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Research article

Digitally topological groups

Sang-Eon Han*

Department of Mathematics Education, Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju-City Jeonbuk, 54896, Korea

* **Correspondence:** Email : sehan@jbnu.ac.kr; Tel: +82-63-270-4449.

Abstract: The purpose of the paper is to study digital topological versions of typical topological groups. In relation to this work, given a digital image $(X, k), X \subset \mathbb{Z}^n$, we are strongly required to establish the most suitable adjacency relation in a digital product $X \times X$, say G_{k^*}, that supports both G_{k^*}-connectedness of $X \times X$ and (G_{k^*}, k)-continuity of the multiplication $\alpha : (X \times X, G_{k^*}) \to (X, k)$ for formulating a digitally topological k-group (or DT-k-group for brevity). Thus the present paper studies two kinds of adjacency relations in a digital product such as a C_{k^*}- and G_{k^*}-adjacency. In particular, the G_{k^*}-adjacency relation is a new adjacency relation in $X \times X \subset \mathbb{Z}^{2n}$ derived from (X, k). Next, the paper initially develops two types of continuities related to the above multiplication α, e.g., the (C_{k^*}, k)- and (G_{k^*}, k)-continuity. Besides, we prove that while the (C_{k^*}, k)-continuity implies the (G_{k^*}, k)-continuity, the converse does not hold. Taking this approach, we define a DT-k-group and prove that the pair $(SC_k^{n,l}, *)$ is a DT-k-group with a certain group operation $*$, where $SC_k^{n,l}$ is a simple closed k-curve with l elements in $\mathbb{Z}^n$. Also, the n-dimensional digital space $(\mathbb{Z}^n, 2n, +)$ with the usual group operation "+" on $\mathbb{Z}^n$ is a DT-$2n$-group. Finally, the paper corrects some errors related to the earlier works in the literature.

Keywords: digital topological version of a topological group; DT-k-group; compatible adjacency; C_{k^*}-adjacency; G_{k^*}-adjacency; G_{k^*}-continuity; digital topology

1. Introduction

Motivated by the well-known fifth of 23 problems formulated by David Hilbert [1, 2], the present paper establishes a digital topological version of a typical topological group. Since the present paper is based on some essential notions such as a digital image, a k-path, a digital space, and so on, we first will remind some concepts. In relation to the study of digital images $X \subset \mathbb{Z}^n$, Rosenfeld [3, 4] initially introduced the digital k-connectivity for low dimensional digital images in $\mathbb{Z}^n, n \in \{1, 2, 3\}$. Let us consider a set $X \subset \mathbb{Z}^n, n \in \{1, 2, 3\}$, as a digital image with a certain digital k-connectivity, denoted by

(X, k), as follows: For $X \subset \mathbb{Z}$, we have $(X, 2)$. For $X \subset \mathbb{Z}^2$, we assume $(X, k), k \in \{4, 8\}$. Besides, for $X \subset \mathbb{Z}^3$, we consider (X, k) with $k \in \{6, 18, 26\}$.
Hereinafter, for our purposes, for $\{a, b\} \subset \mathbb{Z}$ with $a \lneq b$, the set $[a, b]_{\mathbb{Z}}$ is assumed to be the set $\{s \in \mathbb{Z} \,|\, a \leq s \leq b\}$.

Motivated by the above Rosenfeld's approach, a generalization of these adjacencies for low dimensional digital images was proposed to study a high dimensional digital image, as follows [5]: For a natural number t with $1 \leq t \leq n$, we say that the distinct points in $\mathbb{Z}^n$

$$p = (p_i)_{i \in [1,n]_{\mathbb{Z}}} \text{ and } q = (q_i)_{i \in [1,n]_{\mathbb{Z}}}$$

are $k(t, n)$-adjacent if at most t of their coordinates differ by ± 1 and the others coincide. Based on this statement, the $k(t, n)$-adjacency relations of $\mathbb{Z}^n, n \in \mathbb{N}$, were formulated in [5, 6], as follows:

$$k := k(t, n) = \sum_{i=1}^{t} 2^i C_i^n, \text{ where } C_i^n = \frac{n!}{(n-i)!\, i!}, \tag{1.1}$$

where the notation " :=" is used to introduce a new terminology.
For instance,

$$(n, t, k) \in \left\{ \begin{array}{l} (4,1,8), (4,2,32), (4,3,64), (4,4,80), \\ (5,1,10), (5,2,50), (5,3,130), (5,4,210), (5,5,242), \text{ and} \\ (6,1,12), (6,2,72), (6,3,232), (6,4,472), (6,5,664), (6,6,728). \end{array} \right\} \tag{1.2}$$

For a set $X \subset \mathbb{Z}^n, n \in \mathbb{N}$, with one of the k-adjacency of (1.1), we call (X, k) a digital image. For a digital image (X, k), assume two points $x, y \in X$. Then we say that a finite sequence $(x_0, x_1, \cdots, x_m) \subset X \subset \mathbb{Z}^n, n \in \mathbb{N}$, is a k-path if x_i is k-adjacent to x_j, where $j = i + 1, i \in [0, m-1]_{\mathbb{Z}}$.

Let us recall the notion of a digital space [7], as follows: A digital space is a kind of a relation set (X, π), where X is a nonempty set and π is a binary symmetric relation on X such that X is π-connected, where we say that X is *π-connected* if for any two elements x and y of X, there is a finite sequence $(x_i)_{i \in [0,l]_{\mathbb{Z}}}$ of elements in X such that $x = x_0$, $y = x_l$ and $(x_j, x_{j+1}) \in \pi$ for $j \in [0, l-1]_{\mathbb{Z}}$.

Assume a digital image (X, k) with a certain group structure on X, say $(X, *)$, where $X \subset \mathbb{Z}^n$. Then a digital topological version of a topological group, called a digitally topological k-group (DT-k-group for brevity) and denoted by $(X, k, *)$, is logically defined as the combination of a group and a digital k-adjacency structure. Then, we strongly need to establish the most suitable adjacency of a digital product $X \times X$, say G_{k^*}, derived from the given k-adjacency of (X, k) to support both G_{k^*}-connectedness of $X \times X$ and (G_{k^*}, k)-continuity of the multiplication $\alpha : (X \times X, G_{k^*}) \to (X, k)$. Indeed this is essential for formulating a DT-k-group structure of $(X, k, *)$. To achieve this initiative, we can consider some adjacencies of a digital product $X \times X$ that need not be typical k-adjacencies of $\mathbb{Z}^{2n}$ in (1.1). In detail, given a digital image $(X, k), X \subset \mathbb{Z}^n$, the most important thing is that we need to establish a certain adjacency of the Cartesian product $X \times X$ that is suitable for formulating a DT-k-group structure based on both a group $(X, *)$ and a digital image (X, k).

Given a digital image (X, k), after introducing two kinds of adjacencies such as a C_{k^*}- and G_{k^*}-adjacency relation in $X \times X$ (see Definitions 3.2 and 4.4 in the present paper), the present paper further develops the notions of (C_{k^*}, k)- and (G_{k^*}, k)-continuity related to the multiplication $(X \times X, G_{k^*}) \to (X, k)$. Note that the new adjacency relation G_{k^*} in $X \times X \subset \mathbb{Z}^{2n}$ need not belong to the set $\{k := k(t,n) \,|\, t \in [1,2n]_{\mathbb{Z}}\}$ that is the set of typical k-adjacencies of $\mathbb{Z}^{2n}$ (see (1.1)). Based on this approach, we can propose a digital version of a typical topological group derived from a certain group $(X,*)$ and a digital image (X,k). Indeed, both the L_C-property in [8] and the C-compatible k-adjacency of a digital product in [9] can contribute to the establishment of a DT-k-group. Given two digital images $(X_i,k_i), i \in \{1,2\}$, a C-compatible k-adjacency of a Cartesian product in [9] and a C_{k^*}-adjacency in the present paper play important roles in studying product properties of some digital topological invariants relating to the research of digital covering spaces and digital homotopy theory [8, 10]. It was motivated by the Cartesian product adjacency of a graph product in typical graph theory [11]. However, it is clear that these two versions have their own features that need not be equivalent to each other (see Remark 3.2). Moreover, given two digital images (X_i,k_i) in $\mathbb{Z}^{n_i}, i \in \{1,2\}$, it was proved that not every $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ has a C_{k^*}-adjacency (see Example 3.1 and Remarks 3.4 and 3.6). Hence, the present paper proposes a new adjacency relation in $X_1 \times X_2$ and it develops two types of continuities for multiplications that are strongly used to formulate DT-k-groups. In relation to this work, we may raise some issues and queries, as follows:

• Is there a digital image (X,k) with a certain group structure on X ?

• Given a digital image (X,k) with a certain group structure, what relation among elements in the Cartesian product $X \times X$ is the most suitable for establishing a DT-k-group structure on (X,k) ?

Then we are strongly required to have a certain relation making the Cartesian product $X \times X$ connected with respect to the newly-established relation in $X \times X$.

• With a newly-developed adjacency of $X \times X$, how can we establish a DT-k-group structure on X derived from the given digital image (X,k) ?

In case this relation is successfully formulated, it can support to get the earlier works in the literature corrected and vivid from the viewpoints of digital topology and digital geometry.

Next, given two digital images $(X_i,k_i), X_i \subset \mathbb{Z}^{n_i}, i \in \{1,2\}$, suppose a digital product $X_1 \times X_2$ with a typical k'- or a G_{k^*}-adjacency addressing the above queries, say $(X_1 \times X_2, k')$ referred to in (1.1) or $(X_1 \times X_2, G_{k^*})$, derived from the given digital images $(X_i,k_i), X_i \subset \mathbb{Z}^{n_i}, i \in \{1,2\}$. Then, we naturally pose the following queries.

• How to introduce the notion of (G_{k^*},k_i)-continuity of a map from $(X_1 \times X_2, G_{k^*})$ to (X_i,k_i) ?

• What differences are there among the typical (k',k_i)-continuity, the (C_{k^*},k_i)-continuity, and the (G_{k^*},k_i)-continuity ?

• Let $SC_k^{n,l}$ be a simple closed k-curve with l elements in $\mathbb{Z}^n$. Then, how to establish a group structure on $SC_k^{n,l}$?

Furthermore, given $SC_k^{n,l}$, we further have the following question.

• How can we formulate a DT-k-group of $SC_k^{n,l}$?

After developing several new notions, we will address the above mentioned topics.

This paper is organized as follows. Section 2 provides some basic notions that will be used in the paper. In Section 3, given a digital image $(X,k), X \subset \mathbb{Z}^n$, after establishing a C_{k^*}-adjacency of the digital product $X \times X$, we define the notion of (C_{k^*},k)-continuity of a map from $(X \times X, C_{k^*})$ to (X,k). Then, we intensively investigate some properties of (C_{k^*},k)-continuity of a map from $(X \times X, C_{k^*})$ to (X,k). In Section 4, given a digital image (X,k), after establishing a new G_{k^*}-adjacency of the digital product $X \times X$, we define the notion of (G_{k^*},k)-continuity of a map from $(X \times X, G_{k^*})$ to (X,k). Also, we compare among the typical (k',k)-continuity, the (C_{k^*},k)-continuity, and the (G_{k^*},k)-continuity, where k' is a adjacency of $X \times X \subset \mathbb{Z}^{2n}$ referred to in (1.1). Section 5 introduces the notion of a DT-k-group and proves that a simple closed k-curve, denoted by $SC_k^{n,l}$, has a group structure with a certain group operation, denoted by $*$, and finally proves that the combined set $(SC_k^{n,l}, *) := (SC_k^{n,l}, k, *)$ consisting of both the group structure and the digital k-connectivity is a DT-k-group. In particular, given a DT-k-group $(SC_k^{n,l}, *)$, we can make each element $x \in SC_k^{n,l}$ as an identity element depending on our needs after relabeling elements of $SC_k^{n,l}$ (in detail, see Remark 5.4(1)). Also, we prove that $(\mathbb{Z}^n, 2n, +)$ is a DT-$2n$-group. Section 6 corrects some errors in the literature. Section 7 refers to some remarks and a further work. In the paper we will start with only a nonempty and k-connected digital image (X, k). In case a digital image (X, k) is not k-connected, it can invoke some trivial cases when studying k-continuous mappings [12]. Besides, given a set X, we usually use the notation $X^\sharp$ to denote the cardinality of the given set X. In addition, since the paper has many notations, for the convenience of readers, using a certain beginning part of each section, we will give a block summarizing some notations which will used in each section.

2. Preliminaries

To develop the notion of a DT-k-group, the adjacencies of $\mathbb{Z}^n, n \in \mathbb{N}$, referred to in (1.1), are strongly required (see Sections 3–6).
In this section, we will use the following notations with several times.
(1) $SC_k^{n,l}$: A simple closed k-curve with l elements in $\mathbb{Z}^n, n \in \mathbb{N} \setminus \{1\}$.
(2) d_k: A function from (X, k) to $\mathbb{N} \cup \{0\}$ inducing a metric on (X, k) (see (2.2) and (2.3)).
(3) $N_k(p, 1)$: A digital k-neighborhood of the given point p in (X, k).
(4) $\mathbb{N}_0$: The set of even natural numbers (see (2.1) and Example 5.1).

Let us now recall some terminology to develop two adjacencies of a digital product. For a digital image (X, k), two points $x, y \in X$ are k-connected (or k-path connected) if there is a finite k-path from x to y in $X \subset \mathbb{Z}^n$ [13]. We say that a digital image (X, k) is k-connected (or k-path connected) if any two points $x, y \in X$ is k-connected (or k-path connected). In a digital image (X, k), it is clear that the two notions of k-connectedness and k-path connectedness are equivalent. Also, a digital image (X, k) with a singleton is assumed to be k-connected for any k-adjacency. Given a k-adjacency relation of (1.1), a *simple k-path* from x to y on $X \subset \mathbb{Z}^n$ is assumed to be the sequence $(x_i)_{i\in[0,l]_\mathbb{Z}} \subset X \subset \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if and only if either $j = i + 1$ or $i = j + 1$ [13] and $x_0 = x$ and $x_l = y$. The *length* of this simple k-path, denoted by $l_k(x, y)$, is the number l. More precisely, $l_k(x_0, x)$ is the length of a shortest simple k-path from x_0 to x. In case there is no k-path between given distinct points x, y in (X, k), we say that $l_k(x, y) = \infty$. Besides, a simple closed k-curve with l elements in $\mathbb{Z}^n$, denoted by $SC_k^{n,l}, 4 \le l \in \mathbb{N}$ [5, 10, 13, 14], is a sequence $(x_i)_{i\in[0,l-1]_\mathbb{Z}}$ in $\mathbb{Z}^n$, where x_i and x_j are k-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$ [13]. Indeed, the number l of $SC_k^{n,l}$ depends on the situation [10] in (2.1) below. In $\mathbb{Z}^n$, we obtain

$$\left\{\begin{array}{l} (1)\text{ in the case } k = 2n (n \neq 2), \text{ we have } l \in \mathbb{N}_0 \setminus \{2\};\\ (2)\text{ in the case } k = 4 (n = 2), \text{ we obtain } l \in \mathbb{N}_0 \setminus \{2, 6\};\\ (3)\text{ in the case } k = 8 (n = 2), \text{ we have } l \in \mathbb{N} \setminus \{1, 2, 3, 5\};\\ (4)\text{ in the case } k = 18 (n = 3), \text{ we obtain } l \in \mathbb{N} \setminus \{1, 2, 3, 5\}; \text{ and}\\ (5)\text{ in the case } k := k(t, n) \text{ such that } 3 \le t \le n,\\ \text{ we have } l \in \mathbb{N} \setminus \{1, 2, 3\}. \end{array}\right\} \tag{2.1}$$

For instance, we have $SC_{26}^{3,5}$, $SC_8^{2,7}$, and so on.

As a matter of fact, the length $l_k(x, y)$ induces a metric function d_k on a k-connected digital image (X, k) [5, 14]. To be specific, assume a function on a k-connected digital image (X, k), as follows:

$$d_k : (X, k) \times (X, k) \to \mathbb{N} \cup \{0\} \tag{2.2}$$

such that

$$d_k(x, x') := \begin{cases} 0, \text{ if } x = x', \text{ and} \\ l_k(x, x'), \text{ if } x \neq x' \text{ and } x \text{ is } k\text{-connected with } x'. \end{cases} \tag{2.3}$$

Owing to (2.2) and (2.3), the map d_k is obviously a function [5, 14] satisfying $d_k(x, x') \geq 1$ whenever $x \neq x'$. Hence it is clear that for a k-connected digital image (X, k), the map d_k of (2.2) is a metric function on (X, k) [10, 14]. Thus, we can represent a digital k-neighborhood of the point x_0 with radius 1 [5, 8] in the following way [14]

$$N_k(x_0, 1) = \{x \in X \mid d_k(x_0, x) \leq 1\}. \tag{2.4}$$

This k-neighborhood will be strongly used to develop the notions of a C_{k^*}- and a G_{k^*}-adjacency of a digital product in Sections 3 and 4 and comparing several adjacencies of digital products in Sections 3–6. To map every k_0-connected subset of (X, k_0) into a k_1-connected subset of (Y, k_1), the paper [4] established the notion of digital continuity. The digital continuity can be represented by using a digital k-neighborhood in (2.4), as follows:

Proposition 2.1. *[5, 14] Let (X, k_0) and (Y, k_1) be digital images on $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every $x \in X$, $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

In Proposition 2.1, in case $n_0 = n_1$ and $k_0 = k_1$, we say that it is k_0-*continuous*.

3. C_{k^*}-adjacencies of digital products and an establishment of the (C_{k^*}, k')-continuity

This section studies the notion of a C_{k^*}-adjacency of a digital product that will be strongly used to develop a digitally topological k-group (or DT-k-group in this paper) in Section 5. Next, we initially establish the notion of (C_{k^*}, k_i)-continuity of a map from $(X_1 \times X_2, C_{k^*})$ to $(X_i, k_i), i \in \{1, 2\}$. More precisely, given two digital images $(X_i, k_i), i \in \{1, 2\}$, first we develop the so-called C_{k^*}-adjacency relation in the digital product $X_1 \times X_2$ derived from the given $(X_i, k_i), i \in \{1, 2\}$, so that we obtain a relation set $(X_1 \times X_2, C_{k^*})$ (see Proposition 3.7).

In this section, we will often use the following notations.

(1) C-compatible k-adjacency (see Definition 3.1).

(2) C_{k^*}-adjacency relation (see Definition 3.2) in a digital space $(X_1 \times X_2, C_{k^*})$.

(3) $N_{C_{k^*}}(p)$: The set of the elements of C_{k^*}-neighbors of the given point p in a digital space $(X_1 \times X_2, C_{k^*})$ (see (3.5) and (3.6)).

(4) $N_{C_{k^*}}(p, 1)$: A C_{k^*}-neighborhood of the given point p in a digital space $(X_1 \times X_2, C_{k^*})$ (see (3.5).

(5) $X^\sharp$: The cardinal number of the given set X. Indeed, to avoid some confusion with the absolute value used in Sections 2 and 5 (in particular, see the proof of Theorem 5.8), we will use the notation $X^\sharp$.

Since this work is associated with both the C-compatible k-adjacency in [9] and the L_C-property in [8] of a digital product, let us recall them as follows: Motivated by the Cartesian product of graphs in [11], various properties of digital products were used in studying digital homotopic properties and digital covering spaces [8, 9]. Using the digital k-neighborhood of (2.4), we define the following:

Definition 3.1. *[9] For two digital images (X_i, k_i) on $\mathbb{Z}^{n_i}, k_i := k(t_i, n_i), i \in \{1, 2\}$, consider the Cartesian product $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$. We say that a k-adjacency of $X_1 \times X_2$ is strongly Cartesian compatible (C-compatible, for brevity) with the given k_i-adjacency, $i \in \{1, 2\}$, if every point (x_1, x_2) in $X_1 \times X_2$ satisfies the following property:*

$$N_k((x_1, x_2), 1) = (N_{k_1}(x_1, 1) \times \{x_2\}) \cup (\{x_1\} \times N_{k_2}(x_2, 1)), \tag{3.1}$$

where the k-adjacency is one of the typical k-adjacency of $\mathbb{Z}^{n_1+n_2}$ stated in (1.1).

As for the C-compatible k-adjacency of Definition 3.1, we can take some k-adjacency of $X_1 \times X_2$ depending on the situation, where $k := k(t, n_1 + n_2)$ for some $t \in [\max\{t_1, t_2\}, n_1 + n_2]_{\mathbb{Z}}$ [9]. At the moment, note that
(1) not every $X_1 \times X_2$ always has a compatible k-adjacency (see Example 3.1) and further,
(2) not every number $t \in [\max\{t_1, t_2\}, n_1 + n_2]_{\mathbb{Z}}$ is used to formulate a compatible k-adjacency of $X_1 \times X_2$ (see Example 3.1).
(3) However, in case there is a C-compatible k-adjacency of $X_1 \times X_2$, at least the number $t = \max\{t_1, t_2\}$ supports the establishment of the C-compatible $k(t, n_1 + n_2)$-adjacency of $X_1 \times X_2$.
For instance, consider the Cartesian product $SC_4^{2,l} \times SC_4^{2,l} \subset \mathbb{Z}^4$ has the only one C-compatible k-adjacency, where $k = k(1, 4) = 8$ instead the other adjacencies of $\mathbb{Z}^4$.
Motivated by this feature, based on Definition 3.1, we now define the following adjacency which is stronger than the adjacency of Definition 3.1.

Definition 3.2. *For two digital images (X_i, k_i) in $\mathbb{Z}^{n_i}, k_i := k(t_i, n_i), i \in \{1, 2\}$, assume the Cartesian product $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ with a C-compatible k-adjacency. After that, we consider only the case*

$$k := k(t, n_1 + n_2),\ t = max\{t_1, t_2\}. \tag{3.2}$$

Equivalently, distinct points $p := (x_1, x_2)$ and $q := (x_1', x_2')$ in $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$, p is k-adjacent to q if and only if

$$\begin{Bmatrix} \text{either } x_1 \text{ is } k_1\text{-adjacent to } x_1' \text{ and } x_2 = x_2', \\ \text{or } x_2 \text{ is } k_2\text{-adjacent to } x_2' \text{ and } x_1 = x_1'. \end{Bmatrix}$$

After that, we take only the case of $k^* := k$ such that $k := k(t, n_1 + n_2), t = max\{t_1, t_2\}$.
Then we say that this k-adjacency is a C_{k^*}-adjacency of $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ derived from the given two digital images $(X_i, k_i), i \in \{1, 2\}$.

As for the C_{k^*}-adjacency of $X_1 \times X_2$ of Definition 3.2, we can concretely say that a compatible $k(t, n_1+n_2)$-adjacency of $X_1 \times X_2$ satisfying the property of (3.2) is equal to the C_{k^*}-adjacency of $X_1 \times X_2$ such that $k^* = k(t, n_1 + n_2)$ and $t = \max\{t_1, t_2\}$. We now use the notation $(X_1 \times X_2, C_{k^*})$ as a relation set to denote the digital product $X_1 \times X_2$ with a C_{k^*}-adjacency. From Definition 3.2, we obtain the relation set $(X_1 \times X_2, C_{k^*})$ derived from the given two digital images (X_i, k_i) in $\mathbb{Z}^{n_i}, k_i := k(t_i, n_i), i \in \{1, 2\}$, depending on the situation.

To represent a Cartesian product of the two digital images $SC_{k_i}^{n_i,l_i}, i \in \{1,2\}$, as a matrix, we use the notation (see (2.1))

$$SC_{k_1}^{n_1,l_1} := (a_i)_{i\in[0,l_1-1]_{\mathbb{Z}}} \text{ and } SC_{k_2}^{n_2,l_2} := (b_j)_{j\in[0,l_2-1]_{\mathbb{Z}}}.$$

Then, take the Cartesian product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} \subset \mathbb{Z}^{n_1+n_2}$ that can be represented as the following matrix:

$$SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2} := (c_{i,j})_{(i,j)\in[0,l_1-1]_{\mathbb{Z}}\times[0,l_2-1]_{\mathbb{Z}}}, \text{ for brevity, } (c_{i,j}) \tag{3.3}$$

where $c_{i,j} := (a_i, b_j)$.

Remark 3.3. *Assume (X_i, k_i) on $\mathbb{Z}^{n_i}, k_i := k(t_i, n_i), i \in \{1,2\}$. Then we observe the following:*
(1) In case there is a compatible k-adjacency of $X_1 \times X_2$, there is always a C_{k^*}-adjacency of $X_1 \times X_2$ such that

$$k = k^* \text{ and } k^* := k(t, n_1 + n_2),\ t = max\{t_1, t_2\}. \tag{3.4}$$

(2) As an example, consider $SC_8^{2,4} \times SC_8^{2,4}$. While there are three types of C-compatible k-adjacencies, $k \in \{32, 64, 80\}$, there is the only C_{32}-adjacency on $SC_8^{2,4} \times SC_8^{2,4}$ because the only number $32 = k(2,4)$ satisfies Definition 3.2.

Let us compare the typical Cartesian product adjacency in [11] and the current C_{k^*}-adjacency.

Remark 3.4. *Each of the compatible k-adjacency and a C_{k^*}-adjacency is a little bit different from the Cartesian product adjacency in graph theory in [11]. More precisely, given any two graphs G_1, G_2, we always have a Cartesian product adjacency of a graph product $G_1 \times G_2$ [11]. However, as stated in Definitions 3.1 and 3.2, for two digital images $(X_i, k_i), i \in \{1,2\}$, not every $X_1 \times X_2$ has a C_{k^*}-adjacency or a compatible k-adjacency of the given digital product (see also Example 3.1(4) below).*

Definition 3.5. *For two digital images (X_i, k_i) in $\mathbb{Z}^{n_i}, i \in \{1,2\}$, assume the Cartesian product $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ with a certain C_{k^*}-adjacency. Given a point $p := (x_1, x_2) \in X_1 \times X_2$, we define the following sets around the point $p \in (X_1 \times X_2, C_{k^*})$, as follows:*

$$\begin{Bmatrix} N_{C_{k^*}}(p) := \{q \in X_1 \times X_2 \,|\, q \text{ is } C_{k^*}\text{-adjacent to } p\}, \text{ and} \\ N_{C_{k^*}}(p, 1) := N_{C_{k^*}}(p) \cup \{p\}. \end{Bmatrix} \tag{3.5}$$

In Definition 3.5, we call the set $N_{C_{k^*}}(p, 1)$ a C_{k^*}-neighborhood of the point $p \in X_1 \times X_2$. Using (3.5), we can represent a C_{k^*}-adjacency relation between distinct points p and q, as follows: Given an $X_1 \times X_2$ with a C_{k^*}-adjacency, for distinct points p and q in $X_1 \times X_2$, we observe that

$$p \text{ is } C_{k^*}\text{-adjacent with } q \text{ if and only if } q \in N_{C_{k^*}}(p). \tag{3.6}$$

In view of Definition 3.2, we observe that not every digital product has a C_{k^*}-adjacency, as follows:

Example 3.1. *(1) $([a,b]_{\mathbb{Z}} \times [c,d]_{\mathbb{Z}}, C_4)$,*
(2) $(SC_4^{2,l} \times [a,b]_{\mathbb{Z}}, C_6)$,

(3) $(SC_4^{2,l} \times SC_4^{2,l}, C_8)$, $(SC_8^{2,6} \times SC_{26}^{3,4}, C_{130})$, *and* $(SC_8^{2,4} \times SC_8^{2,l}, C_{32})$, *and*
(4) none of $SC_4^{2,l} \times SC_8^{2,6} \subset \mathbb{Z}^4$ *and* $SC_8^{2,6} \times SC_8^{2,6} \subset \mathbb{Z}^4$ *has a* C_{k^*}*-adjacency.*
In detail, consider the digital product $SC_4^{2,4} \times SC_8^{2,6} \subset \mathbb{Z}^4$ *in Example 3.1(4), where*

$$\left\{\begin{array}{l} SC_4^{2,4} := (a_i)_{i\in[0,3]_\mathbb{Z}}, a_0 := (0,0), a_1 := (1,0), a_2 := (1,1), a_3 := (0,1) \text{ and} \\ SC_8^{2,6} := (b_j)_{i\in[0,5]_\mathbb{Z}}, b_0 := (0,0), b_1 := (1,-1), b_2 := (2,-1), \\ b_3 := (3,0), b_4 := (2,1), b_5 := (1,1). \end{array}\right\} \tag{3.7}$$

Then any k-adjacency of $\mathbb{Z}^4$ *of (1.2) is not eligible to be a* C_{k^*}*-adjacency for* $SC_4^{2,4} \times SC_8^{2,6} \subset \mathbb{Z}^4$*. In detail, take the point* $c_{1,2} := (a_1, b_2) \in SC_4^{2,4} \times SC_8^{2,6}$*. Then we obtain*

$$N_{32}(c_{1,2}, 1) = (SC_4^{2,4} \times \{b_2\}) \cup ((SC_4^{2,4} \setminus \{a_3\}) \times \{b_1\}) \cup \{c_{1,3}\} \tag{3.8}$$

whose cardinality is 8*, i.e.,* $(N_{32}(c_{1,2}, 1))^\sharp = 8$.
However, we have

$$\left\{\begin{array}{l} (N_4(a_1, 1) \times \{b_2\}) \cup (\{a_1\} \times N_8(b_2, 1)) = \\ (\{a_0, a_1, a_2\} \times \{b_2\}) \cup (\{a_1\} \times \{b_1, b_2, b_3\}) \end{array}\right\} \tag{3.9}$$

whose cardinality is 5*.*
Hence $N_{32}(c_{1,2}, 1)$ *is not equal to* $(N_4(a_1, 1) \times \{b_2\}) \cup (\{a_1\} \times N_8(b_2, 1))$*. Owing to the point* $c_{1,2} \in SC_4^{2,4} \times SC_8^{2,6}$*, the digital product* $SC_4^{2,4} \times SC_8^{2,6}$ *does not have a* C_{32}*-adjacency. As a matter of fact, any typical digital adjacency of* $\mathbb{Z}^4$ *in (1.2) need not be a* C_{k^*}*-adjacency of* $SC_4^{2,4} \times SC_8^{2,6}$*,* $k^* \in \{8, 32, 64, 80\}$ *(see (1.2)) because any* k^**-adjacency of* $\mathbb{Z}^4$ *cannot support the property (3.1) for* $SC_4^{2,4} \times SC_8^{2,6}$*.*

By Definition 3.5, Remark 3.3, and Example 3.1, we obviously obtain the following:

Remark 3.6. *(1) Not every point* $p \in X_1 \times X_2$ *has an* $N_{C_{k^*}}(p, 1)$*. For instance,* $SC_4^{2,4} \times SC_8^{2,6}$ *has several elements p such that* $N_{C_{32}}(p, 1)$ *does not exist (see (3.8) and (3.9)). Besides, the other numbers* $k^* \in \{8, 64, 80\}$ *does not satisfy the property (3.1) either.*
(2) Given two digital images $(X_i, k_i), i \in \{1, 2\}$*, only in case that a digital product* $X_1 \times X_2$ *has a* C_{k^*}*-adjacency, for a point* $p := (x_1, x_2) \in X_1 \times X_2$*, using the properties of (3.1) and (3.5), we obtain*

$$(N_{C_{k^*}}(p, 1))^\sharp = (N_{k_1}(x_1, 1))^\sharp + (N_{k_2}(x_2, 1))^\sharp - 1. \tag{3.10}$$

(2) Consider two digital images (X_i, k_i) *in* $\mathbb{Z}^{n_i}$*,* $i \in \{1, 2\}$*, where* $k_i := k(t_i, n_i)$ *(see (1.1)). Assume* $k_i = 2n_i$*, i.e.,* $t_i = 1, i \in \{1, 2\}$*. Then the digital product* $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ *always has a* C_{k^*}*-adjacency where* $k^* := k(1, n_1 + n_2) = 2(n_1 + n_2)$ *[9]. For instance, see the case* $(SC_4^{2,8} \times SC_4^{2,8}, C_8)$*. Besides, only the* C_{2n}*-adjacency of* $\mathbb{Z}^n$ *exists in terms of the two* $(\mathbb{Z}^{n_1}, 2n_1)$ *and* $(\mathbb{Z}^{n_2}, 2n_2)$*, where* $n = n_1 + n_2$*,* $n_i \in \mathbb{N}, i \in \{1, 2\}$*.*
(3) The adjacency relation C_{k^*} *of the relation set* $(X_1 \times X_2, C_{k^*})$ *is symmetric.*

Based on the product adjacency relation in $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ stated in Definition 3.2, the papers [8,9] studied various properties of digital products with C_{k^*}-adjacencies. Indeed, a digital product with a

C_{k^*}-adjacency $(X_1 \times X_2, C_{k^*})$ is a kind of relation set that is symmetric in $X_1 \times X_2$. Thus we examine if $(X_1 \times X_2, C_{k^*})$ is a kind of digital space. To do this work, we introduce some terminology, as follows:

Based on the C_{k^*}-adjacency of a digital product, motivated by the classical notions in a typical digital image in [13] (see the previous part in Section 2), we now have the following: Given two digital images $(X_i, k_i), X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$, assume a digital product $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ with a certain C_{k^*}-adjacency, i.e., $(X_1 \times X_2, C_{k^*})$. We say that two points $z, w \in X_1 \times X_2$ are C_{k^*}-connected (or C_{k^*}-path connected) if there is a finite C_{k^*}-path $(z_0, z_1, \cdots, z_m) \subset X_1 \times X_2$ from z to w on $X_1 \times X_2$ such that $z_0 = z$ and $z_m = w$, where we say that a C_{k^*}-path from z to w in $X_1 \times X_2$ means a finite sequence $(z_0, z_1, \cdots, z_m) \subset X_1 \times X_2$ such that z_i is C_{k^*}-adjacent to z_j if $j = i + 1, i \in [0, m - 1]_{\mathbb{Z}}$ or $i = j + 1, j \in [0, m - 1]_{\mathbb{Z}}$. A singleton with C_{k^*}-adjacency is assumed to be C_{k^*}-connected. Given a C_{k^*}-adjacency relation in $X_1 \times X_2$, a *simple C_{k^*}-path* from z to w in $X_1 \times X_2$ is assumed to be the C_{k^*}-path $(z_i)_{i\in[0,l]_{\mathbb{Z}}} \subset X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ such that z_i and z_j are C_{k^*}-adjacent if and only if either $j = i + 1, i \in [0, l - 1]_{\mathbb{Z}}$ or $i = j + 1, j \in [0, l - 1]_{\mathbb{Z}}$ and $z_0 = x$ and $z_l = y$. Also, a simple closed C_{k^*}-curve with l elements in $X_1 \times X_2$, denoted by $SC^{n,l}_{C_{k^*}}$, is a sequence $(z_i)_{i\in[0,l-1]_{\mathbb{Z}}}$ in $X_1 \times X_2$, where z_i and z_j are C_{k^*}-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$.

Proposition 3.7. *Given k_i-connected digital images $(X_i, k_i), X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$, $(X_1 \times X_2, C_{k^*})$ is a digital space.*

Proof: By Remark 3.6(3), since the C_{k^*}-adjacency relation in $X_1 \times X_2$ is obviously symmetric, we only examine if $(X_1 \times X_2, C_{k^*})$ is C_{k^*}-connected. Take any distinct points $p := (x_1, x_2)$ and $q := (x'_1, x'_2)$ in $X_1 \times X_2$. Then, without loss of generality, we may assume the case $x_1 \lneq x'_1$ and $x_2 \leq x'_2$ or the case $x_1 \leq x'_1$ and $x_2 \lneq x'_2$. For our purposes, we now take the first case, i.e., $x_1 \lneq x'_1$ and $x_2 \leq x'_2$. Then consider the differences $|x_1 - x'_1| \gneq 0$ and $|x_2 - x'_2| \geq 0$. According to these finite differences, we can take a finite set

$$\{p := p_1, p_2, p_3, \cdots, p_n := q\} \subset X_1 \times X_2 \tag{3.11}$$

such that p_i is C_{k^*}-adjacent to p_{i+1} in $X_1 \times X_2, i \in [1, n - 1]_{\mathbb{Z}}$ and

$$p, q \in \bigcup_{i\in[1,n]_{\mathbb{Z}}} N_{C_{k^*}}(p_i, 1) \subset X_1 \times X_2. \tag{3.12}$$

Owing to (3.11) and (3.12), we can conclude that $(X_1 \times X_2, C_{k^*})$ is C_{k^*}-connected. □

In case there is a C_{k^*}-adjacency of $X_1 \times X_2$ derived from $(X_i, k_i), X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$, let us now introduce the notion of (C_{k^*}, k')-continuity of a map $f : (X_1 \times X_2, C_{k^*}) \to (Y, k')$.

Definition 3.8. *Given two digital images $(X_i, k_i), X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$, assume a digital product $(X_1 \times X_2, C_{k^*})$ and a digital image (Y, k'). A function $f : (X_1 \times X_2, C_{k^*}) \to (Y, k')$ is (C_{k^*}, k')-continuous at a point $p := (x_1, x_2)$ if for any point $q \in X_1 \times X_2$ such that $q \in N_{C_{k^*}}(p)$ (denoted by $p \leftrightarrow_{C_{k^*}} q$), we obtain $f(q) \in N_{k'}(f(p), 1)$ (denoted by $f(p) \Leftrightarrow_{k'} f(q)$). In case the map f is (C_{k^*}, k')-continuous at each point $p \in X_1 \times X_2$, we say that the map f is (C_{k^*}, k')-continuous.*

This continuity will play a crucial role in establishing a *DT-k*-group in Section 5.

Proposition 3.9. *Assume a Cartesian product $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ with a C_{k^*}-adjacency and a typical digital image (Y, k'). A map $f : (X_1 \times X_2, C_{k^*}) \to (Y, k')$ is (C_{k^*}, k')-continuous at a point $p \in X_1 \times X_2$ if and only if*

$$f(N_{C_{k^*}}(p, 1)) \subset N_{k'}(f(p), 1), \tag{3.13}$$

Proof: By Definition 3.8, the proof is completed. □

Example 3.2. *Let $(X, 2n)$ be a $2n$-connected subset of $(\mathbb{Z}^n, 2n)$. Then each of the typical projection maps $P_i : (X \times X, C_{4n}) \to (X, 2n)$ is a $(C_{4n}, 2n)$-continuous map, $i \in \{1, 2\}$.*

Remark 3.10. *Consider digital images (X_i, k_i) in $\mathbb{Z}^{n_i}$, $i \in \{1, 2\}$, and (Y, k') in $\mathbb{Z}^m$. Given a map from $X_1 \times X_2$ to Y, the (C_{k^*}, k')-continuity of a map not always exist because the existence of $N_{C_{k^*}}(p, 1) \subset X_1 \times X_2$ depends on the situation. However, given the (C_{k^*}, k')-continuity of a map, the (C_{k^*}, k')-continuity is equal to the (k^*, k')-continuity of the given map.*

According to Remark 3.10, since the existence of a C_{k^*}-adjacency for a digital product depends on the situation, we now propose the following result that is the C_{k^*}-adjacency version of the C-compatible k-adjacency studied in [9] (see Theorem 3.8 of [9]) and Remark 3.3(1), as follows:

Theorem 3.11. *Given $SC_{k_i}^{n_i,l_i}$, $i \in \{1, 2\}, k_i := k(t_i, n_i)$ from (1.1), assume $k_i \neq 2n_i, i \in \{1, 2\}$ and $t_1 \leq t_2$. Then we obtain the following cases supporting a C_{k^*}-adjacency for $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$.*

(Case 1) Consider the case $t_1 = t_2$ and $t_1 \neq n_1$, i.e., $k_1 \neq 3^{n_1} - 1$. For each element $y_j \in SC_{k_2}^{n_2,l_2} := (y_j)_{j \in [0, l_2 - 1]_{\mathbb{Z}}}$, assume the number of different coordinates of every pair of the consecutive points y_j and $y_{j+1 (mod\, l_2)}$ in $SC_{k_2}^{n_2,l_2}$ is constant as the number t_2 instead of "at most t_2". Then the product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a C_{k^*}-adjacency such that $k^* := k(t_1, n_1 + n_2)$.
(Case 2) In case $t_1 = n_1$, i.e., $k_1 = 3^{n_1} - 1$, assume that for each element $y_j \in SC_{k_2}^{n_2,l_2} := (y_j)_{j \in [0, l_2 - 1]_{\mathbb{Z}}}$, the number of different coordinates of every pair of the consecutive points y_j and $y_{j+1 (mod\, l_2)}$ is constant as the number t_2 instead of "at most t_2". Then the product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a C_{k^*}-adjacency such that $k^* := k(t_2, n_1 + n_2)$.
(Case 3) In case $t_i = n_i, i \in \{1, 2\}$, i.e., $k_i = 3^{n_i} - 1$ (or $t_i \notin [0, n_i - 1]_{\mathbb{Z}}$), then we can consider two cases:
(Case 3-1) Assume that for each element $y_j \in SC_{k_2}^{n_2,l_2} := (y_j)_{j \in [0, l_2 - 1]_{\mathbb{Z}}}$, the number of different coordinates of every pair of the consecutive points y_j and $y_{j+1 (mod\, l_2)}$ is constant as the number t_2 instead of "at most t_2". Then the product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a C_{k^*}-adjacency such that $k^* := k(t_2, n_1 + n_2)$.
(Case 3-2) Assume that for each element $x_i \in SC_{k_1}^{n_1,l_1} := (x_i)_{i \in [0, l_1 - 1]_{\mathbb{Z}}}$, the number of different coordinates of every consecutive points x_i and $x_{i+1 (mod\, l_1)}$ in $SC_{k_1}^{n_1,l_1}$ is constant as the number n_1 instead of "at most n_1" and for each element $y_j \in SC_{k_2}^{n_2,l_2} := (y_j)_{j \in [0, l_2 - 1]_{\mathbb{Z}}}$, the number of different coordinates of every pair of the consecutive points y_j and $y_{j+1 (mod\, l_2)}$ is constant as the number t_2 instead of "at most t_2".
Then the product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a C_{k^*}-adjacency such that $k^* := k(t_2, n_1 + n_2)$ (see Definition 3.2).

Proof: After comparing with the assertion of Theorem 3.8 of [9], based on Definitions 3.1 and 3.2, we need to only prove the (Case 3-2). Owing to Remark 3.3(1), since the C-compatible $k^* := k(t_2, n_1 + n_2)$-adjacency implies a C_{k^*}-adjacency, this assertion holds. □

By Remark 3.6(2), we obtain the following:

Corollary 3.12. *In Theorem 3.11, in case $k_i = 2n_i, i \in \{1, 2\}$, there is only a $C_{2n_1 + 2n_2}$-adjacency of $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$.*

4. Developments of a G_{k^*}-adjacency relation in a digital product $X_1 \times X_2$ derived from two digital images $(X_i, k_i), i \in \{1,2\}$, and the (G_{k^*}, k')-continuity

This section initially develops a G_{k^*}-adjacency relation in a digital product and establishes the notion of (G_{k^*}, k_i)-continuity of a map from a digital product $(X_1 \times X_2, G_{k^*})$ to $(X_i, k_i), i \in \{1,2\}$. These notions will be strongly used to develop a DT-k-group in Section 5.
In this section, we will use the following notations with several times.
(1) G_{k^*}-adjacency (see Definition 4.1).
(2) MSC_{18}: The minimal simple 18-curve with 6 elements in $\mathbb{Z}^3$ with 26-contractibility (see (4.2)).
(3) $N_{G_{k^*}}(p)$: The set of the elements of G_{k^*}-neighbors of the given point p in a digital space $(X_1 \times X_2, G_{k^*})$ (see (4.3)).
(4) $N_{G_{k^*}}(p,1)$: A G_{k^*}-neighborhood of the given point p in a digital space $(X_1 \times X_2, G_{k^*})$ (see (4.4)).
Given two digital images (X_i, k_i) in $\mathbb{Z}^{n_i}, i \in \{1,2\}$, using the G_{k^*}-adjacency relation in a digital product $X_1 \times X_2$ (see Definition 4.1 below) derived from the given $(X_i, k_i), i \in \{1,2\}$, we obtain the relation set $(X_1 \times X_2, G_{k^*})$. Then, we first establish the notion of (G_{k^*}, k_i)-continuity of a map from $(X_1 \times X_2, G_{k^*})$ to $(X_i, k_i), i \in \{1,2\}$. This approach is a generalization of the (C_{k^*}, k_i)-continuity of a map from $(X_1 \times X_2, C_{k^*})$ to $(X_i, k_i), i \in \{1,2\}$, studied in Section 3. Let us establish the new G_{k^*}-adjacency relation of a digital product that will be strongly used for formulating a DT-k-group in Section 5. Using the condition (3.2), we can define the following:

Definition 4.1. *Assume two digital images $(X_i, k_i := k_i(t_i, n_i)), X_i \subset \mathbb{Z}^{n_i}, i \in \{1,2\}$. For distinct points $p, q \in X_1 \times X_2$, we say that the point $p := (x_1, x_2) \in X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ is related to $q := (x_1', x_2') \in X_1 \times X_2$ if*

$$\begin{cases} \text{(1) in case } x_2 = x_2', x_1 \text{ is } k_1\text{-adjacent to } x_1', \text{ and} \\ \text{(2) in case } x_1 = x_1', x_2 \text{ is } k_2\text{-adjacent to } x_2'. \end{cases} \tag{4.1}$$

After that, considering this relation under the $k^* := k$-adjacency of $\mathbb{Z}^{n_1+n_2}$, where $k := k(t, n_1+n_2), t = max\{t_1, t_2\}$, we say that these two related points p and q are G_{k^*}-adjacent in $X_1 \times X_2$ derived from the given $(X_i, k_i := k_i(t_i, n_i)), X_i \subset \mathbb{Z}^{n_i}, i \in \{1,2\}$. Besides this adjacency is called a G_{k^*}-adjacency of $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ derived from the given two digital images $(X_i, k_i), i \in \{1,2\}$. In addition, we use the notation $(X_1 \times X_2, G_{k^*})$ to denote this digital product $X_1 \times X_2$ with the G_{k^*}-adjacency (or a digital space $(X_1 \times X_2, G_{k^*})$).

Remark 4.2. *(1) In Definition 4.1, in case the given two points p and q are G_{k^*}-adjacent in $X_1 \times X_2$, they should be $k^* := k(t, n_1+n_2)$-adjacent such that they satisfy only the condition of (4.1) of Definition 4.1, i.e., $t = max\{t_1, t_2\}$ and the adjacency k^* is one of the digital connectivity of $\mathbb{Z}^{n_1+n_2}$ stated in (1.1). This implies that a G_{k^*}-adjacency relation may not be equal to a k^*-adjacency one between two points in $X_1 \times X_2$. Namely, the G_{k^*}-adjacency relation in $X_1 \times X_2$ is a new one in $X_1 \times X_2$ that need not be equal to a certain k-adjacency relation in $\mathbb{Z}^{n_1+n_2}$ of (1.1).*
(2) Comparing the G_{k^*}-adjacency relation in $X_1 \times X_2$ and the adjacency relation of a product graph in [11], we obviously make a distinction from each other.

We use the pair $(X_1 \times X_2, G_{k^*})$ to denote this digital product $X_1 \times X_2$ with a G_{k^*}-adjacency. Owing to Definition 4.1, the G_{k^*}-adjacency is always determined by (or derived from) the given k_i-adjacency of $(X_i, k_i := k_i(t_i, n_i)), X_i \subset \mathbb{Z}^{n_i}, i \in \{1,2\}$. After comparing among the adjacency relations of Definitions

3.2 and 4.1, and the typical k-adjacency of $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$, we can make a distinction among them, as follows:

Remark 4.3. *(1) The G_{k^*}-adjacency relation of Definition 4.1 is broader than the C_{k^*}-adjacency of Definition 3.2. More precisely, as stated in Definition 3.2, given two digital images $(X_i, k_i := k(t_i, n_i))$ on $\mathbb{Z}^{n_i}, i \in \{1,2\}$ (see Definition 3.2), not every $X_1 \times X_2$ has a C_{k^*}-adjacency. However, according to Definition 4.1, we always have a G_{k^*}-adjacency relation in the digital product $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$.*
(2) Two k^*-adjacent points in $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ need not be G_{k^*}-adjacent. However, the converse holds. By Definition 4.1, two G_{k^*}-adjacent points in $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ are k^*-adjacent.
(3) We strongly stress on the number $k^* := k(t, n_1 + n_2)$ of a G_{k^*}-adjacency relation. Note that the number t is equal to $max\{t_1, t_2\}$ to determine the number $k^* := k(t, n_1 + n_2)$ for the G_{k^*}-adjacency of $X_1 \times X_2$, where $k_i := k_i(t_i, n_i)), i \in \{1,2\}$. Namely, the number k^* of a G_{k^*}-adjacency absolutely depends on the given $(X_i, k_i := k_i(t_i, n_i)), i \in \{1,2\}$ and the number $t = max\{t_1, t_2\}$. For instance, consider $SC_8^{2,4} \times SC_8^{2,6}$. Then we have only the G_{32}-adjacency relation in the digital product $SC_8^{2,4} \times SC_8^{2,6}$.

As a special case of Definition 4.1, we define the following:

Definition 4.4. *Given a digital image $(X, k := k(t, n)), X \subset \mathbb{Z}^n$, the number $k^* := k(t, 2n)$ for a G_{k^*}-adjacency of $X \times X$ is determined by the number t of $(X, k := k(t, n))$ such that any two G_{k^*}-adjacent points in $X \times X$ should only satisfy the condition (4.1) of Definition 4.1.*

This G_{k^*}-adjacency of $X \times X$ with the condition of $k^* := k(t, n)$ plays a crucial role in establishing a DT-k-group in Section 5 (see Definition 5.5).

Remark 4.5. *(1) In Definition 4.4, the number k^* of G_{k^*} is assumed in $X \times X \subset \mathbb{Z}^{2n}$ that is different from the number k of the k-adjacency of the given digital image $(X, k := k(t, n)), X \subset \mathbb{Z}^n$.*
(2) Given two digital images $(X_i, k_i), i \in \{1,2\}$, according to the situation, i.e., either $(X_1, k_1) \neq (X_2, k_2)$ (see Definition 4.1) or $(X_1, k_1) = (X_2, k_2)$ (see Definition 4.4), we will follow Definitions 4.1 or 4.4 when taking a choice of G_{k^*}.
(3) In view of Definition 4.4, given $(X, k := k(t, n))$, there is at least $k^* := k(t, 2n)$ establishing a G_{k^*}-adjacency of $X \times X$ (see Remark 4.3(1)). For instance, assume $SC_8^{2,6} := (b_j)_{j \in [0,5]_{\mathbb{Z}}}$ in (3.7) (see also Figure 3(a)). Let $c_{i,j} := (b_i, b_j) \in SC_8^{2,6} \times SC_8^{2,6}$. Then consider the point $c_{2,2}$ in $SC_8^{2,6} \times SC_8^{2,6}$. While the point $c_{1,1} := (b_1, b_1)$ is typically 32-adjacent to $c_{2,2}$, it is not G_{32}-adjacent to $c_{2,2}$. However, any G_{32}-adjacent elements in $SC_8^{2,6} \times SC_8^{2,6} \subset \mathbb{Z}^4$ are 32-adjacent.

In the case of $SC_k^{n,l} \times SC_k^{n,l}$, we have some features of a G_{k^*}-adjacency compared to a C_{k^*}-one. To be specific, while $SC_8^{2,6} \times SC_8^{2,6}$ has at least a G_{32}-adjacency, it does not have any C_{k^*}-adjacency, $k^* \in \{32, 64, 80\}$.

Example 4.1. *(1) Given $(\mathbb{Z}, 2)$, $(\mathbb{Z}^2, G_4)$ exists.*
(2) $(SC_k^{n,l} \times [a, b]_{\mathbb{Z}}, G_{k^*})$ exists, where $k^* = k(t, n + 1)$ is determined by the number t of $k := k(t, n)$. For instance, we obtain $(SC_4^{2,8} \times [0, 1]_{\mathbb{Z}})$ with G_6-adjacency (see Figure 1(a)) and $(SC_8^{2,6} \times [0, 1]_{\mathbb{Z}})$ with G_{18}-adjacency (see Figure 1(b)).

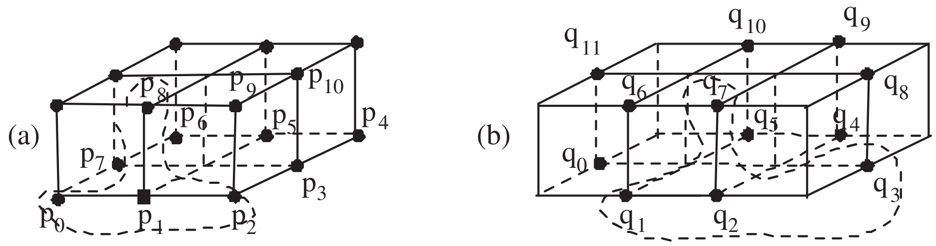

Figure 1. Configuration of the G_6-adjacency of $SC_4^{2,8} \times [0,1]_{\mathbb{Z}}$ and the G_{18}-adjacency $SC_8^{2,6} \times [0,1]_{\mathbb{Z}}$. In (a), each of the points p_0, p_2 and p_8 is G_6-adjacent to the point p_1. In (b), each of the points q_1, q_3 and q_7 is G_{18}-adjacent to the point q_2.

Lemma 4.6. *Given two $SC_{k_i}^{n_i,l_i}, i \in \{1,2\}$, there is always a G_{k^*}-adjacency of the digital product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$, where $k^* := k(t, n_1 + n_2), t = max\{t_1, t_2\}$ and $k_i := k_i(t_i, n_i), i \in \{1,2\}$. However, this G_{k^*}-adjacency need not be equal to a C_{k^*}-adjacency.*

Proof: By Definition 4.1, with the hypothesis, we always have a G_{k^*}-adjacency of $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ (see also Remark 4.3). To be specific, the G_{k^*}-adjacency of $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ is determined by the number $t = max\{t_1, t_2\}$, where $k^* := k(t, n_1 + n_2)$.
However, as mentioned in Remark 4.3, since not every $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ has a C_{k^*}-adjacency, the G_{k^*}-adjacency of $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ need not imply the C_{k^*}-adjacency of it. For instance, as mentioned in Remark 4.5(3), while we have a G_{32}-adjacency for $SC_8^{2,6} \times SC_8^{2,6}$, any k^*-adjacency, $k^* \in \{32, 60, 80\}$, cannot be a C_{k^*}-adjacency of it. □

In view of Definitions 3.2, 4.1, and 4.4, and Remark 4.3, after comparing the G_{k^*}-adjacency with the C_{k^*}-adjacency relation, we observe that a G_{k^*}-adjacency relation is relatively weaker and softer than a C_{k^*}-one. As a generalization of Lemma 4.6, we obtain the following:

Corollary 4.7. *In case there is a C_{k^*}-adjacency of $X_1 \times X_2$ derived from (X_i, k_i) on $\mathbb{Z}^{n_i}, i \in \{1,2\}$, a C_{k^*}-adjacency of $X_1 \times X_2$ implies a G_{k^*}-adjacency of $X_1 \times X_2$ of Definition 4.1. However, in general, a G_{k^*}-adjacency in $X_1 \times X_2$ of Definition 4.1 need not imply a C_{k^*}-adjacency in $X_1 \times X_2$.*

As a special case of $SC_{18}^{3,6}$, let us recall the digital image $MSC_{18} \subset \mathbb{Z}^3$ that is 26-contractible [8,9] (see Figure 2). For instance, we may take the set with an 18-adjacency as follows:

$$MSC_{18} := \left\{ \begin{array}{l} b_0 = (0,0,0), b_1 = (1,-1,0), b_2 = (1,-1,1), \\ b_3 = (2,0,1), b_4 = (1,1,1), b_5 = (1,1,0). \end{array} \right\} \quad (4.2)$$

Then, MSC_{18} is 26-contractible [8,9]. Owing to this feature, the set MSC_{18} has been often called a minimal simple closed 18-curve in $\mathbb{Z}^3$ [8]. In Example 4.2 below, we will take a G_{k^*}-adjacency relation in $MSC_{18} \times MSC_{18}$. Motivated by Theorem 3.11 and Lemma 4.6, and Definitions 3.2 and 4.1, we obtain the following:

Corollary 4.8. *Given two $SC_{k_i}^{n_i,l_i}, i \in \{1,2\}$, assume that there is a C_{k^*}-adjacency of the digital product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ derived from the given $SC_{k_i}^{n_i,l_i}, i \in \{1,2\}$, where $k^* := k(t, n_1 + n_2), t = max\{t_1, t_2\}$ and $k_i := k_i(t_i, n_i), i \in \{1,2\}$. Then the C_{k^*}-adjacency in the digital product $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$ is equivalent to the G_{k^*}-adjacency in $SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}$.*

Example 4.2. *(1)* $SC_8^{2,4} \times SC_8^{2,6} \subset \mathbb{Z}^4$ *has both a* C_{32}*-adjacency and a* G_{32}*-adjacency.*
(2) $MSC_{18} \times MSC_{18} \subset \mathbb{Z}^6$ *does not have any* C_{k^*}*-adjacency such that* $k^* \in \{72, 232, 472, 664, 728\}$ *(see Figure 2 and (1.2)).*
(3) $MSC_{18} \times MSC_{18} \subset \mathbb{Z}^6$ *has a* G_{72}*-adjacency (see Figure 2 and (1.2)).*

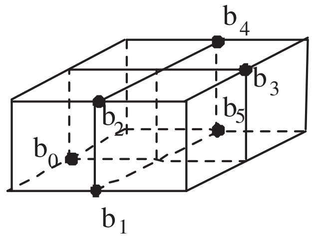

Figure 2. Configuration of MSC_{18} in [8].

Let us further characterize the G_{k^*}-adjacency relation using a certain neighborhood of a point of $X_1 \times X_2$. Based on the G_{k^*}-adjacency of Definition 4.1, we now establish the following G_{k^*}-neighborhood of a given point of $X_1 \times X_2$.

Definition 4.9. *Given two digital images* $(X_i, k_i := k(t_i, n_i)), X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$*, assume the Cartesian product* $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ *with a* G_{k^*}*-adjacency. For a point* $p \in X_1 \times X_2$*, we define*

$$N_{G_{k^*}}(p) := \{q \in X_1 \times X_2 \,|\, q \text{ is } G_{k^*}\text{-adjacent to } p\} \tag{4.3}$$

and

$$N_{G_{k^*}}(p, 1) := N_{G_{k^*}}(p) \cup \{p\}. \tag{4.4}$$

Then we call $N_{G_{k^*}}(p, 1)$ *a* G_{k^*}*-neighborhood of* p*.*

Corollary 4.10. *In view of (4.4), for a digital product with a* G_{k^*}*-adjacency* $(X_1 \times X_2, G_{k^*})$ *and a point* $p := (x_1, x_2) \in X_1 \times X_2$*, we have the following:*

$$N_{G_{k^*}}(p, 1) = (N_{k_1}(x_1, 1) \times \{x_2\}) \cup (\{x_1\} \times N_{k_2}(x_2, 1)). \tag{4.5}$$

Based on Definition 4.4, we have the following:

Example 4.3. *(1) Given a finite digital line* $([0, l]_{\mathbb{Z}}, 2)$*, assume the set* $X := [0, l]_{\mathbb{Z}} \times [0, l]_{\mathbb{Z}} \subset \mathbb{Z}^2$*. Then we can take the* G_{k^*}*-adjacency on* $X \subset \mathbb{Z}^2$ *derived from* $([0, l]_{\mathbb{Z}}, 2)$ *such that* $k^* := k(1, 2)$*, i.e.,* $k^* = 4$*. Besides, each of these* $N_{G_{k^*}}(p, 1) \subset X$ *is equal to the* $N_4(p, 1) \subset X$*.*
(2) In $SC_8^{2,6} \times SC_8^{2,6}$ *(see the elements of* $SC_8^{2,6}$ *in (3.7) and Definition 4.4), we obtain*

$$N_{G_{32}}(c_{2,2}, 1) \neq N_{32}(c_{2,2}, 1),$$

because

$$(N_{G_{32}}(c_{2,2}, 1))^\sharp = 5 \text{ and } (N_{32}(c_{2,2}, 1))^\sharp = 6. \tag{4.6}$$

(3) No $N_{C_{32}}(c_{2,2}, 1)$ *exists because* $SC_8^{2,6} \times SC_8^{2,6}$ *does not have* C_{32}*-adjacency (see the points* $c_{1,1}$ *and* $c_{2,2}$ *of* $SC_8^{2,6} \times SC_8^{2,6}$*).*

Based on Remark 4.3 and the property of (4.4), we obtain the following:

Remark 4.11. *Owing to the structure of (4.3), with the hypothesis stated in Definition 4.9, we obtain the following:*
(1) $N_{G_{k^*}}(p, 1)$ *always exists in* $X_1 \times X_2$*, where the number* t *of* $k := k(t, n_1 + n_2)$ *is equal to the number "*$max\{t_1, t_2\}$*".*
(2) $N_{G_{k^*}}(p)$ *need not be equal to* $N^*_{k^*}(p)$*, where* $N^*_{k^*}(p) := \{q \in X_1 \times X_2 \,|\, q$ *is* k^**-adjacent to* $p\}$*, where the number* k^* *is the digital connectivity of* $X_1 \times X_2$ *stated in (1.1).*
(3) Not every $N_{k^*}(p, 1)$ *is always equal to* $N_{G_{k^*}}(p, 1), p \in X_1 \times X_2$.

Let us characterize $N_{G_{k^*}}(p)$ with some examples.

Example 4.4. *Let us consider the digital images* $X_1 := SC_8^{2,6} := (b_j)_{j\in[0,5]_{\mathbb{Z}}}$ *in (3.7) (see also Figure 3(a)) and* $(X_2 := [0, 1]_{\mathbb{Z}}, 2)$*. Then, for a point* $p_1 := (b_1, 0) \in X_1 \times X_2$ *(see Figure 3(b)), we can consider an* $N_{G_{18}}(p_1, 1)$ *(see Figure 3(c)) in the digital product* $(X_1 \times X_2, G_{18})$ *(see Figure 3(b)). Then, for the point* $p_1 := (b_1, 0) = (1, -1, 0)$*, we obtain the following (see Figure 3(c)):*

$$N_{G_{18}}(p_1, 1) = \{p_0, p_1, p_2, p_6 := (b_1, 1)\},$$

where $p_0 := (b_0, 0), p_2 := (b_2, 0)$ *(see Figure 3(b)). Then, we obviously observe that while the point* $p_7 := (b_2, 1)$ *is* 18*-adjacent to* p_1*, it is not* G_{18}*-adjacent to* p_1 *(compare the objects in Figure 3(c) and (d)).*

Remark 4.12. *Given two digital images* $(X_i, k_i := k(t_i, n_i)), i \in \{1, 2\}$*, assume a* G_{k^*}*-adjacency at a point* $p := (x_1, x_2) \in X_1 \times X_2$*. Then, we have the following identity*

$$(N_{G_{k^*}}(p, 1))^\sharp = (N_{k_1}(x_1, 1))^\sharp + (N_{k_2}(x_2, 1))^\sharp - 1.$$

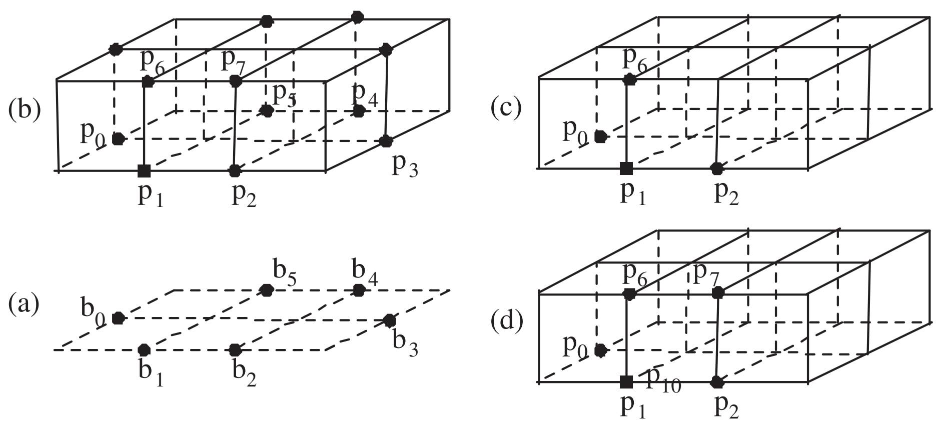

Figure 3. Given two digital images $X_1 := SC_8^{2,6}$ in (a) (see also (3.7)) and $X_2 := [0, 1]_{\mathbb{Z}}$, the digital product $X_1 \times X_2$ with a G_{18}-adjacency is assumed as the object of (b). Besides, for the point $p_1 \in X_1 \times X_2$ in (b), the set $N_{G_{18}}(p_1, 1)$ is described in (c). Based on this approach, we observe $N_{G_{18}}(p_1, 1) \neq N_{18}(p_1, 1)$ because $N_{G_{18}}(p_1, 1) = \{p_0, p_1, p_2, p_6 := (b_1, 1)\}$ and $N_{18}(p_1, 1) = N_{G_{18}}(p_1, 1) \cup \{p_7 := (b_2, 1)\}$ in (d) (see Remark 4.12).

Owing to the symmetric relation of a G_{k^*}-adjacency, we can obtain the following: Given two digital images (X_i, k_i) in $\mathbb{Z}^{n_i}, i \in \{1, 2\}$, assume the Cartesian product $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ with a certain G_{k^*}-adjacency. We say that two points $z, w \in X_1 \times X_2$ are G_{k^*}-connected (or G_{k^*}-path connected) if there is a finite G_{k^*}-path $(z_0, z_1, \cdots, z_m) \subset X_1 \times X_2$ from z to w on $X_1 \times X_2$ such that $z_0 = z$ and $z_m = w$, where we say that a G_{k^*}-path from z to w in $X_1 \times X_2$ means a finite sequence $(z_0, z_1, \cdots, z_m) \subset X_1 \times X_2$ such that z_i is G_{k^*}-adjacent to z_j if $j = i + 1, i \in [0, m-1]_{\mathbb{Z}}$ or $i = j + 1, j \in [0, m-1]_{\mathbb{Z}}$. We say that a digital product $(X_1 \times X_2, G_{k^*})$ is G_{k^*}-connected (or G_{k^*}-path connected) if any two points $z, w \in X_1 \times X_2$ are G_{k^*}-connected (or G_{k^*}-path connected). A singleton with G_{k^*}-adjacency, it is assumed to be G_{k^*}-connected. Given a G_{k^*}-adjacency relation in $X_1 \times X_2$, a *simple G_{k^*}-path* from z to w in $X_1 \times X_2$ is assumed to be the G_{k^*}-path $(z_i)_{i\in[0,l]_{\mathbb{Z}}} \subset X_1 \times X_2$ such that z_i and z_j are G_{k^*}-adjacent if and only if either $j = i + 1, i \in [0, l-1]_{\mathbb{Z}}$ or $i = j + 1, j \in [0, l-1]_{\mathbb{Z}}$ and $z_0 = x$ and $z_l = y$. Also, a simple closed G_{k^*}-curve with l elements in $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$, denoted by $SC^{n,l}_{G_{k^*}}, 4 \le l \in \mathbb{N}$, is a sequence $(z_i)_{i\in[0,l-1]_{\mathbb{Z}}}$ in $X_1 \times X_2$, where z_i and z_j are G_{k^*}-adjacent if and only if $|i - j| = \pm 1 (mod\, l)$.

In view of these notions, we can take the following:

Remark 4.13. *Given an $(X_1 \times X_2, G_{k^*})$ derived from $(X_i, k_i), i \in \{1, 2\}$, we have the following:*
(1) While a G_{k^*}-path implies a k^*-path, the converse does not hold, where the k^*-adjacency is one of the typical adjacency of (1.1).
(2) $SC^{n,l}_{G_{k^*}}$ need not be equal to $SC^{n,l}_{k^*}$. For instance, based on the digital products $(SC^{2,8}_8 \times [0, 1]_{\mathbb{Z}}, G_{18})$ and $(SC^{2,8}_8 \times [0, 1]_{\mathbb{Z}}, 18)$, let us consider two digital images with a G_{18}- and an 18-adjacency such as $SC^{3,8}_{G_{18}}$ and $SC^{3,8}_{18}$, respectively. As shown in Figure 4, assume a digital product $SC^{2,8}_8 \times [0, 1]_{\mathbb{Z}}$ with a G_{18}-adjacency (see Figure 4(a) and Figure 3(a)). It is clear that the set $SC^{3,8}_{G_{18}} := (x_0, x_1, \cdots, x_7)$ in Figure 4(b) is an $SC^{3,8}_{18}$, where $x_0 := p_0, x_1 := p_1, x_2 := p_2, x_3 := p_3, x_4 := p_8, x_5 := p_9, x_6 := p_{10}, x_7 := p_{11}$. However, $SC^{3,8}_{18}$ in (c) is not an $SC^{3,8}_{G_{18}}$ (see the objects in Figure 4 (a) and (c)).

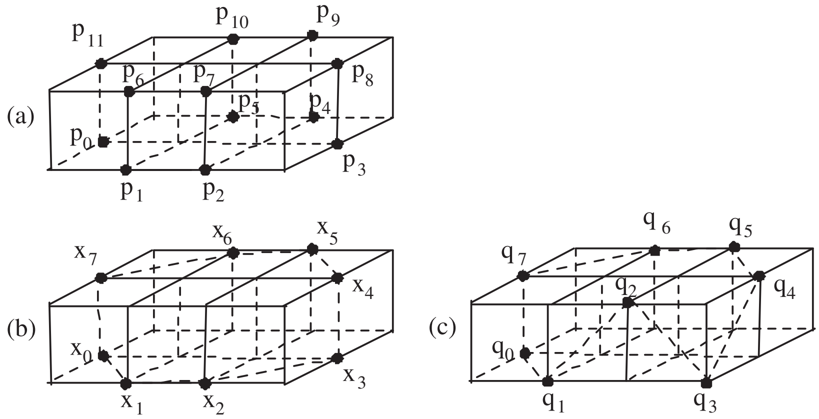

Figure 4. Based on the digital product $SC^{2,8}_8 \times [0, 1]_{\mathbb{Z}}$ with a G_{18}-adjacency (see (a)) or an 18-adjacency (see (a)), comparison between $SC^{3,8}_{G_{18}} := (x_0, x_1, \cdots, x_7)$ in Figure 4(b) and an $SC^{3,8}_{18}$ in (c), where $x_0 := p_0, x_1 := p_1, x_2 := p_2, x_3 := p_3, x_4 := p_8, x_5 := p_9, x_6 := p_{10}, x_7 := p_{11}$ in (b) and $SC^{3,8}_{18} := (q_0, q_1, \cdots, q_7)$ in (c). To be specific, while $SC^{3,8}_{18}$ in (b) is an $SC^{3,8}_{G_{18}}$, $SC^{3,8}_{18}$ in (c) is not an $SC^{3,8}_{G_{18}}$ because the points q_1 and q_2 in (c) are not G_{18}-adjacent.

Given a G_{k^*}-adjacency relation in a Cartesian product, we also have a certain digital space [7] associated with a G_{k^*}-adjacency relation, as follows:

Proposition 4.14. *Given k_i-connected digital images $(X_i, k_i), X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$, the relation set $(X_1 \times X_2, G_{k^*})$ is a digital space.*

Proof: Since the relation G_{k^*}-adjacency in $X_1 \times X_2$ is symmetric, we examine if $(X_1 \times X_2, G_{k^*})$ is G_{k^*}-connected. Take any distinct points $p := (x_1, x_2)$ and $q := (x_1', x_2')$ in $X_1 \times X_2$. Then, without loss of generality, we may assume the case $x_1 \lneq x_1'$ and $x_2 \leq x_2'$ or the case $x_1 \leq x_1'$ and $x_2 \lneq x_2'$. For the purpose of this study, we may take the first case, i.e., $x_1 \lneq x_1'$ and $x_2 \leq x_2'$. Then consider the differences $|\, x_1 - x_1'| \gneq 0$ and $|\, x_2 - x_2'| \geq 0$. According to the size of these finite differences, we can take a finite set

$$\{p := p_1, p_2, p_3, \cdots, p_n := q\} \subset X_1 \times X_2 \tag{4.7}$$

such that p_i is G_{k^*}-adjacent to p_{i+1} in $X_1 \times X_2, i \in [1, n-1]_{\mathbb{Z}}$ and

$$p, q \in \bigcup_{i \in [1,n]_{\mathbb{Z}}} N_{G_{k^*}}(p_i, 1) \subset X_1 \times X_2 \qquad (4.8).$$

Owing to (4.7) and (4.8), we can conclude that $(X_1 \times X_2, G_{k^*})$ is G_{k^*}-connected. □

Based on the relation set $(X_1 \times X_2, G_{k^*})$, we obtain the following:

Lemma 4.15. *Assume the digital space $(X_1 \times X_2, G_{k^*})$ derived from two digital images $(X_i, k_i), X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$. Then, for a point $p \in X_1 \times X_2$, we always obtain $N_{G_{k^*}}(p, 1) \subset N_{k^*}(p, 1)$. However, $N_{G_{k^*}}(p, 1)$ need not be equal to $N_{k^*}(p, 1)$, i.e., $(N_{G_{k^*}}(p, 1))^\sharp \leq (N_{k^*}(p, 1))^\sharp$.*

Proof: For any point $q \in N_{G_{k^*}}(p, 1)$, according to the property (2.4) and Remark 4.3 and 4.11, we obtain $q \in N_{k^*}(p, 1)$. However, in view of Example 4.4 as a counterexample, we can disprove $N_{k^*}(p, 1) \subset N_{G_{k^*}}(p, 1)$. Naively, consider the digital product $(SC_8^{2,8} \times X_2, G_{18})$ in Example 4.4. Then we can confirm $N_{G_{18}}(p, 1) \subsetneq N_{18}(p, 1)$ (see Figure 3(c) and (d)). □

Based on Definition 4.1, we obtain the following:

Proposition 4.16. *Given two digital images (X_i, k_i), $X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$, assume a Cartesian product $X_1 \times X_2$ with a C_{k^*}-adjacency. Then, for a point $p \in X_1 \times X_2$, while $q \in N_{C_{k^*}}(p, 1)$ implies $q \in N_{G_{k^*}}(p, 1)$, the converse does not hold.*

Proof: With the hypothesis, it is clear that $q \in N_{C_{k^*}}(p, 1)$ implies $q \in N_{G_{k^*}}(p, 1)$. However, the converse does not hold with the following counterexample. Consider the digital product $SC_4^{2,4} \times SC_8^{2,6} := (c_{i,j})$, where $c_{i,j} := (a_i, b_j)$ and $SC_4^{2,4} := (a_i)_{i \in [0,3]_{\mathbb{Z}}}$ and $SC_8^{2,6} := (b_j)_{j \in [0,5]_{\mathbb{Z}}}$ in (3.7). Then, for the point $c_{2,2}$, we obviously have $N_{G_{32}}(c_{2,2}, 1)$ (see Definition 4.1) that consists of five elements. However, no $N_{C_{32}}(c_{2,2}, 1)$ exists because the element $c_{2,2}$ does not have any C_{32}-adjacent to a point in the product $SC_4^{2,4} \times SC_8^{2,6} \subset \mathbb{Z}^4$. □

Corollary 4.17. *Based on Definition 4.1, assume two digital images (X_i, k_i), $X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$, and the Cartesian product $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$. Not every point $p \in X_1 \times X_2$ always has an $N_{C_{k^*}}(p, 1)$. However, in case there is a C_{k^*}-adjacency on $X_1 \times X_2$, we obtain*

$$N_{k^*}(p, 1) = N_{C_{k^*}}(p, 1) = N_{G_{k^*}}(p, 1).$$

Namely, each of the C_{k^*}- and the G_{k^*}-adjacency is equal to the typical k^*-adjacency of (1.1).

Proof: By Definitions 3.5 and 4.1 and the properties (3.5) and (4.4), the proof is completed. □

Based on the G_{k^*}-adjacency of a digital product, let us introduce the concept of (G_{k^*}, k')-continuity of a map $f : (X_1 \times X_2, G_{k^*}) \to (Y, k')$.

Definition 4.18. *Given two digital images (X_i, k_i), $X_i \subset \mathbb{Z}^{n_i}, i \in \{1, 2\}$, consider the digital space $(X_1 \times X_2, G_{k^*})$ and a digital image (Y, k'). A function $f : (X_1 \times X_2, G_{k^*}) \to (Y, k')$ is (G_{k^*}, k')-continuous at a point $p := (x_1, x_2)$ if for any point $q \in X_1 \times X_2$ such that $q \in N_{G_{k^*}}(p)$ (denoted by $p \leftrightarrow_{G_{k^*}} q$), we obtain $f(q) \in N_{k'}(f(p), 1)$ (denoted by $f(p) \Leftrightarrow_{k'} f(q)$). In case the map f is (G_{k^*}, k')-continuous at each point $p \in X_1 \times X_2$, we say that the map f is (G_{k^*}, k')-continuous.*

The (G_{k^*}, k')-continuity of Definition 4.18 can be represented by using both a G_{k^*}-neighborhood and a digital k'-neighborhood, as follows:

Proposition 4.19. *Consider a Cartesian product $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$ with a G_{k^*}-adjacency and a typical digital image (Y, k'). A map $f : (X_1 \times X_2, G_{k^*}) \to (Y, k')$ is (G_{k^*}, k)-continuous at the point $p \in X_1 \times X_2$ if and only if*

$$f(N_{G_{k^*}}(p, 1)) \subset N_{k'}(f(p), 1). \tag{4.9}$$

A map $f : (X_1 \times X_2, G_{k^*}) \to (Y, k')$ is (G_{k^*}, k)-continuous if and only if for every point $p \in X_1 \times X_2$ we have

$$f(N_{G_{k^*}}(p, 1)) \subset N_{k'}(f(p), 1).$$

As a special case of Proposition 4.19, based on Definition 4.4, we can consider the following:

Corollary 4.20. *Given a digital image $(X, k), X \subset \mathbb{Z}^n$. Consider a Cartesian product $X \times X \subset \mathbb{Z}^{n_1+n_2}$ with a G_{k^*}-adjacency. Consider a map $f : (X \times X, G_{k^*}) \to (X, k)$. For a point $p := (x_1, x_2) \in X \times X$, the map f is (G_{k^*}, k)-continuous at the point p if and only if*

$$f(N_{G_{k^*}}(p, 1)) \subset N_k(f(p), 1). \tag{4.10}$$

A map $f : (X \times X, G_{k^*}) \to (X, k)$ is (G_{k^*}, k)-continuous at every point $p \in X \times X$, then the map f is (G_{k^*}, k)-continuous.

Example 4.5. *(1) Assume a digital product $(SC_8^{2,6} \times [0, 1]_{\mathbb{Z}}) := \{p_i \mid i \in [0, 11]_{\mathbb{Z}}\}$ with a G_{18}-adjacency. Consider the map*

$$g : (X, G_{18}) \to (\mathbb{Z}, 2)$$

defined by (see Figure 5(1))

$$\begin{Bmatrix} g(\{p_0, p_{11}\}) = \{0\}, g(\{p_1, p_6\}) = \{1\}, g(\{p_2, p_7\}) = \{2\}, \\ g(\{p_3, p_8\}) = \{3\}, g(\{p_4, p_9\}) = \{2\}, g(\{p_5, p_{10}\}) = \{1\}. \end{Bmatrix}$$

Then the map g is a $(G_{18}, 2)$-continuous map.
(2) Assume a digital product $(SC_8^{2,6} \times [0, 1]_{\mathbb{Z}})$ with a G_{18}-adjacency and a subset $X := \{x_i \mid i \in [0, 7]_{\mathbb{Z}}\}$ (see Figure 5(2)(a)) and $Y := \{y_i \mid i \in [0, 7]_{\mathbb{Z}}\}$ (see Figure 5(2)(b)) with an 18-adjacency which is different from (X, G_{18}). Consider the map $f : (X, G_{18}) \to (Y, 18)$ defined by $f(x_i) = y_i, i \in [0, 7]_{\mathbb{Z}}$. Then the map f is a $(G_{18}, 18)$-continuous map because $f(N_{G_{18}}(x_i, 1)) \subset N_{18}(f(x_i), 1)$.

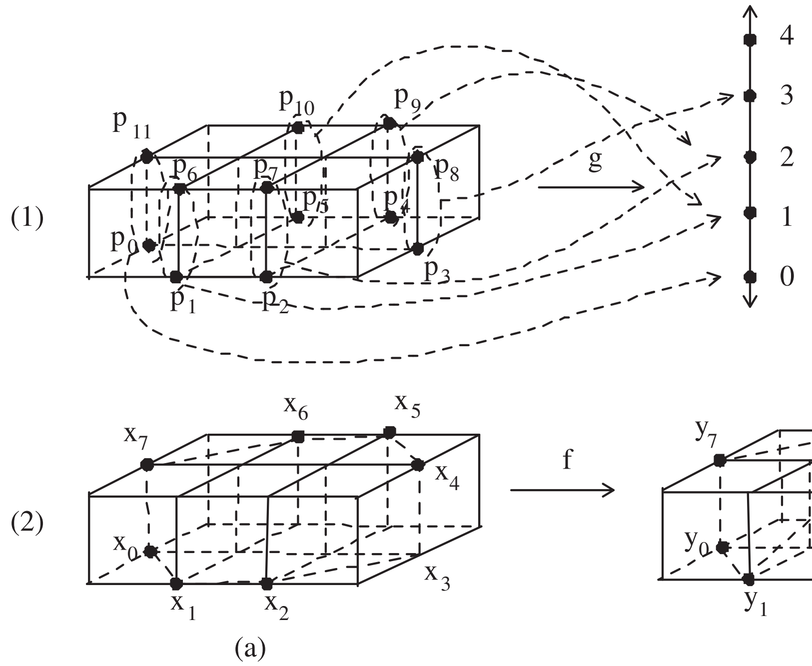

Figure 5. (1) Configuration of the $(G_{18}, 2)$-continuity of the given map g of (1). (2) Configuration of the $(G_{18}, 18)$-continuity of the given map f from $X := \{x_i \,|\, i \in [0, 7]_{\mathbb{Z}}\}$ in (a) to the set $Y := \{y_i \,|\, i \in [0, 7]_{\mathbb{Z}}\}$ in (b) defined by $f(x_i) = y_i, i \in [0, 7]_{\mathbb{Z}}$ (see Example 4.5).

Corollary 4.21. *Let $(X, 2n)$ be a $2n$-connected subset of $(\mathbb{Z}^n, 2n)$. Then each of the typical projection maps $P_i : (X \times X, G_{4n}) \to (X, 2n)$ is a $(G_{4n}, 2n)$-continuous map, $i \in \{1, 2\}$, such that the G_{4n}-adjacency is equal to the typical $4n$-adjacency.*

With some hypothesis of the G_{k^*}-adjacency of $X \times X$, the (G_{k^*}, k)-continuity of Corollary 4.20 will play a crucial role in establishing a certain continuity of a multiplication for formulating a DT-k-group (see Definition 5.5). Let us compare the (G_{k^*}, k')-continuity and the typical (k, k')-continuity.

Theorem 4.22. *While the (G_{k^*}, k')-continuity implies the typical (k^*, k')-continuity, the converse does not hold.*

Proof: By Definition 4.18 and Lemma 4.15, the proof is completed. □

Corollary 4.23. *While the (C_{k^*}, k')-continuity implies the (G_{k^*}, k')-continuity, the converse does not hold.*

Proof: By Propositions 3.9, 4.16, and 4.19, and Corollary 4.20 the proof is completed. □

Corollary 4.24. *In case there is a C_{k^*}-adjacency of $X_1 \times X_2$, the (C_{k^*}, k')-, the (G_{k^*}, k')-, and the (k^*, k')-continuity are equivalent to the other.*

Proof: By Corollary 4.17, the proof is completed. □

In view of Definitions 3.2 and 4.1, we obtain the following:

Remark 4.25 (Advantages of the G_{k^*}-adjacency of a digital product)**.** *Given two digital images (X, k_1) and (Y, k_2), there is always a G_{k^*}-adjacency derived from the two given digital images. However, an* *existence of C_{k^*}-adjacency of a digital product $X \times Y$ depends on the situation. Thus the G_{k^*}-adjacency of a digital product will be used in establishing a digital topological version of a typical topological group in Section 5. Furthermore, since the G_{k^*}-adjacency is a generalization of the C_{k^*}-adjacency of a digital product, some strong utilities of the G_{k^*}-adjacency can be considered in establishing DT-k-group structures (see Definition 5.5).*

5. A development of a DT-k-group with the most suitable adjacency for a digital product $X \times X$ from (X, k)

This section introduces the notion of a DT-k-group derived from a digital image (X, k) with a certain group structure $(X, *)$. Before proceeding with this work, given a digital image (X, k), we now recall some differences between the C_{k^*}-adjacency and the G_{k^*}-adjacency of a digital product $X \times X$ mentioned in Remarks 4.3 and 4.3. Naively, a G_{k^*}-adjacency of a digital product $X \times X$ is a generalization of a C_{k^*}-adjacency of it (see Remark 4.25). As mentioned in Remark 4.5(3), given a digital image $(X, k := k(t, n)), X \subset \mathbb{Z}^n$, we always have at least a certain G_{k^*}-adjacency of a digital product $X \times X$, where $k^* := k(t, 2n)$ is determined by the number t of $(X, k := k(t, n))$. Hence, in relation to the establishment of a DT-k-group, we will follow only this G_{k^*}-adjacency of a digital product $X \times X$ unless stated otherwise.

In this section, we will use the following notations with several times.

(1) $(SC_k^{n,l}, *)$: A digitally k-group with the given binary operation $*$ on $SC_k^{n,l}$ (see Proposition 5.3).

(2) $(\mathbb{Z}^n, 2n, +)$: A digitally $2n$-group with the given binary operation $+$ on $\mathbb{Z}^n$ (see Theorem 5.8)).

(3) $\mathbb{N}_1$: The set of odd natural numbers (see Example 5.1).

Remark 5.1. *When studying DT-k-groups $(X, k, *)$, for a digital image $(X, k := k(t, n)), X \subset \mathbb{Z}^n$, we will recall the following notions from Definitions 3.8 and 4.18 that will be essentially used in this section, e.g., the G_{k^*}- and C_{k^*}-adjacency of a digital product $X \times X \subset \mathbb{Z}^{2n}$ and the related continuities.*

(1) We will take only the G_{k^*}-adjacency of a digital product $X \times X$ such that $k^* := k(t, 2n)$ is determined by the number t of $(X, k := k(t, n))$, so that this G_{k^*}-adjacency always exists. Furthermore, for each point $p \in X \times X$, $N_{G_{k^*}}(p, 1)$ is uniquely determined (see Definitions 3.5 and 4.9 and the properties of (3.1) and (4.5), Example 4.2(3), and Remark 4.11(1)). Hence this G_{k^*}-adjacency, $k^* := k(t, 2n)$, is enough to establish the notion of a DT-k-group derived from a digital image (X, k) with a certain group structure $(X, *)$.

(2) In relation to a digital space $(X \times X, C_{k^*})$ derived from $(X, k := k(t, n)), X \subset \mathbb{Z}^n$ (see Proposition 3.7), we also take only $k^* := k(t, 2n)$, where the number t of $k^* := k(t, 2n)$ is exactly equal to the number t of $(X, k := k(t, n))$.

(3) Based on this approach, we will take a G_{k^*} (resp. C_{k^*})-neighborhood of a given point in a digital space $(X \times X, G_{k^*})$ (resp. $(X \times X, C_{k^*})$). Hence, the (G_{k^*}, k) (resp., (C_{k^*}, k))-continuity of Definition 4.18 (resp. Definition 3.8) is considered (see also Definition 4.4) for formulating a DT-k-group using only the $k^* := k(t, 2n)$-adjacency, where $k^* := k(t, 2n)$ is induced by the number t of $(X, k := k(t, n))$.

Lemma 5.2. *The set $\mathbb{Z}^{2n}, n \in \mathbb{N}$, has a G_{4n}-adjacency derived from $(\mathbb{Z}^n, 2n)$ such that this G_{4n}-adjacency is equal to the C_{4n}-one derived from $(\mathbb{Z}^n, 2n)$, i.e., $G_{4n} = 4n = C_{4n}$.*

Proof: By Definitions 3.2 and 4.1, and Corollary 4.17, the proof is completed. To be specific, take a point $p := (p_1, p_2) \in \mathbb{Z}^n \times \mathbb{Z}^n = \mathbb{Z}^{2n}$. Since

$$N_{G_{4n}}(p, 1) = (N_{2n}(p_1, 1) \times \{p_2\}) \cup (\{p_1\} \times N_{2n}(p_2, 1))$$

and this $N_{G_{4n}}(p, 1)$ is equal to $N_{C_{4n}}(p, 1)$, the proof is completed. □

Let us establish a group structure on the digital image $SC_k^{n,l}$.

Proposition 5.3. *Given an $SC_k^{n,l} := (x_i)_{i\in[0,l-1]_\mathbb{Z}}$ for any k-adjacency of $\mathbb{Z}^n$, we have a group structure on $SC_k^{n,l}$ with the following operation $*$.*

$$* : SC_k^{n,l} \times SC_k^{n,l} \to SC_k^{n,l}$$

given by

$$*(x_i, x_j) = x_i * x_j = x_{i+j(mod\, l)}. \tag{5.1}$$

Then we denote by $(SC_k^{n,l}, *)$ the above group.

Proof: First, the operation "$*$" is well-defined on $SC_k^{n,l}$ as a binary operation for establishing a group structure on $SC_k^{n,l}$. Second, based on the property (5.1), the operation "$*$" is associative. Third, the element x_0 is the identity element and for two elements $x_i, x_j \in SC_k^{n,l}$

$$x_i * x_j = x_0 \text{ if and only if } j = l - i(mod\, l). \tag{5.2}$$

Hence, each element $x_i(\neq x_0)$ uniquely has x_{l-i} as the inverse element and the element x_0 has the inverse itself. □

Example 5.1. *(1) Given $SC_k^{n,l} := (x_i)_{i\in[0,l-1]_\mathbb{Z}}$ with $l \in \mathbb{N}_0$, there are only two elements such as x_0 and $x_{\frac{l}{2}}$ in $SC_k^{n,l}$ such that $(x_0)^{-1} = x_0$ and $(x_{\frac{l}{2}})^{-1} = x_{\frac{l}{2}}$, where x^{-1} means the inverse element of x (see the two elements x_0, x_3 of $SC_8^{2,6}$).*
(2) Given $SC_k^{n,l} := (x_i)_{i\in[0,l-1]_\mathbb{Z}}$ with $l \in \mathbb{N}_1$, there is only one element such as x_0 in $SC_k^{n,l}$ whose inverse is itself (see the element x_0 of $SC_{26}^{3,5}$).

Remark 5.4. *(1) Given an $SC_k^{n,l} := (x_i)_{i\in[0,l-1]_\mathbb{Z}}$, according to our needs, we can relabel the elements of $SC_k^{n,l}$ to obtain a new type of $SC_k^{n,l} := (y_i)_{i\in[0,l-1]_\mathbb{Z}}$. Then the element y_0 is the identity element of the group $(SC_k^{n,l} := (y_i)_{i\in[0,l-1]_\mathbb{Z}}, *)$.*
(2) The group $(SC_k^{n,l}, *)$ in Proposition 5.3 is abelian.

Based on the G_{k^*}-adjacency of $X \times X$ and the (G_{k^*}, k)- as well as the (C_{k^*}, k)-continuity stated in Remark 5.1(1) and (3), we now define the following.

Definition 5.5. *A digitally topological k-group, denoted by $(X, k, *)$ and called a DT-k-group for brevity, is a digital image $(X, k := k(t, n))$ combined with a group structure on $X \subset \mathbb{Z}^n$ using a certain binary operation $*$ such that for $(x, y) \in X^2$ the multiplication*

$$\alpha : (X^2, G_{k^*}) \to (X, k) \text{ given by } \alpha(x, y) = x * y \text{ is } (G_{k^*}, k)\text{-continuous} \tag{5.3}$$

and the inverse map

$$\beta : (X, k) \to (X, k) \text{ given by } \beta(x) = x^{-1} \text{ is } k\text{-continuous}, \tag{5.4}$$

where the number $k^* := k(t, 2n)$ of the G_{k^*}-adjacency of (5.3) is determined by only the number t of the $k := k(t, n)$-adjacency of the given digital image $(X, k := k(t, n))$.

In Definition 5.5, as for the G_{k^*}-adjacency of $X \times X$, we strongly recall the requirement in Remark 5.1(1) and (3).

Remark 5.6. *In view of Definition 5.5, a DT-k-group,* $(X, k, *)$ *has the two structures such as the digital image* (X, k) *and the certain group structure* $(X, *)$ *satisfying the properties of (5.3) and (5.4).*

By Corollary 4.17, we have the following:

Corollary 5.7. *In case there is a* C_{k^*}*-adjacency of* $X\times X$*, i.e., a digital space* $\exists\,(X\times X, C_{k^*})$*, the condition* "$(G_{k^*}, k)$*-continuous" of (5.3) of Definition 9 can be replaced by* "(C_{k^*}, k)*-continuous" because a* C_{k^*}*-adjacency of* $X \times X$ *implies a* G_{k^*}*-adjacency of it (see Corollaries 4.6 and 4.25). For instance, for the case* $SC_8^{2,4}$*,* $SC_8^{2,4} \times SC_8^{2,4}$ *can be assumed to be a digital space* $(SC_8^{2,4} \times SC_8^{2,4}, C_{32})$*. Thus we have a digital space* $(SC_8^{2,4} \times SC_8^{2,4}, G_{32})$ *(see Remark 4.3(1)). Hence the condition* "$(G_{32}, 8)$*-continuous" of (5.3) of Definition 5.5 may be replaced by* "$(C_{32}, 8)$*-continuous" or* "$(32, 8)$*-continuous" because* $G_{32} = 32 = C_{32}$.

Theorem 5.8. $(\mathbb{Z}^n, 2n, +)$ *is a DT-2n-group.*

Proof: First, $(\mathbb{Z}^n, +)$ is a group with the following operation [15]. For two elements $p := (p_1, \cdots, p_n), q := (q_1, \cdots, q_n) \in \mathbb{Z}^n$, we define

$$\left\{\begin{aligned} &+ : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n \text{ given by} \\ &p + q := (p_1 + q_1, \cdots, p_n + q_n). \end{aligned}\right\} \tag{5.5}$$

Then, the operation "+" is a binary operation on $\mathbb{Z}^n$ supporting the group $(\mathbb{Z}^n, +)$ because it is associative, and it has the identity element $0_n := (0, \cdots, 0)$ with n-tuples and the inverse element of an element p, denoted by p^{-1}, is equal to $-p$ [15].
Using Example 3.2, Definitions 3.8 and 4.18, and Propositions 3.9 and 4.19 and Corollary 4.20, let us propose the DT-$2n$-group structure of $(\mathbb{Z}^n, 2n, +)$. To be specific, by Lemma 5.2, we have both a G_{4n}- and a C_{4n}-adjacency of $\mathbb{Z}^n \times \mathbb{Z}^n$ induced by the given $(\mathbb{Z}^n, 2n)$ such that $G_{2n} = C_{2n} = 2n$. Hence, by Corollary 4.17, we may take $G_{4n} = 4n$ to support the $(G_{4n}, 2n)$-continuity of the multiplication (see (3.13) and (4.9))

$$\alpha : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n \tag{5.6}$$

given by $\alpha(p, q) := p + q$ defined in (5.5).
To be precise, take any distinct points $P := (p, q),\ Q := (p', q')$ in $\mathbb{Z}^n \times \mathbb{Z}^n$ such that

$$Q \in N_{G_{4n}}(P, 1) = N_{4n}(P, 1) \text{ (see Lemma 5.2).}$$

More precisely, assume the two points P, Q such that

$$\left\{\begin{aligned} &Q = (p', q') \in N_{G_{4n}}(P, 1) = N_{4n}(P, 1) \subset \mathbb{Z}^{2n}, \\ &\text{where } N_{G_{4n}}(P, 1) = (N_{2n}(p, 1) \times \{q\}) \cup (\{p\} \times N_{2n}(q, 1)). \end{aligned}\right\} \tag{5.7}$$

Then we may consider the two cases: The following four points that are components of the given two points $P, Q \in \mathbb{Z}^n$,

$$p := (x_i)_{i\in[1,n]_\mathbb{Z}}, q := (y_j)_{j\in[1,n]_\mathbb{Z}}, p' := (x'_i)_{i\in[1,n]_\mathbb{Z}}, \text{ and } q' := (y'_j)_{j\in[1,n]_\mathbb{Z}},$$

satisfy one of the following two cases.
(1) For the points p and p' in $\mathbb{Z}^n$, owing to (5.7), there is only one $i_0 \in [1,n]_{\mathbb{Z}}$ such that

$$\begin{Bmatrix} x_{i_0} \neq x'_{i_0} \text{ with } |x_{i_0} - x'_{i_0}| = 1, \\ \text{for } i \in [1,n]_{\mathbb{Z}} \setminus \{i_0\},\ x_i = x'_i, \text{ and} \\ y_j = y'_j \text{ for any } j \in [1,n]_{\mathbb{Z}}. \end{Bmatrix} \tag{5.8}$$

(2) For the points q and q' in $\mathbb{Z}^n$, there is only one $j_0 \in [1,n]_{\mathbb{Z}}$ such that

$$\begin{Bmatrix} y_{j_0} \neq y'_{j_0} \text{ with } |y_{j_0} - y'_{j_0}| = 1, \\ \text{for } j \in [1,n]_{\mathbb{Z}} \setminus \{j_0\},\ y_j = y'_j, \text{ and} \\ x_i = x'_i \text{ for any } i \in [1,n]_{\mathbb{Z}}. \end{Bmatrix} \tag{5.9}$$

Let us investigate these two cases more precisely.
(Case 1) Based on the above case (1), consider the mapping of the two points P and Q by the above map α, i.e.,

$$\begin{Bmatrix} \alpha(P) = \alpha(p,q) := p + q = (x_i + y_i)_{i \in [1,n]_{\mathbb{Z}}} \text{ and} \\ \alpha(Q) = \alpha(p',q') := p' + q' = (x'_i + y'_i)_{i \in [1,n]_{\mathbb{Z}}}. \end{Bmatrix} \tag{5.10}$$

Owing to the properties of (5.8) and (5.9), the property (5.10) implies that

$$\alpha(Q) \in N_{2n}(\alpha(P), 1) \text{ because } |\alpha(P) - \alpha(Q)| = 1,$$

implying that the map α is $(G_{4n}, 2n)$-continuous at the point $P \in \mathbb{Z}^n \times \mathbb{Z}^n$ (see Definition 4.18 and Corollary 4.20).
(Case 2) With the above case (2), after considering the mapping of the two points P and Q by the above map α in (5.10), using a method similar to the approach of (Case 1), we obtain

$$|\alpha(P) - \alpha(Q)| = 1 \text{ so that } \alpha(Q) \in N_{2n}(\alpha(P), 1),$$

implying that the map α is $(G_{4n}, 2n)$-continuous at the point $P \in \mathbb{Z}^n \times \mathbb{Z}^n$ (see Definitionv4.18 and Corollary 4.20).

For instance, let us show the DT-4-group structure of $(\mathbb{Z}^2, 4, +)$, as follows: Assume the two points

$$P := (p,q) \text{ and } Q := (p',q') \text{ in } \mathbb{Z}^2 \times \mathbb{Z}^2,$$

such that

$$Q \in N_{G_8}(P,1) = N_8(P,1) \subset \mathbb{Z}^4,$$

where $p := (x_i)_{i \in [1,2]_{\mathbb{Z}}}$, $q := (y_j)_{j \in [1,2]_{\mathbb{Z}}}$, $p' := (x'_i)_{i \in [1,2]_{\mathbb{Z}}}$, and $q' := (y'_j)_{j \in [1,2]_{\mathbb{Z}}}$. Then, we consider the following two cases as mentioned above.

As for the (Case 1) above, in case there is only one $i_0 \in [1,2]_{\mathbb{Z}}$ such that $x_{i_0} \neq x'_{i_0}$ with $|x_{i_0} - x'_{i_0}| = 1$ and for $i \in [1,2]_{\mathbb{Z}} \setminus \{i_0\}$, we have $x_i = x'_i$, and $y_j = y'_j$, $j \in \{1,2\}$ (see (5.8) and (5.9)). Then consider the mapping of the two points P and Q by the above map α such that

$$\begin{Bmatrix} \alpha(P) = \alpha(p, q) := p + q = (x_i + y_i)_{i\in[1,2]_{\mathbb{Z}}} \text{ and} \\ \alpha(Q) = \alpha(p', q') := p' + q' = (x'_i + y'_i)_{i\in[1,2]_{\mathbb{Z}}}. \end{Bmatrix}$$

Then we obtain

$$|\alpha(P) - \alpha(Q)| = 1 \text{ so that we have } \alpha(Q) \in N_4(\alpha(P), 1).$$

Hence the map α is $(G_8, 4)$-continuous at the point $P \in \mathbb{Z}^2 \times \mathbb{Z}^2$ (see Corollary 4.20).

As for the (Case 2) above, in case there is only one $j_0 \in [1, 2]_{\mathbb{Z}}$ such that $y_{j_0} \neq y'_{j_0}$ with $|y_{j_0} - y'_{j_0}| = 1$ and for $j \in [1, 2]_{\mathbb{Z}} \setminus \{j_0\}$, we have $y_j = y'_j$, and $x_i = x'_i, i \in \{1, 2\}$ (see (5.8) and (5.9)). Then, after considering the mapping of the two points P and Q by the above map α using a method similar to the approach above, we obtain

$$|\alpha(P) - \alpha(Q)| = 1 \text{ so that we obtain } \alpha(Q) \in N_4(\alpha(P), 1) \subset \mathbb{Z}^2,$$

implying that the map α is $(G_8, 4)$-continuous at the point $P \in \mathbb{Z}^2 \times \mathbb{Z}^2$ (see Corollary 4.20).
By Lemma 5.2, this $(G_8, 4)$-continuity of α is exactly equal to $(8, 4)$-continuity of it.
Besides, there is also the $2n$-continuity of the inverse map

$$\beta : \mathbb{Z}^n \to \mathbb{Z}^n$$

given by

$$\beta(p) = -p.$$

Naively, for any point $p \in \mathbb{Z}^n$, by Proposition 2.1, we obtain

$$\beta(N_{2n}(p, 1)) \subset N_{2n}(\beta(p), 1),$$

implying that $(\mathbb{Z}^n, 2n, +)$ is a DT-$2n$-group. □

Regarding the continuity of (5.6), note that the points (x, x) and (y, y) are not G_{2n}-adjacent in $\mathbb{Z}^n \times \mathbb{Z}^n$, where $x := (0, 0, \cdots, 0)$ and $y := (1, 0, \cdots, 0)$ in $\mathbb{Z}^n$ (see Definition 4.1).

Remark 5.9. *In Theorem 5.8, by Lemma 5.2, the $(G_{4n}, 2n)$-continuity of the map α of (5.6) is exactly equal to $(4n, 2n)$-continuity of α (see the (Case 1) and (Case 2) in the proof of Theorem 5.8). For instance, consider a multiplication from $(\mathbb{Z}^2, G_4) \to (\mathbb{Z}, 2)$ (see Figure 6). Then it is clear that it is $(G_4, 2)$-continuous (see Figure 6), which supports a DT-2-group of $(\mathbb{Z}, 2, +)$. To be specific, for convenience, for each $i \in \mathbb{Z}$, let $X_i := \{(x, y) \,|\, y = -x + i, x, y \in \mathbb{Z}\} \subset \mathbb{Z}^2$. Then the G_4-adjacency is equal to the C_4-adjacency so we obtain each of G_4- and C_4-adjacency is equal to the 4-adjacency of $\mathbb{Z}^2$. For instance, consider the multiplication $\alpha : (\mathbb{Z}^2, G_4) \to (\mathbb{Z}, 2)$ is defined as $\alpha(X_i) = i$. Then the map α is clearly $(G_4, 2)$-continuous.*

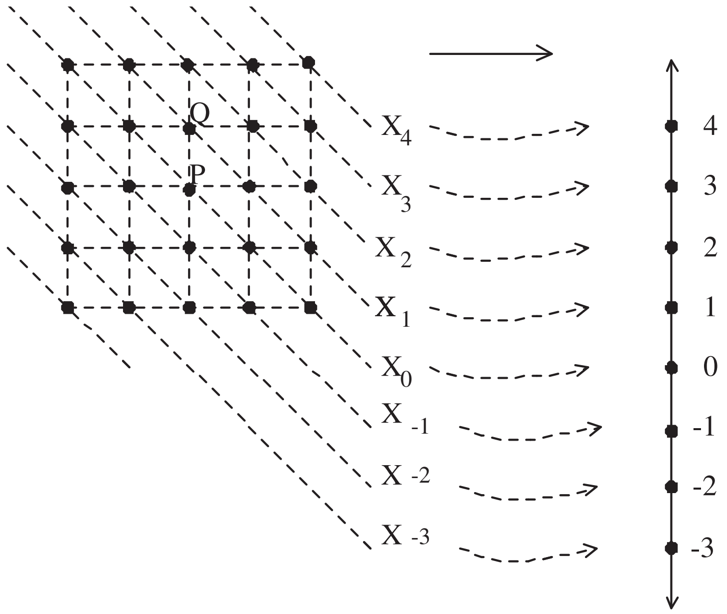

Figure 6. Configuration of the $(G_4, 2)$-continuity of the multiplication from $(\mathbb{Z}^2, G_4) \to (\mathbb{Z}, 2)$ related to being the DT-2-group of $(\mathbb{Z}, 2, +)$, where $P = (0, 0)$ and $Q = (0, 1)$ (see Remark 5.9).

Based on Definition 4.4 and (3.13) and (4.9), Remark 4.25, and Corollary 4.20, let us establish a DT-k-group structure of $(SC_k^{n,l}, *)$ derived from a G_{k^*}-adjacency of the digital product $(SC_k^{n,l} \times SC_k^{n,l}, G_{k^*})$.

Proposition 5.10. *$(SC_k^{n,l}, *)$ is a DT-k-group for any k-adjacency of $\mathbb{Z}^n$.*

Proof: By Proposition 5.3, $(SC_k^{n,l}, *)$ is a group, where $k := k(t, n)$. Let us assume a G_{k^*}-adjacency on the Cartesian product $SC_k^{n,l} \times SC_k^{n,l}$ such that $k^* := k(t, 2n)$.

Naively, we obtain the relation set (see Proposition 4.14)

$$(SC_k^{n,l} \times SC_k^{n,l}, G_{k^*}), \tag{5.11}$$

where the number $k^* := k(t, 2n)$ of the G_{k^*}-adjacency is determined by the number t of $k := k(t, n)$ from $SC_k^{n,l}$. For the purpose of this study, given $SC_k^{n,l} := (x_i)_{i \in [0, l-1]_{\mathbb{Z}}}$, assume the set $SC_k^{n,l} \times SC_k^{n,l}$ as an $(l \times l)$-matrix as follows:

$$[c_{i,j}], \text{ where } c_{i,j} := (x_i, x_j) \in SC_k^{n,l} \times SC_k^{n,l}.$$

Based on the structure of (5.11), let us further assume the map

$$\begin{Bmatrix} \alpha : SC_k^{n,l} \times SC_k^{n,l} \to SC_k^{n,l} \text{ given by} \\ \alpha(x_i, x_j) := x_i * x_j := x_{i+j(mod\, l)}. \end{Bmatrix} \tag{5.12}$$

Consider each point

$$\begin{Bmatrix} p := c_{i,j} := (x_i, x_j) \in N_{G_{k^*}}(p, 1) = \{c_{i,j}, c_{i \pm 1(mod\, l), j}, c_{i, j \pm 1(mod\, l)}\} \\ \subset SC_k^{n,l} \times SC_k^{n,l}. \end{Bmatrix} \tag{5.13}$$

Then, owing to the existence of a G_{k^*}-adjacency on the Cartesian product $SC_k^{n,l} \times SC_k^{n,l}$, for any $p = (x_i, x_j) \in SC_k^{n,l} \times SC_k^{n,l}$ (see Remark 5.1(1)), we obviously have the set $N_{G_{k^*}}(p, 1)$ (see (5.13)) such that

$$\alpha(N_{G_{k^*}}(p, 1)) \subset N_k(\alpha(p), 1) = N_k(x_{i+j(mod\, l)}, 1),$$

implying the (G_{k^*}, k)-continuity of the map α (see Corollary 4.20).
Next, let us assume the map

$$\begin{Bmatrix} \beta : SC_k^{n,l} \to SC_k^{n,l} \text{ given by, for any element } x_i \in SC_k^{n,l} \\ \beta(x_i) := (x_i)^{-1} = x_{l-i(mod\, l)}. \end{Bmatrix} \tag{5.14}$$

Then we now prove that the map β is also k-continuous. To be precise, for any element $x_i \in SC_k^{n,l}$, take the set $N_k(x_i, 1)$. Then, owing to the map β, by Proposition 2.1, we have

$$\beta(N_k(x_i, 1)) \subset N_k(\beta(x_i), 1) = N_k(x_{l-i(mod\, l)}, 1),$$

implying that the map β is k-continuous. □

Regarding the continuity of (5.12), note that the points (x_0, x_0) and (x_1, x_1) are not G_{k^*}-adjacent in $SC_k^{n,l} \times SC_k^{n,l}$ (see Definition 4.1). By Corollaries 4.17, 4.20, and 5.7, we obtain the following:

Corollary 5.11. *In case the digital product $SC_k^{n,l} \times SC_k^{n,l}$ has a C_{k^*}-adjacency (see Theorem 3.11), i.e., $\exists\,(SC_k^{n,l} \times SC_k^{n,l}, C_{k^*})$, $(SC_k^{n,l}, *)$ is a DT-k-group using the C_{k^*}-adjacency. Then, the multiplication of α related to this DT-k-group of $(SC_k^{n,l}, *)$ is (k^*, k)-continuous.*

Proof: By Corollaries 4.18 and 5.7, and Proposition 5.10, the proof is completed. □

In a *DT*-k-group $(X, k, *)$, in case the group $(X, *)$ is abelian, we say that the *DT*-k-group $(X, k, *)$ is abelian.

Remark 5.12. *(1) There are various types of $SC_k^{n,l}$, e.g., $SC_{18}^{3,6}$ that is not 26-contractible and MSC_{18}, that lead to DT-k-groups of them.*
(2) The DT-k-group $(SC_k^{n,l}, *)$ in Proposition 5.10 is abelian

Example 5.2. *(1) $(SC_4^{2,4}, *)$ is an abelian DT-4-group.*
(2) $(SC_8^{2,6}, *)$ is an abelian DT-8-group.
(3) $(SC_{26}^{3,5}, *)$ is an abelian DT-26-group.
(4) $(MSC_{18}, *)$ is an abelian DT-18-group.

Remark 5.13. *A finite digital plane $(X, k), X \subset \mathbb{Z}^n$, need not be a DT-k-group.*

6. Remarks on the earlier approach to a digital topological version of a topological group in the literature of [16]

Motivated by the typical topological group [12, 17], the paper [16] tried to formulate a digital version of a topological group called a "topological k-group". Then, the paper [16] used the notion of a minimal k-adjacency derived from the conditions of (6.1) below for supporting a kind of continuity

of a multiplication associated with a topological k-group. This approach is quite different from the current one in the present paper. Furthermore, in case we follow the approach in [16], we will come across some fatal errors or the obtained results are trivial cases. Besides, the paper [16] referred to several examples and some properties related to a topological k-group. However, since the paper [16] started with a very insufficient, incorrect, and rough adjacency for a digital product, the obtained results related to the study of a topological k-group (see Section 4 of [16]) are mainly incorrect. More precisely, the paper [16] used the so-called the minimal adjacency of a digital product [16] that is incorrect or trivial, as follows:
Given two digital images $(X_i, k_i := k(t_i, n_i))$ in $\mathbb{Z}^{n_i}, i \in \{1, 2\}$, the paper [16] defined the so-called "minimal adjacency", k_*, for a Cartesian product $X_1 \times X_2$, and denote by $(X \times X, k_*)$. Then the k_*-adjacency was derived from the following approach.

Given the Cartesian product $X_1 \times X_2 \subset \mathbb{Z}^{n_1+n_2}$, the paper [16] says that two points (x_1, x_2), (x_1', x_2') in $X_1 \times X_2$ are "minimal k_*-adjacent" to each other if they satisfy "one of the following conditions"

$$\left\{\begin{array}{l} (1)\,(x_1, x_2) \text{ is equal to } (x_1', x_2'), \text{ or} \\ (2)\,x_1 \text{ is } k_1\text{-adjacent to } x_1' \text{ and } x_2 = x_2', \text{ or} \\ (3)\,x_2 \text{ is } k_2\text{-adjacent to } x_2' \text{ and } x_1 = x_1', \text{ or} \\ (4)\,x_1 \text{ is } k_1\text{-adjacent to } x_1' \text{ and } x_2 \text{ is } k_2\text{-adjacent to } x_2'. \end{array}\right\} \tag{6.1}$$

In particular, first of all, we recall the conditions (2)–(4) are exactly equal to the conditions for establishing a normal k-adjacency of a digital product in [5]. Hence we need to cite it appropriately. Besides, based on the conditions of (6.1) formulating a minimal k_*-adjacency in [16] to establish a topological k-group, the paper [16] requires "one of the four conditions" to establish the so-called "minimal k_*-adjacency" of a digital product. Unfortunately, the approach using one the conditions of (6.1) leads to either trivial or incorrect results with the following reason. Based on the adjacency determined by the conditions of (6.1) suggested in [16], for our purposes, let us assume $t_1 \leq t_2$ in $(X_i, k_i := k(t_i, n_i))$ in $\mathbb{Z}^{n_i}, i \in \{1, 2\}$. Then, let us examine if the requirement of "one of the four conditions" of (6.1) is meaningful as a condition for establishing a topological k-group.

Remark 6.1. *(1) Let us assume only the first condition (1) of (6.1). Then we have a reflexive relation in the digital product, implying that the adjacency invokes a discrete case up to k-adjacency of a digital product. Namely, every point has only the reflexive self-adjacency that invokes a discrete relation in $X_1 \times X_2$ from the viewpoint of digital k-connectivity. Indeed, a discrete relation in a digital image is useless because every self-map of the digital product with any k-adjacency of (1.1) is continuous.*
(2) In case we follow only the second condition (2) of (6.1), it might not satisfy the continuity of a multiplication of $\alpha : X \times X \to X$ (see the case of MSC_{18}). For instance, in the case of MSC_{18}, even though the notion of a "minimal adjacency of a digital product" in [16], according to [16], we may take a 72-adjacency of the digital product $MSC_{18} \times MSC_{18} \subset \mathbb{Z}^6$ for the $(72, 18)$-continuity of the multiplication $MSC_{18} \times MSC_{18} \to MSC_{18}$. Then we obviously see that this multiplication $MSC_{18} \times MSC_{18} \to MSC_{18}$ cannot be $(72, 18)$-continuous (see Proposition 2.1).
(3) In case we take only one of the conditions (3)–(4) of (6.1), it also might not satisfy the continuity of a multiplication of $\alpha : X \times X \to X$ (see the case of MSC_{18} with the method used in (2) above).
Thus the approach using the condition of (6.1) cannot be suitable for establish a topological k-group.

Unlike the approach in [16], by Proposition 5.10, using the G_{72}-adjacency of $MSC_{18}\times MSC_{18} \subset \mathbb{Z}^6$, the pair $(MSC_{18}, *)$ is an abelian 18-connected topological 18-group. However, it is not related to the conditions stated in (6.1). In view of Definition 4.4, the set of elements that are G_{72}-adjacent in $MSC_{18} \times MSC_{18}$ is a proper subset of elements that are 72-adjacent in $MSC_{18} \times MSC_{18}$.

Example 6.1. *The DT-18-group $(MSC_{18}, *)$ using (5.12) guarantees the assertion of Remark 6.1.*

7. Some remarks and a further work

After developing several notions such as a C_{k^*}- and a G_{k^*}-adjacency of a digital product $X \times X$ derived from a given digital image (X, k) and a C_{k^*}- and a G_{k^*}-neighborhood of a point in $(X \times X, C_{k^*})$ and $(X\times X, G_{k^*})$ respectively, we established two types of continuities of a multiplication $(X\times X, C_{k^*}) \to (X, k)$ or $(X \times X, G_{k^*}) \to (X, k)$. Based on this approach, we finally formulated a DT-k-group. Besides, the paper gave various examples of DT-k-groups with some special kinds of binary operations.
As a further work, we can classify DT-k-groups in terms of a certain isomorphism from the viewpoint of DT-k-group theory.

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Conflicts of interest

The author declares no conflict of interest.

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