Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 30(6): 2283–2302
DOI: 10.3934/era.2022116
Received: 23 September 2021
Revised: 19 December 2021
Accepted: 03 January 2022
Published: 21 April 2022

Research article

Heteroclinic orbits between static classes of time periodic Tonelli Lagrangian systems

Guowei Yu*

Chern Institute of Mathematics and LPMC, Nankai University, Tianjin, China

* **Correspondence:** Email: yugw@nankai.edu.cn.

Abstract: In this paper, we show the existence of heteroclinic orbits between two different static classes of the Aubry set of a Tonelli Lagrangian when the Aubry set has only finite static classes (which is a generic condition in the sense of Mañé) and the Mañé set satisfies certain isolated condition.

Keywords: Lagrangian systems; heteroclinic orbits; action minimizers; Aubry-Mather theory

1. Introduction

Let M be a closed connected smooth Riemannian manifold, and $L \in C^2(TM \times \mathbb{R}, \mathbb{R})$ be a Lagrangian satisfying the following conditions introduced by Mather [1]:

1. **Periodicity**: L is 1-periodic in time, i.e., $L(x, v, t) = L(x, v, t + 1)$ for all $(x, v, t) \in TM \times \mathbb{R}$;
2. **Positive definiteness**: $\partial^2 L/\partial v^2(x, v, t)$ is positive definite, as a quadratic form, for all $(x, v, t) \in TM \times \mathbb{R}$;
3. **Superlinear growth**: $\lim_{\|v\|_x \to +\infty} \frac{L(x,v,t)}{\|v\|_x} = +\infty$, in each fiber, i.e., for each $A \in \mathbb{R}$, there exists $B(A) \in \mathbb{R}$ such that $L(x, v, t) \geq A\|v\|_x - B(A)$, for all $(x, v, t) \in TM \times \mathbb{R}$.
4. **Completeness**: Every orbit of the Euler-Lagrange flow introduced by L is defined for all time.

Remark 1.1. $\|\cdot\|_x$ *denotes the norm associated to the Riemannian metric on M. Since M is compact, condition* (3) *in the above definition is independent of which Riemannian metric is chosen on M. For any $x, y \in M$, we use $d(x, y)$ denotes the distance defined by the Riemannian metric.*

For any compact interval $[a, b] \subset \mathbb{R}$, we denote $C([a, b], M)$ by the set of all absolutely continuous curves defined on $[a, b]$. The action A_L of L on any $\gamma \in C([a, b], M)$ is defined by

$$A_L(\gamma) := \int_a^b L(d\gamma(t), t)dt, \text{ where } d\gamma(t) = (\gamma(t), \dot{\gamma}(t)).$$

Given $x, y \in M$, we set $C_{[a,b]}(x, y) := \{\gamma \in C([a, b], M) : \gamma(a) = x, \gamma(b) = y\}$ and $C_T(x, y) := C_{[0,T]}(x, y)$ for any $T > 0$. The extremals of A_L in $C_{[a,b]}(x, y)$ are solutions of the Euler-Lagrange equation which in local coordinates is given by

$$\frac{d}{dt}\frac{\partial L}{\partial v}(\gamma(t), \dot\gamma(t), t) = \frac{\partial L}{\partial x}(\gamma(t), \dot\gamma(t), t). \tag{EL}$$

Furthermore we assume L is a critical Lagrangian. A Lagrangian is called critical if the infimum of the actions of all closed curves is 0. Every Tonelli Lagrangian can be made critical by adding a proper constant, see [2].

For any $(x, a), (y, b) \in M \times \mathbb{R}$ with $a < b$, we define

$$h((x, a), (y, b)) = \inf\{\int_a^b L(d\gamma(t), t)dt : \gamma \in C_{[a,b]}(x, y)\}.$$

The infimum above is actually a minimum, see Theorem 2.1.

For any $(x, \alpha), (y, \beta) \in M \times \mathbb{T}$, we define

$$h^\infty((x, \alpha), (y, \beta)) = \liminf_{T\in\mathbb{Z}^+, T\to+\infty} h((x, \alpha), (y, \beta + T)),$$

$$\Phi((x, \alpha), (y, \beta)) = \inf_{T\in\mathbb{Z}^+} h((x, \alpha), (y, \beta + T)).$$

The following propositions of the functions we just defined will be needed in this paper and we refer the interested readers to [3] for their proofs.

Proposition 1.1. *1. The function $h : M\times\mathbb{R}\times M\times\mathbb{R} \to \mathbb{R}; ((x, a), (y, b)) \to h((x, a), (y, b))$ is Lipschitz continuous and bounded on $\{b - a \geq 1\}$.*

2. For any $\alpha, \beta \in \mathbb{T}$, $h^\infty((*, \alpha), (*, \beta)) : M \times M \to \mathbb{R}$ is Lipschitz continuous and satisfies triangle inequality

$$h^\infty((x, \alpha), (y, \beta)) + h^\infty((y, \beta), (z, \rho)) \geq h^\infty((x, \alpha), (z, \rho)),$$

for any $(x, \alpha), (y, \beta), (z, \rho) \in M \times \mathbb{T}$.

3. We set $\widetilde{d} : M \times \mathbb{T} \times M \times \mathbb{T} :\to \mathbb{R}$ as

$$\widetilde{d}((x, \alpha), (y, \beta)) = \Phi((x, \alpha), (y, \beta)) + \Phi((y, \beta), (x, \alpha)),$$

then $\widetilde{d}$ is non-negative.

Let $I \subset \mathbb{R}$ be an interval of time, a curve $\gamma \in C(I, M)$ is called *semi-static* if

$$A_L(\gamma|_{[a,b]}) = \Phi((\gamma(a), a \ mod\ 1), (\gamma(b), b \ mod\ 1)), \quad \forall [a, b] \subset I.$$

When $I = \mathbb{R}$, we say γ is a global semi-static curve. When $I = [a, +\infty)$ (or $I = (-\infty, a]$) for some $a \in \mathbb{R}$, we say γ is a forward (or backward) semi-static curve. Obviously a semi-static curve is a solution of (EL).

If γ satisfies

$$A_L(\gamma|_{[a,b]}) = -\Phi((\gamma(b), b \ mod\ 1), (\gamma(a), a \ mod\ 1)), \quad \forall [a, b] \subset I,$$

we say it is *static*. When $I = \mathbb{R}$, we say γ is a global static curve. By Proposition 1.1 it is not hard to see a static curve must be semi-static.

We define the Mañé set $\widetilde{\mathcal{N}} \subset TM \times \mathbb{T}$ and Aubry set $\widetilde{\mathcal{A}} \subset TM \times \mathbb{T}$ as

$$\widetilde{\mathcal{N}} := \{(d\gamma(t), t \ mod\ 1) : \gamma \in C(\mathbb{R}, M) \text{ is global semi-static.}\};$$

$$\widetilde{\mathcal{A}} := \{(d\gamma(t), t \ mod\ 1) : \gamma \in C(\mathbb{R}, M) \text{ is global static.}\}.$$

Similarly we define

$$\widetilde{\mathcal{N}}^+ := \{(d\gamma(t), t \ mod\ 1) : \gamma \text{ is forward semi-static. }\};$$

$$\widetilde{\mathcal{N}}^- := \{(d\gamma(t), t \ mod\ 1) : \gamma \text{ is backward semi-static. }\}.$$

It is easy to see

$$\widetilde{\mathcal{A}} \subset \widetilde{\mathcal{N}} \subset \widetilde{\mathcal{N}}^{\pm}.$$

Let $\pi : TM \times \mathbb{T} \to M \times \mathbb{T}$ be the usual projection. We set $\mathcal{N} = \pi(\widetilde{\mathcal{N}})$ and $\mathcal{A} = \pi(\widetilde{\mathcal{A}})$. The famous *Mather's graph theorem* (see [1]) tells us $\pi|_{\widetilde{\mathcal{A}}} : \widetilde{\mathcal{A}} \to \mathcal{A}$ is bijective and its inverse $(\pi|_{\widetilde{\mathcal{A}}})^{-1} : \mathcal{A} \to \widetilde{\mathcal{A}}$ is Lipschitz.

Following Mather's graph theorem, for any $(x, \alpha) \in \mathcal{A} \subset M \times \mathbb{T}$, there is a unique global static curve, denoted by $\gamma_{(x,\alpha)}$, satisfying $\gamma_{(x,\alpha)}(\alpha) = x$. Furthermore using $\widetilde{d}$ defined in Proposition 1.1, we can define an equivalence relation on $\widetilde{\mathcal{A}}$ by saying $(x, v, a), (y, w, b) \in \widetilde{\mathcal{A}}$ are equivalent iff $\widetilde{d}(\pi(x, v, a), \pi(y, w, b)) = \widetilde{d}((x, a), (y, b)) = 0$.

By this equivalence relation, we break $\widetilde{\mathcal{A}}$ into classes will be called *static classes*. Let $\widetilde{\mathbb{A}}$ be the set of static classes, through the entire paper $\widetilde{\Lambda}, \widetilde{\Omega}, \widetilde{\Gamma}$ and $\widetilde{\Delta}$ with or without super-index will be used to represent static classes.

Under the assumption that

$$\widetilde{\mathbb{A}} \text{ contains only finite elements,} \qquad (*_1)$$

we will prove the following two theorems.

Remark 1.2. $(*_1)$ *is a generic condition in the sense of Mañé, see [4].*

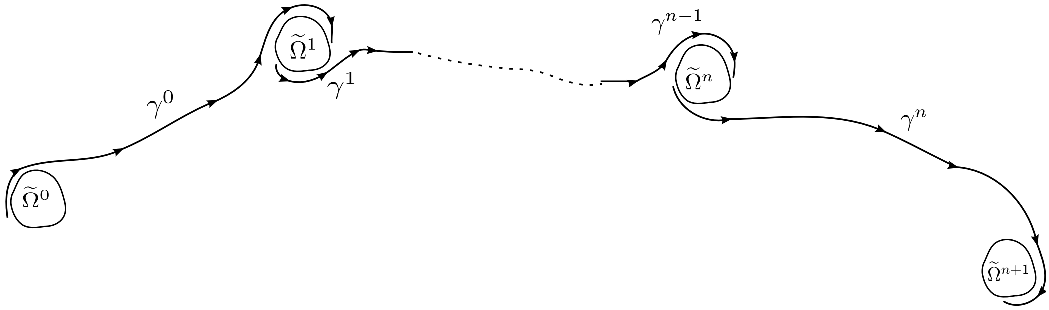
Theorem 1.1. *For any two different static classes $\widetilde{\Lambda}^1, \widetilde{\Lambda}^2 \in \widetilde{\mathbb{A}}$, one of the following must be true:*

1. *There is a global semi-static curve γ with the α-limit set $\alpha(d\gamma) \subset \widetilde{\Lambda}^1$ and the ω-limit set $\omega(d\gamma) \subset \widetilde{\Lambda}^2$;*
2. *There is a finite set of static classes $\{\widetilde{\Omega}^1, \dots, \widetilde{\Omega}^n\} \subset \widetilde{\mathbb{A}} \setminus \{\widetilde{\Lambda}^1, \widetilde{\Lambda}^2\}$ and global semi-static curves $\gamma^i : i = 0, \dots, n$ with $\alpha(d\gamma^i) \subset \widetilde{\Omega}^i$ and $\omega(d\gamma^i) \subset \widetilde{\Omega}^{i+1}$ for $i = 0, \dots, n$, where $\widetilde{\Omega}^0 = \widetilde{\Lambda}^1$ and $\widetilde{\Omega}^{n+1} = \widetilde{\Lambda}^2$.*

By $\alpha(d\gamma)$ (or $\omega(d\gamma)$), we mean the α-limit set (or ω-limit set) of the orbit $\{(\gamma(t), \dot{\gamma}(t), t \ mod\ 1) : t \in \mathbb{R}\}$ of the Euler-Lagrange flow. Similarly we let $\alpha(\gamma)$ (or $\omega(\gamma)$) be the α-limit set (or ω-limit set) of $\{\gamma(t) : t \in \mathbb{Z}\}$ in M.

When L is time-independent the above theorem has been proved by Contreras and Paternain in [5].

Remark 1.3. *Through the entire paper, for any subset $U \subset M$ and $\delta > 0$, by $U(\delta)$ we mean $U(\delta) := \{x \in M : d(x, U) \leq \delta\}$.*



Let $\mathcal{N}_0 := \mathcal{N} \cap M \times \{0\}$ and $\mathcal{A}_0 := \mathcal{A} \cap M \times \{0\}$. Under an further assumption

$$\mathcal{N}_0 \setminus \mathcal{A}_0(\delta) \text{ is totally isolated, for some } \delta > 0 \text{ small enough.} \tag{$*_2$}$$

We will prove along a chain of heteroclinic orbits obtained in Theorem 1.1, there is a real heteroclinic orbit connecting the two given static classes.

Theorem 1.2. *If $(*_1), (*_2)$ are true, for any two different static classes $\widetilde{\Lambda}^1, \widetilde{\Lambda}^2 \in \widetilde{\mathbb{A}}$, there is a curve $\gamma \in C^2(\mathbb{R}, M)$, such that $(d\gamma(t), t \bmod 1)$ is an orbit of the Euler-Lagrange system introduced by L with $\alpha(d\gamma) \subset \widetilde{\Lambda}^1$ and $\omega(d\gamma) \subset \widetilde{\Lambda}^2$.*

Similar variational method has been used in [6] and [7], where the authors established the existence of homoclinic orbits to the Aubry set under various conditions different from ours.

2. Preliminary

By *Mather's graph theorem*, we can break $\mathcal{A}$ into a set of equivalent classes $\mathbb{A} := \{\Lambda = \pi(\widetilde{\Lambda}) : \widetilde{\Lambda} \in \widetilde{\mathbb{A}}\}$. By abusing of notation, they will also be called static classes and capital Greek letters Λ, Ω with or without super-index will be reserved to represent such static classes throughout this paper.

For all the sets $\widetilde{\mathcal{A}}, \widetilde{\mathcal{N}}, \widetilde{\Lambda}$ $(\mathcal{A}, \mathcal{N}, \Lambda)$, by putting a sub-index $t \in \mathbb{T}$ to them, we mean it is their intersection with $TM \times \{t\}$($M \times \{t\}$), for example $\widetilde{\mathcal{A}}_t = \widetilde{\mathcal{A}} \cap TM \times \{t\}$, $\mathcal{A}_t = \mathcal{A} \cap M \times \{t\}$.

For any $n \in \mathbb{N}$, we define $h^n : M \times M \to \mathbb{R}$ by

$$h^n(x, y) = h((x, 0), (y, n)),$$

and $h^\infty : M \times M \to \mathbb{R}$, $\Phi : M \times M \to \mathbb{R}$ by

$$h^\infty(x, y) = \liminf_{n\in\mathbb{N}, n\to+\infty} h^n(x, y);$$

$$\Phi(x, y) = \Phi((x, 0), (y, 0)).$$

In some cases, we need to shift the time parameterization of a given curve for which we introduce the following operator.

Definition 2.1. *Given $a < b \in \mathbb{R}$ for any $c \in \mathbb{R}$, we define an operator $\tau_c : C([a, b], M) \to C([a + c, b + c], M)$ by*

$$\tau_c(\gamma)(t) = \gamma(t - c), \text{ for any } \gamma \in C([a, b], M) \text{ and } t \in [a + c, b + c].$$

The variational study of Tonelli Lagrangian L depends on some standard results proved by Mather in [1].

Lemma 2.1. *Given a real number K and a compact interval $[a,b]$,*

$$\{\gamma \in C([a,b],M) : A_L(\gamma) \leq K\}$$

is compact for the topology of uniform convergence.

Theorem 2.1. *(**Tonelli Theorem**) Given two points $x, y \in M$ and a compact interval $[a,b]$, there is a $\gamma \in C_{[a,b]}(x,y)$ with $A_L(\gamma) = h((x,a),(y,b))$ and γ is a C^2 solution of (EL).*

If a curve $\gamma \in C_{[a,b]}(x,y)$ satisfies $A_L(\gamma) = h((x,a),(y,b))$ we will call it a *minimizer*. Obviously a semi-static curve is a minimizer, but a minimizer is not necessarily a semi-static curve.

The next lemma is well-known to experts, however we can not locate a complete proof in the literature, therefore we give one at here.

Lemma 2.2. *Given any $p, q \in M$ and two sequences of positive integers $\{T_k^1\} \nearrow +\infty, \{T_k^2\} \nearrow +\infty$, if $\{\gamma_k \in C_{[-T_k^1,T_k^2]}(p,q)\}$ is a sequence of minimizers, i.e., $A_L(\gamma_k) = h^{T_k^1+T_k^2}(\gamma_k(-T_k^1), \gamma_k(T_k^2))$, for any $k \in \mathbb{N}$, satisfying*

$$\lim_{k\to+\infty} A_L(\gamma_k) = \liminf_{n\to+\infty} h^n(p,q) = h^\infty(p,q),$$

then there is a global semi-static curve γ, such that γ_k converges to γ uniformly on any compact interval along a subsequence.

Proof. By Proposition 1.1 and Lemma 2.1, it is not hard to see for any $T > 0$, along a subsequence γ_k converges uniformly to a $\gamma_T \in C([-T,T],M)$ on $[-T,T]$. Apply this to a sequence of positive integers $\{T_n\} \nearrow +\infty$, then by a diagonal extraction, we can find a subsequence of γ_k, which we rename as γ_k, and a $\gamma \in C(\mathbb{R},M)$, such that γ_k converges uniformly to γ on any compact interval.

We claim for any $T \in \mathbb{Z}^+$, $A_L(\gamma|_{[-T,T]}) = \Phi(\gamma(-T),\gamma(T))$, so γ is a global semi-static curve.

Assume this is not true, then there is a $T \in \mathbb{Z}^+$, such that

$$\varepsilon = A_L(\gamma|_{[-T,T]}) - \Phi((\gamma(-T),\gamma(T)) > 0. \tag{2.1}$$

Since $\gamma_k|_{[-T,T]}$ converges uniformly to $\gamma|_{[-T,T]}$, the lower semi-continuity of A_L implies

$$A_L(\gamma_k|_{[-T,T]}) \geq A_L(\gamma|_{[-T,T]}) - \frac{\varepsilon}{4} \text{ for } k \text{ large enough.} \tag{2.2}$$

By the definition of Φ there is a $S \in \mathbb{Z}^+$, such that

$$h^S(\gamma(-T),\gamma(T)) \leq \Phi(\gamma(-T),\gamma(T)) + \frac{\varepsilon}{4}. \tag{2.3}$$

Because h^S is Lipschitz,

$$|h^S(\gamma_k(-T),\gamma_k(T)) - h^S(\gamma(-T),\gamma(T))| \leq C[d(\gamma_k(-T),\gamma(-T)) + d(\gamma_k(T),\gamma(T))].$$

As $\lim_{k\to+\infty} d(\gamma_k(\pm T),\gamma(\pm T)) = 0$, for k large enough we have

$$|h^S(\gamma_k(-T),\gamma_k(T)) - h^S(\gamma(-T),\gamma(T))| \leq \frac{\varepsilon}{4}. \tag{2.4}$$

By **Tonelli Theorem**, for any $k \in \mathbb{N}$, there are $\xi_k \in C_{[0,S]}(\gamma_k(-T), \gamma_k(T))$ with

$$A_L(\xi_k) = h^S(\gamma_k(-T), \gamma_k(T)),$$

so by (2.4),

$$A_L(\xi_k) \leq h^S(\gamma(-T), \gamma(T)) + \frac{\varepsilon}{4}. \tag{2.5}$$

Combine this with (2.1), (2.2) and (2.3), we get

$$\begin{aligned} A_L(\xi_k) &\leq \Phi(\gamma(-T), \gamma(T)) + \frac{\varepsilon}{2} \\ &= A_L(\gamma|_{[-T,T]}) - \frac{\varepsilon}{2} \\ &\leq A_L(\gamma_k|_{[-T,T]}) - \frac{\varepsilon}{4}, \end{aligned}$$

for k large enough.

We define a new sequence of curves $\{\widetilde{\gamma}_k \in C_{T_k^1+T_k^2-2T+S}(p,q)\}$ by

$$\widetilde{\gamma}_k(t) = \begin{cases} \gamma_k(t - T_k^1) & \text{if } t \in [0, T_k^1 - T] \\ \xi_k(t - T_k^1 + T) & \text{if } t \in [T_k^1 - T, T_k^1 - T + S] \\ \gamma_k(t - T_k^1 + 2T - S) & \text{if } t \in [T_k^1 - T + S, T_k^1 + T_k^2 - 2T + S]. \end{cases}$$

For all k large enough, $A_L(\widetilde{\gamma}_k) \leq A_L(\gamma_k) - \frac{\varepsilon}{4}$. Since $\{T_k^2 + T_k^1 - 2T + S\}$ goes to infinity as $k \to +\infty$, we have

$$\begin{aligned} h(p,q) &\leq \liminf_{k\to+\infty} h^{T_k^2+T_k^1-2T+S}(p,q) \leq \liminf_{k\to+\infty} A_L(\widetilde{\gamma}_k) \\ &\leq \lim_{k\to+\infty} A_L(\gamma_k) - \frac{\varepsilon}{4} = h(p,q) - \frac{\varepsilon}{4}, \end{aligned}$$

which is absurd and we proved our claim. □

Remark 2.1. *In the previous lemma, if we assume $\{T_k^1\}$ (resp. $\{T_k^2\}$) is bounded, by the same arguments it is not hard to see there is a forward semi-static curve (resp. backward semi-static curve) γ, such that, γ_k converges uniformly to γ on any compact interval in the domain of γ along a subsequence.*

In this paper a stronger version of Lemma 2.2 will be needed.

Lemma 2.3. *Given any $p, q \in M$ and a sequence of positive integers $\{T_k\} \nearrow +\infty$, let $\{\gamma_k \in C_{T_k}(p,q)\}$ be a sequence of minimizers, i.e., $A_L(\gamma_k) = h((p,0),(q,T_k))$, satisfies $\lim_{k\to+\infty} A_L(\gamma_k) = h^\infty(p,q)$.*

If $\{a_k\} \nearrow +\infty, \{b_k\} \nearrow +\infty$ are two sequence of positive integers satisfying

1. *$0 \leq a_k \leq b_k \leq T_k$, for all $k \in \mathbb{N}$;*
2. *$b_k - a_k \to +\infty$ as $k \to +\infty$;*
3. *there are $x, y \in M$, such that along a subsequences $d(\gamma_k(a_k), x) \to 0$ and $d(\gamma_k(b_k), y) \to 0$ as $k \to +\infty$,*

then along a subsequence, $\lim_{k\to+\infty} A_L(\gamma_k|_{[a_k,b_k]}) = h^\infty(x,y)$.

Proof. Noticing that $\{A_L(\gamma_k|_{[a_k,b_k]})\}$ has a finite upper bound, passing to a subsequence if necessary, we can say

$$\lim_{k\to+\infty} h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k)) = \lim_{k\to+\infty} A_L(\gamma_k|_{[a_k,b_k]}) = B < +\infty.$$

We claim $B = h^\infty(x,y)$ and the lemma follows immediately from this claim.

By Lipschitz continuity of h^n,

$$|h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k)) - h^{b_k-a_k}(x,y)| \le C(d(\gamma_k(a_k),x) + d(\gamma_k(b_k),y)),$$

by passing to a proper subsequence, we can say $d(\gamma_k(a_k),x) + d(\gamma_k(b_k),y)) \to 0$ as $k \to +\infty$, then

$$\lim_{k\to+\infty} h^{b_k-a_k}(x,y) = \lim_{k\to+\infty} h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k)).$$

Hence,

$$B = \lim_{k\to+\infty} h^{b_k-a_k}(x,y) \ge \liminf_{n\to+\infty} h^n(x,y) = h^\infty(x,y).$$

Assume $\varepsilon = B - h^\infty(x,y) > 0$, then there are non-negative integers $a'_k < b'_k$ with $b'_k - a'_k \to +\infty$ as $k \to +\infty$ satisfying

$$\lim_{k\to+\infty} h^{b'_k-a'_k}(x,y) = h^\infty(x,y) = B - \varepsilon.$$

Then for k large enough, we have

$$h^{b'_k-a'_k}(x,y) \le B - \frac{3}{4}\varepsilon. \tag{2.6}$$

On the other hand, $B = \lim_{k\to+\infty} h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k))$, so for k large enough

$$h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k)) \ge B - \frac{\varepsilon}{4}. \tag{2.7}$$

Combine (2.6) and (2.7), we get

$$h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k)) \ge h^{b'_k-a'_k}(x,y) + \frac{\varepsilon}{2}. \tag{2.8}$$

Again by the Lipschitz continuity of h^n and $d(\gamma_k(a_k),x) + d(\gamma_k(b_k),y)) \to 0$ as $k \to +\infty$, for k large enough we have

$$h^{b'_k-a'_k}(\gamma_k(a_k),\gamma_k(b_k)) \le h^{b'_k-a'_k}(x,y) + \frac{\varepsilon}{4}. \tag{2.9}$$

(2.8) and (2.9) imply

$$h^{b_k-a_k}(\gamma_k(a_k),\gamma_k(b_k)) \ge h^{b'_k-a'_k}(\gamma_k(a_k),\gamma_k(b_k)) + \frac{\varepsilon}{4}.$$

Now following the same argument as in the proof of Lemma 2.2, we can construct a new sequence $\{\gamma_k^* \in C_{T_k-(b_k-a_k)+(b'_k-a'_k)}(p,q)\}$ with

$$h^\infty(p,q) \le \liminf_{k\to+\infty} A_L(\gamma_k^*) \le \liminf_{k\to+\infty} A_L(\gamma_k) - \frac{\varepsilon}{4} = h^\infty(p,q) - \frac{\varepsilon}{4},$$

which is a contradiction.

□

In next lemma we will strengthen the triangle inequality of h^∞ given in Proposition 1.1.

Lemma 2.4. *Given a $\Lambda \in \mathbb{A}$ and a compact set $U \subset M$ with $\Lambda_0 \cap U = \emptyset$, then we can find an $\varepsilon > 0$ small enough and a $\delta = \delta(\varepsilon) > 0$, such that*

$$h^\infty(x, y) + h^\infty(y, z) \geq h^\infty(x, z) + \varepsilon, \quad \forall x, z \in \Lambda_0(\delta), \quad \forall y \in U.$$

Proof. Because $\Lambda_0 \cap U = \emptyset$ and U is compact, there is an $\varepsilon' > 0$, such that

$$\widetilde{d}(p, y) = h^\infty(p, y) + h^\infty(y, p) \geq \varepsilon' > 0, \quad \forall p \in \Lambda_0, \quad \forall y \in U.$$

Then for any $q \in \Lambda_0$, as h^∞ satisfies triangle inequality,

$$h^\infty(p, y) + h^\infty(y, q) + h^\infty(q, p) \geq h^\infty(p, y) + h^\infty(y, p) \geq \varepsilon',$$

so

$$h^\infty(p, y) + h^\infty(y, q) \geq -h^\infty(q, p) + \varepsilon' = h^\infty(p, q) + \varepsilon', \tag{2.10}$$

the last equality is because of $h^\infty(p, q) + h^\infty(q, p) = \widetilde{d}(p, q) = 0$, as $p, q \in \Lambda_0$ are in the same static class.

Let $\varepsilon = \frac{\varepsilon'}{4}$ and $\delta = \frac{\varepsilon}{8C}$, then for any $x, z \in \Lambda_0(\delta)$, there are $p', q' \in \Lambda_0$ with $d(x, p') \leq \delta$ and $d(z, q') \leq \delta$. By the Lipschitz continuity of h^∞, we have

$$h^\infty(x, y) \geq h^\infty(p', y) - C\delta,$$

$$h^\infty(y, z) \geq h^\infty(y, q') - C\delta,$$

$$h^\infty(x, z) \leq h(p', q') + 2C\delta,$$

combine these with (2.10), we get

$$\begin{aligned} h^\infty(x, y) + h^\infty(y, z) &\geq h^\infty(p', y) + h^\infty(y, q') - 2C\delta \geq \; h(p', q') + \varepsilon' - 2C\delta \\ &\geq h^\infty(x, z) - 2C\delta + 2\varepsilon - 2C\delta = h^\infty(x, z) + 2\varepsilon - 4C\delta \\ &= h^\infty(x, z) + \frac{3}{2}\varepsilon \end{aligned}$$

□

Given an arbitrary $(x, \alpha) \in M \times \mathbb{T}$, by Fathi's weak KAM theory [8], there are at least one forward semi-static curve $\gamma^+ \in C^2([a, +\infty), M)$ with $\gamma^+(a) = x$, $a \; mod \; 1 = \alpha$ and one backward semi-static curve $\gamma^- \in C^2((-\infty, a], M)$ with $\gamma^-(a) = x$, $a \; mod \; 1 = \alpha$.

One of the most important feature of these forward (or backward) semi-static curves is that the must asymptotic to a unique static class of $\widetilde{\mathcal{A}}$.

Lemma 2.5. *If γ is a forward (or backward) semi-static curve, then there is a unique static class $\widetilde{\Lambda} \in \widetilde{\mathbb{A}}$, such that $\omega(d\gamma)$ (or $\alpha(d\gamma)$) $\subset \widetilde{\Lambda}$.*

Generally speaking there may be more than one forward (or backward) semi-static curves starting from (or ending at) a given point $(x, \alpha) \in M \times \mathbb{T}$, however if (x, α) belongs to $\mathcal{A}$, then they are unique.

Lemma 2.6. *If $(x,\alpha) \in \mathcal{A}$, then there is a unique $v \in T_xM$, such that $(x,v,\alpha) \in \widetilde{\mathcal{A}} \subset \widetilde{\mathcal{N}}^{\pm}$.*

As a consequence there is a unique forward semi-static curve $\gamma^+ \in C([\alpha,+\infty),M)$ with $\gamma^+(\alpha) = x$ and a unique backward semi-static curve $\gamma^- \in C((-\infty,\alpha],M)$ with $\gamma^-(\alpha) = x$, and γ^+,γ^- must be static, with

$$\gamma^+(t) = \gamma_{(x,\alpha)}(t), \ \text{if } t \in [\alpha,+\infty);$$

$$\gamma^-(t) = \gamma_{(x,\alpha)}(t), \ \text{if } t \in (-\infty,\alpha],$$

where $\gamma_{(x,\alpha)}$ is the unique global static curve passing (x,α).

The proofs of the above two lemmas can be found in [2,5,8], which we will not repeat here.

3. Heteroclinic chain

We will give the proof of Theorem 1.1 in this section. By assumption $(*_1)$, we can fix a $\delta^* > 0$ small enough, such that for any two different static classes $\Lambda^1,\Lambda^2 \in \mathbb{A}$, $\Lambda^1_0(\delta^*) \cap \Lambda^2_0(\delta^*) = \emptyset$. In order to distinguish the two different cases in Theorem 1.1, we introduce the following definition

Definition 3.1. *Given a $\delta > 0$ and a curve $\gamma \in C([a,b],M)$, for any two static classes $\Lambda^1,\Lambda^2 \in \mathbb{A}$, we defined the following set*

$$K(\gamma,\delta,\Lambda^1,\Lambda^2) := \{\Lambda \in \mathbb{A} \setminus \{\Lambda^1,\Lambda^2\} : \min_{t\in[a,b]\cap\mathbb{Z}} d(\gamma(t),\Lambda_0) \le \delta\}.$$

A similar idea was used by Maxwell [9] and Rabinowitz [10] on a special class of Hamiltonian systems including periodic forced multiple pendulum under time-reversibility assumption.

For the remainder of this section, we fix two arbitrary static classes $\Lambda^1 \ne \Lambda^2 \in \mathbb{A}$, and two points $p \in \Lambda^1_0, q \in \Lambda^2_0$. By **Tonelli Theorem**, there is a sequence of minimizers $\{\gamma_k \in C_{[-T^1_k,T^2_k]}(p,q)\}$ satisfying

1. $\{T^1_k \in \mathbb{Z}^+\} \nearrow +\infty, \{T^2_k \in \mathbb{Z}^+\} \nearrow +\infty$ as $k \to +\infty$;
2. $\lim_{k\to+\infty} A_L(\gamma_k) = \lim_{k\to+\infty} h^{T^2_k+T^1_k}(p,q) = h^{\infty}(p,q)$.

Let's consider the collection of sets $K(\gamma_k,\delta,\Lambda^1,\Lambda^2)$, for all $k \in \mathbb{N}$ and $\delta > 0$, there are two possibilities:

Case 1: There is a $\delta_0 > 0$ small enough, such that for each $m \in \mathbb{N}$, there is a $k > m$ with $K(\gamma_k,\delta_0,\Lambda^1,\Lambda^2) = \emptyset$;

Case 2: For each $\delta > 0$, there is a $m \in \mathbb{N}$, such that $K(\gamma_k,\delta,\Lambda^1,\Lambda^2) \ne \emptyset, \quad \forall k > m.$

First we shall assume case 1 hold, then by passing $\{\gamma_k\}$ to a subsequence, we may assume $K(\gamma_k,\delta_0,\Lambda^1,\Lambda^2) = \emptyset$ for all $k \in \mathbb{N}$.

Proposition 3.1. *If* Case 1 *hold, there is a global semi-static curve γ satisfying*

$$\alpha(d\gamma) \subset \widetilde{\Lambda}^1 \text{ and } \omega(d\gamma) \subset \widetilde{\Lambda}^2.$$

Proof. For each $k \in \mathbb{N}$, we set $S_k := \min\{t \in [-T^1_k,T^2_k] \cap \mathbb{Z} : d(\gamma_k(t),\Lambda^1_0) > \delta^*\}$. It is not hard to see, for each $k \in \mathbb{N}$, S_k is a well-defined integer and $d(\gamma_k(t),\Lambda^1_0) \le \delta^*$, for all $t \in [-T^1_k,S_k) \cap \mathbb{Z}$.

Lemma 3.2. *Both $\{S_k + T^1_k\}$ and $\{T^2_k - S_k\}$ go to infinity as k goes to infinity.*

Proof. (**Lemma 3.2**) We will only give the detailed proof for $S_k + T_k^1 \to +\infty$, while $T_k^2 - S_k \to +\infty$ can be proven similarly.

Suppose $\{S_k + T_k^1\}$ is bounded, then passing $\{\gamma_k\}$ to a subsequence, we can say $S_k + T_k^1 \equiv T \in \mathbb{Z}^+$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, we define $\gamma_k^* \in C([0, T_k^1 + T_k^2], M)$ by $\gamma_k^* = \tau_{T_k^1}(\gamma_k)$, where $\tau_{T_k^1}$ is defined as in Definition 2.1, i.e., we shift the time parameterization on γ_k forward by T_k^1.

Replace γ_k^* by a proper subsequence, by Remark 2.1, there is a forward semi-static curve $\gamma^* \in C([0, +\infty), M)$, such that γ_k^* converges to γ^* on any compact interval of $[0, +\infty)$. Since

$$d(\gamma^*(T), \Lambda_0^1) = \lim_{k\to+\infty} d(\gamma_k^*(T), \Lambda_0^1) = \lim_{k\to+\infty} d(\gamma_k(S_k), \Lambda_0^1) \geq \delta^*,$$

$\gamma^*(T) \notin \Lambda_0^1$. At the same time $\gamma^*(0) = \lim \gamma_k^*(0) = p \in \Lambda_0^1$, then γ^* can not be a static curve.

Because $p \in \Lambda_0^1 \subset \mathbb{A}_0$, there must be a global static curve ξ with $\xi(0) = p$. As a result we have two different forward semi-static curves starting from p and this is a contradiction to Lemma 2.6. □

By Lemma 3.2, without loss of generality, we may assume $S_k \equiv 0$ for all $k \in \mathbb{N}$. Passing γ_k to a subsequence, by Lemma 2.2, there is a global semi-static curve γ, such that γ_k converges uniformly to γ on any compact interval.

Since $d(\gamma_k(t), \Lambda_0^1) \leq \delta^*$, for all $t \in [-T_k^1, 0) \cap \mathbb{Z}$ and $k \in \mathbb{N}$, we have $d(\gamma(t), \Lambda_0^1) \leq \delta^*$, for all $t \in (-\infty, 0) \cap \mathbb{Z}$.

By Lemma 2.5, there is a unique $\widetilde{\Lambda} \in \widetilde{\mathbb{A}}$, such that $\alpha(d\gamma) \subset \widetilde{\Lambda}$, hence we must have $\widetilde{\Lambda} = \widetilde{\Lambda}^1$ and $\alpha(d\gamma) \subset \widetilde{\Lambda}^1$.

Because *Case 1* is true, either $\omega(d\gamma) \subset \widetilde{\Lambda}^2$ or $\omega(d\gamma) \subset \widetilde{\Lambda}^1$ must be true. Assume $\omega(d\gamma) \subset \widetilde{\Lambda}^1$, then there is a sequence of positive integers $\{T_j\} \nearrow +\infty$ with $\lim_{j\to+\infty} d(\gamma(T_j), x) = 0$ for some $x \in \Lambda_0^1$.

On the other hand we can find a subsequence $\{\gamma_{k_j} \in C_{[-T_{k_j}^1, T_{k_j}^2]}(p, q)\}$ of $\{\gamma_k\}$ with $T_j \leq T_{k_j}^2$ for all $j \in \mathbb{N}$ and $d(\gamma_{k_j}(T_j), \gamma(T_j))$ approaches to 0 as $j \to +\infty$. Therefore $d(\gamma_{k_j}(T_j), x)$ approaches to 0 as $j \to +\infty$.

Replacing $\{\gamma_{k_j}\}$ by a proper subsequence, by Lemma 2.3

$$h^\infty(p, x) = \lim_{j\to+\infty} A_L(\gamma_{k_j}|_{[-T_{k_j}^1, T_j]}). \tag{3.1}$$

Since $\{T_{k_j}^1\}$ and $\{T_j\}$ goes to infinity, as $j \to +\infty$, both sequences $\{\gamma_{k_j}|_{[-T_{k_j}^1, 0]}\}$ and $\{\gamma_{k_j}|_{[0, T_j]}\}$ satisfy conditions of Lemma 2.3, passing $\{\gamma_{k_j}\}$ to a proper subsequence, we have

$$h^\infty(p, \gamma(0)) = \lim_{j\to+\infty} A_L(\gamma_{k_j}|_{[-T_{k_j}^1, 0]}); \quad h^\infty(\gamma(0), x) = \lim_{j\to+\infty} A_L(\gamma_{k_j}|_{[0, T_j]}). \tag{3.2}$$

By (3.1) and (3.2), $h^\infty(p, \gamma(0)) + h^\infty(\gamma(0), x) = h^\infty(p, x)$.

However it is easy to see $d(\gamma(0), \Lambda_0^1) \geq \delta^*$, at the same time $p, x \in \Lambda_0^1$, by Lemma 2.4, $h^\infty(p, \gamma(0)) + h^\infty(\gamma(0), x) > h^\infty(p, x)$, which is a contradiction and we finished our proof. □

Now we assume *Case 2* is true, because of $(*_1)$, by passing $\{\gamma_k\}$ to a subsequence, we can say that for a sequence of positive real numbers $\{\delta_k \in (0, \delta^*)\} \searrow 0$, there is a finite set of static classes $\{\Omega^1, \ldots, \Omega^n\} \subset \mathbb{A} \setminus \{\Lambda^1, \Lambda^2\}$ satisfying $K(\gamma_k, \delta_k, \Lambda^1, \Lambda^2\} \equiv \{\Omega^1, \ldots, \Omega^n\}$ for all $k \in \mathbb{N}$.

We set $\Omega^0 = \Lambda^1$ and $\Omega^{n+1} = \Lambda^2$.

Proposition 3.3. *If* Case 2 *is true, there is a chain of global semi-static curves $\gamma^i : i = 0, \ldots, n$ satisfying $\alpha(d\gamma^i) \subset \widetilde{\Omega}^i$ and $\omega(d\gamma^i) \subset \widetilde{\Omega}^{i+1}$ for $i = 0, \ldots, n$.*

Proof. By Definition 3.1, for every $i = 1, \ldots, n$ and $k \in \mathbb{N}$, we can find an $S_k^i \in (-T_k^1, T_k^2) \cap \mathbb{Z}$ with $d(\gamma_k(S_k^i), \Omega_0^i) \leq \delta_k$ and we set $S_k^0 = -T_k^1$ and $S_k^{n+1} = T_k^2$ for all $k \in \mathbb{N}$.

Without loss of generality, we can assume $S_k^i \leq S_k^{i+1}$ for all $i = 0, \ldots, n$ and $k \in \mathbb{N}$. Furthermore by passing $\{\gamma_k\}$ to a proper subsequence , we can say there are $x_i \in \Omega_0^i$ for $i = 0, \ldots, n+1$ with

$$\lim_{k \to +\infty} d(\gamma_k(S_k^i), x_i) = 0,$$

where $x_0 = p$ and $x_{n+1} = q$.

Lemma 3.4. *For every $i = 0, \ldots, n$, $\{S_k^{i+1} - S_k^i\} \to +\infty$ as $k \to +\infty$.*

Proof. (**Lemma 3.4**) It can be proven similarly as we did in the proof of Lemma 3.2 and we will not repeat it here. □

Lemma 3.5. *There is a $\widetilde{\delta} \in (0, \delta^*)$ small enough, such that for every k large enough*

$$K(\gamma_k|_{[S_k^i, S_k^{i+1}]}, \widetilde{\delta}, \Omega^i, \Omega^{i+1}) = \emptyset, \forall i = 0, \ldots, n.$$

Proof. (**Lemma 3.5**) If not, without loss of generality we can assume there are $0 \leq i_0 \neq i_1 \leq n$ and a sequence of positive numbers $\{\widetilde{\delta}_k \in (0, \delta^*)\} \searrow 0$ satisfying

$$\Omega^{i_1} \in K(\gamma_k|_{[S_k^{i_0}, S_k^{i_0+1}]}, \widetilde{\delta}_k, \Omega^{i_0}, \Omega^{i_0+1})$$

for all k large enough.

By the definition of K, we must have $i_1 \neq i_0 + 1$. Let's assume $i_1 > i_0 + 1$ (the case of $i_1 < i_0$ can be proven similarly), then there is a sequence of integers $\{S_k \in (S_k^{i_0}, S_k^{i_0+1})\}$, such that $d(\gamma_k(S_k), \Omega_0^{i_1}) \leq \widetilde{\delta}_k$.

First we will show that $\{S_k^{i_0+1} - S_k\}$ approaches positive infinity as k goes to $+\infty$. If this is not true, replacing $\{\gamma_k\}$ by a proper subsequence, we may assume $S_k^{i_0+1} - S_k \equiv S \in \mathbb{R}$ for all $k \in \mathbb{N}$.

We define a new sequence of minimizers $\{\gamma_k^* = \tau_{-S_k}(\gamma_k)\}$, then

$$d(\gamma_k^*(0), \Omega_0^{i_1}) = d(\tau_{-S_k}(\gamma_k)(0), \Omega_0^{i_1}) = d(\gamma_k(S_k), \Omega_0^{i_1}) \leq \widetilde{\delta}_k.$$

Since $\widetilde{\delta}_k \to 0$ as $k \to +\infty$, passing $\{\gamma^*\}$ to a proper subsequence, we can say $\lim_{k \to +\infty} \gamma_k^*(0) = x$ for some $x \in \Omega_0^{i_1}$.

Again passing $\{\gamma_k^*\}$ to a proper subsequence, by Lemma 2.2, there is a global semi-static curve γ^* such that γ_k^* converges to γ^* on any compact interval. Hence

$$\gamma^*(0) = \lim_{k \to +\infty} \gamma_k^*(0) = x \in \Omega_0^{i_1}.$$

On the other hand

$$\begin{aligned}\gamma^*(S) &= \lim_{k \to +\infty} \gamma_k^*(S) = \lim_{k \to +\infty} \gamma_k(S + S_k) \\ &= \lim_{k \to +\infty} \gamma_k(S_k^{i_0+1}) = x_{i_0+1} \in \Omega_0^{i_0+1}.\end{aligned}$$

As $\Omega_0^{i_0+1} \cap \Omega_0^{i_1} = \emptyset$, γ^* can not be a static curve. Since $(x, 0) \in \Omega_{i_1} \subset \mathcal{A}$, there is a global static curve $\gamma_{(x,0)}$ with $\gamma_{(x,0)}(0) = x$. Obviously $\gamma_{(x,0)} \neq \gamma^*$ and we have two different forward semi-static curves starting from x, which is a contradiction to Lemma 2.6. Therefore $\{S_k^{i_0+1} - S_k\} \to +\infty$ as $k \to +\infty$.

At the same time by Lemma 3.4, $\{S_k^{i_1} - S_k^{i_0+1}\} \to +\infty$ as $k \to +\infty$. Noticing that $\{\gamma_k|_{[S_k, S_k^{i_1}]}\}$ is a sequence of minimizers with

$$\lim_{k\to+\infty} \gamma_k(S_k) = x, \lim_{k\to+\infty} \gamma(S_k^{i_0+1}) = x_{i_0+1} \text{ and } \lim_{k\to+\infty} \gamma_k(S_k^{i_1}) = x_{i_1},$$

by Lemma 2.3, passing $\{\gamma_k\}$ to a proper subsequence, we have

$$\begin{aligned} h^\infty(x, x_{i_1}) &= \lim_{k\to+\infty} A_L(\gamma_k|_{[S_k, S_k^{i_1}]}) \\ &= \lim_{k\to+\infty} A_L(\gamma_k|_{[S_k, S_k^{i_0+1}]}) + \lim_{k\to+\infty} A_L(\gamma_k|_{[S_k^{i_0+1}, S_k^{k_1}]}) \\ &= h^\infty(x, x_{i_0+1}) + h^\infty(x_{i_0+1}, x_{i_1}). \end{aligned}$$

However $x, x_{i_1} \in \Omega_0^{i_1}$ and $x_{i_0+1} \in \Omega_0^{i_0+1}$, by Lemma 2.4,

$$h^\infty(x, x_{i_1}) < h^\infty(x, x_{i_0+1}) + h^\infty(x_{i_0+1}, x_{i_1}),$$

which is absurd, so we are done. □

Now we resume our proof of Proposition 3.3.

For each $i = 0, \ldots, n$, by Lemma 3.4 and Lemma 3.5, we showed that $\{\gamma|_{[S_k^i, S_k^{i+1}]}\}$ is a sequence of minimizers satisfying the following:

1. $S_k^{i+1} - S_k^i \to +\infty$ as $k \to +\infty$;
2. $\lim_{k\to+\infty} d(\gamma_k(S_k^i), x_i) = 0$ and $\lim_{k\to+\infty} d(\gamma_k(S_k^{i+1}), x_{i+1}) = 0$;
3. $K(\gamma_k|_{[S_k^i, S_k^{i+1}]}, \widetilde{\delta}, \Omega^i, \Omega^{i+1}) = \emptyset$, for all k large enough.

Let $\{\gamma_k^i = \tau_{c_k}(\gamma|_{[S_k^i, S_k^{i+1}]})\}$, where $c_k = -\frac{S_k^i + S_k^{i+1}}{2}$, then following the same argument as in the proof of Proposition 3.1, we can show that along a subsequence $\{\gamma_k^i\}$ converges uniformly on any compact interval to a global semi-static curve γ^i satisfying $\alpha(d\gamma^i) \subset \widetilde{\Omega}^i$ and $\omega(d\gamma^i) \subset \widetilde{\Omega}^{i+1}$.

Hence $\{\gamma^i : i = 0, \ldots, n\}$ form a chain of heteroclinic orbits as we wanted and $\{x_i : i = 0, \ldots, n\}$ satisfies $x_i \in \omega(\gamma^{i-1}) \cap \alpha(\gamma^i) \cap \Omega_0^i$ for $i = 1, \ldots, n$, and $x_0 \in \alpha(\gamma^0) \cap \Omega_0^0$, $x_{n+1} \in \omega(\gamma^n) \cap \Omega_0^{n+1}$.

Furthermore we have

$$h^\infty(x_i, x_{i+1}) = \lim_{k\to+\infty} A_L(\gamma_k|_{[S_k^i, S_k^{i+1}]}), \text{ for all } i = 0, \ldots, n, \tag{3.3}$$

Hence,

$$h^\infty(p, q) = \lim_{k\to+\infty} A_L(\gamma_k|_{[S_k^0, S_k^{n+1}]}) = \sum_{i=0}^{n} \lim_{k\to+\infty} A_L(\gamma_k|_{[S_k^i, S_k^{i+1}]}) = \sum_{i=0}^{n} h^\infty(x_i, x_{i+1}). \tag{3.4}$$

□

Obviously Theorem 1.1 follows directly from Proposition 3.1 and Proposition 3.3.

4. Heteroclinic Orbits

This section will be devoted to the proof of Theorem 1.2. In the previous section we proved that for any two different static classes, there is a chain of heteroclinic orbits connecting them. Under the assumption ($*_2$), we will show that there is a real heteroclinic orbit connecting those two static classes along such a chain of heteroclinic orbits.

Such a heteroclinic orbits will be found as a constraint minimizer of the action of Lagrangian L. Using the minimizing properties of static curves and semi-static curves, we will show that the constraint minimizers we find will not bump up to the boundary conditions we posted and it is a real orbit.

First we will introduce some technical lemmas.

Lemma 4.1. *There is a $\hat{\delta} \in (0, \delta^*)$ small enough, such that if $p, q \in \Lambda_0(\hat{\delta})$, for some $\Lambda \in \mathbb{A}$, then*

$$h^\infty(p, x) + h^\infty(x, q) = h^\infty(p, q), \quad \forall x \in \Lambda_0, \tag{4.1}$$

and

$$h^\infty(p, x) + h^\infty(x, q) > h^\infty(p, q), \text{ if } x \in \mathcal{A}_0 \setminus \Lambda_0. \tag{4.2}$$

Proof. Choose an arbitrary $p, q \in \Lambda_0(\delta^*)$, first let's show $h^\infty(p, x) + h^\infty(x, q) = h^\infty(p, x') + h^\infty(x', q)$ if $x, x' \in \Lambda_0$. By the triangle inequality of h^∞,

$$\begin{aligned} h^\infty(p, x) + h^\infty(x, q) &\leq h^\infty(p, x') + h^\infty(x', x) + h^\infty(x, x') + h^\infty(x', q) \\ &= h^\infty(p, x') + h^\infty(x', q), \end{aligned}$$

the last equality is due to the fact that x, x' are contained in the same static class, which means

$$h^\infty(x, x') + h^\infty(x', x) = \widetilde{d}(x, x') = 0.$$

Therefore $h^\infty(p, x) + h^\infty(x, q) \leq h^\infty(p, x') + h^\infty(x', q)$, the other direction of the inequality can be proven similarly.

First we will prove (4.1) under the assumption that both p, q are contained in Λ_0.

For any $x \in \Lambda_0$, by triangle inequality

$$h^\infty(p, x) + h^\infty(x, q) \geq h^\infty(p, q).$$

On the other hand, since q, x are contained in the same static class Λ_0, $\widetilde{d}(q, x) = h^\infty(q, x) + h^\infty(x, q) = 0$, then

$$h^\infty(p, q) - h^\infty(x, q) = h^\infty(p, q) + h^\infty(q, x) \geq h^\infty(p, x),$$

so

$$h^\infty(p, q) \geq h^\infty(p, x) + h^\infty(x, q).$$

As a result, we proved equality (4.1) under the assumption $p, q \in \Lambda_0$.

Now we will drop our previous assumption. Without loss of generality, let's say $q \notin \Lambda_0$. Then there is a sequence of minimizers $\{\gamma_k \in C_{[0,T_k]}(p, q)\}$ with $\{T_k \in \mathbb{Z}^+\} \nearrow +\infty$, and a forward semi-static curve $\gamma \in C([0, +\infty), M)$, such that γ_k converges uniformly to γ on any compact sub-interval of $[0, +\infty)$.

Then $\omega(\gamma) \subset \Omega$ for some $\Omega \in \mathbb{A}$. We can find a sequence of positive integers $\{S_k\} \nearrow +\infty$ and a $x \in \Omega_0$, such that $\lim_{k\to+\infty} d(\gamma(S_k), x) = 0$, then

$$\lim_{k\to+\infty} d(\gamma_k(S_k), x) = 0$$

. 0 We claim $T_k - S_k \to +\infty$ as $k \to +\infty$, if this is not true, passing to a subsequence, we may assume $T_k - S_k \equiv T \in \mathbb{Z}^+$. Let $\xi_k \in C_{[-T_k,0]}(p, q); \xi_k = \tau_{-T_k}(\gamma_k)$ be a new sequence of minimizers.

Then passing to a subsequence ξ_k converges uniformly to a backward semi-static curve ξ on any compact sub-interval of $(-\infty, 0]$.

Noticing that $\xi_k(-T) = \xi_k(S_k - T_k) = \gamma_k(S_k)$ approaches to x as k goes to $+\infty$, so $\xi(-T) = \lim_{k\to+\infty} \xi_k(-T) = x \in \Omega_0$. Then by Lemma 2.6, ξ must be part of a global static curve and $(\xi(t), 0) \in \mathcal{A}_0$ for any $t \in (-\infty, 0] \cap \mathbb{Z}$. However $\xi(0) = \lim \xi_k(0) = \lim \gamma_k(T_k) = q \notin \Lambda_0$, which is a contradiction.

As we just showed, when k goes to $+\infty$, $\{T_k - S_k\}$ approaches to $+\infty$, $\{S_k\}$ approaches to $+\infty$ and $\gamma_k(S_k)$ approaches to x, by Lemma 2.3, passing $\{\gamma_k\}$ to a proper subsequence, we have

$$\begin{aligned} h^\infty(p, q) &= \lim_{k\to+\infty} A_L(\gamma_k|_{[0,T_k]}) \\ &= \lim_{k\to+\infty} A_L(\gamma_k|_{[0,S_k]}) + \lim_{k\to+\infty} A_L(\gamma_k|_{[S_k,T_k]}) \\ &= h^\infty(p, x) + h^\infty(x, q). \end{aligned}$$

Now we will show that for a small enough $\hat{\delta} > 0$, we must have $\Omega = \Lambda$, then $x \in \Lambda_0$ and (4.1) follows immediately from that.

Assume $\Omega \neq \Lambda$, then $x \in \mathcal{A}_0 \setminus \Lambda_0$, which is a compact set without any intersection with Λ_0, then by Lemma 2.4, we can find an $\hat{\delta} \in (0, \delta^*)$, such that for any $p, q \in \Lambda_0(\hat{\delta})$ and $y \in \mathcal{A}_0 \setminus \Lambda_0$, we have

$$h^\infty(p, y) + h^\infty(y, q) > h^\infty(p, q).$$

This is a contradiction to to what we just proved. Therefore we must have $\Omega = \Lambda$.

Finally (4.2) follows directly from Lemma 2.4. □

In section 2, we have mentioned that for any $(x, \alpha) \in M \times \mathbb{T}$, there is at least one forward semi-static curve γ^+ (or backward semi-static curve γ^-) starting from (or ending at) (x, α) with its ω-limit set (or α-limit set) contained in a unique static class. Generally we can not determine which static class it will approach to, however by the above lemma we can determine where it asymptotic to in some special case.

Lemma 4.2. *Let $\hat{\delta}$ be defined as in Lemma 4.1, if $p \in \Lambda_0(\hat{\delta})$, $q \in \Lambda_0$ for some $\Lambda \in \mathbb{A}$ and $\{\gamma_k \in C_{T_k}(p, q)\}$ is a sequence of minimizers with $\lim_{k\to+\infty} A_L(\gamma_k) = h^\infty(p, q)$ and $\{T_k\} \nearrow +\infty$, then there is a forward semi-static curve $\gamma^+ \in C^2([0, +\infty), M)$ satisfying the following conditions*

1. *$\gamma^+(0) = p$;*
2. *$\{\gamma_k\}$ converges uniformly to γ^+ along a subsequence on any compact interval;*
3. *$\omega(d\gamma^+) \subset \widetilde{\Lambda}$.*

Proof. By Remark 2.1, there is a forward semi-static curve γ satisfying conditions (1), (2), we claim it also satisfies condition (3). If not, by Lemma 2.5, there is another static class $\Omega \neq \Lambda$, such that $\omega(d\gamma) \subset \widetilde{\Omega}$.

Then there is a $x \in \Omega_0$ and a sequence of integers $\{S_k \in (0, T_k)\} \nearrow +\infty$, such that

$$\lim_{k\to+\infty} d(\gamma(S_k), x) = 0.$$

Hence passing $\{\gamma_k\}$ to a subsequence, we have

$$\lim_{k\to+\infty} d(\gamma_k(S_k), x) = 0.$$

Similar to the proof of Lemma 4.1, we have $\{T_k - S_k\} \to +\infty$ as $k \to +\infty$.

Since $\{T_k - S_k\} \to +\infty$, $\{S_k\} \to +\infty$ and $\lim_{k\to+\infty} d(\gamma_k(S_k), x) = 0$ as $k \to +\infty$, Lemma 2.3 tells us that along a subsequence of $\{\gamma_k\}$,

$$h^\infty(p, x) = \lim_{k\to+\infty} A_L(\gamma_k|_{[0,S_k]});$$

$$h^\infty(x, q) = \lim_{k\to+\infty} A_L(\gamma_k|_{[S_k,T_k]}).$$

Therefore

$$h^\infty(p, q) = \lim_{k\to+\infty} A_L(\gamma_k|_{[0,T_k]}) = h^\infty(p, x) + h^\infty(x, q). \tag{4.3}$$

If $x \in \Omega_0 \neq \Lambda_0$, Lemma 4.1 tells us

$$h^\infty(p, q) < h^\infty(p, x) + h^\infty(x, q),$$

which is a contradiction to (4.3). As a result we proved our claim and γ is the forward semi-static curve γ^+ we are looking for. □

Remark 4.1. *It is not hard to see that similar argument can to be used to show the existence of a backward semi-static curve γ^- with $\gamma^-(0) = p$ and $\alpha(d\gamma^-) \subset \widetilde{\Lambda}$.*

Lemma 4.3. *Given a global semi-static curve γ, if $p \in \alpha(\gamma) \cap \mathcal{A}_0$ and $q \in \omega(\gamma) \cap \mathcal{A}_0$, then*

$$h^\infty(p, q) = h^\infty(p, \gamma(S)) + A_L(\gamma|_{[S,T]}) + h^\infty(\gamma(T), q), \textit{ for any } S \leq T \in \mathbb{Z}.$$

Proof. For arbitrary $p \in \alpha(\gamma) \cap \mathcal{A}_0$ and $q \in \omega(\gamma) \cap \mathcal{A}_0$, there are two sequences of integers $\{S_k\} \searrow -\infty$ and $\{T_k\} \nearrow +\infty$, such that $\lim_{k\to+\infty} \gamma(S_k) = p$ and $\lim_{k\to+\infty} \gamma(T_k) = q$.

Following Lemma 2.3, without loss of generality, we can say

$$h^\infty(p, q) = \lim_{k\to+\infty} A_L(\gamma|_{[S_k,T_k]}),$$

$$h^\infty(p, \gamma(S)) = \lim_{k\to-\infty} A_L(\gamma|_{[S_k,S]}),$$

$$h^\infty(\gamma(T), q) = \lim_{k\to+\infty} A_L(\gamma|_{[T,T_k]}).$$

Therefore

$$\begin{aligned} h^\infty(p, q) &= \lim_{k\to+\infty} [A_L(\gamma|_{[S_k,S]}) + A_L(\gamma|_{[S,T]}) + A_L(\gamma|_{[T,T_k]})] \\ &= \lim_{k\to+\infty} A_L(\gamma|_{[S_k,S]}) + A_L(\gamma|_{[S,T]}) + \lim_{k\to+\infty} A_L(\gamma|_{[T,T_k]}) \\ &= h^\infty(p, \gamma(S)) + A_L(\gamma|_{[S,T]}) + h^\infty(\gamma(T), q). \end{aligned}$$

□

Lemma 4.4. *Given a global semi-static curve γ, if there are two compact sets $U, V \subset M$ satisfying*

1. $\gamma(S) \in U,\ \gamma(T) \in V$ *for some* $S \leq T \in \mathbb{Z}$;
2. $\partial U \cap \mathcal{N}_0 = \emptyset, \quad \partial V \cap \mathcal{N}_0 = \emptyset$, *where* $\partial U, \partial V$ *are the boundaries of* U, V.

Then for any $p \in \alpha(\gamma) \cap \mathcal{A}_0$, $q \in \omega(\gamma) \cap \mathcal{A}_0$ *and* $(x, y) \in U \times V.$, *we have*

$$\Delta(p, x, y, q; \gamma) := h^\infty(p, x) + h^{T-S}(x, y) + h^\infty(y, q) - h^\infty(p, q) \geq 0.$$

Furthermore, if $x \in U \setminus \mathcal{N}_0$ *or* $y \in V \setminus \mathcal{N}_0$, *then*

$$\Delta(p, x, y, q; \gamma) > 0$$

and there is a $\varepsilon > 0$ *such that*

$$\min\{\Delta(p, x, y, q; \gamma) : (x, y) \in U \times V \setminus Int(U) \times Int(V)\} \geq \varepsilon,$$

where $Int(U), Int(V)$ *are the interiors of* U, V.

Proof. Assume there are $x \in U$, $y \in V$ with $\Delta(p, x, y, q; \gamma) < 0$, then

$$h^\infty(p, x) + h^{T-S}(x, y) + h^\infty(y, q) < h^\infty(p, q). \tag{4.4}$$

By Remark 2.1, we can find two sequences of minimizers

$$\{\xi_k \in C_{[0,S_k]}(p, x)\} \text{ with } \{S_k \in \mathbb{Z}\} \nearrow +\infty;$$

$$\{\zeta_k \in C_{[0,T_k]}(y, q)\} \text{ with } \{T_k \in \mathbb{Z}\} \nearrow +\infty,$$

satisfying

$$h^\infty(p, x) = \lim_{k\to+\infty} A_L(\xi_k); \quad h^\infty(y, q) = \lim_{k\to+\infty} A_L(\zeta_k).$$

Let $\eta \in C_{[S,T]}(x, y)$ be a minimizer, i.e., $A_L(\eta) = h^{T-S}(x, y)$. We define a new sequence of curves $\{\gamma_k \in C_{[-S_k, T-S+T_k]}(p, q)\}$ by

$$\gamma_k(t) = \begin{cases} \xi_k(t + S_k), & \text{if } t \in [-S_k, 0]; \\ \eta(t + S), & \text{if } t \in [0, T - S]; \\ \zeta_k(t - T + S), & \text{if } t \in [T - S, T - S + T_k]. \end{cases}$$

Obviously $S_k + T_k + T - S \to +\infty$ as $k \to +\infty$, and

$$\lim_{k\to+\infty} A_L(\gamma_k) = h^\infty(p, x) + h^{T-S}(x, y) + h^\infty(y, q).$$

However,

$$\begin{aligned} h^\infty(p, q) &= \liminf_{n\to+\infty} h^n(p, q) \leq \liminf_{k\to+\infty} h^{S_k+T_k+T-S}(p, q) \\ &\leq \lim_{k\to+\infty} A_L(\gamma_k) = h^\infty(p, x) + h^{T-S}(x, y) + h^\infty(y, q) \\ &< h^\infty(p, q), \end{aligned}$$

where the last inequality follows from (4.4). However this is absurd and we proved the first part of the lemma.

For the second part, without loss of generality we can assume there are $x \in U \setminus \mathcal{N}_0$ and $y \in V$ with $\Delta(p, x, y, q; \gamma) = 0$. Let $\{\gamma_k\}$ be defined as above then

$$\lim_{k\to+\infty} A_L(\gamma_k) = h^\infty(p, x) + h^{T-S}(x, y) + h^\infty(y, q) = h^\infty(p, q). \tag{4.5}$$

Although γ_k is not necessary a minimizer, by (4.5), it is not hard to see, for any $T' \in \mathbb{Z}^+$, $\{A_L(\gamma_k|_{[-T',T']})\}$ has a finite upper bound, and the argument in the proof of Lemma 2.2 will still hold. Therefore along a subsequence $\{\gamma_k\}$ converges uniformly to a global semi-static curve γ^* on any compact interval. Which means

$$\gamma^*(0) = \lim_{k\to+\infty} \gamma_k(0) = \eta(S) = x \in \mathcal{N}_0,$$

and this is a contradiction to our assumption.

Hence $\Delta(p, x, y, q; \gamma) > 0$, if $x \in U \setminus \mathcal{N}_0$ or $y \in V \setminus \mathcal{N}_0$. By the Lipschitz continuity of h^n and h^∞, there is a $\varepsilon > 0$ such that

$$\min\{\Delta(p, x, y, q; \gamma) : (x, y) \in U \times V \setminus Int(U) \times Int(V)\} \geq \varepsilon > 0.$$

□

Now we are ready to prove Theorem 1.2.

Proof. (**Theorem 1.2**) We will follow the notations from the previous section, let $p \in \Lambda^1_0$, $q \in \Lambda^2_0$ and $\{\gamma_k \in C_{[-T^1_k,T^2_k]}(p, q)\}$ is a sequence of minimizers satisfying

$$h^\infty(p, q) = \lim_{k\to+\infty} A_L(\gamma_k).$$

Again there are two different cases as we discussed in the previous section.

If *Case 1* is true, nothing needs to be done here.

If *Case 2* is true, by the proof of Proposition 3.3, we have a set of finite static classes $\{\Omega^1, \ldots, \Omega^n\} \subset \mathbb{A} \setminus \{\Lambda^1, \Lambda^2\}$, a chain of global semi-static curves $\{\gamma^i : i = 1, \ldots, n\}$ and $\{x_i \in \Omega^i_0 : i = 0, 1, \ldots, n + 1\}$, where $x_0 = p$, $x_{n+1} = q$, satisfying

1. $\alpha(d\gamma^i) \subset \widetilde{\Omega}^i$ and $\omega(d\gamma^i) \subset \widetilde{\Omega}^{i+1}$, for $i = 0, \ldots, n$, where $\Omega^0 = \Lambda^1$ and $\Omega^{n+1} = \Lambda^2$;
2. $x_i \in \omega(\gamma^{i-1}) \cap \alpha(\gamma^i) \cap \Omega^i_0$, for $i = 1, \ldots, n$, and $x_0 = p \in \alpha(\gamma^0) \cap \Omega^0_0$, $x_{n+1} \in \omega(\gamma^n) \cap \Omega^{n+1}_0$.
3. $h^\infty(x_0, x_{n+1}) = \sum_{i=0}^n h^\infty(x_i, x_{i+1})$.

Let $\hat{\delta} > 0$ be defined as in Lemma 4.1, for each γ^i, we can choose $S^i < T^i \in \mathbb{Z}$ satisfying

$$\gamma^i(S^i) \in Int(\Omega^i_0(\hat{\delta})) \text{ and } \gamma^i(T^i) \in Int(\Omega^{i+1}_0(\hat{\delta})).$$

Then by assumption $(*_2)$, for each $i = 0, \ldots, n$ there are compact sets $U^i, V^i \subset M$ satisfying

$$\gamma^i(S^i) \in Int(U^i), \quad U^i \subset \Omega^i_0(\hat{\delta}), \quad \partial U^i \cap \mathcal{N}_0 = \emptyset;$$

$$\gamma^i(T^i) \in Int(V^i), \quad V^i \subset \Omega^{i+1}_0(\hat{\delta}), \quad \partial V^i \cap \mathcal{N}_0 = \emptyset.$$

For each $(Y,Z) := \{(y_0,z_0),\dots,(y_n,z_n)\} \in \prod_{i=0}^n U^i \times V^i$, we define a function J by

$$J(Y,Z) := h^\infty(x_0,y_0) + \sum_{i=0}^{n} h^{T^i-S^i}(y_i,z_i) + \sum_{i=0}^{n-1} h^\infty(z_i,y_{i+1}) + h^\infty(z_n,x_{n+1}). \tag{4.6}$$

For each $i = 0,\dots,n-1$, $z_i \in V^i \subset \Omega_0^{i+1}(\hat{\delta})$ and $y_{i+1} \in U^{i+1} \subset \Omega_0^{i+1}(\hat{\delta})$, by Lemma 4.1,

$$h^\infty(z_i,y_{i+1}) = h^\infty(z_i,x_{i+1}) + h^\infty(x_{i+1},y_{i+1}).$$

Hence the function J can be rewritten as

$$J(Y,Z) = \sum_{i=0}^{n}[h^\infty(x_i,y_i) + h^{T^i-S^i}(y_i,z_i) + h^\infty(z_i,x_{i+1})]. \tag{4.7}$$

For any $\tau = (\tau_i)_{i=0}^{n-1}$ with each $\tau_i \in \mathbb{Z}^+$ define a $J_\tau : \prod_{i=0}^n U^i \times V^i \to \mathbb{R}$, by

$$J_\tau(Y,Z) = h^\infty(x_0,y_0) + \sum_{i=0}^{n} h^{T^i-S^i}(y_i,z_i) + \sum_{i=0}^{n-1} h^{\tau_i}(z_i,y_{i+1}) + h^\infty(z_n,x_{n+1}). \tag{4.8}$$

Let

$$c_\tau := \inf\{J_\tau(Y,Z) : (Y,Z) = \{(y_0,z_0),\dots,(y_n,z_n)\} \in \prod_{i=0}^{n} U^i \times V^i\},$$

by the compactness of U^i, V^i and Lipschitz continuity of h^n, h^∞, it is easy to see the above infimum is in fact a minimum.

Lemma 4.5. *There is a τ, such that if $(Y',Z') = \{(y_0',z_0'),\dots,(y_n',z_n')\} \in \prod_{i=0}^n U^i \times V^i$ satisfies $J_T(Y',Z') = c_T$, then*

$$(Y',Z') = \{(y_0',z_0'),\dots,(y_n',z_n')\} \in \prod_{i=0}^{n} Int(U^i) \times Int(V^i).$$

We postpone the proof of the above lemma for a moment. Now there is a

$$(Y',Z') = \{(y_0',z_0'),\dots,(y_n',z_n')\} \in \prod_{i=0}^{n} Int(U^i) \times Int(V^i)$$

with $J_\tau(Y',Z') = c_\tau$. Meanwhile for each $i = 0,\dots,n$, there is a minimizer $\zeta_i \in C_{[0,T^i-S^i]}(y_i',z_i')$ with $A_L(\zeta_i) = h^{T^i-S^i}(y_i',z_i')$, and for each $j = 0,\dots,n-1$, there is a minimizer $\eta_j \in C_{[0,\tau_j]}(z_j',y_{j+1}')$ such that $A_L(\eta_j) = h^{\tau_j}(z_{j+1}',y_{j+1}')$.

By Lemma 4.2, there is a forward semi-static $\gamma^+ \in C([0,+\infty),M)$ with $\gamma^+(0) = z_n'$ and $\omega(d\gamma^+) \subset \widetilde{\Omega}^{n+1}$, and a backward semi-static curve $\gamma^- \in C((-\infty,0],M)$ with $\gamma^-(0) = y_0'$ and $\alpha(d\gamma^-) \subset \widetilde{\Omega}^0$.

Gluing these curves together by the following order

$$\gamma^- * \zeta_0 * \eta_0 * \cdots * \eta_{n-1} * \zeta_n * \gamma^+,$$

we get a new curve $\gamma \in C(\mathbb{R},M)$ with $\alpha(d\gamma) \subset \widetilde{\Lambda}^1$ and $\omega(d\gamma) \subset \widetilde{\Lambda}^2$.

By the standard variational argument, it is not hard to see γ is a classical solution of (EL) and we are done.

□

Proof of Lemma 4.5. Set $\{\gamma^i(S^i), \gamma^i(T^i)\} := \{(\gamma^0(S^0), \gamma^0(T^0)), \dots, (\gamma^n(S^n), \gamma^n(T^n))\}$. Since $\{\gamma^i(S^i), \gamma^i(T^i)\} \in \prod_{i=0}^n Int(U^i) \times Int(V^i)$, it is enough to show, there is a τ such that

$$J_\tau(Y, Z) > J_\tau(\{\gamma^i(S^i), \gamma^i(T^i)\}),$$

for all $(Y, Z) = \{(y_0, z_0), \dots, (y_n, z_n)\} \in \prod_{i=0}^n U^i \times V^i \setminus \prod_{i=0}^n Int(U^i) \times Int(V^i)$. Using the expression of J given in (4.7), we get

$$\begin{aligned} J(Y, Z) - J(\{\gamma^i(S^i), \gamma^i(T^i)\}) = &\sum_{i=0}^n \{h^\infty(x_i, y_i) + h^{T^i - S^i}(y_i, z_i) + h^\infty(z_i, x_{i+1}) \\ &- [h^\infty(x_i, \gamma^i(S^i)) + h^{T^i - S^i}(\gamma^i(S^i), \gamma^i(T^i)) + h^\infty(\gamma^i(T^i), x_{i+1})]\}. \end{aligned}$$

Meanwhile by Lemma 4.3, for each $i = 0, \dots, n$,

$$h^\infty(x_i, \gamma^i(S^i)) + h^{T^i - S^i}(\gamma^i(S^i), \gamma^i(T^i)) + h^\infty(\gamma^i(T^i), x_{i+1}) = h^\infty(x_i, x_{i+1}).$$

Therefore

$$J(Y, Z) - J(\{\gamma^i(S^i), \gamma^i(T^i)\}) = \sum_0^n \Delta(x_i, y_i, z_i, x_{i+1}; \gamma^i).$$

By Lemma 4.4, there is an $\varepsilon > 0$ independent of the choice of (Y, Z), such that

$$J(Y, Z) - J(\{\gamma^i(S^i), \gamma^i(T^i)\}) \geq \varepsilon. \tag{4.9}$$

By the compactness of U^i, V^i and Lipschitz continuity of h^n, h^∞, there is a $\tau' \in \mathbb{Z}^+$, such that for all $i = 0, \dots, n-1$, when $\tau_i \geq \tau'$,

$$h^{\tau_i}(z_i, y_{i+1}) - h^\infty(z_i, y_{i+1}) \geq -\frac{\varepsilon}{4(n-1)}.$$

Then (4.6) and (4.8) imply

$$J_\tau(Y, Z) \geq J(Y, Z) - \frac{\varepsilon}{4}. \tag{4.10}$$

Meanwhile for each $i = 0, \dots, n-1$, we can always find a $\tau_i \geq \tau'$, such that

$$h^{\tau_i}(\gamma^i(T_i), \gamma^{i+1}(S^{i+1})) - h^\infty(\gamma^i(T^i), \gamma^{i+1}(s^{i+1}) \leq \frac{\varepsilon}{4(n-1)}.$$

Again by (4.6) and (4.8), we get

$$J_\tau(\{\gamma^i(S^i), \gamma^i(T^i)\}) \leq J(\{\gamma^i(S^i), \gamma^i(T^i)\}) + \frac{\varepsilon}{4}. \tag{4.11}$$

Combining the above inequality with (4.9) and (4.10), we have

$$J_\tau(Y, Z) \geq J(Y, Z) - \frac{\varepsilon}{4} \geq J(\{\gamma^i(S^i), \gamma^i(T^i)\}) + \frac{3\varepsilon}{4} \geq J_\tau(\{\gamma^i(S^i), \gamma^i(T^i)\}) + \frac{\varepsilon}{2}.$$

This finishes our proof of the lemma.

□

Acknowledgments

This work is supported by the National Key R&D Program of China (2020YFA0713303), NSFC (No. 12171253), the Fundamental Research Funds for the Central Universities and Nankai Zhide Fundation.

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. J. N. Mather, Action minimizing invariant measures for positive definite Lagrangian systems, *Math. Z.*, **207** (1991), 169–207. https://doi.org/10.1007/BF02571383
2. G. Contreras, J. Delgado, R. Iturriaga, Lagrangian flows: the dynamics of globally minimizing orbits, II. *Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society*, **28** (1997), 155–196. https://doi.org/10.1007/BF01233390
3. P. Bernard, Connecting orbits of time dependent Lagrangian systems, *Ann. Inst. Fourier*, **52** (2002), 1533–1568. https://doi.org/10.5802/aif.1924
4. P. Bernard, G. Contreras, A generic property of families of Lagrangian systems, *Ann. Math.*, **167** (2008), 1099–1108. https://doi.org/10.4007/annals.2008.167.1099
5. G. Contreras, G. P. Paternain, Connecting orbits between static classes for generic Lagrangian systems, *Topology*, **41** (2002), 645–666. https://doi.org/10.1016/S0040-9383(00)00042-2
6. X. Cui, C. Q. Cheng, W. Cheng, Existence of infinitely many homoclinic orbits to Aubry sets for positive definite Lagrangian systems, *J. Differ. Equ.*, **214** (2005), 176–188. https://doi.org/10.1016/j.jde.2004.08.008
7. Y. Zheng, C. Q. Cheng, Homoclinic orbits of positive definite Lagrangian systems, *J. Differ. Equ.*, **229** (2006), 297–316. https://doi.org/10.1016/j.jde.2006.03.007
8. A. Fathi, Weak KAM Theorem in Lagrangian Dynamics, book to appear, Cambridge University Press.
9. T. O. Maxwell, Heteroclinic chains for a reversible Hamiltonian system, *Nonlinear Anal.*, **28** (1997), 871–887. https://doi.org/10.1016/0362-546X(95)00193-Y
10. P. H. Rabinowitz, Connecting orbits for a reversible Hamiltonian system, *Ergod. Theory Dyn. Syst.*, **20** (2000), 1767–1784. https://doi.org/10.1017/S0143385700000985
11. J. N. Mather, Variational construction of connecting orbits, *Ann. Inst. Fourier*, **43** (1993), 1349–1386.

AIMS Press