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Research article

Multi-shockpeakons for the stochastic Degasperis-Procesi equation

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Abstract: The deterministic Degasperis-Procesi equation admits weak multi-shockpeakon solutions of the form

$$u(x,t) = \sum_{i=1}^{n} m_i(t) e^{-|x-x_i(t)|} - \sum_{i=1}^{n} s_i(t)\mathrm{sgn}(x - x_i(t)) e^{-|x-x_i(t)|},$$

where $\mathrm{sgn}(x)$ denotes the signum function with $\mathrm{sgn}(0) = 0$, if and only if the time-dependent parameters $x_i(t)$ (positions), $m_i(t)$ (momenta) and $s_i(t)$ (shock strengths) satisfy a system of $3n$ ordinary differential equations. We prove that a stochastic perturbation of the Degasperis-Procesi equation also has weak multi-shockpeakon solutions if and only if the positions, momenta and shock strengths obey a system of $3n$ stochastic differential equations.

Keywords: the stochastic Degasperis-Procesi equation; the deterministic Degasperis-Procesi equation; multiplicative noise; Stratonovich stochastic process; multi-peakons; multi-shockpeakons

1. Introduction

Consider the ab-family of Equations [1]

$$\partial_t u - \partial_{xxt} u + \partial_x (a(u, \partial_x u)) = \partial_x \left(b'(u) \frac{(\partial_x u)^2}{2} + b(u)\partial_{xx} u \right). \tag{1.1}$$

The family (1.1) contains interesting deterministic equations, such as those studied by [2].

The first celebrated member of (1.1) is the well-known deterministic Camassa-Holm (CH) equation [3,4] ($b(u) = u$ and $a(u, u_x) = \frac{3}{2}u^2$). The existence and classification of weak travelling wave solutions of the CH equation were considered in [5]. Stochastic perturbations of the CH equation were studied in [6–10].

If $b(u) = u$ and $a(u, u_x) = u^3$, Eq (1.1) becomes the deterministic modified Camassa-Holm (mCH) equation

$$\partial_t u - \partial_{xxt} u = u\partial_{xxx} u + 2\partial_x u \partial_{xx} u - 3u^2 \partial_x u. \tag{1.2}$$

Observe that the transformation $u(x,t) = \tilde{u}(\xi, t)$, $\xi = x + ct$, $c \in \mathbb{R}$, reduces (1.2) to the following modified Dullin-Gottwald-Holm (mDGH) equation

$$\partial_t \tilde{u} - \partial_{\xi\xi t} \tilde{u} - \tilde{u}\partial_{\xi\xi\xi}\tilde{u} - 2\partial_\xi \tilde{u} \partial_{\xi\xi}\tilde{u} + 3\tilde{u}^2 \partial_\xi \tilde{u} = -c\partial_\xi \tilde{u} + c\partial_{\xi\xi\xi}\tilde{u}.$$

Travelling waves for Eq (1.2) were found via computational methods by [11]. Wave breaking, classification of traveling waves and explicit elliptic peakons for the mCH equation (1.2) were analysed in [12]. An stochastic perturbation of the Dullin-Gottwald-Holm equation [13] was studied in [14].

The particular case of (1.1), where $b(u) = u$ and $a(u, u_x) = 2u^2 - \frac{(\partial_x u)^2}{2}$, corresponds to the deterministic Degasperis-Procesi (DP) equation [15]

$$\partial_t u - \partial_{xxt} u = u\partial_{xxx} u + 3\partial_x u \partial_{xx} u - 4u\partial_x u \tag{1.3}$$

or, alternatively, to the hyperbolic-elliptic formulation

$$\begin{cases} \partial_t u + \partial_x \left(\frac{u^2}{2}\right) + \partial_x p = 0, \\ (1 - \partial_{xx})\, p = \frac{3}{2}u^2, \end{cases} \tag{1.4}$$

which is used to define the weak solutions of the DP equation [16]. In fact, inverting

$$m = (1 - \partial_{xx})u$$

as $u = g * m$ where

$$g(x) = \frac{1}{2}e^{-|x|}, \tag{1.5}$$

(1.4) can be expressed as a conservation law [17]:

$$\partial_t u + \partial_x \left[\frac{1}{2}u^2 + g * (\frac{3}{2}u^2)\right] = 0. \tag{1.6}$$

Weak solutions are functions which satisfy (1.6) in the usual distributional sense.

The DP equation (1.6) admits weak solutions representing a wave train of discontinuous solitons called shockpeakons [18], given by

$$u(x,t) = 2\sum_{i=1}^{n} m_i(t) g(x - x_i(t)) + 2\sum_{i=1}^{n} s_i(t) g'(x - x_i(t)), \tag{1.7}$$

where $g(x-y) = \frac{1}{2}e^{-|x-y|}$, and

$$g'(x) = -\frac{1}{2}\operatorname{sgn}(x) e^{-|x|}, \tag{1.8}$$

with the convention $g'(0) = 0$.

The n-shockpeakon (1.7) is a weak solution of the nonlocal DP equation (1.6) if and only if the time-dependent parameters x_i (positions), m_i (momenta), and s_i (shock strengths), $i = 1, ..., n$, satisfy the following dynamical system of $3n$ ODEs [18]:

$$\frac{dx_i}{dt} = u(x_i), \quad \frac{dm_i}{dt} = 2s_i u(x_i) - 2m_i\{\partial_x u(x_i)\}, \quad \frac{ds_i}{dt} = -s_i\{\partial_x u(x_i)\}, \tag{1.9}$$

where

$$u(x_i) := u(x_i(t), t) = 2\sum_{k=1}^{n} m_k g(x_i - x_k) + 2\sum_{k=1}^{n} s_k g'(x_i - x_k) \tag{1.10}$$

and

$$\{\partial_x u(x_i)\} = \{u_x(x_i)\} := 2\sum_{k=1}^{n} m_k g'(x_i - x_k) + 2\sum_{k=1}^{n} s_k g(x_i - x_k). \tag{1.11}$$

Remark 1.1. *Of course, if $s_i = 0$, for all $i = 1, 2, ..., n$, then the n-shockpeakon ansatz (1.7) reduces to the ordinary n-peakon of the DP equation [19, 20]*

$$u(x,t) = 2\sum_{i=1}^{n} m_i(t) g(x - x_i(t)), \tag{1.12}$$

where g is given by (1.5). The $2n$ DP multipeakon ODEs are understood in the case where the $m_i, i = 1, .., n$, are positive.

The DP equation is also completely integrable, possesses a Lax pair, a bi-Hamiltonian structure, and an infinite hierarchy of symmetries and conservation laws [21]. A method for the classification of all traveling wave solutions for some dispersive nonlinear wave equations that encompasses the DP equation (1.3) was presented in [5]. Global existence, L^1-stability and uniqueness results for weak solutions in $L^1(\mathbb{R}) \cap BV(\mathbb{R})$ and in $L^2(\mathbb{R}) \cap L^4(\mathbb{R})$ with an additional entropy condition were obtained in [22]. Here, $BV(\mathbb{R})$ is the space of functions with bounded variation. The peakon-antipeakon interactions and shock waves in the DP equation were studied in [18, 23, 24]. In [18], the author showed that a jump discontinuity forms when a peakon collides with an antipeakon, and that the entropy weak solution in this case is described by a shockpeakon. Stochastic perturbations of the DP equation were studied in [1, 25]. In [1], the authors considered the Cauchy problem for a stochastic (additive) perturbation of the DP equation (1.3) with the initial conditions in the class $L^2(\mathbb{R}) \cap L^{2+\epsilon}(\mathbb{R})$, for any small $\epsilon > 0$, and established the existence of a global pathwise solution, via kinetic theory [26]. In [25], the authors studied the global well-posedness of a stochastic dynamic driven by a linear and multiplicative noise, in the space of sample paths $C([0,\infty), H^s(\mathbb{R}))$, $s > 3/2$.

This work is concerned with the existence of multi-shockpeakons of the stochastic Degasperis-Procesi (SDP) equation, in the one-dimensional domain $\mathbb{R}$, with a multiplicative noise.

The stochastic evolution differential equations are given by

$$du = -(u\partial_x u + \partial_x p)\ dt - \sum_{j=1}^{3n} \xi^j(t)(\partial_x u) \circ dW_t^j, \tag{1.13}$$

$$(1-\partial_{xx})\,p \;=\; \frac{3}{2}u^2, \tag{1.14}$$

where $x \in \mathbb{R}$. Here $u = u(x,t)$ denotes the velocity of the fluid, $\{\xi^j\}_{j=1}^{3n}$ is a set of prescribed functions depending only on the time variable, the symbol $\circ$ denotes a Stratonovich stochastic process and $\{W_t^j\}_{j=1}^{3n}$ is a set of Brownian motions.

Equations (1.13) and (1.14) can be reformulated into the following form:

$$du = -\partial_x\left[\frac{1}{2}u^2 + g*(\frac{3}{2}u^2)\right]dt - \sum_{j=1}^{3n}\xi^j(\partial_x u)\circ dW_t^j. \tag{1.15}$$

We say that u is a weak solution to (1.15) if it satisfies the following integral equation

$$\iint \phi du dx = \iint \partial_x\phi\left[\frac{1}{2}u^2 + g*(\frac{3}{2}u^2)\right]dtdx - \sum_{j=1}^{3n}\iint \phi[\xi^j(\partial_x u)]\circ dW_t^j dx, \tag{1.16}$$

for any test function $\phi(\cdot,t)\in C_0^\infty(\mathbb{R})$.

We seek weak solutions of the SDP equation (1.15) of the form

$$u(x,t) \;=\; \sum_{i=1}^{n}u^i(x,t) = 2\sum_{i=1}^{n}m_i(t)g(x-x_i(t)) + 2\sum_{i=1}^{n}s_i(t)g'(x-x_i(t)). \tag{1.17}$$

Here we define u^i to be the contribution from a single shockpeakon, $x_i(t), m_i(t)$ and $s_i(t)$, $i = 1,2,...,n$, are the positions, momenta and shock strengths, respectively, g and g' are given by (1.5) and (1.8), respectively, and we take $g'(0) = 0$.

Remark 1.2. *Of course, if $s_i = 0$ for all $i = 1,2,...,n$, then the multi-shockpeakon (1.17) reduces to the n-peakon*

$$u(x,t) \;=\; 2\sum_{i=1}^{n}m_i(t)g(x-x_i(t)), \tag{1.18}$$

where g is given by (1.5).

2. Multi-shockpeakon soliton solutions for the SDP equation (1.15)

Let us state here the main result of this paper:

Theorem 2.1. *The shockpeakon (1.17) is a weak solution of the SDP equation (1.15) if and only if the stochastic process for $(x_1(t),...,x_n(t),m_1(t),...,m_n(t),s_1(t),...,s_n(t))$ is given by the following system of 3n stochastic differential equations (SDEs)*

$$\begin{aligned} dx_i(t) &= u(x_i)dt + \sum_{j=1}^{3n} \xi^j(t) \circ dW_t^j && (2.1)\\ dm_i(t) &= [2s_i u(x_i) - 2m_i\{\partial_x u(x_i)\}]\, dt, && (2.2)\\ ds_i(t) &= -s_i\{\partial_x u(x_i)\}dt, && (2.3) \end{aligned}$$

where

$$u(x_i) = 2\sum_{k=1}^{n} m_k g(x_i - x_k) + 2\sum_{k=1}^{n} s_k g'(x_i - x_k) \quad (2.4)$$

and

$$\{u_x(x_i)\} = \{\partial_x u(x_i)\} := 2\sum_{k=1}^{n} m_k g'(x_i - x_k) + 2\sum_{k=1}^{n} s_k g(x_i - x_k), \quad (2.5)$$

$i = 1, 2, ..., n$.

Proof. Suppose that

$$\begin{aligned} dx_i(t) &= a_i(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t), s_1(t), ..., s_n(t), t)dt \\ &+ \sum_{j=1}^{3n} b_{ij}(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t), s_1(t), ..., s_n(t), t) \circ dW_t^j \end{aligned} \quad (2.6)$$

$$\begin{aligned} dm_i(t) &= c_i(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t), s_1(t), ..., s_n(t), t)dt \\ &+ \sum_{j=1}^{3n} d_{ij}(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t), s_1(t), ..., s_n(t), t) \circ dW_t^j \end{aligned} \quad (2.7)$$

$$\begin{aligned} ds_i(t) &= e_i(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t), s_1(t), ..., s_n(t), t)dt \\ &+ \sum_{j=1}^{3n} f_{ij}(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t), s_1(t), ..., s_n(t), t) \circ dW_t^j, \end{aligned} \quad (2.8)$$

$i = 1, 2, ..., n$, are the stochastic differential equations for the evolution of $x_i(t), m_i(t), s_i(t)$.

In what follows, we will use the abbreviations $g(x - x_i(t))$ as g_i, $g'(x - x_i(t))$ as g_i', $\delta_i := \delta(x - x_i(t))$, $i = 1, ..., n$, (δ is the Dirac delta distribution) and for $i = 1, ..., n$, $j = 1, ..., 3n$,

$$a_i(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t), s_1(t), ..., s_n(t), t) = a_i,$$
$$b_{ij}(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t), s_1(t), ..., s_n(t), t) = b_{ij},$$
$$c_i(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t), s_1(t), ..., s_n(t), t) = c_i,$$
$$d_{ij}(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t), s_1(t), ..., s_n(t), t) = d_{ij},$$
$$e_i(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t), s_1(t), ..., s_n(t), t) = e_i,$$
$$f_{ij}(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t), s_1(t), ..., s_n(t), t) = f_{ij}.$$

We will look for solutions of the SDP equation (1.15) of the form (1.17) with $x_i(t), m_i(t), s_i(t)$ obeying (2.6), (2.7) and (2.8), respectively, the functions a_i, c_i, e_i, $i = 1, ..., n$, b_{ij}, d_{ij} and f_{ij}, $i = 1, ..., n$, $j = 1, ..., 3n$, satisfying all the necessary conditions for the existence and uniqueness of solutions to (2.6)–(2.8) and their extendability to a given time interval $[t_0, t_1]$ with $t_1 > t_0 \geq 0$.

Taking the differential of (1.17) and substituting in equations (2.6)–(2.8) we obtain

$$\begin{aligned}
du &= \sum_{i=1}^{n}\left[\frac{\partial u^i}{\partial x_i}dx_i + \frac{\partial u^i}{\partial m_i}dm_i + \frac{\partial u^i}{\partial s_i}ds_i\right] \\
&= 2\sum_{i=1}^{n} m_i(t)dg(x - x_i(t)) + 2\sum_{i=1}^{n} g(x - x_i(t))dm_i(t) \\
&+ 2\sum_{i=1}^{n} s_i(t)dg'(x - x_i(t)) + 2\sum_{i=1}^{n} g(x - x_i(t))ds_i(t) \\
&= 2\sum_{i=1}^{n} m_i(t)dg(x - x_i(t)) + 2\sum_{i=1}^{n} g(x - x_i(t))[c_i dt + \sum_{j=1}^{3n} d_{ij} \circ dW_t^j] \\
&+ 2\sum_{i=1}^{n} s_i(t)dg'(x - x_i(t)) + 2\sum_{i=1}^{n} g(x - x_i(t))[e_i dt + \sum_{j=1}^{3n} f_{ij} \circ dW_t^j]
\end{aligned}$$

$$\begin{aligned}
&= 2\sum_{i=1}^{n} m_i(t)\mathrm{sgn}(x - x_i(t))g(x - x_i(t))a_i dt \\
&+ 2\sum_{i=1}^{n} m_i(t)\mathrm{sgn}(x - x_i(t))g(x - x_i(t))\sum_{j=1}^{3n} b_{ij} \circ dW_t^j \\
&+ 2\sum_{i=1}^{n} g(x - x_i(t))[c_i dt + \sum_{j=1}^{3n} d_{ij} \circ dW_t^j] \\
&+ 2\sum_{i=1}^{n} s_i(t)[\frac{\partial}{\partial x_i} g'(x - x_i(t))a_i]dt
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 2\sum_{i=1}^{n} s_i(t)\frac{\partial}{\partial x_i} g'(x-x_i(t))\sum_{j=1}^{3n} b_{ij}\circ dW_t^j \\
&+ \quad 2\sum_{i=1}^{n} g'(x-x_i(t))[e_i dt + \sum_{j=1}^{3n} f_{ij}\circ dW_t^j]
\end{aligned}
$$

$$
\begin{aligned}
= &\quad +2\sum_{i=1}^{n} m_i(t)\mathrm{sgn}(x-x_i(t))g(x-x_i(t))a_i dt \\
+ &\quad 2\sum_{i=1}^{n} m_i(t)\mathrm{sgn}(x-x_i(t))g(x-x_i(t))\sum_{j=1}^{3n} b_{ij}\circ dW_t^j \\
+ &\quad 2\sum_{i=1}^{n} g(x-x_i(t))[c_i dt + \sum_{j=1}^{3n} d_{ij}\circ dW_t^j] \\
+ &\quad 2\sum_{i=1}^{n} s_i(t)g(x-x_i(t))(-1+2\delta(x-x_i(t)))a_i dt \\
+ &\quad 2\sum_{i=1}^{n} s_i(t)g(x-x_i(t))(-1+2\delta(x-x_i(t)))\sum_{j=1}^{3n} b_{ij}\circ dW_t^j \\
+ &\quad 2\sum_{i=1}^{n} g'(x-x_i(t))[e_i dt + \sum_{j=1}^{3n} f_{ij}\circ dW_t^j].
\end{aligned} \tag{2.9}
$$

Furthermore, using formula (A3) in [18],

$$
u^2 \quad = \quad 4\sum_{k,l=1}^{n}(m_k m_l g_k g_l + s_k s_l g'_k g'_l + m_k s_l g_k g'_l + s_k m_l g'_k g_l). \tag{2.10}
$$

This implies that

$$
\begin{aligned}
\partial_x u^2 \quad = &\quad \sum_{k,l=1}^{n} 4\{(m_k m_l + s_k s_l)(g_k g'_l + g'_k g_l) + (m_k s_l + s_k m_l)(g_k g_l + g'_k g'_l) \\
- &\quad 4[s_k s_l (g'_k \delta_l + g'_l \delta_k) + m_k s_l g_k \delta_l + m_l s_k g_l \delta_k]\}
\end{aligned} \tag{2.11}
$$

(using formula (A4) in [18]).

From (2.10) and formula (A7) in [18] we have

$$
2\partial_x p \quad = \quad 2g' * [\frac{3}{2}u^2] = \sum_{k,l=1}^{n} 4(-(m_k m_l + s_k s_l)(g_k g'_l + g'_k g_l)
$$

$$
\begin{aligned}
&- \quad 4(m_k s_l + s_k m_l)(g_k g_l + g_k' g_l')) + 2\sum_{k,l=1}^{n} e^{-|x_k-x_l|}((m_k m_l + s_k s_l)(g_k' + g_l') \\
&+ \quad m_k s_l(4g_k - g_l) + s_k m_l(-g_k + 4g_l)) + \\
&+ \quad 2\sum_{k,l=1}^{n} \operatorname{sgn}(x_k - x_l) e^{-|x_k-x_l|}((2m_k m_l - s_k s_l)(g_k - g_l) + (m_k s_l + s_k m_l)(g_k' - g_l')).
\end{aligned} \tag{2.12}
$$

From (2.9), (2.11) and (2.12) we obtain

$$
\begin{aligned}
0 \quad = \quad & 2du + \partial_x u^2 dt + 2(\partial_x p)dt + 2\sum_{j=1}^{3n} \xi^j(\partial_x u) \circ dW_t^j = 2du + 2\sum_{j=1}^{3n} \xi^j(\partial_x u) \circ dW_t^j \\
- \quad & 8\left[\sum_{k=1}^{n}\left(\sum_{l=1}^{n} s_k g_l'(x_k) + \sum_{l=1}^{n} m_l g_l(x_k)\right) s_k \delta_k\right] dt \\
+ \quad & 2\left[\sum_{k=1}^{n}\left(2m_k \sum_{l=1}^{n} m_l e^{-|x_k-x_l|} + 2s_k \sum_{l=1}^{n} s_l e^{-|x_k-x_l|}\right) g_k'\right] dt \\
+ \quad & 2\left[\sum_{k=1}^{n}\left(4m_k \sum_{l=1}^{n} s_l e^{-|x_k-x_l|} - 2s_k \sum_{l=1}^{n} m_l e^{-|x_k-x_l|}\right) g_k\right] dt \\
+ \quad & 2\left[\sum_{k=1}^{n}\left(4m_k \sum_{l=1}^{n} m_l \operatorname{sgn}(x_k - x_l) e^{-|x_k-x_l|} - 2s_k \sum_{l=1}^{n} s_l \operatorname{sgn}(x_k - x_l) e^{-|x_k-x_l|}\right) g_k\right] dt \\
+ \quad & 2\left[\sum_{k=1}^{n}\left(2m_k \sum_{l=1}^{n} s_l \operatorname{sgn}(x_k - x_l) e^{-|x_k-x_l|} + 2s_k \sum_{l=1}^{n} m_l \operatorname{sgn}(x_k - x_l) e^{-|x_k-x_l|}\right) g_k'\right] dt \\
= \quad & 2du + 2\sum_{j=1}^{3n} \xi^j(\partial_x u) \circ dW_t^j \\
-4 \quad & \left[\sum_{k=1}^{n} s_k u(x_k) g_k \delta_k - \sum_{k=1}^{n}(2m_k\{u_x(x_k)\} - 2s_k u(x_k)) g_k - \sum_{k=1}^{n}(s_k\{u_x(x_k)\} + m_k u(x_k)) g_k'\right] dt \\
= \quad & \left[\sum_{i=1}^{n}(-4m_i a_i + 4e_i) g_i'\right] dt \\
+ \quad & \left[\sum_{i=1}^{n}(4c_i - 4s_i a_i + 4s_i \delta_i a_i) g_i\right] dt \\
- \quad & \left[4\sum_{i=1}^{n} s_i u(x_i) g_i \delta_i - 4\sum_{i=1}^{n}(2m_i\{u_x(x_i)\} - 2s_i u(x_i)) g_i - 4\sum_{i=1}^{n}(s_i\{u_x(x_i)\} + m_i u(x_i)) g_i'\right] dt \\
+ \quad & 4\sum_{i=1}^{n} m_i(t) g_i' \sum_{j=1}^{3n} \xi^j \circ dW_t^j + 4\sum_{i=1}^{n} s_i(t) g_i \sum_{j=1}^{3n} \xi^j \circ dW_t^j - 8\sum_{i=1}^{n} s_i(t) \delta_i g_i \sum_{j=1}^{3n} \xi^j \circ dW_t^j \\
+ \quad & 4\sum_{i=1}^{n} g_i \sum_{j=1}^{3n} d_{ij} \circ dW_t^j - 4\sum_{i=1}^{n} g_i s_i \sum_{j=1}^{3n} b_{ij} \circ dW_t^j - 4\sum_{i=1}^{n}(m_i g_i') \sum_{j=1}^{3n} b_{ij} \circ dW_t^j
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 4\sum_{i=1}^{n} g_i' \sum_{j=1}^{3n} f_{ij} \circ dW_t^j + 8\sum_{i=1}^{n} s_i g_i \delta_i \sum_{j=1}^{3n} b_{ij} \circ dW_t^j \\
&= \quad \left[\sum_{i=1}^{n} (-4m_i a_i + 4e_i) g_i'\right] dt \\
&+ \quad \left[\sum_{i=1}^{n} (4c_i - 4s_i a_i + 4s_i \delta_i a_i) g_i\right] dt \\
&- \quad \left[4\sum_{i=1}^{n} s_i u(x_i) g_i \delta_i - 4\sum_{i=1}^{n} (2m_i\{u_x(x_i)\} - 2s_i u(x_i)) g_i - 4\sum_{i=1}^{n} (s_i\{u_x(x_i)\} + m_i u(x_i)) g_i'\right] dt \\
&+ \quad 4\sum_{i=1}^{n}\sum_{j=1}^{3n} [m_i(\xi^j - b_{ij}) + f_{ij}] g_i' \circ dW_t^j + 4\sum_{i=1}^{n}\sum_{j=1}^{3n} [s_i(\xi^j - b_{ij}) + d_{ij}] g_i \circ dW_t^j \\
&+ \quad 8\sum_{i=1}^{n}\sum_{j=1}^{3n} (b_{ij} - \xi^j) s_i g_i \delta_i \circ dW_t^j.
\end{aligned}
\tag{2.13}
$$

In order to verify that (1.17) is a weak solution of the SDP equation (1.15) we will substitute it and (2.13) into (1.16) to obtain

$$
\begin{aligned}
2\iint \phi \sum_{i=1}^{n} [-m_i g_i' a_i + s_i g_i (2\delta_i - 1) a_i + g_i c_i + g_i' e_i] dt dx \\
- \iint \partial_x \phi \left[\frac{1}{2}u^2 + g * (\frac{3}{2}u^2)\right] dt dx \\
= -2\iint \phi \sum_{i=1}^{n}\sum_{j=1}^{3n} [m_i(\xi^j - b_{ij}) + f_{ij}] g_i' \circ dW_t^j dx \\
-2\iint \phi \sum_{i=1}^{n}\sum_{j=1}^{3n} [s_i(\xi^j - b_{ij}) + d_{ij}] g_i \circ dW_t^j dx \\
+4\iint \phi \sum_{i=1}^{n} s_i \delta_i g_i \sum_{j=1}^{3n} (\xi^j - b_{ij}) \circ dW_t^j dx.
\end{aligned}
\tag{2.14}
$$

We must show that the deterministic and stochastic parts of the above equation will both be equal to zero.

Consider the multi-shockpeakon solution for u from (1.17). This multi-shockpeakon is a weak solution to the deterministic equation, and therefore the left-hand side of (2.14) is zero. Moreover from (2.13) the left-hand side of (2.14) is zero if and only if $a_i = u(x_i)$, $c_i = 2s_i u(x_i) - 2m_i\{\partial_x u(x_i)\}$ and $e_i = -s_i\{\partial_x u(x_i)\}$, $i = 1, ..., n$, where $u(x_i)$ and $\{\partial_x u(x_i)\}$ are given by (2.4) and (2.5) respectively since $\{\delta_i, g_i, g_i'\}_{i=1}^n$ is a linearly independent set. We also have

$$
-[m_i(\xi^j - b_{ij}) + f_{ij}] g_i' - [s_i(\xi^j - b_{ij}) + d_{ij}] g_i + 2s_i \delta_i g_i (\xi^j - b_{ij}) = 0
$$

almost everywhere if and only if $b_{ij} = \xi^j$, $i = 1, 2, ..., n$, $j = 1, ..., 3n$, and $d_{ij} = f_{ij} = 0$, $i = 1, 2, ..., n$, $j = 1, ..., 3n$, since $\{\delta_i, g_i, g_i'\}_{i=1}^n$ is a linearly independent set. □

From the above Theorem we deduce the following results:

Corollary 2.2. *The n-peakon (1.18) is a weak solution of the SDP equation (1.15) if and only if the stochastic process for* $(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t))$*, is given by the following system of* $2n$ *SDEs*

$$\begin{aligned} dx_i(t) &= u(x_i)dt + \sum_{j=1}^{3n} \xi^j \circ dW_t^j, \\ dm_i(t) &= [-2m_i\{u_x(x_i)\}]\, dt, \end{aligned}$$

$i = 1, 2, ..., n.$

Corollary 2.3. *The stochastic process for*

$$(x_1(t), ..., x_n(t), m_1(t), ..., m_n(t), s_1(t), ..., s_n(t))$$

(2.1)-(2.3) becomes the deterministic system of $3n$ *ODEs (1.9) if and only if* $\xi^j = 0$, $j = 1, ..., 3n$.

Remark 2.4. *From Corollary 2.3 above and Theorem 2.1 in [18] it follows that the weak multi-shockpeakon of the form (1.17), with* x_i, m_i *and* s_i, $i = 1, .., n$, *satisfying the system (2.1)–(2.3), with* $\xi^j = 0$, $j = 1, .., 3n$, *is a solution of the deterministic DP equation in the weak form (1.6) (Equation (1.15) with* $\xi^j = 0$, $j = 1, ..., 3n$*).*

Remark 2.5. *From (2.2), (2.4) and (2.5) and Proposition 4.1 in [27], the multi-shockpeakon (1.17) conserves momentum.*

3. Example

Letting $n = 1$ in (2.1)–(2.3) and choosing $\xi^j(t) = constant = \xi^j(t_0)$, $j = 1, 2, 3$, we see that the dynamics of a single shockpeakon is described by the stochastic equations

$$\begin{aligned} dx_1(t) &= m_1 dt + \sum_{j=1}^{3} \xi^j(t_0) \circ dW_t^j && (3.1) \\ dm_1(t) &= 0 && (3.2) \\ ds_1(t) &= -s_1^2 dt. && (3.3) \end{aligned}$$

Thus $m_1(t) = m_1(t_0)$, and therefore

$$x_1(t) = x_1(t_0) + m_1(t_0)(t - t_0) + \xi^j(t_0) \sum_{j=1}^{3} (W^j(t) - W^j(t_0)).$$

The equation (3.3) is equivalent to $s_1 \equiv 0$ or $\frac{d}{dt}(1/s_1) = 1$; Consequently

$$s_1(t) = \frac{s_1(t_0)}{1 + (t - t_0)s_1(t_0)}.$$

It follows that

$$\begin{aligned} u(x,t) &= m_1(t_0)e^{-|x-(x_1(t_0)+m_1(t_0)(t-t_0)+\xi^j(t_0)\sum_{j=1}^{3}(W^j(t)-W^j(t_0)))|} \\ &+ 2\frac{s_1(t_0)}{1+(t-t_0)s_1(t_0)}g'(x-(x_1(t_0)+m_1(t_0)(t-t_0)+\xi^j(t_0)\sum_{j=1}^{3}(W^j(t)-W^j(t_0))), \end{aligned}$$

where g' is given by (1.8).

3.1. Numerical simulations

In this section, numerical simulations are used to illustrate the effect of the stochastic term in the example above (See Figure 1–10). We take $\xi^j \equiv 1$, $j = 1, 2, 3$, and use that the increments $W^j(t + \Delta t) - W^j(t)$, j=1,2,3, have a normal distribution with zero expected value and variance equal to Δt.

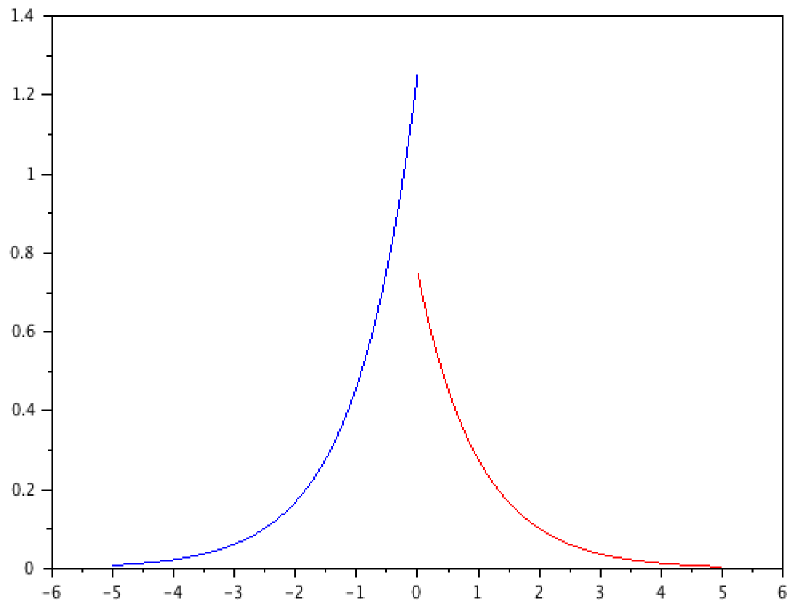

Figure 1. Shockpeakon $u(x, t_0)$ with position $x_1(t_0) = 0$, momentum $m_1(t_0) = 1$ and shock strength $s_1(t_0) = 1/4$.

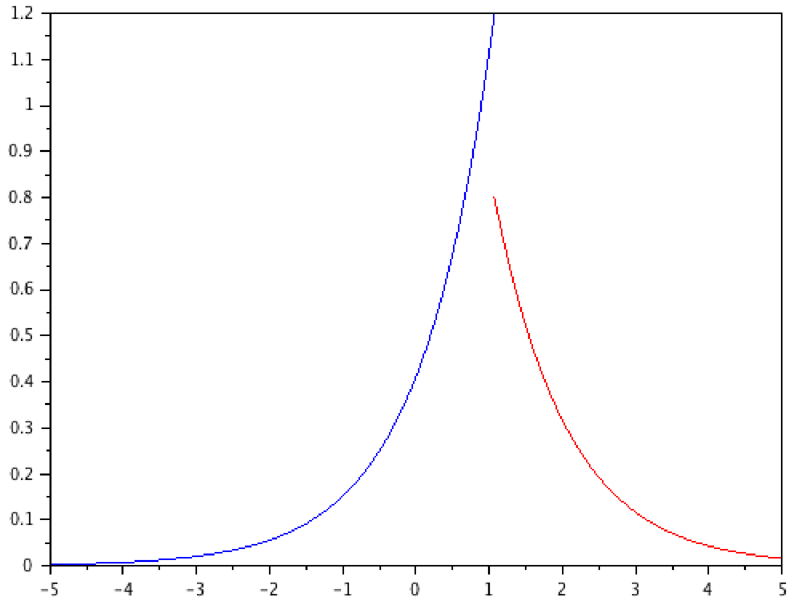

Figure 2. Shockpeakon $u(x, t = 2)$ with position $x_1(t_0 = 1) = 0$, momentum $m_1(t_0 = 1) = 1$ and shock strength $s_1(t_0 = 1) = 1/4$.

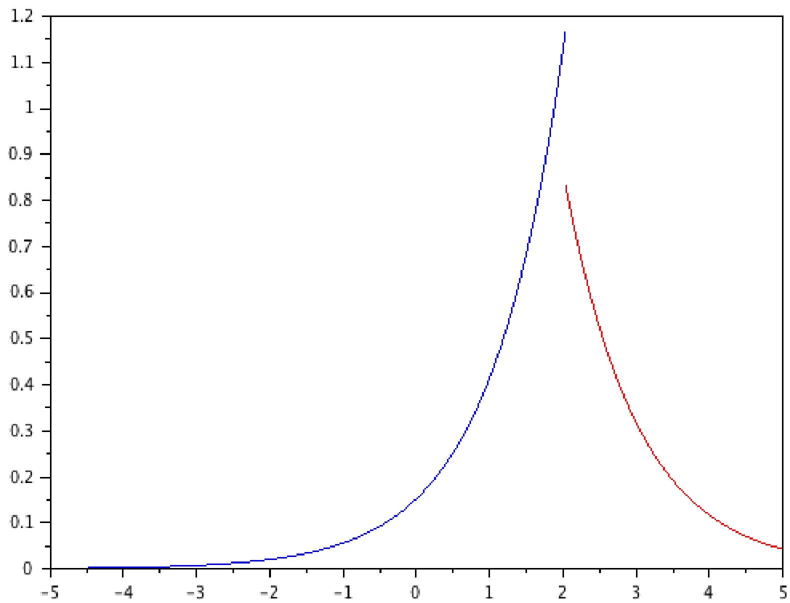

Figure 3. Shockpeakon $u(x, t = 3)$ with position $x_1(t_0 = 1) = 0$, momentum $m_1(t_0 = 1) = 1$ and shock strength $s_1(t_0 = 1) = 1/4$.

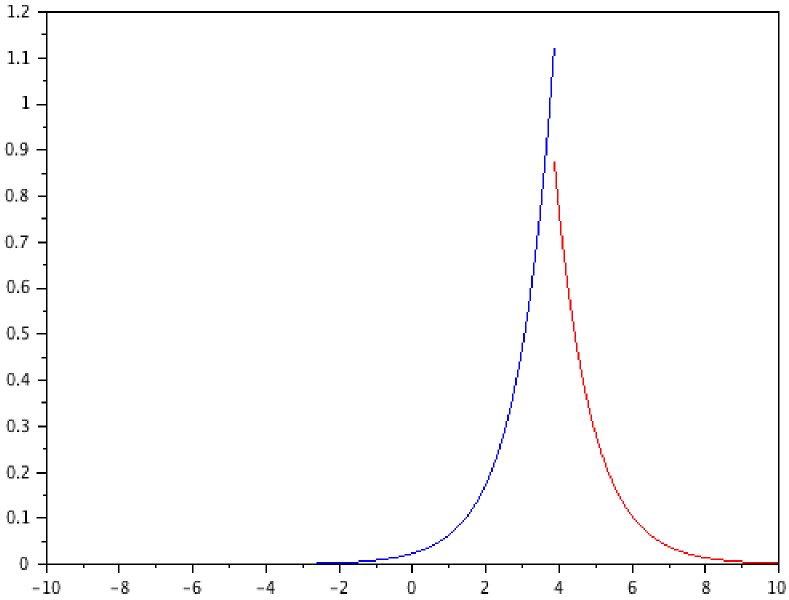

Figure 4. Shockpeakon $u(x, t = 5)$ with position $x_1(t_0 = 1) = 0$, momentum $m_1(t_0 = 1) = 1$ and shock strength $s_1(t_0 = 1) = 1/4$.

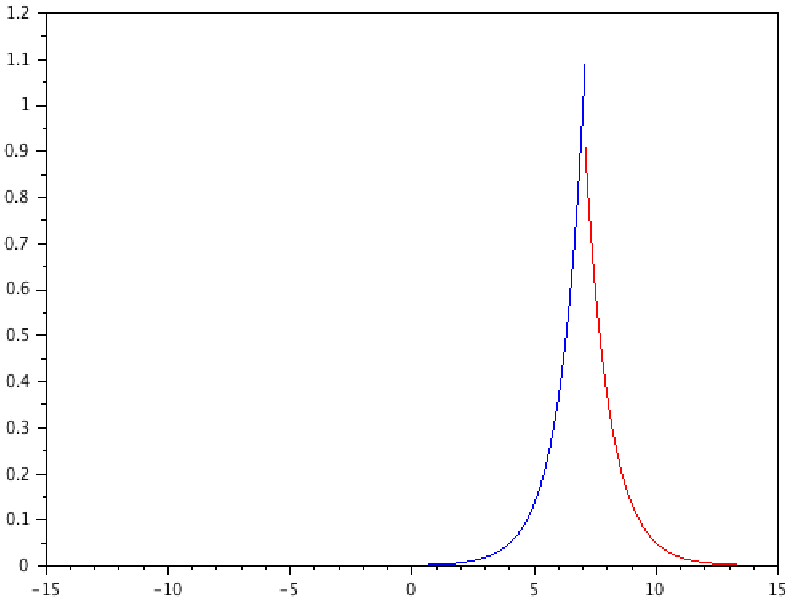

Figure 5. Shockpeakon $u(x, t = 8)$ with position $x_1(t_0 = 1) = 0$, momentum $m_1(t_0 = 1) = 1$ and shock strength $s_1(t_0 = 1) = 1/4$.

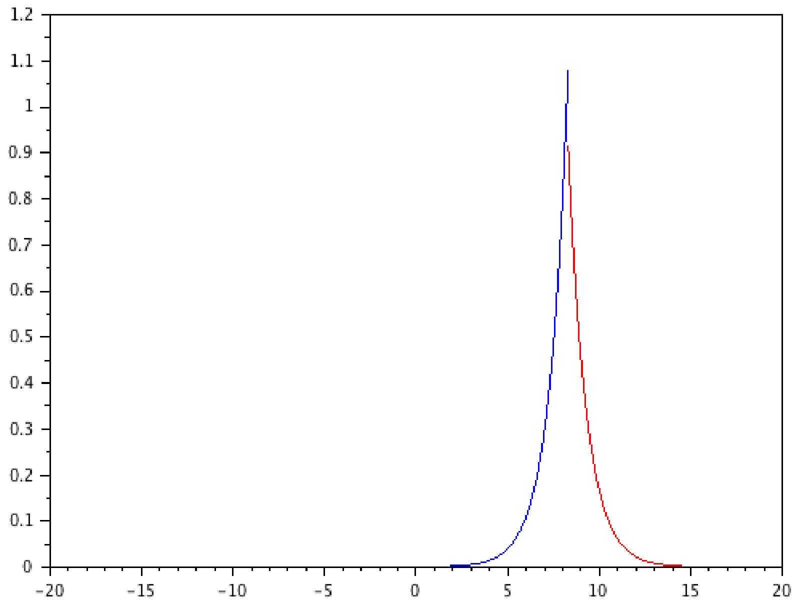

Figure 6. Shockpeakon $u(x, t = 9)$ with position $x_1(t_0 = 1) = 0$, momentum $m_1(t_0 = 1) = 1$ and shock strength $s_1(t_0 = 1) = 1/4$.

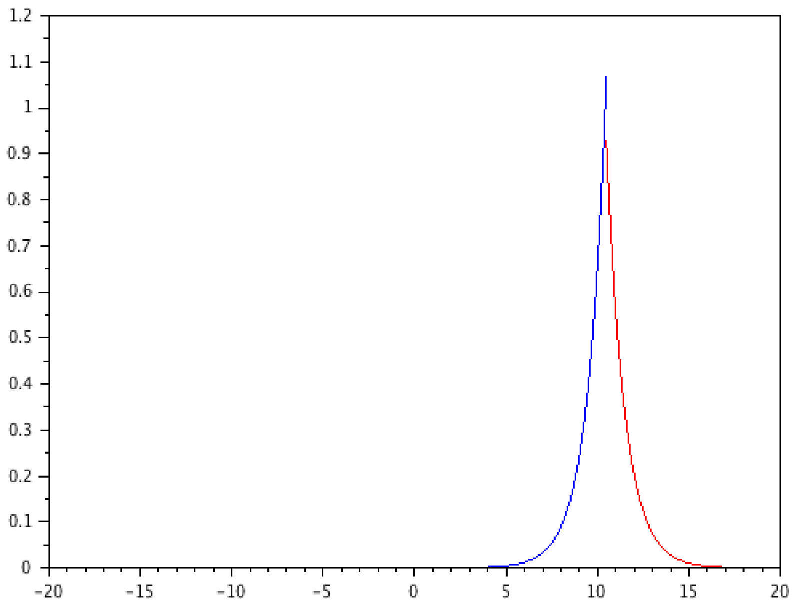

Figure 7. Shockpeakon $u(x, t = 11)$ with position $x_1(t_0 = 1) = 0$, momentum $m_1(t_0 = 1) = 1$ and shock strength $s_1(t_0 = 1) = 1/4$.

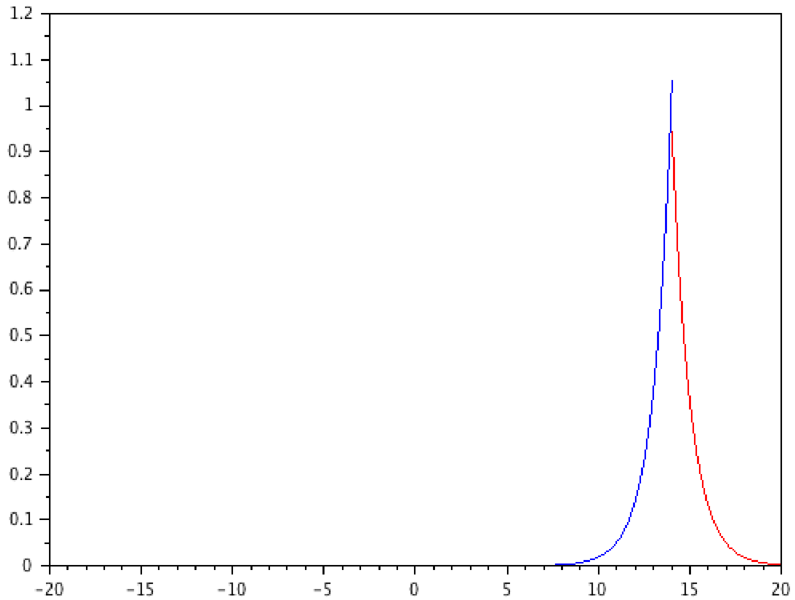

Figure 8. Shockpeakon $u(x, t = 15)$ with position $x_1(t_0 = 1) = 0$, momentum $m_1(t_0 = 1) = 1$ and shock strength $s_1(t_0 = 1) = 1/4$.

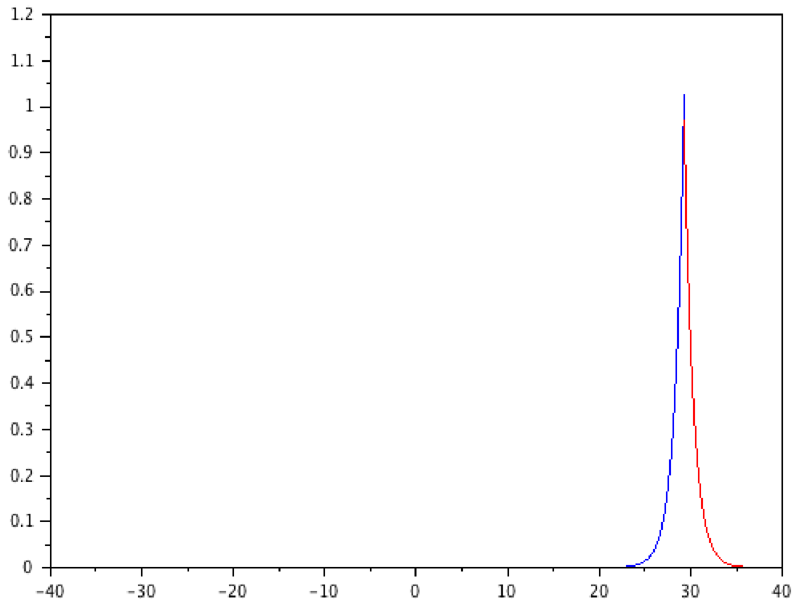

Figure 9. Shockpeakon $u(x, t = 30)$ with position $x_1(t_0 = 1) = 0$, momentum $m_1(t_0 = 1) = 1$ and shock strength $s_1(t_0 = 1) = 1/4$.

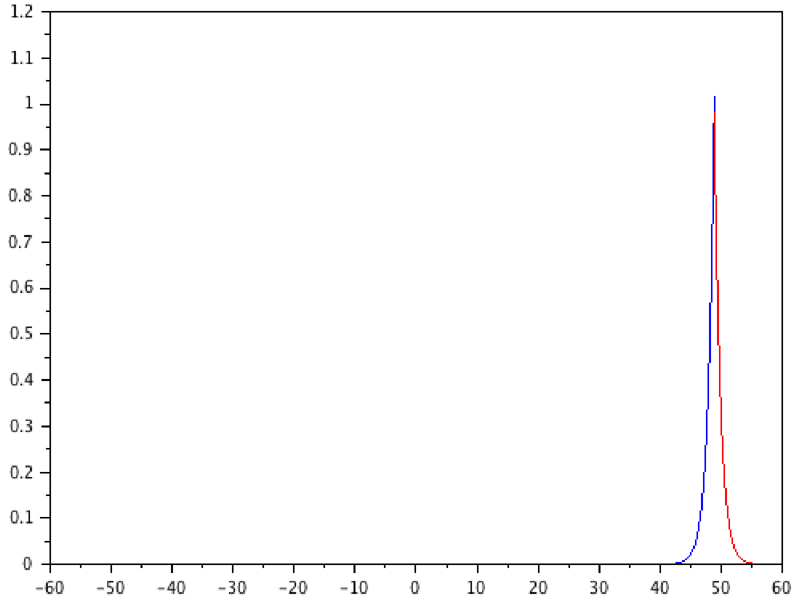

Figure 10. Shockpeakon $u(x, t = 50)$ with position $x_1(t_0 = 1) = 0$, momentum $m_1(t_0 = 1) = 1$ and shock strength $s_1(t_0 = 1) = 1/4$.

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Conflict of interest

The author declares there is no conflicts of interest.

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