (3.11) holds with $b < \lfloor\alpha^m\rfloor$. Since the last element of S is $\lceil\alpha^m\rceil$, we obtain by Lemma 3.1 that S is followed by the constant segment $(\ell, \ell, \ldots, \ell)$ of length $\ell - 1$, where $\ell = \lfloor\alpha^m\rfloor$. Therefore (3.11) implies that

$$\mathrm{Diff}(B(\alpha^m)) = (S, S, S, \ldots, S, \ell, \ell, \ldots, \ell, x, \ldots), \tag{3.12}$$

where $x = \lfloor\alpha^m\rfloor$ or $\lceil\alpha^m\rceil$, and we need to show that $x = \lceil\alpha^m\rceil$ so that the segment $(\ell, \ell, \ldots, \ell, x)$ in (3.12) is indeed S.

Before proceeding further, let us explain the idea to be used in the proof. By the definition of $\mathrm{Diff}(B(\alpha^m))$, the segment such as S implies that there exists a sequence

$$a, a + d, a + 2d, \ldots, a + (k-1)d, a + kd + 1,$$

where $a = \lfloor n\alpha^m\rfloor$ for some $n \in \mathbb{N}$, $d = \lfloor\alpha^m\rfloor = k$, $a + jd = \lfloor(n + j)\alpha^m\rfloor$ for $j = 1, 2, \ldots, k - 1$, and $a + kd + 1 = \lfloor(n + k)\alpha^m\rfloor$. If S appears as the first segment, then we can choose $n = 1$; but the above argument can be used whenever S appears. Although $d = k$, we think of d as the difference and k as the number of terms.

Repeatedly applying the above argument to (3.12), we see that there exists a sequence

$$\begin{aligned}&a, a + d, \ldots, a + kd + 1, a + (k + 1)d + 1, \ldots, a + (2k - 1)d + 1,\\ &a + 2kd + 2, \ldots, a + bkd + b, a + bkd + b + \ell, a + bkd + b + 2\ell, \ldots,\\ &a + bkd + b + (\ell - 1)\ell, a + bkd + b + (\ell - 1)\ell + x,\end{aligned} \tag{3.13}$$

where $a = \lfloor\alpha^m\rfloor = d = k = \ell$, $a + d = \lfloor 2\alpha^m\rfloor$, $a + 2d = \lfloor 3\alpha^m\rfloor$, ..., $a + bkd + b = \lfloor(bk + 1)\alpha^m\rfloor$, ..., $a + bkd + b + (\ell - 1)\ell + x = \lfloor(bk + \ell + 1)\alpha^m\rfloor$. Subtracting the last element in (3.13) by the first element in (3.13) and substituting $d = k$ and $\ell = k$, we see that

$$bk^2 + b + k^2 - k + x = \lfloor(bk + k + 1)\alpha^m\rfloor - \lfloor\alpha^m\rfloor. \tag{3.14}$$

Writing $\alpha^m = k + \{\alpha^m\}$ and letting $z = \lfloor(bk + k + 1)\{\alpha^m\}\rfloor$, we see that the right-hand side of (3.14) is $(bk + k + 1)k + z - k$, which implies that $b - k + x = z$. To calculate z, we recall from Lemma 2.4 that $\{\alpha^m\} = -\beta^m$ and $k\{\alpha^m\} = 1 - \beta^{2m}$. Therefore $z = \left\lfloor b + 1 - (\beta^m + \beta^{2m} + b\beta^{2m})\right\rfloor$. Since $b \le \lfloor\alpha^m\rfloor - 1 < \alpha^m - 1$, we see that

$$\beta^m + \beta^{2m} + b\beta^{2m} < \beta^m + \beta^{2m} + (\alpha^m - 1)\beta^{2m} = 0.$$

Therefore $z \ge b + 1$. Since $b - k + x = z$, we obtain $x \ge k + 1 = \lceil\alpha^m\rceil$. Since $x = \lfloor\alpha^m\rfloor$ or $\lceil\alpha^m\rceil$, we conclude that $x = \lceil\alpha^m\rceil$, as required.

Next, suppose that (3.11) holds with $b = \lfloor\alpha^m\rfloor$. Similar to the previous case, we obtain by Lemma 3.1 that S is followed by the constant segment $(\ell, \ell, \ldots, \ell)$ of length $\ell - 1$, and therefore (3.11) implies

$$\mathrm{Diff}(B(\alpha^m)) = (S, S, S, \ldots, S, \ell, \ell, \ldots, \ell, x, y, \ldots), \tag{3.15}$$

where $\ell = \lfloor \alpha^m \rfloor$, the number of S in (3.15) is $b = \lfloor \alpha^m \rfloor$, and x, y are $\lfloor \alpha^m \rfloor$ or $\lceil \alpha^m \rceil$. By Lemma 3.1, we know that x, y cannot be both $\lceil \alpha^m \rceil$. By Lemma 3.3, x, y cannot be both $\lfloor \alpha^m \rfloor$. Therefore

$$(x = \lfloor \alpha^m \rfloor \text{ and } y = \lceil \alpha^m \rceil) \text{ or } (x = \lceil \alpha^m \rceil \text{ and } y = \lfloor \alpha^m \rfloor) \tag{3.16}$$

If $x = \lfloor \alpha^m \rfloor$ and $y = \lceil \alpha^m \rceil$, then the segment $(\ell, \ell, \ldots, \ell, x, y)$ in (3.15) is S_0, and we are done. By (3.16), it is enough to show that $x \neq \lceil \alpha^m \rceil$. Applying the same argument to (3.15) instead of (3.12), we can still obtain the sequence as in (3.13). Then (3.14) holds and the calculation after (3.14) still works, and therefore $b - k + x = z$. Since we now have $b = k$, we obtain $x = z$. The first part of the calculation of z in the previous case still works too. Therefore

$$x = z = \left\lfloor b + 1 - (\beta^m + \beta^{2m} + b\beta^{2m}) \right\rfloor. \tag{3.17}$$

Since $b = \lfloor \alpha^m \rfloor$, we obtain by Lemma 2.4 that

$$b\beta^{2m} = (b\{\alpha^m\})\{\alpha^m\} = (1 - \beta^{2m})(-\beta^m) = -\beta^m + \beta^{3m},$$

and therefore $\beta^m + \beta^{2m} + b\beta^{2m} = \beta^{2m} + \beta^{3m} \in (0, 1)$. So (3.17) implies that $x = b \neq \lceil \alpha^m \rceil$, as required. This completes the proof.

Next, we give an analogue of Theorem 3.4 when m is even.

Theorem 3.5. *Let $m \geq 4$ be an even integer. Then the first $\lceil \alpha^m \rceil^2 - 1$ elements of* $\mathrm{Diff}(B(\alpha^m))$ *can be written as*

$$\mathrm{Diff}(B(\alpha^m)) = (T_0, T, T, \ldots, T, T_0, \lceil \alpha^m \rceil, \ldots), \tag{3.18}$$

where T and T_0 are the segments given in (3.3) *and* (3.4)*, and there are exactly $\lfloor \alpha^m \rfloor - 1$ of T appearing after the first T_0 in* (3.18)*.*

Proof. Since the proof of this theorem uses the same idea as that of Theorem 3.4, we sometimes skip some details. For convenience, let $\ell = \lfloor \alpha^m \rfloor$ and $u = \lceil \alpha^m \rceil$. By Lemma 3.2, the first ℓ elements of $\mathrm{Diff}(B(\alpha^m))$ is the segment T_0. By Lemma 3.1, we know that $\lceil \alpha^m \rceil$ follows the segment T_0, so in particular, $\lceil \alpha^m \rceil$ follows the second T_0 appearing in (3.18). Therefore, to prove (3.18), it is enough to show that T follows the first T_0 and if we write $\mathrm{Diff}(B(\alpha^m))$ as

$$\mathrm{Diff}(B(\alpha^m)) = (T_0, \underbrace{T, T, T, \ldots, T}_{b \text{ copies of } T}, \ldots), \tag{3.19}$$

then T follows $T^{(b)}$ if $1 \leq b < \ell - 1$, and T_0 follows $T^{(b)}$ if $b = \ell - 1$, where $T^{(b)}$ is the b copies of T as defined in (3.9) and written in (3.19).

Step 1. By Lemma 3.1, we know that T_0 is followed by the constant segment $(u, u, \ldots, u)$ of length $\ell - 1$. So we can write

$$\mathrm{Diff}(B(\alpha^m)) = (T_0, u, u, \ldots, u, x, y, \ldots), \tag{3.20}$$

where $x, y \in \{\ell, u\}$. To show that $x = u$ and $y = \ell$, we apply the same argument as in the proof of Theorem 3.4 to obtain from (3.20) the sequence

$$a, a + d, a + 2d, \ldots, a + (k-1)d, a + kd - 1, a + kd - 1 + u, \ldots,$$

$$a + kd - 1 + (\ell - 1)u, a + kd - 1 + (\ell - 1)u + x, \tag{3.21}$$

where $a = \lfloor \alpha^m \rfloor$, $d = \lceil \alpha^m \rceil$, $k = \lfloor \alpha^m \rfloor$, $a + d = \lfloor 2\alpha^m \rfloor$, ..., $a + (k-1)d = \lfloor k\alpha^m \rfloor$, $a + kd - 1 = \lfloor (k+1)\alpha^m \rfloor$, ..., $a + kd - 1 + (\ell - 1)u + x = \lfloor (k + \ell + 1)\alpha^m \rfloor$. Subtracting the last element in (3.21) by the first element in (3.21), we see that

$$kd - 1 + (\ell - 1)u + x = \lfloor (k + \ell + 1)\alpha^m \rfloor - \lfloor \alpha^m \rfloor. \tag{3.22}$$

To calculate the right-hand side of (3.22), we first apply Lemma 2.4 to obtain

$$\begin{aligned} \lfloor \{k\alpha^m\} + \{\ell\alpha^m\} + \{\alpha^m\} \rfloor &= \lfloor 1 - k\beta^m + 1 - \ell\beta^m + 1 - \beta^m \rfloor \\ &= \left\lfloor 3 - (2 - \beta^m + \beta^{2m}) \right\rfloor = 1. \end{aligned}$$

Thus the right-hand side of (3.22) is equal to

$$\lfloor k\alpha^m \rfloor + \lfloor \ell\alpha^m \rfloor + \lfloor \{k\alpha^m\} + \{\ell\alpha^m\} + \{\alpha^m\} \rfloor = \lfloor k\alpha^m \rfloor + \lfloor \ell\alpha^m \rfloor + 1. \tag{3.23}$$

Substituting $k = \ell$, $d = \ell + 1$, $u = \ell + 1$ in (3.22) and applying (3.23) and Lemma 2.4, we obtain

$$2\ell^2 + \ell - 2 + x = 2\lfloor \ell\alpha^m \rfloor + 1 = 2(\ell^2 + \lfloor \ell\{\alpha^m\} \rfloor) + 1 = 2(\ell^2 + \ell - 1) + 1,$$

which implies $x = \ell + 1 = u$, as required. By Lemma 3.3, y cannot be u. So $y = \ell$ and the segment $(u, u, \ldots, u, x, y)$ in (3.20) is T, as desired.

Step 2. Next, suppose that (3.19) holds with $b < \ell - 1$. Again, we know that T is followed by the constant segment $(u, u, \ldots, u)$ of length $\ell - 1$. Therefore (3.19) implies that

$$\text{Diff}(B(\alpha^m)) = (T_0, T, T, \ldots, T, u, u, \ldots, u, x, y, \ldots), \tag{3.24}$$

where $x, y \in \{\ell, u\}$ and we only need to show that $x = u$ so that the segment $(u, u, \ldots, u, x, y)$ in (3.24) is indeed T. For convenience, let

$$s = a + kd - 1 + ((k+1)d - 1)b.$$

Then we obtain from (3.24) the sequence

$$\begin{aligned} &a, a + d, \ldots, a + (k-1)d, a + kd - 1, \ldots, a + 2kd - 1, \\ &a + (2k+1)d - 2, \ldots, s, s + u, \ldots, s + (\ell - 1)u, s + (\ell - 1)u + x, \end{aligned} \tag{3.25}$$

where $a = \lfloor \alpha^m \rfloor$, $d = u$, $k = \ell$, and $s + (\ell - 1)u + x = \lfloor (bk + k + b + \ell + 1)\alpha^m \rfloor$. Subtracting the last element in (3.25) by the first element in (3.25) and substituting $d = u = k + 1$ and $\ell = k$, we obtain

$$bk^2 + 2bk + 2k^2 + k - 2 + x = \lfloor (bk + 2k + b + 1)\alpha^m \rfloor - \lfloor \alpha^m \rfloor. \tag{3.26}$$

Writing $\alpha^m = k + \{\alpha^m\}$ and letting $z = \lfloor (bk + 2k + b + 1)\{\alpha^m\} \rfloor$, we see that the right-hand side of (3.26) is $bk^2 + 2k^2 + bk + z$, and so (3.26) reduces to $bk + k - 2 + x = z$. By Lemma 2.4, we see that

$$k\{\alpha^m\} = (L_m - 1)(1 - \beta^m) = L_m - 1 - \beta^m(\alpha^m + \beta^m - 1) = L_m - 2 + \beta^m - \beta^{2m},$$

and therefore

$$\begin{aligned} z &= \lfloor (b+2)k\{\alpha^m\} + (b+1)(1-\beta^m) \rfloor \\ &= \left\lfloor (b+2)(L_m-2) + (b+2)(\beta^m - \beta^{2m}) + (b+1)(1-\beta^m) \right\rfloor \\ &= (b+2)(L_m-2) + b + \left\lfloor 1 + \beta^m - (b+2)\beta^{2m} \right\rfloor. \end{aligned} \tag{3.27}$$

Observe that we have not used the assumption $1 \le b < \ell - 1$ in any calculation above. So far, we only use b for the number of T appearing in (3.19). This observation will be used in Step 3 too. We now consider the last term in (3.27). We have the equivalence

$$\begin{aligned} 1 + \beta^m - (b+2)\beta^{2m} \ge 1 &\Leftrightarrow 1 \ge (b+2)\beta^m \\ &\Leftrightarrow \alpha^m \ge b+2 \\ &\Leftrightarrow b \le \lfloor \alpha^m \rfloor - 2. \end{aligned} \tag{3.28}$$

It is easy to see that $1 + \beta^m - (b+2)\beta^{2m} < 2$. Therefore (3.27) and (3.28) imply

$$z = (b+2)(L_m-2) + b + 1 = (b+2)(k-1) + b + 1 = bk + 2k - 1.$$

Since $bk + k - 2 + x = z$, it follows that $x = k + 1 = u$, as desired.
Step 3. This step is similar to Step 2. Suppose (3.19) holds with $b = \ell - 1$. As already observed in Step 2, we did not use the assumption $b < \ell - 1$ before the end of Step 2. Therefore (3.24) to (3.28) still hold in this case. But we now have $b = \ell - 1$, so (3.28) implies that $1 + \beta^m - (b+2)\beta^{2m} < 1$, and it is easy to verify by applying Lemma 2.4 that $1 + \beta^m - (b+2)\beta^{2m} > 0$. Therefore (3.27) implies $z = (b+2)(L_m-2) + b = bk + 2k - 2$. Since $bk + k - 2 + x = z$, we obtain $x = k = \lfloor \alpha^m \rfloor$, as desired. This completes the proof.

4. Sumsets associated with $B(\alpha^m)$

In this section, we give various results on sumsets associated with $B(\alpha^m)$. We begin with a simple but useful result for general Beatty sequences as follows.

Theorem 4.1. *If $x > 1$ is an irrational number, $h \in \mathbb{N}$, and the h-fold sumset $hB(x)$ contains $\lceil x \rceil$ consecutive integers, then $(h+1)B(x)$ is cofinite. More precisely, if $hB(x)$ contains consecutive integers $m, m+1, \ldots, m + \lfloor x \rfloor$, then $(h+1)B(x)$ contains every integer that is larger than or equal to $m + \lfloor x \rfloor$.*

Proof. Suppose $hB(x)$ contains consecutive integers $m, m+1, \ldots, m+k$, where $k = \lfloor x \rfloor$. For each $i = 0, 1, 2, \ldots, k$, let $A_i = (m+i) + B(x)$. Then $\bigcup_{i=0}^{k} A_i$ is a subset of $(h+1)B(x)$. So it is enough to show that

$$\bigcup_{i=0}^{k} A_i = [m+k, \infty) \cap \mathbb{N}. \tag{4.1}$$

If $y \in \bigcup_{i=0}^{k} A_i$, then $y \in A_i$ for some i, and so $y = m + i + \lfloor nx \rfloor$ for some $n \in \mathbb{N}$, and therefore $y \ge m + \lfloor x \rfloor$. Conversely, suppose $y \in \mathbb{N}$ and $y \ge m + k$. Since

$$[m+k, \infty) = \bigcup_{n=1}^{\infty} [m + \lfloor nx \rfloor, m + \lfloor (n+1)x \rfloor),$$

we see that $m + \lfloor nx \rfloor \leq y < m + \lfloor (n+1)x \rfloor$ for some $n \in \mathbb{N}$. Recall that $\lfloor (n+1)x \rfloor - \lfloor nx \rfloor \leq \lceil x \rceil$ for every $n \in \mathbb{N}$. Therefore $y = m + \lfloor nx \rfloor + j$ for some $j = 0, 1, 2, \ldots, \lceil x \rceil - 1$. Thus $y \in A_j \subseteq \bigcup_{i=0}^{k} A_i$. This completes the proof.

In the proof of the next theorem, we write $[a, b]$ to denote the interval of integers, that is,

$$[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}.$$

For example, if A is a set of an arithmetic progression $a, a+d, a+2d, \ldots, a+kd$, then $A = a + d * [0, k]$ and the h-fold sumset $hA = ha + d * [0, hk]$. We would like to show that $\lceil \alpha^m \rceil B(\alpha^m)$ is always cofinite for all $m \geq 1$. If $m \leq 2$, then we already have a proof in [6]. If $m \geq 3$ and m is odd, then we have a short proof as follows.

Theorem 4.2. *Let $m \geq 3$ be an odd integer and $h = \lfloor \alpha^m \rfloor$. Then $hB(\alpha^m)$ contains $\lceil \alpha^m \rceil$ consecutive integers. In addition, $(h+1)B(\alpha^m)$ is cofinite, or more precisely, $(h+1)B(\alpha^m)$ contains every integer that is larger than or equal to $h^3 + h\left\lfloor (h^2+1)\alpha^m \right\rfloor = h^4 + h^3 + 2h^2$.*

Proof. Let $k = d = \lfloor \alpha^m \rfloor$. We consider d as the first k terms in the segment S_0 and $k+1$ the number of terms in S_0. By Theorem 3.4, Diff$(B(\alpha^m))$ contains the segment S_0. This implies that $B(\alpha^m)$ contains the segment $a, a+d, a+2d, \ldots, a+kd, a+(k+1)d+1$, where $a = \lfloor n\alpha^m \rfloor$ for some $n \in \mathbb{N}$. Let $A_1 = a + d * [0, k]$, $b = a + (k+1)d + 1$, and $A = A_1 \cup \{b\}$. Then $hA = \bigcup_{\ell=0}^{h-1} (\ell b + (h-\ell)A_1) \cup \{hb\}$. Let $0 \leq \ell < h$ be integers. Then

$$\begin{aligned} \ell b + (h-\ell)A_1 &= \ell b + (h-\ell)a + d * [0, (h-\ell)k] \\ &= \ell a + \ell(k+1)d + \ell + (h-\ell)a + d * [0, (h-\ell)k] \\ &= \ell + \ell(k+1)d + ha + d * [0, (h-\ell)k] \\ &= \ell + ha + d * [\ell(k+1), hk + \ell]. \end{aligned} \tag{4.2}$$

Since $\ell \leq h-1$ and $h = k$, we have $hk - \ell(k+1) \geq hk - (h-1)(k+1) = 1$. Therefore $\ell(k+1) \leq hk - 1 \leq hk + \ell$. By (4.2), $\ell b + (h-\ell)A_1$ contains the integer x_ℓ where $x_\ell = \ell + ha + d(hk-1)$, and therefore $x_\ell \in hA$. Since $0 \leq \ell < h$ is arbitrary, we obtain the consecutive integers $x_0, x_1, x_2, \ldots, x_{h-1}$ in hA. Furthermore, hA_1 contains the integer $x_h =: ha + dhk$ which is equal to $1 + x_{h-1}$. Therefore $x_0, x_1, x_2, \ldots, x_{h-1}, x_h$ are consecutive integers in hA. Since $A \subseteq B(\alpha^m)$, we see that $hB(\alpha^m)$ contains $\lceil \alpha^m \rceil$ consecutive integers $x_0, x_1, \ldots, x_h$. Therefore we obtain by Theorem 4.1 that $(h+1)B(\alpha^m)$ is cofinite and contains every integer that is larger than or equal to x_h.

So it only remains to write x_h in the desired form. First, since $x_h = ha + dhk$ and $d = k = h$, we obtain $x_h = ha + h^3$. Secondly, we see from Theorem 3.4 that there are h^2 elements in Diff$(B(\alpha^m))$ appearing before the segment S_0. Therefore $a = \lfloor n\alpha^m \rfloor = \left\lfloor (h^2+1)\alpha^m \right\rfloor$. Hence $x_h = h^3 + h\left\lfloor (h^2+1)\alpha^m \right\rfloor$.

Alternatively, we can write $a = \left\lfloor (h^2+1)\alpha^m \right\rfloor$ in terms of h only. Since the first k^2 elements of Diff$(B(\alpha^m))$ are $S^{(k)}$, it follows that the (k^2+1)-th element of $B(\alpha^m)$ is $\lfloor \alpha^m \rfloor + (kd+1)k = k^3 + 2k = h^3 + 2h$. Therefore $x_h = ha + h^3 = h(h^3 + 2h) + h^3 = h^4 + h^3 + 2h^2$. This completes the proof.

Theorem 4.3. *Let $m \geq 4$ be an even integer and $h = \lfloor \alpha^m \rfloor$. Then $hB(\alpha^m)$ contains $\lceil \alpha^m \rceil$ consecutive integers and $(h+1)B(\alpha^m)$ contains every integer that is larger than or equal to*

$$(h-1)\left\lfloor (h+1)^2\alpha^m \right\rfloor + 2h = h^4 + 2h^3 - h^2 - h + 1.$$

Proof. Let $a = \lfloor \alpha^m \rfloor = k$ and $d = \lceil \alpha^m \rceil$. We consider a as the first term in $B(\alpha^m)$, k the number of terms in the segment T_0, and d the first k elements of the segment T and also the first $k-1$ elements of the segment T_0. By Theorem 3.5, we obtain the first $\lceil \alpha^m \rceil^2 - 1$ elements of $\mathrm{Diff}(B(\alpha^m))$, and so we know the first $\lceil \alpha^m \rceil^2$ terms of $B(\alpha^m)$. We write these $\lceil \alpha^m \rceil^2 = (k+1)^2$ terms of $B(\alpha^m)$ as entries of a matrix $\left[a_{ij}\right]$ where $0 \le i, j \le k$, $a_{0,0} = \lfloor \alpha^m \rfloor$ is the first term, $a_{0,1} = \lfloor 2\alpha^m \rfloor$ is the second term, …, and $a_{k,k} = \left\lfloor (k+1)^2 \alpha^m \right\rfloor$ is the $(k+1)^2$-th term of $B(\alpha^m)$. For clarity, we write $a_{i,j}$ instead of a_{ij}. So, for instance, we have

$$\begin{aligned} &a_{0,0} = a, a_{0,1} = a + d, \ldots, a_{0,k-1} = a + (k-1)d, a_{0,k} = a + kd - 1, \\ &a_{1,0} = a + (k+1)d - 1, \ldots, a_{k,k} = a + (k^2 + 2k)d - k - 1. \end{aligned}$$

We write the entries in column C_0, C_1, C_2 and C_{k-2}, C_{k-1}, C_k separately as follows:

$$\begin{array}{ccccc} & (C_0) & (C_1) & (C_2) & \cdots \\ (R_0) & a & a+d & a+2d & \cdots \\ (R_1) & a+(k+1)d-1 & a+(k+2)d-1 & a+(k+3)d-1 & \cdots \\ (R_2) & a+2(k+1)d-2 & a+(2k+3)d-2 & a+(2k+4)d-2 & \cdots \\ (R_3) & a+3(k+1)d-3 & a+(3k+4)d-3 & a+(3k+5)d-3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ (R_{k-1}) & a+(k-1)(k+1)d-(k-1) & a+k^2d-(k-1) & a+(k^2+1)d-(k-1) & \cdots \\ (R_k) & a+k(k+1)d-k & a+(k^2+k+1)d-k & a+(k^2+k+2)d-k & \cdots \end{array}$$

$$\begin{array}{ccccc} & \cdots & (C_{k-2}) & (C_{k-1}) & (C_k) \\ (R_0) & \cdots & a+(k-2)d & a+(k-1)d & a+kd-1 \\ (R_1) & \cdots & a+(2k-1)d-1 & a+2kd-1 & a+(2k+1)d-2 \\ (R_2) & \cdots & a+3kd-2 & a+(3k+1)d-2 & a+(3k+2)d-3 \\ (R_3) & \cdots & a+(4k+1)d-3 & a+(4k+2)d-3 & a+(4k+3)d-4 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ (R_{k-1}) & \cdots & a+(k^2+k-3)d-(k-1) & a+(k^2+k-2)d-(k-1) & a+(k^2+k-1)d-k \\ (R_k) & \cdots & a+(k^2+2k-2)d-k & a+(k^2+2k-1)d-k-1 & a+(k^2+2k)d-k-1 \end{array}$$

There are two patterns that are helpful in the following calculation. First, in each row R_i for $i = 0, 1, 2, \ldots, k-1$, we have $a_{i,j} + d = a_{i,j+1}$ for $0 \le j \le k-2$, and $a_{i,k-1} + d - 1 = a_{i,k}$, and in row R_k, we have $a_{k,j} + d = a_{k,j+1}$ for $0 \le j \le k-3$, $a_{k,k-2} + d - 1 = a_{k,k-1}$, and $a_{k,k-1} + d = a_{k,k}$. Secondly, in each column C_j for $0 \le j \le k$ and $j \ne k-1$, we have

$$a_{i,j} + (k+1)d - 1 = a_{i+1,j} \quad \text{for all } i = 0, 1, 2, \ldots, k-1,$$

and in column C_{k-1}, we have

$$\begin{aligned} a_{i,k-1} + (k+1)d - 1 &= a_{i+1,k-1} \quad \text{for all } i = 0, 1, 2, \ldots, k-2, \text{ and} \\ a_{k-1,k-1} + (k+1)d - 2 &= a_{k,k-1}. \end{aligned}$$

These two patterns are used throughout the remaining proof without further reference.

Let $A = \{a_{i,j} \mid 0 \le i, j \le k\}$ be the set of the first $(k+1)^2$ elements of $B(\alpha^m)$. We construct consecutive integers $x_1, x_2, \ldots, x_{k+1}$ that are in the h-fold sumset hA as follows. Let $x_1 = (h-1)a_{k,k} + a_{0,0}$, which is clearly an element of hA. Next, let $x_2 = x_1 + 1$. To write x_2 as the sum of h elements of A, we observe that for $0 \le j \le k-3$,

$$a_{k,k} + a_{0,j} + 1 = (a_{k,k} - d - (d-1)) + (a_{0,j} + d + d) = a_{k,k-2} + a_{0,j+2}.$$

Therefore $x_2 = (h-2)a_{k,k} + a_{k,k} + a_{0,0} + 1 = (h-2)a_{k,k} + a_{k,k-2} + a_{0,2}$. In general, for $j = 1, 2, 3, \ldots, \left\lfloor \frac{k+1}{2} \right\rfloor$, let

$$x_j = (h-j)a_{k,k} + (j-1)a_{k,k-2} + a_{0,2j-2}. \tag{4.3}$$

Then, for each $j = 1, 2, \ldots, \left\lfloor \frac{k+1}{2} \right\rfloor$, we see that x_j is a sum of $h - j + j - 1 + 1 = h$ elements of A, and so $x_j \in hA$. In addition, for $2 \le j \le \left\lfloor \frac{k+1}{2} \right\rfloor$, we have

$$x_j - x_{j-1} = (-a_{k,k} + a_{k,k-2}) + (a_{0,2j-2} - a_{0,2j-4}) = (-2d + 1) + 2d = 1.$$

Therefore $x_1, x_2, x_3, \ldots, x_{\left\lfloor \frac{k+1}{2} \right\rfloor}$ are consecutive integers in hA. Next, we divide the construction into two cases according to the parity of k.

Case 1. k is even. So we already have consecutive integers $x_1, x_2, \ldots, x_{\frac{k}{2}}$, and we need to construct $x_{\frac{k}{2}+1}, x_{\frac{k}{2}+2}, \ldots, x_k, x_{k+1}$. Let $x_{\frac{k}{2}+1} = x_{\frac{k}{2}} + 1$. Observe that

$$\begin{aligned} a_{k,k} + a_{0,k-2} + 1 &= (a_{k,k} - ((k+1)d - 1) - (d-1)) + (a_{0,k-2} + ((k+1)d - 1) + d) \\ &= a_{k-1,k-1} + a_{1,k-1}. \end{aligned} \tag{4.4}$$

By (4.3), we have $x_{\frac{k}{2}} = \frac{k}{2}a_{k,k} + \left(\frac{k}{2} - 1\right)a_{k,k-2} + a_{0,k-2}$. This and (4.4) imply that

$$\begin{aligned} x_{\frac{k}{2}+1} = x_{\frac{k}{2}} + 1 &= \left(\frac{k}{2} - 1\right)a_{k,k} + \left(\frac{k}{2} - 1\right)a_{k,k-2} + (a_{k,k} + a_{0,k-2} + 1) \\ &= \left(\frac{k}{2} - 1\right)a_{k,k} + \left(\frac{k}{2} - 1\right)a_{k,k-2} + a_{k-1,k-1} + a_{1,k-1}, \end{aligned}$$

which is a sum of $\frac{k}{2} - 1 + \frac{k}{2} - 1 + 1 + 1 = k = h$ elements of A, and so it is an element of hA. In general, for $1 \le j \le \frac{k}{2}$, let

$$x_{\frac{k}{2}+j} = \left(\frac{k}{2} - j\right)a_{k,k} + \left(\frac{k}{2} - j\right)a_{k,k-2} + (2j-1)a_{k-1,k-1} + a_{2j-1,k-1}. \tag{4.5}$$

Clearly, $x_{\frac{k}{2}+j}$ is a sum of $\frac{k}{2} - j + \frac{k}{2} - j + 2j - 1 + 1 = k = h$ elements of A. In addition, for $2 \le j \le \frac{k}{2}$, we have

$$\begin{aligned} x_{\frac{k}{2}+j} - x_{\frac{k}{2}+j-1} &= -a_{k,k} - a_{k,k-2} + 2a_{k-1,k-1} + a_{2j-1,k-1} - a_{2j-3,k-1} \\ &= -(a_{k-1,k-1} + (d-1) + ((k+1)d - 1)) \\ &\quad - (a_{k-1,k-1} - d + ((k+1)d - 1)) + 2a_{k-1,k-1} + 2((k+1)d - 1) \\ &= 1. \end{aligned}$$

This shows that $x_1, x_2, \ldots, x_{\frac{k}{2}}, x_{\frac{k}{2}+1}, x_{\frac{k}{2}+2}, \ldots, x_k$ are consecutive integers contained in hA. It remains to construct x_{k+1}. Let $x_{k+1} = x_k + 1$. By (4.5), we know x_k, and so

$$\begin{aligned} x_{k+1} &= (k-1)a_{k-1,k-1} + a_{k-1,k-1} + 1 \\ &= (k-1)(a_{k-1,k-1} + (k+1)d - 2) + (a_{k-1,k-1} - (k-1)((k+1)d - 1)) + k \\ &= (k-1)a_{k,k-1} + a_{0,k-1} + k \\ &= (k-1)a_{k,k-1} + a_{0,k}, \end{aligned}$$

which is a sum of $k - 1 + 1 = k = h$ elements of A. Hence, we obtain consecutive integers $x_1, x_2, x_3, \ldots, x_{k+1}$ in hA, as desired.

Case 2. k is odd. So we already have $\frac{k+1}{2}$ consecutive integers $x_1, x_2, x_3, \dots, x_{\frac{k+1}{2}}$ in hA. Therefore we need to construct $x_{\frac{k+1}{2}+1}, x_{\frac{k+1}{2}+2}, \dots, x_{k+1}$. Let $x_{\frac{k+1}{2}+1} = x_{\frac{k+1}{2}} + 1$. By (4.3), we know $x_{\frac{k+1}{2}}$, and so we know that

$$\begin{aligned} x_{\frac{k+1}{2}+1} &= \left(\frac{k-1}{2}\right)a_{k,k} + \left(\frac{k-1}{2}\right)a_{k,k-2} + a_{0,k-1} + 1 \\ &= \left(\frac{k-1}{2} - 1\right)a_{k,k} + \left(\frac{k-1}{2} - 1\right)a_{k,k-2} + 2a_{k-1,k-1} \\ &\quad + a_{k,k} + a_{k,k-2} - 2a_{k-1,k-1} + a_{0,k-1} + 1. \end{aligned} \tag{4.6}$$

For $0 \le j \le k-3$, we have

$$\begin{aligned} &a_{k,k} + a_{k,k-2} - 2a_{k-1,k-1} + a_{j,k-1} + 1 \\ &\quad = (a_{k-1,k-1} + (d-1) + (k+1)d - 1) + (a_{k-1,k-1} - d + (k+1)d - 1) \\ &\qquad - 2a_{k-1,k-1} + (a_{j,k-1} + 1) \\ &\quad = a_{j,k-1} + 2((k+1)d - 1) \\ &\quad = a_{j+2,k-1}. \end{aligned} \tag{4.7}$$

From (4.6) and (4.7), we obtain

$$x_{\frac{k+1}{2}+1} = \left(\frac{k-1}{2} - 1\right)a_{k,k} + \left(\frac{k-1}{2} - 1\right)a_{k,k-2} + 2a_{k-1,k-1} + a_{2,k-1},$$

which is a sum of $\frac{k-1}{2} - 1 + \frac{k-1}{2} - 1 + 2 + 1 = k = h$ elements of A. In general, for each $j = 1, 2, 3 \dots, \frac{k-1}{2}$, let

$$x_{\frac{k+1}{2}+j} = \left(\frac{k-1}{2} - j\right)a_{k,k} + \left(\frac{k-1}{2} - j\right)a_{k,k-2} + 2ja_{k-1,k-1} + a_{2j,k-1}. \tag{4.8}$$

Clearly, $x_{\frac{k+1}{2}+j}$ is a sum of $\frac{k-1}{2} - j + \frac{k-1}{2} - j + 2j + 1 = k = h$ elements of A for every $j = 1, 2, 3, \dots, \frac{k-1}{2}$. In addition, for $2 \le j \le \frac{k-1}{2}$, we obtain from (4.8) and (4.7) that

$$\begin{aligned} x_{\frac{k+1}{2}+j} - x_{\frac{k+1}{2}+j-1} &= -(a_{k,k} + a_{k,k-2} - 2a_{k-1,k-1}) + a_{2j,k-1} - a_{2j-2,k-1} \\ &= -(a_{k,k} + a_{k,k-2} - 2a_{k-1,k-1} + a_{2j-2,k-1} + 1) + a_{2j,k-1} + 1 \\ &= 1. \end{aligned}$$

This shows that $x_1, x_2, \dots, x_k$ are consecutive integers in hA. So it remains to construct x_{k+1}. Let $x_{k+1} = x_k + 1$. By (4.8), we know x_k, and so we obtain

$$\begin{aligned} x_{k+1} &= (k-1)a_{k-1,k-1} + a_{k-1,k-1} + 1 \\ &= (k-1)(a_{k-1,k-1} + (k+1)d - 2) + (a_{k-1,k-1} - (k-1)((k+1)d - 1)) + k \\ &= (k-1)a_{k,k-1} + a_{0,k-1} + k \\ &= (k-1)a_{k,k-1} + a_{0,k}, \end{aligned}$$

which is a sum of $k - 1 + 1 = k = h$ elements of A. Hence, we obtain consecutive integers $x_1, x_2, x_3, \dots, x_{k+1}$, as desired.

In both cases, we obtain $\lceil \alpha^m \rceil$ consecutive integers in hA. Since $A \subseteq B(\alpha^m)$, we see that $hB(\alpha^m)$ contains $\lceil \alpha^m \rceil$ consecutive integers. By Theorem 4.1, $(h+1)B(\alpha^m)$ is cofinite and contains every integer that is larger than or equal to

$$x_{k+1} = x_1 + k = (h-1)a_{k,k} + a_{0,0} + k.$$

Recall that $a_{k,k}$ is the $(k+1)^2$-th element of $B(\alpha^m)$ and $a_{0,0}$ is the first element of $B(\alpha^m)$, and $k = h$. Therefore $a_{k,k} = \left\lfloor (h+1)^2\alpha^m \right\rfloor$, $a_{0,0} = \lfloor \alpha^m \rfloor = h$, and $x_{k+1} = (h-1)\left\lfloor (h+1)^2\alpha^m \right\rfloor + 2h$. Alternatively, we know from the list of $a_{i,j}$ that $a_{0,0} = a = h$, $a_{k,k} = a + (k^2+2k)d - k - 1 = (h^2+2h)(h+1) - 1$, and therefore $x_{k+1} = h^4 + 2h^3 - h^2 - h + 1$. This completes the proof.

We can use the argument from the proof of Theorem 4.3 to get a sharper version of Theorem 4.2 as follows.

Theorem 4.4. *Let $m \geq 3$ be an odd integer and $h = \lfloor \alpha^m \rfloor$. Then $(h+1)B(\alpha^m)$ contains every integer that is larger than or equal to $2h^3 + 2h$.*

Proof. Since the proof of this theorem is similar to that of Theorem 4.3, we skip some details. Let $a = \lfloor \alpha^m \rfloor = k = d$ and consider a as the first term in $B(\alpha^m)$, k the number of terms in the segment S, and d the first k terms of S_0 (and the first $k-1$ terms of S). By Theorem 3.4, we can write the first $\lfloor \alpha^m \rfloor^2 + \lceil \alpha^m \rceil + 1$ elements of $B(\alpha^m)$ as entries of the matrix $\left[a_{ij}\right]$ as follows.

$$\begin{array}{cccccc}
 & (C_0) & (C_1) & (C_2) & \cdots & (C_{k-1}) \\
(R_0) & a & a+d & a+2d & \cdots & a+(k-1)d \\
(R_1) & a+kd+1 & a+(k+1)d+1 & a+(k+2)d+1 & \cdots & a+(2k-1)d+1 \\
(R_2) & a+2kd+2 & a+(2k+1)d+2 & a+(2k+2)d+2 & \cdots & a+(3k-1)d+2 \\
(R_3) & a+3kd+3 & a+(3k+1)d+3 & a+(3k+2)d+3 & \cdots & a+(4k-1)d+3 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
(R_{k-1}) & a+(k-1)kd+k-1 & \vdots & \vdots & \cdots & \vdots \\
(R_k) & a+k^2d+k & \vdots & \vdots & \cdots & a+(k^2+k-1)d+k \\
(R_{k+1}) & a+(k^2+k)d+k & a+(k^2+k+1)d+k+1 & X & \cdots & X
\end{array}$$

For instance, in row R_0, we have

$$a_{0,0} = a, a_{0,1} = a+d, \ldots, a_{0,k-1} = a+(k-1)d,$$

and in row R_{k+1}, we have

$$a_{k+1,0} = a + (k^2+k)d + k, a_{k+1,1} = a + (k^2+k+1)d + k + 1,$$

and for $j = 2, 3, \ldots, k-1$, the number $a_{k+1,j}$ is unspecified and is not used in the proof. The patterns that are helpful in the following calculation are as follows.

First, in each row R_i for $i = 0, 1, \ldots, k$, we have $a_{i,j} + d = a_{i,j+1}$ for $0 \leq j \leq k-2$ and $a_{i,k-1} + d + 1 = a_{i+1,0}$ if $i \neq k$, and $a_{k,k-1} + d = a_{k+1,0}$. In row R_{k+1}, we have $a_{k+1,0} + d + 1 = a_{k+1,1}$. Secondly, in each column C_j for $j = 0, 1, 2, \ldots, k-1$, we have $a_{i,j} + kd + 1 = a_{i+1,j}$ for $i = 0, 1, 2, \ldots, k-1$, and $a_{k,0} + kd = a_{k+1,0}$, and $a_{k,1} + kd + 1 = a_{k+1,1}$.

Let $A = \{a_{i,j} \mid 0 \leq i \leq k \text{ and } 0 \leq j \leq k-1\} \cup \{a_{k+1,0}, a_{k+1,1}\}$ be the first $k^2 + k + 2$ elements of $B(\alpha^m)$ given above. We construct consecutive integers $x_0, x_1, \ldots, x_k$ as follows. Let

$$x_0 = a_{k+1,0} + (h-2)a_{0,k-1} + a_{0,k-2}$$

and for $j = 1, 2, \dots, k-2$, let

$$x_j = a_{k+1,1} + (h-2-j)a_{0,k-1} + (j-1)a_{0,0} + a_{j-1,k-1} + a_{0,k-2-j}.$$

It is easy to see that x_j is a sum of h elements of A for every $j = 0, 1, 2, \dots, k-2$. In addition, $x_1 - x_0$ is equal to

$$(a_{k+1,1} - a_{k+1,0}) + (a_{0,k-3} - a_{0,k-2}) = d + 1 - d = 1.$$

Furthermore, for $j = 2, 3, \dots, k-2$, we have

$$\begin{aligned} x_j - x_{j-1} &= -a_{0,k-1} + a_{0,0} + (a_{j-1,k-1} - a_{j-2,k-1}) + (a_{0,k-2-j} - a_{0,k-1-j}) \\ &= -(k-1)d + (kd+1) + (-d) = 1. \end{aligned}$$

Therefore $x_0, x_1, x_2, \dots, x_{k-2}$ are consecutive integers in hA. Next, let $x_{k-1} = 1 + x_{k-2}$, which is equal to

$$\begin{aligned} &1 + a_{k+1,1} + (h-2)a_{0,0} + a_{k-3,k-1} \\ &\quad = (a_{k+1,1} - (kd+1) - d) + (h-2)a_{0,0} + (a_{k-3,k-1} + kd + 1 + d + 1) \\ &\quad = a_{k,0} + (h-2)a_{0,0} + a_{k-1,0}, \end{aligned}$$

and therefore $x_{k-1} \in hA$. Next, let $x_k = 1 + x_{k-1}$, which is equal to

$$\begin{aligned} &a_{k,0} + kd + (h-2)(a_{0,0} + (k-1)d) + (a_{k-1,0} - (k-1)(kd+1) + (k-1)d) \\ &\quad = a_{k+1,0} + (h-2)a_{0,k-1} + a_{0,k-1}, \end{aligned}$$

and thus $x_k \in hA$. Hence $x_0, x_1, x_2, \dots, x_k$ are consecutive integers in hA. By Theorem 4.1, $(h+1)B(\alpha^m)$ contains every integer that is larger than or equal to x_k. We have

$$\begin{aligned} x_k &= a_{k+1,0} + (h-1)a_{0,k-1} \\ &= a + (k^2 + k)d + k + (k-1)(a + (k-1)d) \\ &= 2k^3 + 2k = 2h^3 + 2h. \end{aligned}$$

This completes the proof.

Remark 2. The integer $x = x(m, h) = h^4 + 2h^3 - h^2 - h + 1$ in Theorem 4.3, and the integer $x = x(m, h) = 2h^3 + 2h$ in Theorem 4.4 are best possible when $m \leq 5$. For example, if $m = 3$, then $x(m, h)$ in Theorem 4.4 is equal to $2h^3 + 2h = 136$, and so $5B(\alpha^3)$ contains every integer that is larger than or equal to 136. Then we can either straightforwardly check or use a computer to verify that $135 \notin 5B(\alpha^m)$, and thus 136 is best possible.

More generally, for a cofinite proper subset A of $\mathbb{N}$, let $G(A) = \mathbb{N} \setminus A$, $g(A) = |G(A)|$, $f(A) = \max G(A)$, and $c(A) = f(A) + 1$. Although this may be slightly different from a more restrictive definition in algebraic geometry or numerical semigroup theory, we may call $G(A)$ the set of gaps of A, $g(A)$ the genus of A, $f(A)$ the Frobenius number of A, and $c(A)$ the conductor of A. Then Theorems 4.3 and 4.4 actually give some information on the genus, the Frobenius number, and the conductor of $\lceil \alpha^m \rceil B(\alpha^m)$. We record these as a corollary.

Corollary 1. *Let m be a positive integer and $h(m) = \lfloor \alpha^m \rfloor$. Let $g(m)$, $f(m)$, $c(m)$ be the genus, the Frobenius number, and the conductor of the $(h+1)$-fold sumset $(h+1)B(\alpha^m)$, respectively, as defined in Remark 2. Then the following statements hold:*

$$\begin{aligned}
&g(1) = 2, f(1) = 3, c(1) = 4,\\
&g(2) = 11, f(2) = 26, c(2) = 27,\\
&g(3) = 47, f(3) = 135, c(3) = 136,\\
&g(4) = 251, f(4) = 1686, c(4) = 1687,\\
&g(5) = 747, f(5) = 2683, c(5) = 2684.
\end{aligned}$$

Furthermore, if $m \geq 6$ and m is even, then $f(m) \leq h^4 + 2h^3 - h^2 - h$; if $m \geq 6$ and m is odd, then $f(m) \leq 2h^3 + 2h - 1$. In particular, the values of $f(m)$ and $c(m)$ obtained from (or implied by) Theorems 4.3 and 4.4 are best possible for $3 \leq m \leq 5$. In fact, simply substituting $h = h(1) = \lfloor \alpha \rfloor = 1$ in Theorem 4.4 and $h = h(2) = \lfloor \alpha^2 \rfloor = 2$ in Theorem 4.3, and comparing them with the results in [6], we see that they are also best possible for $m = 1, 2$.

Proof. When $m = 1$ or 2, we obtain this corollary from Theorems 3.1 and 3.8 in [6]. For $3 \leq m \leq 5$, we apply Theorems 4.3 and 4.4 to obtain an explicit constant $N = N(m)$ depending only on m such that the $\lceil \alpha^m \rceil$-fold sumset $\lceil \alpha^m \rceil B(\alpha^m)$ contains every integer that is larger than or equal to N. Then we can either straightforwardly check or use a computer to verify that $N - 1$ is not in $\lceil \alpha^m \rceil B(\alpha^m)$. So N is the smallest integer such that $[N, \infty) \cap \mathbb{Z}$ is contained in $\lceil \alpha^m \rceil B(\alpha^m)$. This completes the proof.

We conclude this paper with the following conjecture.

Conjecture 1. *Let $h = \lfloor \alpha^m \rfloor$ for each $m \geq 3$. Then the integers $h^4 + 2h^3 - h^2 - h + 1$ and $2h^3 + 2h$ given in Theorems 4.3 and 4.4 are best possible for all $m \geq 3$. In other words, the Frobenius number $f(m)$ of $\lceil \alpha^m \rceil B(\alpha^m)$ satisfies*

$$f(m) = \begin{cases} h^4 + 2h^3 - h^2 - h, & \text{if } m \text{ is even;} \\ 2h^3 + 2h - 1, & \text{if } m \text{ is odd.} \end{cases}$$

Conjecture 2. *For each $m \in \mathbb{N}$, the number $\lceil \alpha^m \rceil$ is the smallest positive integer such that $\lceil \alpha^m \rceil B(\alpha^m)$ is cofinite.*

Question 1. *Numerical evidence suggests (but does not prove) that there are infinitely many Fibonacci numbers that are not an element of $\lfloor \alpha^m \rfloor B(\alpha^m)$. Is this true in a more general situation? If $\alpha > 0 > \beta$, $|\alpha| > \max\{1, |\beta|\}$, and α, β are roots of the characteristic polynomial $x^2 - ax - b$ of the Lucas sequence of the first kind $(U_n(a,b))_{n\geq 1}$, is there an interesting connection between the sumsets of $B(\alpha^m)$ and U_n? Here U_n is defined by the recurrence relation $U_0 = 0$, $U_1 = 1$, and $U_n = aU_{n-1} + bU_{n-2}$ for $n \geq 2$, where a, b are fixed integers, $(a,b) = 1$, and $a^2 + 4b > 0$.*

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Conflict of interest

The authors declare there is no conflicts of interest.

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