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***Research article***

# Homoclinic solutions of discrete $p$-Laplacian equations containing both advance and retardation

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**Abstract:** We consider a $2m$th-order nonlinear $p$-Laplacian difference equation containing both advance and retardation. Using the critical point theory, we establish some new and weaker criteria on the existence of homoclinic solutions with mixed nonlinearities.

**Keywords:** homoclinic solution; $p$-Laplacian; mixed nonlinearity; critical point theory

## 1. Introduction

Consider the following $2m$th-order nonlinear $p$-Laplacian difference equation containing both advance and retardation

$$(-1)^m\Delta^m(r_{n-m}\phi_p(\Delta^m u_{n-m})) + \omega_n u_n = f(n, u_{n+1}, u_n, u_{n-1}), \quad n \in \mathbb{Z}. \tag{1.1}$$

Here $p > 1$ is a real number, $\phi_p(s) = |s|^{p-2}s$ for all $s \in \mathbb{R}$, $\Delta$ is the forward difference operator defined by $\Delta u_k = u_{k+1} - u_k$, $\Delta^j u_k = \Delta(\Delta^{j-1}u_k)$ for $j \geq 2$, $\{r_n\}$ and $\{\omega_n\}$ are real positive $T$-periodic sequences for a positive integer $T$, $f \in C(\mathbb{Z} \times \mathbb{R}^3, \mathbb{R})$ with $f$ being $T$-periodic in the first variable.

Special cases of (1.1) are produced, for example, when we look for standing waves of the discrete nonlinear Schrödinger (DNLS) equation,

$$i\dot{\psi}_n = -\Delta^2\psi_{n-1} + v_n\psi_n - f_n(\psi_n), \quad n \in \mathbb{Z}.$$

Assume that the nonlinearity is gauge invariant, i.e.,

$$f_n(e^{i\theta}u) = e^{i\theta}f_n(u), \quad \theta \in \mathbb{R}.$$

Since solitons are spatially localized time-periodic solutions and decay to zero at infinity, $\psi_n$ has the form

$$\psi_n = u_n e^{-i\omega t} \text{ and } \lim_{|n|\to\infty} \psi_n = 0,$$

where $\{u_n\}$ is a real valued sequence and $\omega \in \mathbb{R}$ is the temporal frequency. Then we arrive at the nonlinear equation

$$-\Delta^2 u_n + v_n u_n - \omega u_n = f_n(u_n), \quad n \in \mathbb{Z}. \tag{1.2}$$

Clearly, (1.2) is a special form of (1.1) with $m = 1$ and $p = 2$ but without advance or retardation.

We assume that $f(n, 0, 0, 0) = 0$ for each $n \in \mathbb{Z}$, then $\{u_n\} = \{0\}$ is a solution of (1.1), which is called the trivial solution. As usual, we say that a solution $u = \{u_n\}$ of (1.1) is homoclinic (to 0) if $\lim_{|n|\to\infty} u_n = 0$. In addition, if $\{u_n\} \neq \{0\}$, then $u$ is called a nontrivial homoclinic solution.

Critical point theory was introduced into discrete systems by Guo-Yu [1] in 2003 to study the existence of periodic and subharmonic solutions. It has been proved to be a powerful tool for studying the existence of homoclinic solutions for discrete nonlinear systems [2]. Among them, the theory of difference equations has been widely used to examine discrete models appearing in many fields [3, 4]. In recent years, the existence of homoclinic and heteroclinic solutions and boundary value problems for various difference equations have been investigated by many researchers [5–14]. For example, some researchers have studied the following nonlinear difference equation with a coercive weight function

$$-\Delta(a_k \phi_p(\Delta u_{k-1})) + b_k \phi_p(u_k) = \lambda f(k, u_k), \quad k \in \mathbb{Z}, \tag{1.3}$$

where $\lambda$ is a positive real parameter, $a, b : \mathbb{Z} \to (0, +\infty)$. By means of critical point theory, Iannizzotto and Tersian [6] have proved the existence of at least two nontrivial homoclinic solutions when $\lambda$ is big enough of (1.3). Moreover, infinitely many homoclinic solutions were obtained in [12] by employing Nehari manifold methods, and in [11] by applying the fountain theorem.

In particular, difference equations containing both advance and retardation have important background and applications in the field of cybernetics and biological mathematics [15, 16]. Thus they have received considerably attention. For some recent works, we refer readers to [7, 10, 17, 18] and references therein. For instances, by using the mountain pass theorem and periodic approximations, Shi *et al.* [10] studied the existence of a nontrivial homoclinic orbit of

$$\Delta\left(\phi_p(\Delta u_{n-1})\right) - q_n \phi_p(u_n) + f(n, u_{n+M}, u_n, u_{n-M}) = 0, \quad n \in \mathbb{Z},$$

where $M$ is a given nonnegative integer. Kong [7] employed the critical point theory to study the existence of at least three homoclinic solutions for the following $p$-Laplacian difference equation with both advance and retardation

$$(-1)^n \Delta^n \left(a(k-n)\phi_p\left(\Delta^n u(k-n)\right)\right) + b(k)\phi_p(u(k)) = \lambda f(k, u(k+1), u(k), u(k-1)),$$

$k \in \mathbb{Z}$, where $\lambda$ is a positive real parameter, $a, b : \mathbb{Z} \to (0, +\infty)$. Unlike the problem we studied, in this article, the author requires that $b(k)$ is unbounded.

Inspired by the above interesting research, we shall attempt to establish the new sufficient conditions on the existence of nontrivial homoclinic solutions for more general nonlinear terms of (1.1), see remarks 1 and 2 for details. To wit, we have

**Theorem 1.1** *Assume that there exists a function* $F \in C^1(\mathbb{Z} \times \mathbb{R}^2, \mathbb{R})$ *having the following properties with* $p > 2$.

$(T_1)$ *For* $n \in \mathbb{Z}, v_1, v_2, v_3 \in \mathbb{R}$, $F(n+T, v_1, v_2) = F(n, v_1, v_2)$ *and*

$$\partial_2 F(n-1, v_2, v_3) + \partial_3 F(n, v_1, v_2) = f(n, v_1, v_2, v_3)$$

*where we denote by*

$$\partial_2 F(n, v_2, v_3) = \frac{\partial F(n, v_2, v_3)}{\partial v_2} \quad \text{and} \quad \partial_3 F(n, v_1, v_2) = \frac{\partial F(n, v_1, v_2)}{\partial v_2};$$

$(T_2)$

$$\limsup_{|v_1|+|v_2|\to 0} \frac{F(n, v_1, v_2)}{v_1^2 + v_2^2} = 0;$$

$(T_3)$ $\partial_i F(n, v_1, v_2) = o(|(v_1, v_2)|)$ *as* $(v_1, v_2) \to (0,0)$ *for all* $n \in \mathbb{Z}$, $i = 2, 3$;

$(T_4)$ *There exists a real sequence* $\{a_n\}$ *such that*

$$\liminf_{|v_1|+|v_2|\to\infty} \frac{F(n, v_1, v_2)}{|v_1|^p + |v_2|^p} = a_n \leq \infty;$$

$(T_5)$ $\partial_2 F(n, v_1, v_2)v_1 + \partial_3 F(n, v_1, v_2)v_2 - pF(n, v_1, v_2) > 0$ *for all* $(n, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2 \setminus \{(0,0)\}$.

*If* $pa_n > \bar{r}2^{mp}$ *for each* $n \in \mathbb{Z}$, *then* (1.1) *has at least a nontrivial solution* $u$ *in* $l^2$, *where* $\bar{r} = \max_{n\in\mathbb{Z}}\{r_n\}$.

**Theorem 1.2** *Assume that there exists* $F \in C^1(\mathbb{Z} \times \mathbb{R}^2, \mathbb{R})$ *satisfying* $(T_1), (T_2), (T_3)$ *and the following properties with* $1 < p \leq 2$.

$(T_6)$ *There exists a real sequence* $\{b_n\}$ *such that*

$$\liminf_{|v_1|+|v_2|\to\infty} \frac{F(n, v_1, v_2)}{v_1^2 + v_2^2} = b_n \leq \infty;$$

$(T_7)$ $\partial_2 F(n, v_1, v_2)v_1 + \partial_3 F(n, v_1, v_2)v_2 - 2F(n, v_1, v_2) > 0$ *for all* $(n, v_1, v_2) \in \mathbb{Z} \times \mathbb{R}^2 \setminus \{(0,0)\}$;

$(T_8)$ $\partial_2 F(n, v_1, v_2)v_1 + \partial_3 F(n, v_1, v_2)v_2 - 2F(n, v_1, v_2) \to +\infty$ *as* $|v_1| + |v_2| \to \infty$.

*If* $2b_n > \omega_n$ *for each* $n \in \mathbb{Z}$, *then* (1.1) *has at least a nontrivial solution* $u$ *in* $l^2$.

**Remark 1.** If a solution $\{u_n\}$ of (1.1) is in $l^2$, then $\lim_{|n|\to\infty} u_n = 0$ and $\{u_n\}$ is a homoclinic solution. The condition $(T_4)$ implies that the nonlinearity $F$ can be mixed super $p$-linear with asymptotically p-linear at $\infty$ and $(T_6)$ implies that the nonlinear term $F$ can be mixed superquadratic linear with asymptotically quadratic linear at $\infty$. In some references, the nonlinear $f$ is assumed to be either only superlinear or only asymptotically linear at $\infty$, which plays an important role in establishing the existence of nontrivial homoclinic solutions.

**Remark 2.** If $m = 1$, $r_n \equiv 1$, and $f(n, u_{n+1}, u_n, u_{n-1}) = g(n, u_n)$, then Theorem 1.1 reduces to Theorem 2.2 in [9] when $\phi$-Laplacian is $p$-Laplacian. Moreover, our sufficient conditions are based on the limit superior and limit inferior, which are more applicable.

This rest of the paper is organized as follows. In Section 2, we establish the variational framework associated with (1.1) and cite the Mountain Pass Lemma. Section 3 and Section 4 are devoted to the proofs of Theorem 1.1 and Theorem 1.2, respectively. The paper concludes with an example to illustrate the applicability of the main results.

## 2. The variational structure

We first establish the corresponding variational framework for (1.1).

Let $S$ be the set of all two-sided sequences, that is,

$$S = \{u = \{u_n\} | u_n \in \mathbb{R}, n \in \mathbb{Z}\}.$$

Then $S$ is a vector space with $au + bv = \{au_n + bv_n\}$ for $u, v \in S, a, b \in \mathbb{R}$. For any fixed positive integer $k$, we define the subspace $E_k$ of $S$ as

$$E_k = \{u = \{u_n\} \in S \,|\, u_{n+2kT} = u_n, n \in \mathbb{Z}\}.$$

Obviously, $E_k$ is isomorphic to $\mathbb{R}^{2kT}$ and we identify $u = (u_1, u_2, \cdots, u_{2kT})^* \in E_k$, where * denotes the transpose of a vector. $E_k$ can be equipped with the inner product $(\cdot,\cdot)_k$ and norm $\|\cdot\|_k$ defined respectively by

$$(u, v)_k = \sum_{n=-kT}^{kT-1} u_n v_n, \quad u, v \in E_k$$

and

$$\|u\|_k = \left(\sum_{n=-kT}^{kT-1} u_n^2\right)^{\frac{1}{2}}, \quad u \in E_k.$$

In $E_k$, we also define the equivalent norms $\|\cdot\|_{k\infty}$ by

$$\|u\|_{k\infty} = \max\{|u_n| : -kT \le n \le kT-1\}, \quad u \in E_k$$

and $\|\cdot\|_{kp}$ by

$$\|u\|_{kp} = \left(\sum_{n=-kT}^{kT-1} u_n^p\right)^{\frac{1}{p}}, \quad u \in E_k.$$

By Hölder inequality and Jensen inequality, we have

$$\|u\|_{kp} \le c_k(p)\|u\|_k, \quad u \in E_k, \tag{2.1}$$

where

$$c_k(p) = \begin{cases} (2kT)^{\frac{2-p}{2p}}, & 1 < p < 2, \\ 1, & 2 \le p. \end{cases}$$

For $p \ge 1$, let

$$l^p = \left\{ u = \{u_n\} \in S \;\middle|\; \|u\|_{l^p} = \left(\sum_{n\in\mathbb{Z}} |u_n|^p\right)^{\frac{1}{p}} < \infty \right\}.$$

For simplicity, the inner product and norm in $l^2$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively.

Consider the functional $J_k$ in $E_k$ defined by

$$J_k(u) = \sum_{n=-kT}^{kT-1} \left[\frac{1}{p} r_n |\Delta^m u_n|^p + \frac{1}{2}\omega_n u_n^2 - F(n, u_{n+1}, u_n)\right], \tag{2.2}$$

whose Fréchet derivative is given by

$$\begin{aligned}\langle J_k'(u), v\rangle &= \sum_{n=-kT}^{kT-1}\Big[r_n\phi_p(\Delta^m u_n)\Delta^m v_n + \omega_n u_n v_n - f(n, u_{n+1}, u_n, u_{n-1})v_n\Big]\\ &= \sum_{n=-kT}^{kT-1}\Big[-\Delta\left(r_{n-1}\phi_p(\Delta^m u_{n-1})\right)\Delta^{m-1}v_n + \omega_n u_n v_n - f(n, u_{n+1}, u_n, u_{n-1})v_n\Big]\\ &\cdots\\ &= \sum_{n=-kT}^{kT-1}\Big[(-1)^m\Delta^m\left(r_{n-m}\phi_p(\Delta^m u_{n-m})\right)v_n + \omega_n u_n v_n - f(n, u_{n+1}, u_n, u_{n-1})v_n\Big],\end{aligned} \tag{2.3}$$

for $u, v \in E_k$.

Equation (2.3) implies that (1.1) is the corresponding Euler-Lagrange equation for $J_k$. It is easy to see that the critical points of $J_k$ in $E_k$ are exactly $2kT$-periodic solutions of the difference equation (1.1).

Let $P$ be the $2kT \times 2kT$ matrix corresponding to the quadratic form $\sum_{k=1}^{2kT}(\Delta u_k)^2$ with $u_{2kT+1} = u_1$ for $k \in \mathbb{Z}$, that is,

$$P = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1\\ -1 & 2 & -1 & \cdots & 0 & 0\\ 0 & -1 & 2 & \cdots & 0 & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 2 & -1\\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

By matrix theory, the eigenvalues of $P$ are

$$\lambda_j = 4\sin^2\frac{j\pi}{2kT},\ j = 0, 1, 2, \cdots, 2kT-1.$$

It follows that $\lambda_0 = 0, \lambda_1 > 0, \lambda_2 > 0, \cdots, \lambda_{2kT-1} > 0$. Moreover, $\lambda_{\max} = \max\{\lambda_1, \lambda_2, \cdots, \lambda_{2kT-1}\} = 4$.

For the readers' convenience, we now cite the Mountain Pass Lemma. Let $H$ be a Hilbert space and $C^1(H, \mathbb{R})$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on $H$, $B_r$ be the open ball in $H$ with radius $r$ and center 0, and $\partial B_r$ denote its boundary.

**Definition 2.1** *Let $J \in C^1(H, \mathbb{R})$. A sequence $\{x_j\} \subset H$ is called a Cerami sequence ($(C)$ sequence for short) for $J$ if $J(x_j) \to c$ for some $c \in \mathbb{R}$ and $(1 + \|x_j\|)J'(x_j) \to 0$ as $j \to \infty$. We say $J$ satisfies the Cerami condition ($(C)$ condition for short) if any $(C)$ sequence for $J$ possesses a convergent subsequence.*

**Lemma 2.1** (Mountain Pass Lemma [19]) *If $J \in C^1(H, \mathbb{R})$ and satisfies the following conditions: there exist $e \in H\backslash\{0\}$ and $r \in (0, \|e\|)$ such that $\max\{J(0), J(e)\} < \inf_{u\in\partial B_r} J(u)$. Then there exists a $(C)$ sequence $\{u_n\}$ for the mountain pass level $c$ which is defined by*

$$c = \inf_{h\in\Gamma}\max_{s\in[0,1]} J(h(s)),$$

*where*

$$\Gamma = \{h \in C([0, 1], H) \mid h(0) = 0, h(1) = e\}.$$

Finally, by similar arguments as those in [18], we can obtain the following result.

**Lemma 2.2** *For $u \in E_k$, we have*

$$\Big(\sum_{n=-kT}^{kT-1}(\Delta^m u_n)^2\Big)^{\frac{p}{2}} \le \lambda_{\max}^{\frac{mp}{2}}\|u\|_k^p = 2^{mp}\|u\|_k^p, \ \ n \in \mathbb{Z}.$$

By Lemma 2.2 and (2.1), for $u \in E_k$,

$$\begin{aligned}\frac{1}{p}\sum_{n=-kT}^{kT-1} r_n\,|\Delta^m u_n|^p &\le \frac{\bar{r}}{p}\left[\Big(\sum_{n=-kT}^{kT-1}|\Delta^m u_n|^p\Big)^{\frac{1}{p}}\right]^p\\ &\le \frac{\bar{r}}{p}\left[c_k(p)\Big(\sum_{n=-kT}^{kT-1}(\Delta^m u_n)^2\Big)^{\frac{1}{2}}\right]^p\\ &\le \frac{\bar{r}}{p}c_k^p(p)2^{mp}\|u\|_k^p.\end{aligned}$$

## 3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we need some preparation. Denote $\omega_* = \min_{n\in\mathbb{Z}}\{\omega_n\}$.

**Lemma 3.1** *Under the assumptions of Theorem 1.1, the functional $J_k$ satisfies the $(C)$ condition.*

*Proof.* Let $\{u^{(j)}\} \subset E_k$ be a $(C)$ sequence for $J_k$. We need to show that $\{u^{(j)}\}$ has a convergent subsequence. Since $E_k$ is finite dimensional, it suffices to show that $\|u^{(j)}\|_k$ is bounded. By assumption, $J_k(u^{(j)}) \to c$ for some $c \in \mathbb{R}$ and $(1 + \|u^{(j)}\|_k)J_k'(u^{(j)}) \to 0$ as $j \to \infty$. Then there exists $M > 0$ such that $|J_k(u^{(j)})| \le M$ and $\|(1 + \|u^{(j)}\|_k)J_k'(u^{(j)})\| \le M$ for $j \in \mathbb{N}$. So we have $\|u^{(j)}\|_k\|J_k'(u^{(j)})\| \le \|(1 + \|u^{(j)}\|_k)J_k'(u^{(j)})\| \le M$ for $j \in \mathbb{N}$. Then by (2.2), (2.3) and $(T_5)$, we have

$$\begin{aligned}\sum_{n=-kT}^{kT-1}\left(\Big(\frac{p}{2}-1\Big)\omega_*|u_n^{(j)}|^2\right) &\le \sum_{n=-kT}^{kT-1}\left(\Big(\frac{p}{2}-1\Big)\omega_n|u_n^{(j)}|^2\right)\\ &\le pJ_k(u^{(j)}) - \langle J'(u^{(j)}), u^{(j)}\rangle\\ &\le p|J_k(u^{(j)})| + \|u^{(j)}\|_k\|J_k'(u^{(j)})\|\\ &\le (p+1)M.\end{aligned} \tag{3.1}$$

Choose $\delta > 0$ such that

$$\Big(\frac{p}{2}-1\Big)\omega_* u^2 > (p+1)M \ \text{ for } \ |u| > \delta.$$

This and (3.1) imply that $|u_n^{(j)}| \le \delta$ for $n \in \mathbb{Z}$, that is,

$$\|u^{(j)}\|_{k\infty} \le \delta. \tag{3.2}$$

Since $E_k$ is finite dimensional, $\|\cdot\|_k$ and $\|\cdot\|_{k\infty}$ are equivalent. Then (3.2) implies that $\{\|u^{(j)}\|_k\}$ is bounded. The proof is completed.

**Lemma 3.2** *Under the assumptions of Theorem 1.1, there exists $n_0 \in \mathbb{N}$ such that $J_k$ has at least a* *nonzero critical point $u^{(k)}$ in $E_k$ for each $k \geq n_0$.*

*Proof.* We first show that $J_k$ satisfies conditions of Lemma 2.1. From $(T_2)$, there exists $r > 0$ such that

$$F(n, u_1, u_2) \leq \frac{1}{8}\omega_*(u_1^2 + u_2^2) \text{ for } |u_1| + |u_2| \leq r.$$

Then, for $u \in E_k$ with $\|u\|_k \leq r$,

$$\begin{aligned} J_k(u) &\geq \frac{1}{2}\sum_{n=-kT}^{kT-1} \omega_n u_n^2 - \sum_{n=-kT}^{kT-1} F(n, u_{n+1}, u_n) \\ &\geq \frac{1}{2}\sum_{n=-kT}^{kT-1} \omega_n u_n^2 - \sum_{n=-kT}^{kT-1} \frac{1}{8}\omega_*(u_{n+1}^2 + u_n^2) \\ &\geq \frac{1}{4}\omega_*\|u\|_k^2. \end{aligned}$$

Taking $a = \frac{1}{4}\omega_* r^2$ gives $J_k|_{\partial B_r} \geq a > 0$.

Since $a_n > \frac{\bar{r}}{p}2^{mp}$ for each $n \in \mathbb{Z}$, there exists $\varepsilon \in (0, 1)$ such that

$$2(a_n - \varepsilon)(1 - \varepsilon) > \frac{\bar{r}}{p}2^{mp}.$$

For a given $e = \{e_n\} \in l^2$ with $\sum_{n=-\infty}^{\infty} |e_n|^p = 1$. Let $n_0$ be large enough such that

$$\sum_{n=-n_0T}^{n_0T-1} |e_n|^p \geq 1 - \varepsilon.$$

For $k \geq n_0$, define $e^{(k)} \in E_k$ by

$$e_n^{(k)} = \begin{cases} e_n, & -n_0T \leq n \leq n_0T - 1; \\ 0, & -kT \leq n \leq -n_0T - 1 \text{ or } n_0T \leq n \leq kT - 1. \end{cases}$$

By $(T_4)$, there exists $\mu_0 > r$, such that

$$F(n, \mu e_{n+1}, \mu e_n) \geq (a_n - \varepsilon)\mu^p(|e_{n+1}|^p + |e_n|^p) \text{ for } -n_0T \leq n \leq n_0T - 1 \text{ and } \mu \geq \mu_0.$$

Then, for $\mu \geq \mu_0$,

$$\begin{aligned} J_k(\mu e^{(k)}) &= \sum_{n=-kT}^{kT-1} \left(\frac{1}{p} r_n |\mu \Delta^m e_n^{(k)}|^p + \frac{\omega_n}{2}|\mu e_n^{(k)}|^2 - F(n, \mu e_{n+1}^{(k)}, \mu e_n^{(k)})\right) \\ &\leq \frac{\bar{r}}{p}2^{mp}\mu^p + \sum_{n=-kT}^{kT-1} \left(\frac{\omega_n}{2}|\mu e_n^{(k)}|^2 + (\varepsilon - a_n)\mu^p(|e_{n+1}^{(k)}|^p + |e_n^{(k)}|^p)\right) \\ &\leq \frac{\bar{r}}{p}2^{mp}\mu^p + \sum_{n=-n_0T}^{n_0T-1} \frac{\omega_n}{2}\mu^2 e_n^2 + 2(\varepsilon - a_n)(1 - \varepsilon)\mu^p \\ &\leq \left(\frac{\bar{r}}{p}2^{mp} + 2(\varepsilon - a_n)(1 - \varepsilon)\right)\mu^p + \sum_{n=-n_0T}^{n_0T-1} \frac{\omega_n}{2}\mu^2 e_n^2. \end{aligned}$$

Noticing that $p > 2$ and $\frac{\bar{r}}{p}2^{mp} + 2(\varepsilon - a_n)(1 - \varepsilon) < 0$, there exists $\mu' > \mu_0$ such that

$$J_k(\mu' e^{(k)}) < 0.$$

It can easily be seen that $J_k(0) = 0$. Then we have $r \in (0, \|\mu' e^{(k)}\|_k)$ and

$$\max\{J_k(0), J_k(\mu' e^{(k)})\} = 0 < a \leq \inf_{u \in \partial B_r} J_k(u).$$

Now that we have verified all assumptions of Lemma 2.1, we know $J_k$ possesses a $(C)$ sequence $\{u_j^{(k)}\}$ for the mountain pass level $c_k \geq a$ with

$$c_k = \inf_{h \in \Gamma_k} \max_{s \in [0,1]} J_k(h(s)),$$

where

$$\Gamma_k = \left\{ h \in C([0,1], E_k) \,\middle|\, h(0) = 0, h(1) = \mu' e^{(k)} \right\}.$$

According to Lemma 3.1, $\{u_j^{(k)}\}$ has a convergent subsequence $\{u_{j_m}^{(k)}\}$ such that $u_{j_m}^{(k)} \to u^{(k)}$ as $j_m \to \infty$ for some $u^{(k)} \in E_k$. Since $J_k \in C^1(E_k, \mathbb{R})$, we have

$$J_k(u_{j_m}^{(k)}) \to J_k(u^{(k)}) \text{ and } (1 + \|u_{j_m}^{(k)}\|_k) J'(u_{j_m}^{(k)}) \to (1 + \|u^{(k)}\|_k) J'(u^{(k)})$$

as $j_m \to \infty$. By the uniqueness of the limit, we obtain that $u^{(k)}$ is a critical point of $J_k$ corresponding to $c_k$. Moreover, $u^{(k)}$ is nonzero as $c_k \geq a > 0$.

**Lemma 3.3** *There exist constants $\alpha, \beta, N > 0$ such that*

$$\alpha \leq \|u^{(k)}\|_{k\infty} \leq \beta \text{ and } \|u^{(k)}\|_k \leq N$$

*hold for every critical point $u^{(k)}$ of $J_k$ in $E_k$ with $k \geq n_0$ obtained in Lemma 3.2.*

*Proof.* For $k \geq n_0$, we define $h_k \in \Gamma_k$ as $h_k(s) = s\mu_0 e^{(k)}$ for $s \in [0, 1]$. Similarly to the derivation of [18], we can find

$$\begin{aligned} J_k(u^{(k)}) &\leq \max_{s \in [0,1]} \left\{ J_k(s\mu_0 e^{(k)}) \right\} \\ &\leq \max_{s \in [0,1]} \left\{ \sum_{n=-n_0 T}^{n_0 T - 1} \left( \frac{\bar{r}}{p} |\Delta^m (s\mu_0 e_n)|^p + \frac{\omega_n}{2} (s\mu_0 e_n)^2 - F(n, s\mu_0 e_{n+1}, s\mu_0 e_n) \right) \right\} \\ &\leq \max_{s \in [0,1]} \left\{ \frac{\bar{r}}{p} 2^{mp} \left( \sum_{n=-n_0 T}^{n_0 T - 1} (s\mu_0 e_n)^2 \right)^{\frac{p}{2}} + \sum_{n=-n_0 T}^{n_0 T - 1} \left( \frac{\omega_n}{2} (s\mu_0 e_n)^2 - F(n, s\mu_0 e_{n+1}, s\mu_0 e_n) \right) \right\} \\ &\triangleq M_0. \end{aligned} \tag{3.3}$$

Obviously, $M_0 > 0$ is independent of $k$.

Since $u^{(k)}$ is a critical point of $J_k$, by (2.2), (2.3) and (3.3), we have

$$
\begin{aligned}
pJ_k(u^{(k)}) &= pJ_k(u^{(k)}) - \langle J'(u^{(k)}), u^{(k)} \rangle \\
&= \sum_{n=-kT}^{kT-1} \left( f(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)}) u_n^{(k)} - pF(n, u_{n+1}^{(k)}, u_n^{(k)}) \right) + \sum_{n=-kT}^{kT-1} \left( \left( \frac{p}{2} - 1 \right) \omega_n |u_n^{(k)}|^2 \right) \\
&\le pM_0.
\end{aligned} \tag{3.4}
$$

Choose $\beta > 0$ such that

$$\left(\frac{p}{2} - 1\right)\omega_n u^2 > pM_0 \ \text{ for } \ n \in \mathbb{Z} \ \text{ and } \ |u| > \beta.$$

This combined with (3.4) implies that $|u_n^{(k)}| \le \beta$ for each $n \in \mathbb{Z}$, that is,

$$\|u^{(k)}\|_{k\infty} \le \beta.$$

From (2.3), we have

$$\sum_{n=-kT}^{kT-1} \omega_n (u_n^{(k)})^2 \le \sum_{n=-kT}^{kT-1} f(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)}) u_n^{(k)}. \tag{3.5}$$

By $(T_3)$, there exists $\alpha > 0$ such that

$$\partial_i F(n, v_1, v_2) \le \frac{1}{8}\omega_* \sqrt{v_1^2 + v_2^2} \quad \text{for} \quad |v_1| + |v_2| < 2\alpha, i = 2, 3,$$

which together with (3.5) produces

$$
\begin{aligned}
\sum_{n=-kT}^{kT-1} \omega_n (u_n^{(k)})^2 &\le \sum_{n=-kT}^{kT-1} \left( \partial_2 F(n, u_{n+1}^{(k)}, u_n^{(k)}) u_{n+1}^{(k)} + \partial_3 F(n, u_{n+1}^{(k)}, u_n^{(k)}) u_n^{(k)} \right) \\
&\le \frac{1}{4}\omega_* \sum_{n=-kT}^{kT-1} \left( (u_{n+1}^{(k)})^2 + (u_n^{(k)})^2 \right) \\
&= \frac{1}{2}\omega_* \|u^{(k)}\|_k^2.
\end{aligned} \tag{3.6}
$$

Arguing by a contradiction, we have

$$\|u^{(k)}\|_{k\infty} \ge \alpha.$$

In view of $(T_5)$ and (3.4), we have

$$\omega_* \left(\frac{p}{2} - 1\right) \|u^{(k)}\|_k^2 \le \sum_{n=-kT}^{kT-1} \left( \left(\frac{p}{2} - 1\right) \omega_n |u_n^{(k)}|^2 \right) \le pM_0.$$

Let $N = \sqrt{\frac{pM_0}{\omega_*\left(\frac{p}{2}-1\right)}}$. Then we have

$$\|u^{(k)}\|_k \le N.$$

The proof is complete.

Now, we are ready to prove Theorem 1.1.

According to Lemma 3.2, there exists $n_0 \in \mathbb{N}$ such that for every $k > n_0$, $J_k$ has a critical point $u^{(k)} = \{u_n^{(k)}\} \in E_k$. Moreover, there exists $n_k \in \mathbb{Z}$ such that

$$\alpha \leq \left|u_{n_k}^{(k)}\right| \leq \beta. \tag{3.7}$$

Note that

$$(-1)^m \Delta^m (r_{n-m}\phi_p(\Delta^m u_{n-m}^{(k)})) + \omega_n u_n^{(k)} = f(n, u_{n+1}^{(k)}, u_n^{(k)}, u_{n-1}^{(k)}), \quad n \in \mathbb{Z}. \tag{3.8}$$

By the periodicity of $\{\omega_n\}$ and $f(n, u_{n+1}, u_n, u_{n-1})$, we see that $\{u_{n+T}^{(k)}\}$ is also a solution of (3.8). Without loss of generality, we may assume that $0 \leq n_k \leq T-1$ in (3.7). Moreover, passing to a subsequence of $\{u^{(k)}\}$ if necessary, we can also assume that $n_k = n^*$ for $k \geq n_0$ and some integer $n^*$ between 0 and $T-1$. It follows from (3.7) that we can choose a subsequence, still denoted by $\{u^{(k)}\}$, such that

$$u_n^{(k)} \to u_n \text{ as } k \to \infty, \quad n \in \mathbb{Z}.$$

Then $u = \{u_n\}$ is a nonzero sequence as (3.7) implies $|u_{n^*}| \geq \alpha$. It remains to show that $u = \{u_n\} \in l^2$ and it is a solution of (1.1).

Let

$$A_k = \{n \in \mathbb{Z} \mid |u_{n+1}^{(k)}| < \alpha \text{ and } |u_n^{(k)}| < \alpha, -kT \leq n \leq kT-1\},$$
$$B_k = \{n \in \mathbb{Z} \mid |u_{n+1}^{(k)}| \geq \alpha \text{ or } |u_n^{(k)}| \geq \alpha, -kT \leq n \leq kT-1\}.$$

Since $F(n, u_1, u_2)$ is continuously differentiable in the second and third variables and $T$-periodic in $n$, for $n \in \mathbb{Z}, \alpha \leq |u_1| + |u_2| \leq 2\beta$, let

$$d_1 = \max\{\partial_2 F(n, u_1, u_2)u_1 + \partial_3 F(n, u_1, u_2)u_2\},$$
$$d_2 = \min\left\{\frac{1}{p}(\partial_2 F(n, u_1, u_2)u_1 + \partial_3 F(n, u_1, u_2)u_2) - F(n, u_1, u_2)\right\}.$$

It is clear that $d_1, d_2 > 0$. Thus, for $n \in B_k$,

$$\begin{aligned}&\partial_2 F(n, u_{n+1}^{(k)}, u_n^{(k)})u_{n+1}^{(k)} + \partial_3 F(n, u_{n+1}^{(k)}, u_n^{(k)})u_n^{(k)} \\ \leq &\frac{d_1}{d_2}\left(\frac{1}{p}\left(\partial_2 F(n, u_{n+1}^{(k)}, u_n^{(k)})u_{n+1}^{(k)} + \partial_3 F(n, u_{n+1}^{(k)}, u_n^{(k)})u_n^{(k)}\right) - F(n, u_{n+1}^{(k)}, u_n^{(k)})\right).\end{aligned}$$

This combined with (3.4), (3.5) and (3.6) gives us

$$\begin{aligned}\sum_{n=-kT}^{kT-1} \omega_n (u_n^{(k)})^2 &\leq \sum_{n=-kT}^{kT-1}\left(\partial_2 F(n, u_{n+1}^{(k)}, u_n^{(k)})u_{n+1}^{(k)} + \partial_3 F(n, u_{n+1}^{(k)}, u_n^{(k)})u_n^{(k)}\right) \\ &\leq \frac{1}{2}\omega_* \|u^{(k)}\|_k^2 + \sum_{n \in B_k}\left(\partial_2 F(n, u_{n+1}^{(k)}, u_n^{(k)})u_{n+1}^{(k)} + \partial_3 F(n, u_{n+1}^{(k)}, u_n^{(k)})u_n^{(k)}\right) \\ &\leq \frac{1}{2}\omega_* \|u^{(k)}\|_k^2 + \frac{d_1 M_0}{d_2}.\end{aligned}$$

It follows that

$$\|u^{(k)}\|_k^2 \leq \frac{2d_1 M_0}{d_2\omega_*}. \tag{3.9}$$

Given $\varrho \in \mathbb{N}$, for $k > \max\{\varrho, n_0\}$, it follows from (3.9) that

$$\sum_{n=-\varrho}^{\varrho} (u_n^{(k)})^2 \leq \|u^{(k)}\|_k^2 \leq \frac{2d_1 M_0}{d_2\omega_*}.$$

It is clear that $\sum_{n=-\varrho}^{\varrho} u_n^2 \leq \frac{2d_1 M_0}{d_2\omega_*}$ as $k \to \infty$ and hence $u = \{u_n\} \in l^2$ by the arbitrariness of $\varrho$.

Now, for each $n \in \mathbb{Z}$, letting $k \to \infty$ in (3.8) gives us

$$(-1)^m \Delta^m (r_{n-m}\phi_p(\Delta^m u_{n-m})) + \omega_n u_n = f(n, u_{n+1}, u_n, u_{n-1}), \quad n \in \mathbb{Z},$$

that is, $u = \{u_n\}$ satisfies (1.1).

Consequently, we infer that $u = \{u_n\}$ is a nontrivial solution of (1.1). This completes the proof of Theorem 1.1.

## 4. Proof of Theorem 1.2

The proof of Theorem 1.2 is quite similar to that of Theorem 1.1. But some of the arguments are different. As a result, we provide some details below.

**Lemma 4.1** *Under the assumptions of Theorem 1.2, the functional $J_k$ satisfies the $(C)$ condition.*

*Proof.* Let $\{u^{(j)}\} \subset E_k$ be a $(C)$ sequence for $J_k$. As in the proof of Lemma 3.1, there exists $M > 0$ such that $|J_k(u^{(j)})| \leq M$ and $\|u^{(j)}\|_k \|J_k'(u^{(j)})\| \leq M$ for $j \in \mathbb{N}$. Then by (2.2), (2.3) and $1 < p \leq 2$, we have

$$\begin{aligned} &\sum_{n=-kT}^{kT-1} \left(\partial_2 F(n, u_{n+1}^{(j)}, u_n^{(j)}) u_{n+1}^{(j)} + \partial_3 F(n, u_{n+1}^{(j)}, u_n^{(j)}) u_n^{(j)} - 2F(n, u_{n+1}^{(j)}, u_n^{(j)})\right) \\ \leq& 2J_k(u^{(j)}) - \langle J_k'(u^{(j)}), u^{(j)} \rangle \\ \leq& 2|J_k(u^{(j)})| + \|u^{(j)}\|_k \|J_k'(u^{(j)})\| \\ \leq& 3M. \end{aligned} \tag{4.1}$$

From $(T_8)$, there exists $\delta > 0$ such that

$$\partial_2 F(n, u_1, u_2) u_1 + \partial_3 F(n, u_1, u_2) u_2 - 2F(n, u_1, u_2) > 3M \ \text{ for } \ n \in \mathbb{Z}, \ \ |u_1| + |u_2| > \delta.$$

Then (4.1) and $(T_7)$ imply that $|u_n^{(j)}| \leq \delta$ for $n \in \mathbb{Z}$, that is,

$$\|u^{(j)}\|_{k\infty} \leq \delta. \tag{4.2}$$

Since $E_k$ is finite dimensional, $\|\cdot\|_k$ and $\|\cdot\|_{k\infty}$ are equivalent. Then (4.2) tells us that $\{\|u^{(j)}\|_k\}$ is bounded and hence $\{u^{(j)}\}$ has a convergent subsequence. This completes the proof.

**Lemma 4.2** *Under the assumptions of Theorem 1.2, there exists $n_0 \in \mathbb{N}$ such that $J_k$ has at least a* *nonzero critical point $u^{(k)}$ in $E_k$ for each $k \geq n_0$.*

*Proof.* Proceeding as in the proof of Lemma 3.2, there exist $r > 0$ and $a > 0$ such that $J_k|_{\partial B_r} \geq a > 0$. Since $2b_n > \omega_n$, there exists $d > 0$ such that

$$b_n - \frac{\omega_n}{2} > d \ \text{ for } \ n \in \mathbb{Z}.$$

Let $\varepsilon \in (0, 1)$ satisfy

$$\left(\frac{\bar{r}}{p} 2^{mp} c_k^p(p) + 1\right)\varepsilon < d.$$

There exists $e = \{e_n\} \in l^2$ with $\sum_{n=-\infty}^{\infty} |e_n|^2 = 1$ such that $\sum_{n=-\infty}^{\infty} |\Delta^m e_n|^2 < \varepsilon$. Let $n_0$ be large enough such that

$$\sum_{n=-n_0T}^{n_0T-1} |\Delta^m e_n|^2 < \varepsilon \ \text{ and } \ \frac{1}{2} \leq \sum_{n=-n_0T}^{n_0T-1} e_n^2 \leq 1.$$

For $k \geq n_0$, define $e^{(k)} \in E_k$ by

$$e_n^{(k)} = \begin{cases} e_n, & -n_0T \leq n \leq n_0T - 1; \\ 0, & -kT \leq n \leq -n_0T - 1 \text{ or } n_0T \leq n \leq kT - 1. \end{cases}$$

By $(T_6)$, there exists $\mu_0 > \max\{r, 1\}$ such that

$$F(n, \mu e_{n+1}, \mu e_n) \geq (b_n - \varepsilon)\mu^2(e_{n+1}^2 + e_n^2) \ \text{ for } -n_0T \leq n \leq n_0T - 1 \text{ and } \mu \geq \mu_0.$$

Then, for $\mu \geq \mu_0$,

$$\begin{aligned} J_k(\mu e^{(k)}) &= \sum_{n=-kT}^{kT-1} \left(\frac{1}{p} r_n |\mu \Delta^m e_n^{(k)}|^p + \frac{\omega_n}{2}|\mu e_n^{(k)}|^2 - F(n, \mu e_{n+1}^{(k)}, \mu e_n^{(k)})\right) \\ &\leq \frac{\bar{r}}{p} 2^{mp} c_k^p(p)\varepsilon\mu^p + \sum_{n=-kT}^{kT-1} \left(\frac{\omega_n}{2}|\mu e_n^{(k)}|^2 + (\varepsilon - b_n)\mu^2(|e_{n+1}^{(k)}|^2 + |e_n^{(k)}|^2)\right) \\ &\leq \frac{\bar{r}}{p} 2^{mp} c_k^p(p)\varepsilon\mu^2 + \frac{\omega_n}{2}\mu^2 + (\varepsilon - b_n)\mu^2 \\ &\leq \left[\left(\frac{\bar{r}}{p} 2^{mp} c_k^p(p) + 1\right)\varepsilon - d\right]\mu^2. \end{aligned}$$

Thus

$$J_k(\mu_0 e^{(k)}) \leq \left[\left(\frac{\bar{r}}{p} 2^{mp} c_k^p(p) + 1\right)\varepsilon - d\right]\mu_0^2 < 0.$$

The remaining arguments are the same as those in the proof of Lemma 3.2.

**Lemma 4.3** *There exist $\alpha', \beta' > 0$ such that*

$$\alpha' \leq \|u^{(k)}\|_{k\infty} \leq \beta'$$

*holds for every critical point $u^{(k)}$ of $J_k$ in $E_k$ with $k \geq n_0$ obtained in Lemma 4.2.*

*Proof.* we can find $M_1 > 0$ (independent of $k$) such that $J_k(u^{(k)}) \leq M_1$ for $k \geq n_0$. Since $u^{(k)}$ is a critical point of $J_k$, by (2.2) and (2.3), we have

$$\sum_{n=-kT}^{kT-1} \left(\partial_2 F(n, u_{n+1}^{(k)}, u_n^{(k)}) u_{n+1}^{(k)} + \partial_3 F(n, u_{n+1}^{(k)}, u_n^{(k)}) u_n^{(k)} - 2F(n, u_{n+1}^{(k)}, u_n^{(k)})\right) \leq 2M_1. \tag{4.3}$$

From $(T_8)$, there exists $\beta' > 0$ such that

$$\partial_2 F(n, u_1, u_2) u_1 + \partial_3 F(n, u_1, u_2) u_2 - 2F(n, u_1, u_2) > 2M_1 \quad \text{for} \quad n \in \mathbb{Z}, \quad |u_1| + |u_2| > \beta'.$$

This and (4.3) together imply that $|u_n^{(k)}| \leq \beta'$ for each $n \in \mathbb{Z}$, that is,

$$\|u^{(k)}\|_{k\infty} \leq \beta'.$$

Then similar argumengts as those in the proof of Lemma 3.3 yield

$$\|u^{(k)}\|_{k\infty} \geq \alpha'.$$

Then Theorem 1.2 can be proved in the same manner as that for Theorem 1.1 and hence we omit the details.

## 5. Example

In this section, we give an example to illustrate Theorem 1.1.

**Example 5.1.** Consider the difference equation (1.1), where

$$\begin{aligned} & f(n, v_1, v_2, v_3) \\ =& \theta v_2 \left[\left(2 + \cos\left(\frac{n\pi}{T}\right)\right)\left(v_1^2 + v_2^2\right)^{\frac{\theta}{2}-1} + \left(2 + \cos\left(\frac{(n-1)\pi}{T}\right)\right)\left(v_2^2 + v_3^2\right)^{\frac{\theta}{2}-1}\right], \end{aligned}$$

where $\theta > p > 2$, $T$ is a given positive integer. Take

$$F(n, v_1, v_2) = \left[2 + \cos\left(\frac{n\pi}{T}\right)\right]\left(v_1^2 + v_2^2\right)^{\frac{\theta}{2}}.$$

Then

$$\begin{aligned} & \partial_2 F(n-1, v_2, v_3) + \partial_3 F(n, v_1, v_2) \\ =& \theta v_2 \left[\left(2 + \cos\left(\frac{n\pi}{T}\right)\right)\left(v_1^2 + v_2^2\right)^{\frac{\theta}{2}-1} + \left(2 + \cos\left(\frac{(n-1)\pi}{T}\right)\right)\left(v_2^2 + v_3^2\right)^{\frac{\theta}{2}-1}\right]. \end{aligned}$$

It is easy to see that all the assumptions of Theorem 1.1 are satisfied. Consequently, equation (1.1) has at least a nontrivial solution $u$ in $l^2$.

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## Conflict of interest

The authors declare there is no conflicts of interest.

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