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Research article

A weak Galerkin method for nonlinear stochastic parabolic partial differential equations with additive noise

Hongze Zhu[1,*], **Chenguang Zhou**[2] **and Nana Sun**[3]

[1] School of Science, Shandong Jianzhu University, Jinan 250101, Shandong, China
[2] Faculty of Science, Beijing University of Technology, Beijing 100124, China
[3] Dongjia Primary School, Jinan 250101, Shandong, China

* **Correspondence:** Email: 1137078394@qq.com.

Abstract: In this paper, a weak Galerkin (WG for short) finite element method is used to approximate nonlinear stochastic parabolic partial differential equations with spatiotemporal additive noises. We set up a semi-discrete WG scheme for the stochastic equations, and derive the optimal order for error estimates in the sense of strong convergence.

Keywords: weak Galerkin finite element method; Lipschitz continuity; nonlinear stochastic partial differential equation; Q-Wiener process; additive noise

1. Introduction

In the recent few centuries, partial differential equations have provided powerful scientific tools for describing many mathematical models. However, with the development of science and the further understanding of natural phenomena, the classical differential equations can not explain random phenomena in the nature and other fields well. In this context, the stochastic partial differential equations (SPDEs for short) are brought in as the models for depicting a variety of random phenomena [1]. The general form of such equations is usually formulated as

$$
\begin{aligned}
&du = (Au + F(u))dt + G(u)dW(t),\\
&u(0) = u_0.
\end{aligned}
$$

Here, H is a Hilbert space, and $u(t)$ is an H-valued random process. We denote by $u_0 \in H$ the initial value. A is a linear, self-adjoint, positive definite, not necessarily bounded operator with a compact inverse and densely defined in a subspace of H. F and G are usually nonlinear operators on H. $W(t)$ is an H-valued Q-Wiener process defined in a filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$.

In this paper, we mainly concern the following SPDE with additive noise,

$$\begin{cases} du + (Au + F(u))dt = GdW(t), & in\ \mathcal{D}, \quad 0 \le t \le T, \\ u = 0, & on\ \partial\mathcal{D}, \quad 0 \le t \le T, \\ u(0) = u_0, & in\ \mathcal{D}, \end{cases} \tag{1.1}$$

where $\mathcal{D} \subset \mathbb{R}^2$ is a polygonal domain and the linear operator G is independent of u.

Up to now, different kinds of numerical methods have been applied to solving the form of the SPDE (1.1), such as finite difference methods, finite element methods, discontinuous Galerkin methods, WG methods, etc. In [2], the author presents a finite difference method for stochastic nonlinear second-order boundary-value problems (BVPs) driven by additive noises, and proves that the finite difference solution converges to the solution to the original stochastic BVP at $O(h)$ in the mean-square sense. The stochastic Allen-Cahn equation with additive noise is discretized by means of a spectral Galerkin method in space and a tamed version of the exponential Euler method in time [3]. A discontinuous Galerkin method is applied in [4] for stochastic differential equations driven by additive noise, and convergence analysis is provided. In [5], the authors analyze strong approximation errors of both finite element semi-discretization and spatio-temporal full discretization for the stochastic Allen-Cahn equation driven by additive noise in space dimension $d \le 3$. A stochastic analogue of the local discontinuous Galerkin method is constructed for a stochastic two-point boundary-value problem driven by an additive white noise [6]. In [7], the authors adopt the Argyris finite elements to solve (1.1) with $A = -\Delta^2$ and obtain the optimal order h^β of error estimates with $\beta > 0$. In [8, 9], the linear version of (1.1) is investigated via the WG methods and the optimal order estimates in the sense of strong convergence are derived.

In general, WG methods, firstly proposed by Wang and Ye [10], are newly developed numerical techniques for solving partial differential equations. The essence of this method is the use of weak finite element functions and their weak derivatives computed with a framework that mimics the distribution or generalized functions. Since the method was put forward, it has been applied/extended to different kinds of partial differential equations, such as biharmonic equations [11], Stokes equations [13], linear elasticity equations [22], poroelasticity problems [14], parabolic problems [15, 16], SPDEs [8, 9], etc.

In this paper, we adopt the WG method with a parameter-free stabilization term for solving the SPDE (1.1) with $A = -\Delta$. The main characteristic of this method is that the WG finite element space consists of discontinuous functions, which allows the WG method applied to the general polygonal or polyhedral meshes. This characteristic makes the WG method efficient and highly flexible. The optimal order for strongly convergent error estimates in L_2-norm is studied based on the established semi-discrete WG scheme. As far as we know, this paper is the first to apply the ideas of WG to nonlinear stochastic models for error analysis.

This paper is organized as follows. In Section 2, we provide several definitions and assumptions as the preliminaries for the theoretical analysis. Section 3 introduces the details of WG method and sets up the semi-discrete WG scheme of the nonlinear stochastic model. Several error estimates for the related deterministic problem are supplied in Section 4, which is helpful for the derivation of our later main result. And finally, in Section 5, we derive the optimal order for strongly convergent error estimates in L_2-norm based on the established semi-discrete WG scheme.

2. Definitions and assumptions

In this section, we introduce several definitions and assumptions as a preparation for the later theoretical analysis.

Recall that $\mathcal{D} \subset \mathbb{R}^2$ is a polygonal domain. Unless particularly stated, in this paper, we shall use the standard notations for Sobolev spaces and their associated norms [17]. Let $H = L_2(\mathcal{D})$ whose inner product and norm are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, respectively. Denote by $H^s = H^s(\mathcal{D})$ with norm $\|\cdot\|_s$. Let H_0 and H_0^s be the subspaces of H and H^s, respectively, the elements of which vanish on the boundary $\partial\mathcal{D}$.

Denote by $Q : H \to H$ a linear self-adjoint operator with eigenvalues $\gamma_i > 0$ $(i = 1, 2, \cdots)$ and corresponding normalized eigenfunctions $e_i \in H$ $(i = 1, 2, \cdots)$. Then e_i $(i = 1, 2, \cdots)$ form a family of completely orthonormal bases of the space H. We further assume:

(**H1**): The operator Q is bounded and positive definite.

(**H2**): The operator Q has bounded trace, i.e., $Tr(Q) = \sum_{i=1}^{\infty} \gamma_i < +\infty$.

Since $W(t)$ is a Q-Wiener process defined on a given filtrated probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, $W(t)$ is H-valued. Denote by $\beta_i(t)$ $(i = 1, 2, ...)$ a family of Brownian motions in which the elements are independently and identically distributed. Then the Wiener process $W(t)$ can be written in the form of its Fourier expansion [18]:

$$W(t) = \sum_{i=1}^{\infty} \gamma_i^{1/2} e_i \beta_i(t).$$

Next, we define several operator spaces. Let $L(Q^{1/2}(H), H)$ be the space of bounded linear operators from $Q^{1/2}(H)$ to H and denote by $L_2^0(Q^{1/2}(H), H)$ a subspace of $L(Q^{1/2}(H), H)$ satisfying:

$$L_2^0(Q^{1/2}(H), H) = \{\phi \in L(Q^{1/2}(H), H) : \sum_{l=1}^{\infty} \| \phi Q^{1/2} e_l \|^2 < \infty\}.$$

Let $L(H)$ be the linear bounded operator space from H to H. Then we define a Hilbert-Schmidt operator space $L_{HS}(H) \subset L(H)$, i.e.,

$$L_{HS}(H) = \{\Phi \in L(H) : \sum_{i=1}^{\infty} \| \Phi e_i \|^2 < \infty\}$$

with norm

$$\| \Phi \|_{HS} = (\sum_{i=1}^{\infty} \| \Phi e_i \|^2)^{1/2}, \quad \forall \Phi \in L_{HS}(H).$$

It is not hard to see that for any $\psi \in L_2^0(Q^{1/2}(H), H)$, the operator $\psi Q^{1/2} \in L_{HS}(H)$. Let $\mathbf{E}$ represent the standard mathematical expectation. Then, for any $\psi \in L_2^0(Q^{1/2}(H), H)$, the following isometry equation holds.

$$\mathbf{E} \| \int_0^t \psi(s) dW(s) \|^2 = \int_0^t \| \psi(s) Q^{1/2} \|_{HS}^2 \, ds. \tag{2.1}$$

Additionally, define the space $L_2(\Omega; H)$,

$$L_2(\Omega; H) = \{v : \int_\Omega \| v \|^2 \, dP(\omega) < \infty\} \tag{2.2}$$

with norm

$$\| v \|_{L_2(\Omega;H)} = (\int_\Omega \| v \|^2 \, dP(\omega))^{1/2}, \quad \forall v \in L_2(\Omega; H).$$

Similarly, we can define $L_\infty(\Omega; H)$.

Let $A = -\Delta$. Assume the operator A defined on H_0^2 has real eigenvalues $\lambda_i > 0$ ($i = 1, 2, ...$) with corresponding eigenfunctions $g_i \in H_0^2 \subset H$ ($i = 1, 2, ...$). Then for all $v \in H_0^2$, we have

$$Av = \sum_{i=1}^{\infty} \lambda_i (v, g_i) g_i.$$

For $s > 0$, we define the space $\dot{H}^s$:

$$\dot{H}^s = \{v \in H : \| A^{s/2} v \| = \left(\sum_{i=1}^{\infty} \lambda_i^s (v, g_i)^2 \right)^{1/2} < \infty\}$$

with norm $| \cdot |_s = \| A^{s/2} \cdot \|$. Similar to (2.2), we can define $L_2(\Omega; \dot{H}^s)$.

The lemma below provides the relationship between $\dot{H}^s$ and H_0^s.

Lemma 2.1. *[19, Lemma 3.1] For any $s > 0$, we have*

$$\dot{H}^s = \{v \in H^s : \Delta^j v = 0 \quad on\ \partial\mathcal{D}, \quad j < s/2\}.$$

Moreover, $| \cdot |_s$ is equivalent to $\| \cdot \|_s$ in $\dot{H}^s$, where

$$| v |_s = \begin{cases} \| \Delta^p v \|, & if \quad s = 2p, \\ \| \nabla(\Delta^p v) \|, & if \quad s = 2p + 1. \end{cases} \tag{2.3}$$

Now we write three assumptions for the SPDE (1.1) we consider.

(**H3**): The initial value $u_0 \in L_2(\Omega; \dot{H}^2)$.

(**H4**): The operator $F : H_0 \to \dot{H}^1$ satisfies:

$$| F(u_1) - F(u_2) |_1 \le C \| u_1 - u_2 \|, \ \forall u_1, u_2 \in H_0, \tag{2.4}$$

where C is a positive constant. In this paper, the letter C denotes a generic positive constant which may be different at different occurrences. Furthermore, if we take $u_2 \equiv 0$, then

$$| F(u_1) |_1 \le C(\| u_1 - u_2 \| + | F(u_2) |_1) \le C(\| u_1 \| + 1), \ \forall u_1 \in H_0. \tag{2.5}$$

(**H5**): The operator $G \in L_2^0(Q^{1/2}(H), H)$ satisfies

$$\| A^{1/2} G Q^{1/2} \|_{HS} < \infty. \tag{2.6}$$

Let $E(t) = e^{-tA}$. Then (1.1) admits a unique mild solution of the form [18]:

$$u(t) = E(t)u_0 - \int_0^t E(t-s)F(u(s))ds + \int_0^t E(t-s)GdW(s). \tag{2.7}$$

From [20], the following two lemmas hold true.

Lemma 2.2. *If the assumptions* **H3 – H5** *hold, u is the mild solution of (1.1). Then for* $0 \le t_1 \le t_2 \le T$, *there exists a constant C such that*

$$\| u(t_1) - u(t_2) \|_{L_2(\Omega;H)} \le C \, | t_1 - t_2 |^{1/2}. \tag{2.8}$$

Lemma 2.3. *If the assumptions* **H3 – H5** *hold, u is the mild solution of (1.1). Then*

$$\sup_{s \in [0,T]} \mathbf{E}[\| u(s) \|^2] < \infty. \tag{2.9}$$

3. The WG method

In this section, the details of WG method is introduced, and the semi-discrete WG scheme for the SPDE (1.1) is established.

Let $\mathcal{T}_h$ be a regular partition of the domain $\mathcal{D}$ satisfying the shape regularity requirements A1-A4 in [21]. The boundary of each element $K \in \mathcal{T}_h$ is denoted by ∂K. Let h_K be the diameter of K and $h = \max_{K \in \mathcal{T}_h} h_K$.

We define the weak function space:

$$W(K) \triangleq \{v = \{v_0, v_b\}; v_0 \in L^2(K), v_b \in L^2(\partial K)\}.$$

Here v_0 is the value of v in K and v_b is the value of v on the boundary ∂K.

With an inclusion map $i : H^1(K) \to W(K)$ [21],

$$i(v) = \{v\,|_K, v\,|_{\partial K}\}, \ \forall v \in H^1(K),$$

we can embed $H^1(K)$ into the weak function space $W(K)$.

For any $u, v \in W(K)$, denote by $(u, v)_K$ the standard L^2-inner product in K and by $\|\cdot\|_K$ the corresponding norm. Similarly, $\langle u, v\rangle_{\partial K}$ represents the L^2-inner product on ∂K, and the norm is notated by $\|\cdot\|_{\partial K}$.

Let r be a non-negative integer. For each $K \in \mathcal{T}_h$, let $P_{r+1}(K)$ and $P_{r+1}(\partial K)$ be the sets of polynomials with degree no more than $r+1$ in K and on ∂K, respectively. Define a discrete weak function space $W_{r+1}(K) \subset W(K)$, i.e.,

$$W_{r+1}(K) \triangleq \{v = \{v_0, v_b\}; v_0 \in P_{r+1}(K), v_b \in P_{r+1}(\partial K)\}.$$

For each $K \in \mathcal{T}_h$, let $V(K, r) = [P_r(K)]^2$. Define the discrete weak gradient operator $\nabla_d : W_{r+1}(K) \to V(K, r)$, such that for any $v = \{v_0, v_b\} \in W_{r+1}(K)$, $\nabla_d v \in V(K, r)$ is the unique vector-valued polynomial satisfying:

$$(\nabla_d v, \mathbf{q})_K \triangleq -(v_0, (\nabla \cdot \mathbf{q}))_K + \langle v_b, (\mathbf{q} \cdot \mathbf{n})\rangle_{\partial K}, \quad \forall \mathbf{q} \in V(K, r). \tag{3.1}$$

Now we extend the definition of the weak function space $W(K)$ and the discrete weak function space $W_{r+1}(K)$ to the whole domain $\mathcal{D}$. Denote by $W(\mathcal{D}) = \{v : v\,|_K \in W(K)\}$ the weak function space defined on the domain $\mathcal{D}$. Let $S_h(r+1) \subset W(\mathcal{D})$ be the the discrete weak function space satisfying:

$$S_h(r+1) \triangleq \{v = \{v_0, v_b\}; v\,|_K \in W_{r+1}(K), \forall K \in \mathcal{T}_h\},$$

and denote by $S_h^0(r+1)$ the subspace of $S_h(r+1)$ with vanishing values on the boundary $\partial\mathcal{D}$. If no confusion occurs, we use respectively S_h to denote $S_h(r+1)$, and use S_h^0 to denote $S_h^0(r+1)$ throughout this paper.

We also define the following global vector-valued polynomial space:

$$V(r) \triangleq \{q : \ q\,|_K \in V(K, r)\}.$$

Then we extend the definition of the weak gradient operator ∇_d from each $K \in \mathcal{T}_h$ to the whole domain $\mathcal{D}$. That is to say, for any $v \in S_h(r+1)$, define the operator $\nabla_d : S_h(r+1) \to V(r)$, such that

$$\nabla_d v\,|_K \triangleq \nabla_d(v\,|_K). \tag{3.2}$$

Next, we bring in several locally defined projection operators which are helpful for our theoretical analysis. For each $K \in \mathcal{T}_h$, the L_2-projection operators $Q_0 : L_2(K) \to P_{r+1}(K)$ and $Q_b : L_2(\partial K) \to P_{r+1}(\partial K)$ are defined piecewisely in K and on ∂K, respectively. Define $Q_h u = \{Q_0 u_0, Q_b u_b\} : W(\mathcal{D}) \to S_h(r+1)$ satisfying:

$$Q_h\,|_K\colon W(K) \xrightarrow{L_2} W_{r+1}(K), \quad \forall K \in \mathcal{T}_h.$$

For each $K \in \mathcal{T}_h$, let $R_h : [L_2(\mathcal{D})]^2 \to V(r)$ be the L_2-projection operator defined by

$$R_h\,|_K\colon [L_2(K)]^2 \xrightarrow{L_2} V(K, r), \quad \forall K \in \mathcal{T}_h. \tag{3.3}$$

Now we introduce a bilinear form as follows: For any $u_h, v_h \in S_h$,

$$a_s(u_h, v_h) = \sum_{K\in\mathcal{T}_h} (\nabla_d u_h, \nabla_d v_h)_K + \sum_{K\in\mathcal{T}_h} h_K^{-1}\langle u_0 - u_b, v_0 - v_b\rangle_{\partial K},\ \forall u_h, v_h \in S_h.$$

Take the following elliptic problem into account:

$$\begin{aligned} -\Delta u &= f, \quad in\ \mathcal{D}, \\ u &= 0, \quad on\ \partial\mathcal{D}. \end{aligned} \tag{3.4}$$

The variational formulation for (3.4) is to find $u \in H_0^1(\mathcal{D})$ such that

$$(\nabla u, \nabla v) = (f, v), \quad \forall v \in H_0^1(\mathcal{D}). \tag{3.5}$$

The WG scheme for (3.4) is to find $u_h \in S_h^0$ such that

$$a_s(u_h, v_h) = (f, v_h) = (Q_h f, v_h), \quad \forall v_h \in S_h^0. \tag{3.6}$$

Let $A_h : S_h^0 \to S_h^0$ be the operator satisfying:

$$(A_h u_h, v_h) = a_s(u_h, v_h), \quad \forall v_h \in S_h^0.$$

Here A_h is a linear, self-adjoint, symmetric, positive definite operator. Then, the numerical scheme (3.6) is equivalent to finding $u_h \in S_h^0$ such that

$$A_h u_h = Q_h f. \tag{3.7}$$

Based on [21, Theorem 8.2] and [8, (4.7)], we have the following lemma about the error estimates of the elliptic problem (3.4) with WG method.

Lemma 3.1. *Assuming $u \in H^2(\mathcal{D})$ and $u_h \in S_h$ are the solutions of (3.4) and (3.7), respectively. Then there exists a positive constant C which depends only on the domain $\mathcal{D}$, such that*

$$\| Q_h u - u_h \| \leq Ch^2 \| f \| . \tag{3.8}$$

Define two operators $G = A^{-1}$ and $G_h = A_h^{-1}$. $G : H_0 \to H_0^1$ and $G_h : S_h^0 \to S_h^0$ are the solution operators of (3.4) and (3.7), respectively. In other words, $u = Gf$ and $u_h = G_h Q_h f$ are respectively the solutions of (3.4) and (3.7). It is easy to see that G and G_h are linear, symmetric, positive definite operators. Then (3.8) can be written as

$$\| (Q_hG - G_hQ_h)f \| \leq Ch^2 \| f \| . \tag{3.9}$$

Now we approximate the stochastic problem (1.1) with WG method. The semi-discrete WG scheme for (1.1) is to find an $H-$valued random process $u_h(\cdot, t) \in S_h^0$ with $u_h(0) = Q_hu_0$, such that for any $v_h \in S_h^0$ and $0 \leq t \leq T$,

$$(u_h(t), v_h) - (u_h(0), v_h) + \int_0^t (A_hu_h, v_h)ds + \int_0^t (Q_hF(u_h), v_h)ds = (\int_0^t Q_hGdW(s), v_h). \tag{3.10}$$

In fact, (3.10) is equivalent to

$$\begin{array}{rcl} du_h + A_hu_hdt + Q_hF(u_h)dt &=& Q_hGdW, \quad in\ \mathcal{D},\ 0 \leq t \leq T, \\ u_h(0) &=& Q_hu_0, \quad in\ \mathcal{D}. \end{array} \tag{3.11}$$

Let $E_h(t) = e^{-tA_h}$, $t \geq 0$. It is obvious that the equation (3.11) has a mild solution

$$u_h(t) = E_h(t)Q_hu_0 - \int_0^t E_h(t-s)Q_hF(u_h(s))ds + \int_0^t E_h(t-s)Q_hGdW(s). \tag{3.12}$$

4. Several error estimates for the related deterministic problem

In this section, we provide several error estimates with respect to the related deterministic problem, which are used in the subsequent section.

Lemma 4.1. *[12, Lemma 3.2] For any $\alpha, \beta \in \mathbb{R}$ and $l \geq 0$, we have*

$$| D_t^lE(t)v |_\beta \leq Ct^{-(\beta-\alpha)/2-l} | v |_\alpha,\ t > 0,\ 2l+\beta \geq \alpha, \tag{4.1}$$

and

$$\int_0^t s^\alpha | D_t^lE(s)v |_\beta^2 ds \leq C | v |_{2l+\beta-\alpha-1}^2,\ t > 0,\ \alpha \geq 0, \tag{4.2}$$

where D_t^l is the l-th derivative with respect to t.

Consider the following two problems:

$$u_t + Au = 0, \qquad u(0) = u_0, \tag{4.3}$$

and

$$u_{h,t} + A_hu_h = 0, \qquad u_h(0) = Q_hu_0. \tag{4.4}$$

Then $u = E(t)u_0$, $u_h = E_h(t)Q_hu_0$ are the solutions of (4.3) and (4.4), respectively. Making use of the forms of u and u_h, the error equation is shown as follows:

$$G_he_t + e = \rho, \tag{4.5}$$

where $e(t) = u_h(t) - Q_hu(t)$, $\rho = (Q_hG - G_hQ_h)u_t$ and $e(0) = u_h(0) - Q_hu(0) = 0$.

Indeed, by virtue of (4.3) and (4.4), we supply

$$\begin{aligned} G_h e_t + e &= (G_h u_{h,t} + u_h) - (G_h Q_h u_t + Q_h u) \\ &= G_h(u_{h,t} + A_h u_h) - (G_h Q_h u_t + Q_h u) \\ &= (Q_h G - G_h Q_h) u_t. \end{aligned}$$

According to the error equation (4.5), it follows from [8] that

$$\int_0^t \| e \|^2 \, ds \leq \int_0^t \| \rho \|^2 \, ds, \tag{4.6}$$

and

$$t \| e \|^2 \leq \frac{1}{4} t \| e \|^2 + 4t \| \rho \|^2 + \int_0^t (2s \| \rho_t \| \| e \| + \| e \|^2 + 2 \| \rho \| \| e \|) ds. \tag{4.7}$$

Furthermore, we can derive the following two lemmas.

Lemma 4.2. *For any $t \in [0, T]$, assume that $e(t) \in S_h^0$, $e(0) = 0$ and (4.5) holds. Then there exists a positive constant C, such that*

$$\| e(t) \| \leq C(\sup_{0 \leq s \leq t} s \| \rho_t(s) \| + \sup_{0 \leq s \leq t} \| \rho(s) \|), \quad t \geq 0. \tag{4.8}$$

Proof. Due to (4.7), (4.6) and the mean inequality, we obtain

$$\| e(t) \|^2 \leq C(\sup_{0 \leq s \leq t} s^2 \| \rho_t(s) \|^2 + \sup_{0 \leq s \leq t} \| \rho(s) \|^2), \quad t \geq 0,$$

which completes the proof. □

Lemma 4.3. *Under the same assumptions of Lemma 4.2, for any fixed positive number ϵ, there exists a positive constant C_ϵ depending on ϵ, such that*

$$\| e(t) \| \leq (\epsilon \sup_{0 \leq s \leq t} s \| \rho_t(s) \| + C_\epsilon \sup_{0 \leq s \leq t} \| \rho(s) \|), \quad t \geq 0. \tag{4.9}$$

Proof. The proof is similar to the one in Lemma 4.2. Noticing that

$$2s \| \rho_t \| \| e \| \leq (\epsilon^2 s^2 \| \rho_t \|^2 + \frac{1}{\epsilon^2} \| e \|^2), \tag{4.10}$$

together with (4.7) and (4.6), we finish the proof. □

Notate $F_h(t) = E_h(t) Q_h - Q_h E(t)$. Then we render the following estimate results.

Lemma 4.4. *[8, Lemma 4.4] For $v \in \dot{H}^2$, then we have*

$$\| F_h v \|_{L_\infty([0,T];H)} = \sup_{0 \leq t \leq T} \| F_h v \| \leq C h^2 \mid v \mid_2 . \tag{4.11}$$

If $v \in \dot{H}^1$ and $0 \leq t \leq T$, then

$$\| F_h v \|_{L_2([0,t];H)} = (\int_0^t \| F_h v \|^2 \, ds)^{1/2} \leq C h^2 \mid v \mid_1, \tag{4.12}$$

where C is a positive constant only depending on the domain $\mathcal{D}$.

Theorem 4.1. *Assuming that $u_0 \in \dot{H}^1$ and $t > 0$, then we have*

$$\| \int_0^t F_h u_0 ds \| \leq Ch^2 \| u_0 \|, \tag{4.13}$$

where C is a positive constant only depending on the domain $\mathcal{D}$.

Proof. Denote by $\tilde{e}(t)$ and $\tilde{\rho}(t)$ the integral $\int_0^t e(s)ds$ and $\int_0^t \rho(s)ds$, respectively. Together with $e(0) = 0$, we acquire

$$G_h \tilde{e}_t + \tilde{e} = G_h e(t) + \int_0^t e(s)ds = \int_0^t (G_h e_t + e)ds = \tilde{\rho}.$$

It is easy to check $\tilde{e}(0) = 0$. According to (4.8), we provide

$$\| \tilde{e}(t) \| \leq C(\sup_{0 \leq s \leq t} s \| \rho(s) \| + \sup_{0 \leq s \leq t} \| \tilde{\rho}(s) \|). \tag{4.14}$$

It follows from Lemma 4.1 and (3.9) that

$$s \| \rho(s) \| = s \| (G_h Q_h - Q_h G) u_t(s) \| \leq Ch^2 s \| u_t(s) \| \leq Ch^2 \| u_0 \|, \tag{4.15}$$

and

$$\| \tilde{\rho}(s) \| = \| \int_0^s (G_h Q_h - Q_h G) u_t(\tau) d\tau \| = \| (G_h Q_h - Q_h G)(u(s) - u_0) \| \leq Ch^2 \| u_0 \| . \tag{4.16}$$

Here we notice that $E_h(s) = e^{-sA_h}$ is a bounded operator. Then the proof is completed. □

Theorem 4.2. *Under the same assumptions of Theorem 4.1, we supply*

$$\| F_h u_0 \| \leq Ch^2 t^{-1} \| u_0 \|, \tag{4.17}$$

where C is a positive constant only depending on the domain $\mathcal{D}$.

Proof. Define $\hat{e}(t) = te$, then by (4.5),

$$G_h \hat{e}_t + \hat{e} = G_h e + G_h(te_t) + te = G_h e + t\rho.$$

We denote $\chi(t) = G_h e(t) + t\rho(t)$. It is easy to check $\hat{e}(0) = 0$. Due to (4.9), for any $\epsilon > 0$, we present

$$\| te(t) \| = \| \hat{e}(t) \| \leq \epsilon \sup_{0 \leq s \leq t} s \| \chi_t(s) \| + C_\epsilon \sup_{0 \leq s \leq t} \| \chi(s) \| .$$

For the two parts on the right side of the estimate above, we have

$$\| \chi(s) \| \leq s \| \rho(s) \| + \| G_h e(s) \|,$$

and it follows from (4.5) that

$$\begin{aligned} s \| \chi_t(s) \| &\leq s^2 \| \rho_t(s) \| + s \| \rho(s) \| + s \| G_h e_t(s) \| \\ &\leq s^2 \| \rho_t(s) \| + s \| \rho(s) \| + s(\| \rho(s) \| + \| e(s) \|) \\ &= s^2 \| \rho_t(s) \| + 2s \| \rho(s) \| + \| \hat{e}(s) \| . \end{aligned}$$

Let $\epsilon = 1/2$, then

$$\| \hat{e}(t) \| \leq (1/2) \sup_{0\leq s\leq t} \| \hat{e}(s) \| + C(\sup_{0\leq s\leq t} s^2 \| \rho_t(s) \| + \sup_{0\leq s\leq t} s \| \rho(s) \| + \sup_{0\leq s\leq t} \| G_h e(s) \|).$$

Choose $0 \leq s_0 \leq t$ such that $\| \hat{e}(s_0) \| = \sup_{0\leq s\leq t} \| \hat{e}(s) \|$, then

$$\| \hat{e}(t) \| \leq \| \hat{e}(s_0) \| \leq (1/2) \| \hat{e}(s_0) \| + C(\sup_{0\leq s\leq t} s^2 \| \rho_t(s) \| + \sup_{0\leq s\leq t} s \| \rho(s) \| + \sup_{0\leq s\leq t} \| G_h e(s) \|).$$

Hence

$$t \| e(t) \| = \| \hat{e}(t) \| \leq \| \hat{e}(s_0) \| \leq C(\sup_{0\leq s\leq t} s^2 \| \rho_t(s) \| + \sup_{0\leq s\leq t} s \| \rho(s) \| + \sup_{0\leq s\leq t} \| G_h e(s) \|). \tag{4.18}$$

Now we estimate $\| G_h e(s) \|$. Let $\tilde{e}(t) = \int_0^t e(s)ds$, then

$$G_h e + \tilde{e} = G_h \tilde{e}_t + \tilde{e} = \tilde{\rho}.$$

By virtue of (4.14), we obtain

$$\| G_h e(t) \| \leq \| \tilde{e}(t) \| + \| \tilde{\rho}(t) \| \leq C(\sup_{0\leq s\leq t} s \| \rho(s) \| + \sup_{0\leq s\leq t} \| \tilde{\rho}(s) \|). \tag{4.19}$$

Combining (4.18) with (4.19), we have

$$\| e(t) \| \leq Ct^{-1}(\sup_{0\leq s\leq t} s^2 \| \rho_t(s) \| + \sup_{0\leq s\leq t} s \| \rho(s) \| + \sup_{0\leq s\leq t} \| \tilde{\rho}(s) \|).$$

Because of Lemma 4.1 and (3.9), we render

$$s^2 \| \rho_t(s) \| = s^2 \| (G_h Q_h - Q_h G) u_{tt}(s) \| \leq Ch^2 s^2 \| u_{tt}(s) \| \leq Ch^2 \| u_0 \| .$$

With the help of (4.15) and (4.16), we complete the proof. □

5. The main result

In this section, we derive the optimal order for strongly convergent error estimates between the mild solution (2.7) of the SPDE (1.1) and its semi-discrete WG approximation (3.12) in L_2-norm.

The next lemma plays a very important role in getting our main result.

Lemma 5.1. *[20] For all $C_1, C_2 \geq 0, \alpha > 0$, $t \in [0, T]$, let $\phi : [0, T] \to \mathbb{R}$ be a nonnegative and continuous function. If*

$$\phi(t) = C_1 + C_2 \int_0^t (t-s)^{\alpha-1} \phi(s) ds, \tag{5.1}$$

then there exists a constant $C = C(C_2, T, \alpha)$ such that for all $t \in [0, T]$,

$$\phi(t) \leq CC_1. \tag{5.2}$$

Theorem 5.1. *Let u and u_h be the mild solutions of (1.1) and (3.11), respectively. If the assumptions* **H3 − H5** *in Section 2 hold, then there exists a constant C depending on the domain $\mathcal{D}$ and the upper bound of time T such that*

$$\| u_h(t) - Q_h u(t) \|_{L_2(\Omega;H)} \leq Ch^2.$$

Proof. Making use of (2.7) and (3.12), we write

$$\begin{aligned} u_h(t) - Q_h u(t) &= F_h(t)u_0 + \int_0^t (Q_h E(t-s)F(u(s)) - E_h(t-s)Q_h F(u_h(s))ds)ds \\ &\quad + \int_0^t F_h(t-s)GdW(s) \\ &:= F_1 + F_2 + F_3. \end{aligned}$$

We record the three parts on the right side as F_1, F_2 and F_3, respectively, and estimate each part one by one. Firstly, we can easily obtain $\| F_1 \|_{L_2(\Omega;H)} \leq h^2 \| u_0 \|_{L_2(\Omega;\dot{H}^2)}$ by (4.11).

F_2 can be written as a combination of three parts:

$$\begin{aligned} F_2 &= \int_0^t Q_h E(t-s)F(u(s))ds - \int_0^t E_h(t-s)Q_h F(u_h(s))ds \\ &= \int_0^t E_h(t-s)Q_h(F(u(s)) - F(u_h(s)))ds \\ &\quad - \int_0^t F_h(F(u(s)) - F(u(t)))ds \\ &\quad - \int_0^t F_h(F(u(t)))ds \\ &= I_1 - I_2 - I_3. \end{aligned}$$

Next, we estimate I_1, I_2 and I_3, respectively. $E_h(t-s)$ and Q_h are both bounded operators. Then together with (2.4), we find

$$\| I_1 \|_{L_2(\Omega;H)} \leq C \int_0^t \| u_h(s) - Q_h u(s) \|_{L_2(\Omega;H)} \, ds.$$

By Theorem 4.2, Lemma 2.1, (2.4) and (2.8), we have

$$\begin{aligned} \| I_2 \|_{L_2(\Omega;H)} &\leq \int_0^t \| F_h(F(u(s)) - F(u(t))) \|_{L_2(\Omega;H)} \, ds \\ &\leq Ch^2 \int_0^t (t-s)^{-1} \| F(u(s)) - F(u(t)) \|_{L_2(\Omega;H)} \, ds \\ &\leq Ch^2 \int_0^t (t-s)^{-1} \| F(u(s)) - F(u(t)) \|_{L_2(\Omega;\dot{H}^1)} \, ds \\ &\leq Ch^2 \int_0^t (t-s)^{-1} \| u(s) - u(t) \|_{L_2(\Omega;H)} \, ds \\ &\leq Ch^2 \int_0^t (t-s)^{-1/2} ds \\ &\leq Ct^{1/2}h^2 \end{aligned}$$

$$\leq \quad CT^{1/2}h^2.$$

From Theorem 4.1, Lemma 2.1, (2.5) and (2.9), we supply

$$\begin{aligned}
\| I_3 \|_{L_2(\Omega;H)} &\leq Ch^2 \| F(u(t)) \|_{L_2(\Omega;H)} \\
&\leq Ch^2 \| F(u(t)) \|_{L_2(\Omega;\dot{H}^1)} \\
&\leq Ch^2(\| u(t) \|_{L_2(\Omega;H)} + 1) \\
&\leq Ch^2(\sup_{s\in[0,T]} \| u(s) \|_{L_2(\Omega;H)} + 1).
\end{aligned}$$

Thus,

$$\| F_2 \|_{L_2(\Omega;H)} \leq Ch^2 + C \int_0^t \| u_h(s) - Q_h u(s) \|_{L_2(\Omega;H)} \, ds.$$

Now we consider F_3. With the help of (2.1), we provide

$$\begin{aligned}
\| F_3 \|^2_{L_2(\Omega;H)} &= \mathbf{E} \| \int_0^t F_h(t-s) G dW(s) \|^2 \\
&= \int_0^t \| F_h(t-s) G Q^{1/2} \|^2_{HS} \, ds \\
&= \sum_{l=1}^{\infty} \int_0^t \| F_h(t-s) G Q^{1/2} e_l \|^2 \, ds.
\end{aligned}$$

From (4.12), it follows that

$$\| F_3 \|^2_{L_2(\Omega;H)} \leq C \sum_{l=1}^{\infty} h^4 \, | g Q^{1/2} e_l |_1^2 = Ch^4 \| A^{1/2} g Q^{1/2} \|^2_{HS} \, .$$

Hence,

$$\begin{aligned}
\| u_h(t) - Q_h u(t) \|_{L_2(\Omega;H)} &\leq \| F_1 \|_{L_2(\Omega;H)} + \| F_2 \|_{L_2(\Omega;H)} + \| F_3 \|_{L_2(\Omega;H)} \\
&= Ch^2 + C \int_0^t \| u_h(s) - Q_h u(s) \|_{L_2(\Omega;H)} \, ds.
\end{aligned}$$

By Lemma 5.1 with $\alpha = 1$ and $\phi(t) = \| u_h(t) - Q_h u(t) \|_{L_2(\Omega;H)}$, the proof is completed. □

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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