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ERA, 30(6): 2247–2262.
DOI: 10.3934/era.2022114
Received: 15 August 2021
Revised: 06 December 2021
Accepted: 20 December 2021
Published: 21 April 2022

Research article

Martingale transforms on Banach function spaces

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Abstract: We establish the boundedness of martingale transforms on Banach function spaces by using the Rubio de Francia extrapolation theory and the interpolation theorem by Zygmund. The main result also yields the boundedness of the martingale transforms on rearrangement-invariant Banach function spaces, Orlicz spaces, Lorentz-Karamata spaces, Zygmund spaces, Lebesgue spaces with variable exponents, Morrey spaces with variable exponents and Lorentz-Karamata Morrey spaces.

Keywords: martingale transform; Banach function spaces; Orlicz spaces; Lorentz-Karamata spaces; variable Lebesgue spaces; Morrey spaces

1. Introduction

This paper aims to extend the boundedness of martingale transforms to Banach function spaces. The martingale transform is one of the important topics in probability and martingale function spaces [1–4]. The main results of this paper use the extrapolation theory to obtain the boundedness of the martingale transform on Banach function spaces which includes rearrangement-invariant Banach function spaces, Orlicz spaces, Lorentz-Karamata spaces, Zygmund spaces, Lebesgue spaces with variable exponents and Morrey type spaces such as Morrey spaces with variable exponents and Lorentz-Karamata-Morrey spaces.

We use two different methods, the Rubio de Francia extrapolation theory and the interpolation theorem by Zygmund.

Roughly speaking, the Rubio de Francia extrapolation theory uses the weighted norm inequalities for the martingale transforms to obtain the mapping properties for those Banach function spaces which satisfy some conditions related with the boundedness of maximal function. The interpolation theorem extends the mapping properties to the Zygmund spaces. The Zygmund spaces are rearrangement-invariant Banach function spaces used to capture the mapping properties of operators on the limiting cases of the Lebesgue space L^p for p approaching 1.

This paper is organized as follows. The definition of Banach function spaces on probability spaces and some assumptions for the probability spaces are given in Section 2. The main results are established in Section 3. The applications of the main results on some concrete function spaces such as the Orlicz spaces, the Lorentz-Karamata spaces, the Lebesgue spaces with variable exponents, the Lorentz-Karamata-Morrey spaces and the Morrey spaces with variable exponents are presented in Section 4.

2. Definitions and preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathcal{M}$ be the space of measurable functions on $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$ be the filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ where $(\mathcal{F}_n)_{n\geq 0}$ is a nondecreasing sequence of sub-σ-algebras of. Let $\mathcal{F}_{-1} = \mathcal{F}_0$.

For any martingale $f = (f_n)_{n\geq 0}$ on Ω, write $d_i f = f_i - f_{i-1}$, $i > 0$ and $d_0 f = f_0$. Let $f = (f_n)_{n\geq 0}$ be a uniformly integrable martingale. We identify the martingale f with its pointwise limit f_∞ where the existence of the limit is guaranteed by the uniform integrability. For any integrable function f, the martingale generated by f is given by $f_n = E_n f$ where E_n is the expectation operator associated with $\mathcal{F}_n$, $n \geq 0$.

The maximal function and the truncated maximal function of the martingale f is defined by

$$Mf = \sup_{i\geq 0} |f_i| \quad \text{and} \quad M_n f = \sup_{0\leq i\leq n} |f_i|, \quad n \geq 0,$$

respectively.

For any predictable sequence $v = (v_n)_{n\geq 0}$ and martingale f, the martingale transform T_v is defined as

$$(T_v f)_n = \sum_{k=1}^{n} v_k d_k f, \quad (T_v f)_0 = 0.$$

The celebrated result on the convergence of martingale transform states that whenever f is a bounded L^1 martingale, then $T_v f = ((T_v f)_n)_{n\geq 0}$ converges almost everywhere on $\{x \in \Omega : Mv(x) < \infty\}$.

Definition 2.1. A Banach space $X \subset \mathcal{M}$ is said to be a Banach function space on $(\Omega, \mathcal{F}, \mathbb{P})$ if it satisfies

1. $\|f\|_X = 0 \Leftrightarrow f = 0$ a.e.,
2. $|g| \leq |f|$ a.e. $\Rightarrow \|g\|_X \leq \|f\|_X$,
3. $0 \leq f_n \uparrow f$ a.e. $\Rightarrow \|f_n\|_X \uparrow \|f\|_X$,
4. $L^\infty \hookrightarrow X \hookrightarrow L^1$.

Item (4) of the above definition guarantees that for any measurable set E, $\chi_E \in X$ and $\int_E |f| d\mathbb{P} \leq C\|f\|_X$. In particular, Item (4) assets that for any $f \in X$, $f = (f_n)_{n\geq 0}$ is a bounded L^1 martingale. Therefore, for any $f \in X$, the pointwise limit of $E_n f$, $\lim_{n\to\infty} E_n f = f_\infty$, exists and we identify the martingale $f = (f_n)_{n\geq 0}$ with f_∞.

Moreover, whenever $\|v\|_{L^\infty} = \sup_{n\geq 0} \|v_n\|_{L^\infty} < \infty$, for any $f \in X$, the martingale transform $T_v f = ((T_v f)_n)_{n\geq 0}$ converges almost everywhere on Ω. Thus, we are allowed to identify the martingale transform $T_v f$ with its pointwise limit $(T_v f)_\infty$.

Furthermore, Item (4) guarantees that X is a Banach function space defined in [5, Chapter 1, Definitions 1.1 and 1.3]. Thus, the results in [5] apply to the Banach function space defined in Definition 2.1.

The rearrangement-invariant Banach function spaces are examples of Banach function spaces. The reader is referred to [5, Chapter 2, Definition 4.1] for the definition of rearrangement-invariant Banach function spaces. Particularly, the Lorentz spaces, the Lorentz-Karamata spaces, the Orlicz spaces and the grand Lebesgue spaces are examples of Banach function spaces. The family of Banach function spaces also includes the Lebesgue spaces with variable exponents and the Morrey type spaces [1, 6]. Notice that the Morrey type spaces on $\mathbb{R}^n$ are generally not Banach function spaces on $\mathbb{R}^n$. For the definitions of Morrey type spaces on $\mathbb{R}^n$, the reader is referred to [7, Definition 2.4].

The reader is referred to [1–3, 8] for the mapping properties of martingale transforms on Lebesgue spaces, Hardy spaces, Orlicz spaces, rearrangement-invariant Banach function spaces and Morrey spaces.

We recall the definition of associate space from [5, Chapter 1, Definitions 2.1 and 2.3].

Definition 2.2. Let X be a Banach function space. The associate space of X, X', is the collection of all measurable function f such that

$$\|f\|_{X'} = \sup\left\{ \int_\Omega |fg| d\mathbb{P} : g \in X,\ \|g\|_X \le 1 \right\} < \infty.$$

According to [5, Chapter 1, Theorems 1.7 and 2.2], when X is a Banach function space, X' is also a Banach function space. In addition, the Lorentz-Luxemburg theorem [5, Chapter 1, Theorem 2.7] yields

$$X = (X')'. \tag{2.1}$$

We have the Hölder inequality for X and X', see [5, Chapter 1, Theorem 2.4].

Theorem 2.1. *Let X be a Banach function space. Then, for any $f \in X$ and $g \in X'$, we have*

$$\int_\Omega |fg| d\mathbb{P} \le \|f\|_X \|g\|_{X'}.$$

For any Banach function space X, the weak type Banach function space $w-X$ consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{w-X} = \sup_{\lambda>0} \lambda \left\| \chi_{\{x\in\Omega : |f(x)|>\lambda\}} \right\|_X < \infty.$$

For any $0 < r < \infty$ and Banach lattice X, the r-convexification of X, X^r is defined as

$$X^r = \{f : |f|^r \in X\}.$$

The vector space X^r is equipped with the quasi-norm $\|f\|_{X^r} = \||f|^r\|_X^{1/r}$.

We begin with the assumptions imposed on the filtration $\mathcal{F}$. We assume that every σ-algebra $\mathcal{F}_n$ is regular and generated by finitely or countably many atoms, where $B \in \mathcal{F}_n$ is called an atom if it satisfies the nested property. That is, any $A \subseteq B$ with $A \in \mathcal{F}_n$ satisfying

$$P(A) = P(B) \quad \text{or} \quad P(A) = 0.$$

Denote the set of atoms by $\mathcal{A}(\mathcal{F}_n)$ and write $\mathcal{A} = \cup_{n\ge 0}\mathcal{A}(\mathcal{F}_n)$. Whenever $\mathcal{F} = (\mathcal{F}_n)_{n\ge 0}$ satisfy the above condition, we say that $\mathcal{F}$ is generated by atoms.

The notion of atoms were used in [9] for the studies of the martingale Hardy spaces with variable exponent and in [1, 6] for the studies of the martingale Morrey and Campanato spaces.

Suppose that $\mathcal{F}$ is generated by atoms. For any measurable function f, it is easy to see that

$$Mf(x) = \sup_{A \ni x} \frac{1}{\mathbb{P}(A)} \int_A |f| d\mathbb{P} \tag{2.2}$$

where the supremum is taken over all $A \in \mathcal{A}$ containing x.

Write $M^0 f = |f|$. For any $k \in \mathbb{N}$, let M^k be the k-iteration of M.

Next, we recall the definition of the Muckenhoupt weight functions on probability space. The Muckenhoupt weight functions on $\mathbb{R}^n$ were introduced by Muckenhoupt and the extension of this class of weight functions on probability space was given by Izumisawa and Kazamaki in [10].

In the followings, we recall the definition of the Muckenhoupt weight functions when $\mathcal{F}$ is generated by atoms.

Definition 2.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space where $\mathcal{F}$ is generated by atoms. A nonnegative integrable function ω is said to be an A_p weight if it satisfies

$$[\omega]_{A_p} = \sup_{A \in \mathcal{A}} \left(\frac{1}{\mathbb{P}(A)} \int_A \omega d\mathbb{P} \right) \left(\frac{1}{\mathbb{P}(A)} \int_A \omega^{-\frac{p'}{p}} d\mathbb{P} \right)^{\frac{p}{p'}} < \infty$$

where $p' = \frac{p}{p-1}$. A nonnegative integrable function ω is said to be an A_1 weight if there is a constant $C > 0$ such that for any $A \in \mathcal{A}$

$$\frac{1}{\mathbb{P}(A)} \int_A \omega d\mathbb{P} \leq C\omega, \quad \text{a.e.} \quad \text{on} \quad A. \tag{2.3}$$

The infimum of all such C is denoted by $[\omega]_{A_1}$. We define $A_\infty = \cup_{p \geq 1} A_p$.

We have $A_p \subset A_q$, $1 \leq p \leq q \leq \infty$. It is well known that $\omega \in A_p$, $p \in (1, \infty)$, if and only if for all $n \geq 0$

$$(\omega_n)((\omega^{-\frac{p'}{p}})_n)^{\frac{p}{p'}} < C.$$

When $p = 1$, we have $\omega \in A_1$ if and only if $\mathbb{P}(M_n \omega \leq c\omega_n) = 1$ for all $n \geq 0$.

Here we give a brief outline on the proof of the above characterization of A_1. When $\mathbb{P}(M_n \omega \leq c\omega_n) = 1$ for all $n \geq 0$, by applying $\lim_{n \to \infty}$ on both sides of $M_n \omega \leq c\omega_n$, we obtain (2.3). Whenever ω satisfies (2.3), for any $1 \leq k \leq n$, we have $\omega_k \leq C\omega_n$. By taking supremum over $1 \leq k \leq n$, we get $M_n \omega \leq C\omega_n$.

We now present the weighted norm inequalities for martingale transforms from [8] and [11, Theorem 5.3].

Theorem 2.2. *Let $p \in (1, \infty)$, $\omega \in A_1$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. If $\mathcal{F}$ is generated by atoms and the predictable sequence $v = (v_n)_{n \geq 0}$ satisfies $\|v\|_{L^\infty} = \sup_{n \geq 0} \|v_n\|_{L^\infty} < \infty$, then for any $f \in L^p(\omega)$,*

$$\|T_v f\|_{L^p(\omega)} \leq C[\omega]_{A_1} \|f\|_{L^p(\omega)}. \tag{2.4}$$

for some $C > 0$ and there is a constant $C > 0$ such that for any $f \in L^1(\omega)$,

$$\sup_{\lambda > 0} \lambda \left\| \chi_{\{x \in \Omega : |T_v f(x)| > \lambda\}} \right\|_{L^1(\omega)} \leq C[\omega]_{A_1} (1 + \log(2^\beta [\omega]_{A_1})) \|f\|_{L^1(\omega)}. \tag{2.5}$$

Definition 2.4. Let X be a Banach function space, $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Suppose that $\mathcal{F}$ is generated by atoms. For any Banach function space X, we write $X \in \mathbb{M}$ if the maximal function M is bounded on X. We write $X \in \mathbb{M}'$ if the maximal function M is bounded on X'.

It is well known that the Lebesgue space $L^p \in \mathbb{M} \cap \mathbb{M}'$, $p \in (1, \infty)$. According to the Boyd interpolation theorem [5, Chapter 3, Thereom 5.16], the maximal function is bounded on the rearrangement-invariant Banach function space X whenever the Boyd's indices of X are located in $(0, 1)$. For brevity, the reader is referred to [5, Chapter 3, Definitions 5.12] for the definition of the Boyd's indices. Consequently, some Lorentz spaces, Orlicz spaces, Lorentz-Karamata spaces and grand Lebesgue spaces belong to $\mathbb{M}$, see [2, Section 6] for the Boyd's indices of the above mentioned function spaces.

3. Main results

The main result on the boundedness of martingale transform on Banach function spaces is presented and established in this section. We obtain this result by using the Rubio de Francia extrapolation theory and the Zygmund interpolation theorem. We begin with the definition of an operator used in the Rubio de Francia extrapolation theory.

Definition 3.1. Let X be a Banach function space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Suppose that $\mathcal{F}$ is generated by atoms and $X \in \mathbb{M}$. Define

$$\mathcal{R}_X h = \sum_{k=0}^{\infty} \frac{M^k h}{2^k \|M\|_{X \to X}^k} \tag{3.1}$$

where $\|M\|_{X\to X}$ denote the operator norm of M.

As $X \in \mathbb{M}$, we see that

$$\|\mathcal{R}_X h\|_X \le \sum_{k=0}^{\infty} \frac{\|M^k h\|_X}{2^k \|M\|_{X \to X}^k} \le 2\|h\|_X.$$

That is, $\mathcal{R}_X$ is bounded on X.

The following proposition gives an embedding result for Banach function space into weighted Lebesgue spaces.

Proposition 3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and X be a Banach function space. Suppose that $\mathcal{F}$ is generated by atoms. If there exists a $p \in (1, \infty)$ such that $X^{1/p}$ and $X^{1/p} \in \mathbb{M}'$, then*

$$X \hookrightarrow \bigcap_{h \in (X^{1/p})'} L^p(\mathcal{R}_{(X^{1/p})'} h). \tag{3.2}$$

Proof: As $X^{1/p} \in \mathbb{M}'$, we find that $\mathcal{R}_{(X^{1/p})'}$ is bounded on $(X^{1/p})'$.

Let $f \in X$. For any $h \in (X^{1/p})'$, Theorem 2.1 yields

$$\int_\Omega |f|^p \mathcal{R}_{(X^{1/p})'} h d\mathbb{P} \le \||f|^p\|_{X^{1/p}} \|\mathcal{R}_{(X^{1/p})'} h\|_{(X^{1/p})'} \le C\|f\|_X^p \|h\|_{(X^{1/p})'}.$$

Therefore, $f \in L^p(\mathcal{R}_{(X^{1/p})'} h)$. ■

With the above result, we can get rid of the density or the approximation arguments and obtain the mapping properties of the martingale transform on the entire Banach function space.

We are now ready to present and establish the main result of this paper, the mapping properties of the martingale transforms on Banach function spaces. We follow the ideas from [7] to obtain the following result.

Theorem 3.2. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space and* X *be a Banach function space. Suppose that* $\mathcal{F}$ *is generated by atoms and the predictable sequence* $v = (v_n)_{n\geq 0}$ *satisfies* $\|v\|_{L^\infty} = \sup_{n\geq 0} \|v_n\|_{L^\infty} < \infty$.

1. *If there exists a* $p \in (1, \infty)$ *such that* $X^{1/p}$ *is a Banach function space and* $X^{1/p} \in \mathbb{M}'$*, then there exists a constant* $C > 0$ *such that for any* $f \in X$

$$\|T_v f\|_X \leq C\|f\|_X. \tag{3.3}$$

2. *If* $X \in \mathbb{M}'$*, then there exists a constant* $C > 0$ *such that for any* $f \in X$

$$\|T_v f\|_{w-X} \leq C\|f\|_X. \tag{3.4}$$

Proof: We first prove (3.3). For any $h \in (X^{1/p})'$ with $\|h\|_{(X^{1/p})'} \leq 1$, the definition of $\mathcal{R}_{(X^{1/p})'}$, we see that

$$M\mathcal{R}_{(X^{1/p})'}h \leq \sum_{k=0}^{\infty} \frac{M^{k+1}h}{2^k \|M_{\mathcal{T}}\|^k_{X\to X}} \leq 2\|M\|_{X\to X}\mathcal{R}_{(X^{1/p})'}h.$$

According to (2.3), we have $\mathcal{R}_{(X^{1/p})'}h \in A_1$ with $[\mathcal{R}_{(X^{1/p})'}h]_{A_1} \leq 2\|M\|_{X\to X}$.

In addition, we have

$$\begin{aligned}
\|\mathcal{R}_{(X^{1/p})'}h\|_{(X^{1/p})'} &\leq \sum_{k=0}^{\infty} \frac{\|M^k h\|_{(X^{1/p})'}}{2^k \|M\|^k_{(X^{1/p})'\to(X^{1/p})'}} \\
&\leq \sum_{k=0}^{\infty} \frac{\|M\|^k_{(X^{1/p})'\to(X^{1/p})'}\|h\|_{(X^{1/p})'}}{2^k \|M\|^k_{(X^{1/p})'\to(X^{1/p})'}} \leq 2\|h\|_{(X^{1/p})'}.
\end{aligned}$$

Thus, $\mathcal{R}_{(X^{1/p})'}$ is bounded on $(X^{1/p})'$.

For any $f \in X$, as Theorem 2.2 guarantees that T_v is bounded on $L^p(\mathcal{R}_{(X^{1/p})'}h)$, the embedding (3.2) assures that $T_v f$ is well defined and

$$\int_\Omega |T_v f|^p \mathcal{R}_{(X^{1/p})'}h d\mathbb{P} \leq C \int_\Omega |f|^p \mathcal{R}_{(X^{1/p})'}h d\mathbb{P}$$

for some $C > 0$ independent of h.

Theorem 2.1 and the boundedness of $\mathcal{R}_{(X^{1/p})'}$ give

$$\int_\Omega |T_v f|^p \mathcal{R}_{(X^{1/p})'}h d\mathbb{P} \leq C\||f|^p\|_{X^{1/p}}\|\mathcal{R}_{(X^{1/p})'}h\|_{(X^{1/p})'} \leq C\|f\|^p_X\|h\|_{(X^{1/p})'}.$$

Obviously, the definition of $\mathcal{R}_{(X^{1/p})'}$ guarantees that

$$\int_\Omega |T_v f|^p |h| d\mathbb{P} \leq \int_\Omega |T_v f|^p \mathcal{R}_{(X^{1/p})'}h d\mathbb{P} \leq C\|f\|^p_X\|h\|_{(X^{1/p})'}.$$

By taking supremum over $h \in (X^{1/p})'$ with $\|h\|_{(X^{1/p})'} \leq 1$ on both sides of the above inequalities, Definition 2.2 and (2.1) assert that

$$\begin{aligned}\|T_v f\|_X^p &= \||T_v f|^p\|_{X^{1/p}}\\ &= \sup\left\{\int_\Omega |T_v f|^p |h| d\mathbb{P} : \|h\|_{(X^{1/p})'} \leq 1\right\} \leq C\|f\|_X^p\end{aligned}$$

which gives the boundedness of the martingale transform $T_v : X \to X$.

Next, we prove (3.4). Let $f \in X$. In view of Theorem 2.2, for any $\lambda > 0$, $F_\lambda = \lambda \chi_{\{x\in\Omega: |T_v f(x)|\geq\lambda\}}$ is well defined. Moreover, for any $h \in X'$, Theorems 2.1, 2.2 and the boundedness of $\mathcal{R}_{X'}$ yield a constant independent of λ and h such that

$$\begin{aligned}\int_\Omega F_\lambda |h| d\mathbb{P} &\leq \int_\Omega F_\lambda \mathcal{R}_{X'} h d\mathbb{P} \leq C\int_\Omega |f|\mathcal{R}_{X'} h d\mathbb{P}\\ &\leq C\|f\|_X \|\mathcal{R}_{X'} h\|_{X'} \leq C\|f\|_X \|h\|_{X'}.\end{aligned}$$

By taking supremum over $h \in X'$ with $\|h\|_{X'} \leq 1$ on both sides of the above inequalities, Definition 2.2 and (2.1) assert that

$$\|F_\lambda\|_X \leq C\|f\|_X$$

for some $C > 0$ independent of $\lambda > 0$. By taking supremum over $\lambda > 0$, we obtain

$$\|T_v f\|_{w-X} = \sup_\lambda \lambda \|\chi_{\{x\in\Omega: |T_v f(x)|\geq\lambda\}}\|_X = \sup_{\lambda>0} \|F_\lambda\|_X \leq C\|f\|_X$$

which is (3.4). ∎

Next, we use the Zygmund interpolation theorem to study the mapping properties of the martingale transforms on Zygmund spaces. The Zygmund spaces are used to capture the mapping properties of the martingale transform for the limiting cases of Lebesgue spaces L^p when p is approaching to 1.

Definition 3.2. Let $\alpha \in \mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. The Zygmund space $L^1(\log L)^\alpha$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L^1(\log L)^\alpha} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f|}{\lambda}\right)\left(\log\left(e + \frac{|f|}{\lambda}\right)\right)^\alpha d\mathbb{P}\right\} < \infty.$$

When $\alpha = 0$, we have $L^1(\log L)^0 = L^1$. The reader is referred to [5, Chapter 4, Section 6] for more information of Zygmund spaces.

Theorem 3.3. *Let $\alpha \in (-1, \infty)$. If T is a linear operator, $T : L^p \to L^p$ for some $p \in (1, \infty)$ and $T : L^1 \to w\text{-}L^1$ are bounded, then $T : L^1(\log L)^{\alpha+1} \to L^1(\log L)^\alpha$ is bounded.*

For the proof of the above result, see [5, Corollary 6.15]. Notice the result given in [5, Corollary 6.15] are presented with the assumption that T is of joint weak type $(1, 1, p, p)$. In view of [5, Chapter 4, Proposition 4.2, Definition 4.9 and Theorem 4.11], whenever $T : L^p \to L^p$ for some $p \in (1, \infty)$ and $T : L^1 \to w\text{-}L^1$ are bounded, T is of joint weak type $(1, 1, p, p)$.

In view of Theorems 2.2, the martingale transform T_v satisfies the conditions in Theorem 3.3, therefore, we obtain the following result.

Theorem 3.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. If $\mathcal{F}$ is generated by atoms and the predictable sequence $v = (v_n)_{n\geq 0}$ satisfies $\|v\|_{L^\infty} = \sup_{n\geq 0} \|v_n\|_{L^\infty} < \infty$, then for any $\alpha > -1$, the martingale transform $T_v : L^1(\log L)^{\alpha+1} \to L^1(\log L)^\alpha$ is bounded.*

The above result yields and extends the boundedness of $T_v : L^1(\log L) \to L^1$.

4. Applications

In this section, we apply Theorem 3.2 to some concrete Banach function spaces, namely, the rearrangement-invariant Banach function spaces, the Orlicz spaces, the Lorentz-Karamata spaces, the Lebesgue spaces with variable exponents and the Morrey type spaces built on Banach function spaces.

4.1. Rearrangement-invariant Banach function spaces

For any $f \in \mathcal{M}$, define

$$d_f(\lambda) = P(\{x \in \Omega : |f(x)| > \lambda\}), \quad \lambda > 0.$$

The decreasing rearrangement of f is defined as

$$f^*(t) = \inf\{\lambda : d_f(\lambda) \leq t\}, \quad t \geq 0.$$

Two measurable functions f and g are equimeasurable if for any $\lambda \geq 0$, $d_f(\lambda) = d_g(\lambda)$. We say that a Banach function space X is rearrangement-invariant if for every pair of equimeasurable functions f, g, we have $\|f\|_X = \|g\|_X$.

If $(\Omega, \mathbb{P})$ is a nonatomic measure space, in view of the Luxemburg representation theorem [5, Chapter 2, Theorem 4.10], for any rearrangement-invariant Banach function space X, there exists a norm ρ_X satisfies Items (1)–(3) of Definition 2.1 such that $\|f\|_X = \rho_X(f^*)$. We write $\bar{X}$ for the Banach function space on $[0, 1]$ endowed with the norm ρ_X.

The reader may consult [2, 3] for studies of the martingale function spaces built on rearrangement-invariant function spaces.

For any $s \geq 0$ and Lebesgue measurable function f on $[0, 1]$, define $(D_s f)(t) = f(st)$, $t \in (0, \infty)$. Let $\|D_s\|_{\bar{X}\to\bar{X}}$ be the operator norm of D_s on $\bar{X}$. We recall the definition of Boyd's indices for rearrangement-invariant Banach function spaces from [5, Chapter 3, Definition 5.12].

Definition 4.1. Let $(\Omega, \mathbb{P})$ be a nonatomic measure space and X be a rearrangement-invariant Banach function space. Define the lower Boyd index of X, $\underline{\alpha}_X$, and the upper Boyd index of X, $\bar{\alpha}_X$, by

$$\underline{\alpha}_X = \inf_{t\in(1,\infty)} \frac{\log \|D_{1/t}\|_{\bar{X}}}{\log t} = \lim_{t\to\infty} \frac{\log \|D_{1/t}\|_{\bar{X}}}{\log t},$$
$$\bar{\alpha}_X = \sup_{t\in(0,1)} \frac{\log \|D_{1/t}\|_{\bar{X}}}{\log t} = \lim_{t\to 0^+} \frac{\log \|D_{1/t}\|_{\bar{X}}}{\log t},$$

respectively.

According to [12, Theorem B], we have the following boundedness result of maximal function M on rearrangement-invariant Banach function spaces.

Theorem 4.1. *Let* $(\Omega, \mathbb{P})$ *be nonatomic and* X *be a rearrangement-invariant Banach function space. The maximal function* M *is bounded on* X *if and only if* $\bar{\alpha}_X < 1$.

The results in [12, Theorem B] are for continuous martingale, the proof of the discrete martingale is similar, see also [5, Chapter 3, Theorem 5.17].

The boundedness of the martingale transform for rearrangement-invariant Banach function space X satisfying $0 < \underline{\alpha}_X \leq \bar{\alpha}_X < 1$ follows from Boyd's interpolation theorem [5, Chapter 3, Theorem 5.16], see also [3]. It is also a special case of [2, Theorem 4.4].

Theorem 4.2. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space and* X *be a rearrangement-invariant Banach function space. Suppose that* $\mathcal{F}$ *is generated by atoms and the predictable sequence* $v = (v_n)_{n\geq 0}$ *satisfies* $\|v\|_{L^\infty} = \sup_{n\geq 0} \|v_n\|_{L^\infty} < \infty$.

If $0 < \underline{\alpha}_X \leq \bar{\alpha}_X < 1$, *then there exists a constant* $C > 0$ *such that for any* $f \in X$,

$$\|T_v f\|_X \leq C\|f\|_X.$$

The reader is referred to [2, Theorem 4.4] for a generalization of the preceding result.

Next, we consider the mapping properties of the martingale transform T_v in the limiting case, $0 < \underline{\alpha}_X \leq \bar{\alpha}_X \leq 1$.

Theorem 4.3. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space and* X *be a rearrangement-invariant Banach function space. Suppose that* $\mathcal{F}$ *is generated by atoms and the predictable sequence* $v = (v_n)_{n\geq 0}$ *satisfies* $\|v\|_{L^\infty} = \sup_{n\geq 0} \|v_n\|_{L^\infty} < \infty$.

If $0 < \underline{\alpha}_X \leq \bar{\alpha}_X \leq 1$, *then there exists a constant* $C > 0$ *such that for any* $f \in X$

$$\|T_v f\|_{w-X} \leq C\|f\|_X.$$

Proof: As $\mathcal{F}$ is generated by atoms, $(\Omega, \mathbb{P})$ is a nonatomic measure space. In addition, [5, Chapter 3, (5.33)] shows that $\bar{\alpha}_{X'} = 1 - \underline{\alpha}_X < 1$, therefore, Theorem 4.1 asserts that the Hardy-Littlewood maximal function is bounded on $X \in \mathbb{M}'$. Consequently, Theorem 3.2 yields a constant $C > 0$ such that for any $f \in X$,

$$\|T_v f\|_{w-X} \leq C\|f\|_X. \quad \blacksquare$$

We apply Theorem 4.3 to two concrete rearrangement-invariant Banach function spaces, the Lorentz-Karamata spaces and the Orlicz spaces.

We first consider the Lorentz-Karamata spaces. We recall the definition of the Lorentz-Karamata spaces from [13].

Definition 4.2. A Lebesgue measurable function $b : [1, \infty) \to (0, \infty)$ is called as a slowly varying function if for any $\epsilon > 0$

1. the function $t \to t^\epsilon b(t)$ is equivalent to a non-decreasing function on $[1, \infty)$, and
2. the function $t \to t^{-\epsilon} b(t)$ is equivalent to a non-increasing function on $[1, \infty)$.

For any slowly varying function b, define

$$\gamma_b(t) = b(t^{-1}), \quad 0 < t \leq 1.$$

Definition 4.3. Let $0 < r, p < \infty$, b be a slowly varying function and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. The Lorentz-Karamata space $L_{r,p,b}$ consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{L_{r,p,b}} = \left(\int_0^1 t^{\frac{p}{r}-1} (\gamma_b(t) f^*(t))^p dt \right)^{\frac{1}{p}} < \infty.$$

The reader is referred to [13–15] for the studies of martingale functions spaces built on Lorentz-Karamata spaces. In addition, it had been further extended to the Orlicz-Karamata spaces in [16].

When $1 < r, p < \infty$, the Lorentz-Karamata space $L_{r,p,b}$ is a rearrangement-invariant Banach function space. Furthermore, according to [17, Proposition 6.1], we have $\underline{\alpha}_{L_{r,p,b}} = \bar{\alpha}_{L_{r,p,b}} = \frac{1}{r}$. Notice that the Boyd's indices defined in [17, Proposition 6.1] are reciprocals of the ones defined in Definition 4.1.

When $b = 1$, the Lorentz-Karamata space $L_{r,p,b}$ becomes the Lorentz space $L_{r,p}$.

We now apply Theorem 3.2 to obtain the mapping properties of the martingale transform on $L_{1,p,b}$.

Theorem 4.4. *Let $p \in (1, \infty)$, b be a slowly varying function and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. If $\mathcal{F}$ is generated by atoms and the predictable sequence $v = (v_n)_{n \geq 0}$ satisfies $\|v\|_{L^\infty} = \sup_{n \geq 0} \|v_n\|_{L^\infty} < \infty$, then there exists a constant $C > 0$ such that for any $f \in L_{1,p,b}$,*

$$\|T_v\|_{w-L_{1,p,b}} \leq C \|f\|_{L_{1,p,b}}.$$

When $b = 1$. the above theorem gives the mapping properties of T_v on the Lorentz space $L_{1,p}$.

Next, we consider the Orlicz spaces. A function $\Phi : [0, \infty] \to [0, \infty]$ is called a Young's function if it is convex, left-continuous and $\Phi(0) = 0$.

Definition 4.4. Let Φ be a Young's function. The Orlicz space L^Φ consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{L^\Phi} = \inf \left\{ \lambda : \int_\Omega \Phi(|f|/\lambda) d\mathbb{P} \right\} < \infty.$$

For any Young's function, we write $\Phi \in \triangle_2$ if there exists a constant $C > 0$ such that for any $t \geq 0$

$$\Phi(2t) \leq C\Phi(t).$$

The complementary function (the conjugate function) of Φ is defined as

$$\tilde{\Phi}(t) = \sup\{st - \Phi(s) : s \geq 0\}, \quad t \geq 0.$$

It is well known that the complementary function of $\tilde{\Phi}$ is Φ and

$$(L^\Phi)' = L^{\tilde{\Phi}}.$$

Theorem 3.2 yields the mapping properties of the martingale transform on the Orlicz space L^Φ.

Theorem 4.5. *Let Φ be a Young's function and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Suppose that $\Phi \in \triangle_2$. If $\mathcal{F}$ is generated by atoms and the predictable sequence $v = (v_n)_{n \geq 0}$ satisfies $\|v\|_{L^\infty} = \sup_{n \geq 0} \|v_n\|_{L^\infty} < \infty$, then there exists a constant $C > 0$ such that for any $f \in L^\Phi$,*

$$\|T_v\|_{w-L^\Phi} \leq C \|f\|_{L^\Phi}. \tag{4.1}$$

Proof: As $\Phi \in \triangle_2$, [18, Volume II, Corollary 2.b.5] asserts that $\underline{\alpha}_{L^\phi} > 0$. Notice that the Boyd's indices defined in [18, Volume II, Corollary 2.b.5] are reciprocals of the ones defined in Definition 4.1. Since $1 - \bar{\alpha}_{L^{\tilde{\Phi}}} = \underline{\alpha}_{L^\Phi} > 0$. Thus, Theorem 4.1 guarantees that the maximal function M is bounded on $L^{\tilde{\Phi}}$. That is, $L^\Phi \in \mathbb{M}'$. Theorem 3.2 yields (4.1). ∎

For the martingale transform on exponential Orlicz spaces, the reader is referred to [19].

4.2. Lebesgue spaces with variable exponents

We apply Theorem 3.2 to Lebesgue spaces with variable exponents. Let $\mathcal{P}(\Omega)$ be the collection of all measurable functions $p(\cdot) : \Omega \to (0, \infty)$. For any measurable set $A \subset \Omega$, write

$$p_+(A) = \sup_{x \in A} p(x), \quad p_-(A) = \inf_{x \in A} p(x),$$

$p_+ = p_+(\Omega)$ and $p_- = p_-(\Omega)$.

Definition 4.5. Let $p(\cdot) : \Omega \to [1, \infty]$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}$ consists of all measurable functions f satisfying

$$\|f\|_{L^{p(\cdot)}} = \inf \{\lambda > 0 : \rho(|f(x)|/\lambda) \le 1\} < \infty$$

where

$$\rho(f) = \int_{\{x \in \Omega : p(x) \neq \infty\}} |f(x)|^{p(x)} dx + \|f \chi_{\{x \in \Omega : p(x) = \infty\}}\|_{L^\infty}.$$

We call $p(x)$ the exponent function of $L^{p(\cdot)}$.

According to [20, Theorem 3.2.13], $L^{p(\cdot)}$ is a Banach function space. Furthermore, we have $(L^{p(\cdot)})' = L^{p'(\cdot)}$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ [20, Theorem 3.2.13].

The martingale theory with variable exponents was begun with the fundamental paper [9] by Jiao et al. Now we recall a condition that guarantees the boundedness of the maximal function M on $L^{p(\cdot)}$, see [9, Theorem 3.5] and [21].

Theorem 4.6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Suppose that $\mathcal{F}$ is generated by atoms. If $p(\cdot) \in \mathcal{P}(\Omega)$ satisfies $1 < p_- \le p_+ < \infty$ and*

$$\mathbb{P}(A)^{p_-(A) - p_+(A)} \le K_{p(\cdot)}, \quad \forall A \in \mathcal{A} \tag{4.2}$$

for some $K_{p(\cdot)} > 0$, then there exists a $C > 0$ such that for any $f \in L^{p(\cdot)}$,

$$\|Mf\|_{L^{p(\cdot)}} \le C \|f\|_{L^{p(\cdot)}}.$$

The condition (4.2) is very crucial for the studies of maximal function on Lebesgue spaces with variable exponents defined on probability spaces, it replaces the log-Hölder continuity [20, Defintion 4.1.1] for the study of Hardy-Littlewood maximal function for $L^{p(\cdot)}$ on $\mathbb{R}^n$.

The reader is referred to [9, 22, 23] for more information on the maximal function on martingale function spaces built on Lebesgue spaces with variable exponents.

Lemma 4.7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Suppose that $\mathcal{F}$ is generated by atoms. If $p(\cdot) \in \mathcal{P}(\Omega)$ satisfies $1 < p_- \le p_+ < \infty$ and (4.2), then $p'(\cdot)$ also satisfies (4.2).*

Proof: For any $A \in \mathcal{A}$, since $p'_-(A) = (p_+(A))'$ and $p'_+(A) = (p_-(A))'$, we find that

$$\begin{aligned} \mathbb{P}(A)^{p'_-(A) - p'_+(A)} = \mathbb{P}(A)^{(p_+(A))' - (p_-(A))'} &= \mathbb{P}(A)^{\frac{p_+(A)}{p_+(A)-1} - \frac{p_-(A)}{p_-(A)-1}} \\ &= \mathbb{P}(A)^{\frac{p_-(A) - p_+(A)}{(p_+(A)-1)(p_-(A)-1)}} \end{aligned}$$

$$\leq \mathbb{P}(A)^{\frac{p_-(A)-p_+(A)}{(p_--1)^2}}$$
$$\leq K_{p(\cdot)}^{\frac{1}{(p_--1)^2}}.$$

Thus, $p'(\cdot)$ satisfies (4.2). ∎

We are now ready to establish the boundedness of the martingale transform T_v on $L^{p(\cdot)}$.

Theorem 4.8. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Suppose that $\mathcal{F}$ is generated by atoms and the predictable sequence $v = (v_n)_{n\geq 0}$ satisfies $\|v\|_{L^\infty} = \sup_{n\geq 0} \|v_n\|_{L^\infty} < \infty$.*

If $p(\cdot) \in \mathcal{P}(\Omega)$ satisfies $1 < p_- \leq p_+ < \infty$ and (4.2), then there exists a constant $C > 0$ such that for any $f \in L^{p(\cdot)}$

$$\|T_v f\|_{L^{p(\cdot)}} \leq C\|f\|_{L^{p(\cdot)}}. \tag{4.3}$$

Proof: Take $r \in (1, p_-)$. Write $q(\cdot) = p(\cdot)/r$. Thus, $L^{p(\cdot)/r}$ is a Banach function space.

We find that

$$\mathbb{P}(A)^{q_-(A)-q_+(A)} = \mathbb{P}(A)^{\frac{p_-(A)-p_+(A)}{r}} \leq K_{p(\cdot)}^{\frac{1}{r}}.$$

Therefore, $q(\cdot)$ satisfies (4.2). Lemma 4.7 guarantees that $q'(\cdot) = (p(\cdot)/r)'$ also satisfies (4.2). In addition, $1 < (p(\cdot)/r)'_- \leq (p(\cdot)/r)'_+ < \infty$. Theorem 4.6 assures that the maximal function is bounded on $L^{(p(\cdot)/r)'} = (L^{p(\cdot)/r})'$. Consequently, we are allowed to apply Theorem 3.2 and obtain the boundedness of T_v on $L^{p(\cdot)}$. ∎

4.3. Morrey spaces

We apply Theorem 3.2 to study the martingale transform on Morrey type spaces [1, 6]. We recall the definition of Morrey type spaces from [1, Definition 2.4].

Definition 4.6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, X be a Banach function space. Suppose that $\mathcal{F}$ is generated by atoms and $u : \mathcal{A} \to (0, \infty)$. The Morrey space $M_{X,u}$ consists of all $f \in \mathcal{M}$ satisfying

$$\|f\|_{M_{X,u}} = \sup_{B\in\mathcal{A}} \frac{1}{u(B)} \|\chi_B f\|_X < \infty.$$

When $X = L^p$, $p \in (1, \infty)$ and $u(B) = \mathbb{P}(B)^{\lambda+\frac{1}{p}}$, $\lambda \in \mathbb{R}$, the Morrey space $M_{X,u}$ becomes the Morrey space $L_{p,\lambda}$ introduced and studied in [6].

As the Morrey space $M_{X,u}$ is defined on $\mathcal{F}$ which is generated by atoms, it satisfies the assumptions that $\mathcal{F} = \{\mathcal{F}_n\}$ are generated by at most countable families of atoms given in [1, 6]. Thus, the results in [1, 6] apply to $M_{X,u}$.

A result on the boundedness of martingale transform on Morrey spaces $M_{X,u}$ is given in [1, Theorem 3.4]. It requires the assumption that the martingale transform T_v is bounded on X. As the boundedness of the martingale transform T_v on Banach function space X is obtained in Theorem 3.2, the result in [1, Theorem 3.4] can be refined as follows.

Theorem 4.9. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let X be a Banach function space. Suppose that $\mathcal{F}$ is generated by atoms and $u : \mathcal{T} \to (0, \infty)$. If there exists a $p \in (1, \infty)$ such that*

$X \in \mathbb{M}$, $X^{1/p}$ is a Banach function space, $X^{1/p} \in \mathbb{M}'$ and there exists a constant $C > 0$ such that for any $m \geq 0$ and $B \in \mathcal{A}(\mathcal{F}_m)$,

$$\sum_{j=0}^{m} \frac{u(B_j)}{u(B)} \frac{\|\chi_B\|_X}{\|\chi_{B_j}\|_X} \leq C \tag{4.4}$$

where B_j is the unique element in $\mathcal{A}(\mathcal{F}_{m-j})$ containing B, then $T_v : M_{X,u} \to M_{X,u}$ is bounded.

Proof: As $X^{1/p}$ is a Banach function space and $X^{1/p} \in \mathbb{M}'$, Theorem 3.2 assures that $T_v : X \to X$ is bounded. Therefore, [1, Theorem 3.4] yields the boundedness of $T_v : M_{X,u} \to M_{X,u}$. ■

We apply Theorem 4.9 to the Lorentz-Karamata-Morrey spaces and the Morrey spaces with variable exponent.

We first consider the Lorentz-Karamata-Morrey spaces. Let, $\theta \in [0, 1)$, $r, p \in (1, \infty)$, b be a slowly varying function and $u_\theta(A) = \|\chi_A\|^{\theta}_{L_{r,p,b}}$, $A \in \mathcal{A}$. We denote the Lorentz-Karamata-Morrey space $M_{L_{r,p,b},u_\theta}$ by $M^{\theta}_{r,p,b}$.

We first state some results on the Lorentz-Karamata space from [24]. In view of [24, Proposition 3.4.33], we have

$$\left(\int_0^a t^{\frac{p}{r}-1} (\gamma_b(t))^p dt \right)^{\frac{1}{p}} \approx a^{\frac{1}{r}} \gamma_b(a).$$

Thus, for any $B \in \mathcal{F}$, we get

$$\|\chi_B\|_{L_{r,p,b}} \approx (\mathbb{P}(B))^{\frac{1}{r}} \gamma_b(\mathbb{P}(B)).$$

We state a condition used to study the Morrey spaces. Let $\beta > 0$. If for any $I, J \in \mathcal{A}$ with $I \subset J$, we have

$$\frac{\mathbb{P}(I)}{\mathbb{P}(J)} \leq 1 - 2^{-\beta}, \tag{4.5}$$

then we say that $\mathcal{F}$ is generated by β-atoms.

Consequently, for any $B, D \in \mathcal{A}$ with $B \subseteq D$, there exists a constant $C > 0$ such that

$$\frac{\|\chi_B\|_{L_{r,p,b}}}{\|\chi_D\|_{L_{r,p,b}}} \leq C \frac{(\mathbb{P}(B))^{\frac{1}{r}} \gamma_b(\mathbb{P}(B))}{(\mathbb{P}(D))^{\frac{1}{r}} \gamma_b(\mathbb{P}(D))}.$$

Since b satisfies Item (1) of Definition 4.2, for any $\epsilon \in (0, \frac{1}{r})$, there exists a constant $C > 0$ such that

$$\frac{\|\chi_B\|_{L_{r,p,b}}}{\|\chi_D\|_{L_{r,p,b}}} \leq C \left(\frac{\mathbb{P}(B)}{\mathbb{P}(D)} \right)^{\frac{1}{r}-\epsilon}. \tag{4.6}$$

As a result of the above inequality, we find that

$$\sum_{j=0}^{m} \frac{u_\theta(B_j)}{u_\theta(B)} \frac{\|\chi_B\|_{L_{r,p,b}}}{\|\chi_{B_j}\|_{L_{r,p,b}}} = \sum_{j=0}^{m} \left(\frac{\|\chi_B\|_{L_{r,p,b}}}{\|\chi_{B_j}\|_{L_{r,p,b}}} \right)^{1-\theta} \leq C \sum_{j=0}^{m} \left(\frac{\mathbb{P}(B)}{\mathbb{P}(D)} \right)^{(1-\theta)(\frac{1}{r}-\epsilon)}$$

for some $C > 0$. Whenever $\mathcal{F}$ is generated by β-atoms, (4.5) yields

$$\sum_{j=0}^{m} \frac{u_\theta(B_j)}{u_\theta(B)} \frac{\|\chi_B\|_{L_{r,p,b}}}{\|\chi_{B_j}\|_{L_{r,p,b}}} \leq C \sum_{j=0}^{m} (1 - 2^{-\beta})^{(m-j)(1-\theta)(\frac{1}{r}-\epsilon)} < C$$

for some $C > 0$. The above inequality shows that u_θ satisfies (4.4) with $X = L_{r,p,b}$. We are allowed to use Theorem 4.9 to obtain the boundedness of the martingale transform on the Lorentz-Karamata-Morrey space in the following corollary.

Corollary 4.10. *Let $\theta \in [0,1)$, $\beta \geq 0$, $r, p \in (1,\infty)$, b be a slowly varying function and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. If $\mathcal{F}$ is generated by β-atoms and the predictable sequence $v = (v_n)_{n\geq 0}$ satisfies $\|v\|_{L^\infty} = \sup_{n\geq 0} \|v_n\|_{L^\infty} < \infty$, then the martingale transform T_v is bounded on $M^\theta_{r,p,b}$.*

Proof: As $r \in (1,\infty)$, in view [17, Proposition 6.1], we have $\underline{\alpha}_{L_{r,p,b}} = \bar{\alpha}_{L_{r,p,b}} = \frac{1}{r}$. Theorem 4.1 guarantees that $L_{r,p,b} \in \mathbb{M}$. In addition, for any $q \in (1, \min(r,p))$, we have $(L_{r,p,b})^{1/q} = L_{r/q,p/q,b^q}$ and $\bar{\alpha}_{L'_{r/q,p/q,b^q}} = 1 - \frac{q}{r} < 1$. Thus, $L_{r/q,p/q,b^q}$ is a Banach function space and Theorem 4.1 yields $L_{r/q,p/q,b^q} \in \mathbb{M}'$. The boundedness of the martingale transform $T_v : M^\theta_{r,p,b} \to M^\theta_{r,p,b}$ is guaranteed by Theorem 4.9. ∎

We now turn to the Morrey spaces with variable exponents. Let $p(\cdot) \in \mathcal{P}(\Omega)$ with $1 < p_- \leq p_+ < \infty$ satisfying (4.2). Let $\theta \in [0,1)$. Define $u_\theta(B) = \|\chi_B\|^\theta_{L^{p(\cdot)}}$, $B \in \mathcal{A}$. We denote the Morrey space with variable exponent $M_{L^{p(\cdot)},u_\theta}$ by $M^\theta_{p(\cdot)}$.

For any $B, D \in \mathcal{A}$ with $B \subseteq D$ In view of (2.2), we have

$$\chi_D \frac{\mathbb{P}(B)}{\mathbb{P}(D)} \leq M\chi_B. \tag{4.7}$$

For any $r \in (1, p'_-)$, $L^{p'(\cdot)/r}$ is a Banach function space and $p'(\cdot)/r$ satisfies (4.2). Theorem 4.6 assures that the maximal function M is bounded on $L^{p'(\cdot)/r}$. By applying the norm $\|\cdot\|_{L^{p'(\cdot)/r}}$ on both sides of (4.7),

$$\|\chi_D\|_{L^{p'(\cdot)/r}} \frac{\mathbb{P}(B)}{\mathbb{P}(D)} \leq \|M\chi_B\|_{L^{p'(\cdot)/r}} \leq C\|\chi_B\|_{L^{p'(\cdot)/r}}.$$

That is,

$$\frac{\mathbb{P}(B)}{\mathbb{P}(D)} \leq C\frac{\|\chi_B\|_{L^{p'(\cdot)/r}}}{\|\chi_D\|_{L^{p'(\cdot)/r}}} = C\left(\frac{\|\chi_B\|_{L^{p'(\cdot)}}}{\|\chi_D\|_{L^{p'(\cdot)}}}\right)^r.$$

As $p(\cdot)$ satisfies (4.2), the maximal function is bounded on $L^{p(\cdot)}$. According to [1, Lemma 2.8], we find that for any $A \in \mathcal{A}$, we have

$$\mathbb{P}(A) \leq \|\chi_A\|_{L^{p(\cdot)}}\|\chi_A\|_{L^{p'(\cdot)}} \leq C\mathbb{P}(A).$$

Consequently,

$$\frac{\|\chi_B\|_{L^{p(\cdot)}}}{\|\chi_D\|_{L^{p(\cdot)}}} \leq C\left(\frac{\mathbb{P}(B)}{\mathbb{P}(D)}\right)^{1-\frac{1}{r}}. \tag{4.8}$$

For any $\theta \in [0,1)$, we get

$$\sum_{j=0}^m \frac{u_\theta(B_j)}{u_\theta(B)} \frac{\|\chi_B\|_{L^{p(\cdot)}}}{\|\chi_{B_j}\|_{L^{p(\cdot)}}} = \sum_{j=0}^m \left(\frac{\|\chi_B\|_{L^{p(\cdot)}}}{\|\chi_{B_j}\|_{L^{p(\cdot)}}}\right)^{1-\theta} \leq C\sum_{j=0}^m \left(\frac{\mathbb{P}(B)}{\mathbb{P}(B_j)}\right)^{(1-\theta)(1-\frac{1}{r})}.$$

Whenever $\mathcal{F}$ is generated by β-atoms with $\beta > 0$, we find that

$$\sum_{j=0}^m \frac{u_\theta(B_j)}{u_\theta(B)} \frac{\|\chi_B\|_{L^{p(\cdot)}}}{\|\chi_{B_j}\|_{L^{p(\cdot)}}} \leq C\sum_{j=0}^m (1-2^{-\beta})^{(m-j)(1-\theta)(1-\frac{1}{r})} < C \tag{4.9}$$

for some $C > 0$. Thus, u_θ fulfills (4.4) with $X = L^{p(\cdot)}$. Consequently, we obtain the following result for martingale transform on $M^\theta_{p(\cdot)}$.

Corollary 4.11. *Let $\theta \in [0,1)$, $\beta \geq 0$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy $1 < p_- \leq p_+ < \infty$ and (4.2). If $\mathcal{F}$ is generated by β-atoms and the predictable sequence $v = (v_n)_{n\geq 0}$ satisfies $\|v\|_{L^\infty} = \sup_{n\geq 0} \|v_n\|_{L^\infty} < \infty$, then the martingale transform T_v is bounded on $M^\theta_{p(\cdot)}$.*

Proof: Since $1 < p_- \leq p_+ < \infty$ and $p(\cdot)$ satisfies (4.2), Theorem 4.6 guarantees that $L^{p(\cdot)} \in \mathbb{M}$. For any $r \in (1, p_-)$, $L^{p(\cdot)/r}$ is a Banach function space and Lemma 4.7 assures that $L^{p(\cdot)/r} \in \mathbb{M}'$. In addition, (4.9) asserts that u_θ fulfills (4.4) with $X = L^{p(\cdot)}$. Therefore, Theorem 4.9 yields the boundedness of T_v on $M^\theta_{p(\cdot)}$. ∎

Acknowledgments

The author thanks the reviewers for their valuable suggestions, especially, correcting a mistake on Theorem 4.6.

Conflict of interest

The authors declare there is no conflicts of interest.

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