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Research article

New error bounds for the tensor complementarity problem

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Abstract: This paper discusses new error bounds for the tensor complementarity problem using a P-tensor. A new lower error bound and a global error bound are presented for such a problem. It is proved that the norm of the exact solution of the tensor complementarity problem with a P-tensor has a lower bound and an upper bound. When the order of a tensor is 2, all the results for the tensor complementarity problem obtained reduce to those for the linear complementarity problem.

Keywords: tensor complementarity problem; P-tensor; error bound; linear complementarity problem; P-matrix

1. Introduction

The set of all real rth-order n-dimensional tensors is denoted by $\mathbb{T}_{r,n}$, where $r \geq 3$ and $n \geq 2$ are positive integers. A tensor $\mathscr{A} \in \mathbb{T}_{r,n}$ is called a P-tensor [1], if for each vector $y \in \mathbb{R}^n \setminus \{0\}$, there exists an index $j \in J_n$ such that

$$y_j(\mathscr{A}y^{r-1})_j \geq 0,$$

where J_n defines the index set $\{1, 2, ..., n\}$. For a given vector $q \in \mathbb{R}^n$ and a tensor $\mathscr{A} = (a_{j_1 \dots j_r}) \in \mathbb{T}_{r,n}$ with a multi-array of real entries $a_{j_1 \dots j_r}$, where $j_i \in J_n$ for $i \in \{1, ..., r\}$, the tensor complementarity problem denoted by the TCP($\mathscr{A}, q$), is to find a real vector $y \in \mathbb{R}^n$ such that

$$y \geq 0, \quad q + \mathscr{A}y^{r-1} \geq 0, \quad \textit{and} \quad y^T(q + \mathscr{A}y^{r-1}) = 0, \tag{1.1}$$

where $\mathscr{A}y^{r-1} \in \mathbb{R}^n$ and the jth element of $\mathscr{A}y^{r-1}$ is determined by

$$(\mathscr{A}y^{r-1})_j := \sum_{j_2,\dots,j_r=1}^{n} a_{jj_2 \dots j_r} y_{j_2} \cdots y_{j_r}$$

for $j \in J_n$. When the tensor $\mathscr{A}$ is a matrix, the TCP($\mathscr{A}, q$) reduces to the linear complementarity problem [2], denoted by the LCP(M, q), which is to find a real vector $y \in \mathbb{R}^n$ such that

$$y \geq 0, \quad q + My \geq 0, \quad \textit{and} \quad y^T(q + My) = 0,$$

where $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. Error bounds of the LCP(M, q) have been extensively studied in recent years. For example, Mathias and Pang in [3] introduced fundamental quantities associated with an arbitrary P-matrix and derived global upper and lower error bounds for the approximate solution of the LCP(M, q) with a P-matrix. Global upper and lower error bounds to the LCP(M, q) with a P-matrix were given in [2] by Cottle et al. We refer to [4–7] for more new results on the estimation of error bounds to the LCP(M, q) with a P-matrix. We denote the LCP(M, q) with a P-matrix by the LCP($M, q; p$) and the TCP($\mathscr{A}, q$) with a P-tensor by the TCP($\mathscr{A}, q; p$).

In recent years, the properties of several types of structured tensors in [8,9] were used to investigate some properties about the TCP($\mathscr{A}, q$) solution set. The nonempty of the solution set of the TCP($\mathscr{A}, q$) and the compactness of the solution set of the TCP($\mathscr{A}, q$) were explored in [10,11] by Che et al. Huang et al. [12,13] proved the existence of the solution of the TCP($\mathscr{A}, q$). Yu et al. [14] evaluated the stability features of the TCP($\mathscr{A}, q$) solution set. Zheng et al. [15] extended the results in [3] to the tensors and presented two TCP($\mathscr{A}, q; p$) global error bounds. In addition, some applications of the TCP($\mathscr{A}, q$) were given in [16]. In this paper, we mainly expand the error bounds of the approximate solution of the LCP($M, q; p$) in [2, 7] to the cases of the TCP($\mathscr{A}, q; p$). We also extend the bound of the norm of the exact solution of the LCP($M, q; p$) in [7] to the case of the TCP($\mathscr{A}, q; p$).

The rest of this paper is arranged as follows. Section 2 presents some fundamental concepts and summarizes some related results that will be used later. Some new error bounds of the TCP($\mathscr{A}, q; p$) are given in Section 3. Section 4 drawn some conclusions.

2. Preliminaries

In this section, we summarize some results and introduce some denotations that will be used later. The following results are true for a P-tensor.

Lemma 2.1. *([17]) For any P-tensor $\mathscr{A} \in \mathbb{T}_{r,n}$ and any $q \in \mathbb{R}^n$, the solution set of the TCP($\mathscr{A}, q$) is nonempty and compact.*

Lemma 2.2. *([15, 18]) There does not exist an odd order P-tensor.*

For any tensor $\mathscr{A} \in \mathbb{T}_{r,n}$, the infinite norm of $\mathscr{A}$ is defined as follows:

$$\| \mathscr{A} \|_\infty := \max_{j \in J_n} \sum_{j_2,\ldots,j_r=1}^{n} | a_{jj_2\ldots j_r} | .$$

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an operator, T is said to be positively homogeneous iff $T(ty) = tT(y)$ holds, for any positive t, $y \in \mathbb{R}^n$. Song and Qi in [1] introduced two quantities of the form

$$\alpha(T_{\mathscr{A}}) := \min_{\|y\|_\infty=1} \max_{j\in J_n} y_j(T_{\mathscr{A}}y)_j \tag{2.1}$$

for an arbitrary positive even integer r, and

$$\alpha(F_{\mathscr{A}}) := \min_{\|y\|_\infty=1} \max_{j\in J_n} y_j(F_{\mathscr{A}}y)_j. \tag{2.2}$$

Here $T_{\mathscr{A}} : \mathbb{R}^n \to \mathbb{R}^n$ and $F_{\mathscr{A}} : \mathbb{R}^n \to \mathbb{R}^n$ are two operators that are positively homogenous and defined by

$$T_{\mathscr{A}} y := \begin{cases} \| y \|_2^{2-r} \mathscr{A} y^{r-1}, & y \neq 0, \\ 0, & y = 0, \end{cases} \tag{2.3}$$

and

$$F_{\mathscr{A}} y := (\mathscr{A} y^{r-1})^{[\frac{1}{r-1}]}, \tag{2.4}$$

respectively. Here $y^{[\frac{1}{r-1}]} = (y_1^{\frac{1}{r-1}}, y_2^{\frac{1}{r-1}}, ..., y_n^{\frac{1}{r-1}})^T$ for $y = (y_1, y_2, ..., y_n)^T$. It is true that

$$\alpha(F_{\mathscr{A}}) \| y \|_\infty^2 \leq \max_{j \in J_n} y_j (F_{\mathscr{A}} y)_j = \max_{j \in J_n} y_j (\mathscr{A} y^{r-1})_j^{\frac{1}{r-1}}. \tag{2.5}$$

The following result in [1] gives two criterions of a P-tensor based on $\alpha(F_{\mathscr{A}})$ and $\alpha(T_{\mathscr{A}})$.

Lemma 2.3. *([1]) Let $\mathscr{A} \in \mathbb{T}_{r,n}$, $\mathscr{A}$ is a P-tensor iff $\alpha(T_{\mathscr{A}}) > 0$. When r is even, it is especially true that $\mathscr{A}$ is a P-tensor iff $\alpha(F_{\mathscr{A}}) > 0$.*

3. New error bounds to the TCP($\mathscr{A}, q$)

In this section, new error bounds of the approximate solution of (1.1) are proposed. Denote the solution set of (1.1) by the set of the form

$$S := \{ y \in \mathbb{R}^n \mid y \geq 0, q + \mathscr{A} y^{r-1} \geq 0, y^T (q + \mathscr{A} y^{r-1}) = 0 \}, \tag{3.1}$$

then according to Lemma 2.1, S is nonempty and compact when $\mathscr{A} \in \mathbb{T}_{r,n}$ is a P-tensor. Thus for any $v \in \mathbb{R}^n$, there is a vector $\tilde{y} \in S$ such that

$$\| v - \tilde{y} \|_\infty = \min_{y \in S} \| v - y \|_\infty . \tag{3.2}$$

We also need the results as follows.

Lemma 3.1. *For an arbitrary positive integer r, it is true that*

$$\| \mathscr{A} y^{r-1} \|_\infty \leq \| y \|_\infty^{r-1} \| \mathscr{A} \|_\infty. \tag{3.3}$$

Proof. In fact

$$\begin{aligned} \| \mathscr{A} y^{r-1} \|_\infty &= \max_{j \in J_n} | (\mathscr{A} y^{r-1})_j | \\ &= \max_{j \in J_n} | \sum_{j_2, ..., j_r = 1}^{n} a_{j j_2 \ldots j_r} y_{j_2} \ldots y_{j_r} | \\ &\leq \max_{j \in J_n} \sum_{j_2, ..., j_r = 1}^{n} | a_{j j_2 \ldots j_r} | \| y \|_\infty^{r-1} \\ &= \| y \|_\infty^{r-1} \| \mathscr{A} \|_\infty . \end{aligned}$$

This completes the proof.

Theorem 3.1. *Suppose $q \in \mathbb{R}^n$, $\mathscr{A} \in \mathbb{T}_{r,n}$ is a P-tensor. Let $\tilde{y}$ be the unique solution of the TCP($\mathscr{A}, q$). We have*

$$\frac{\| \tilde{v}_{\tilde{y}} \|_\infty}{max\{1, \| \mathscr{A} \|_\infty^{\frac{1}{r-1}}\}} \leq \| v - \tilde{y} \|_\infty, \forall v \in \mathbb{R}^n, \tag{3.4}$$

where $\tilde{v}_{\tilde{y}}$ is defined by

$$\tilde{v}_{\tilde{y}} := min\{v, (\mathscr{A}(v - \tilde{y})^{r-1})^{[\frac{1}{r-1}]} + (q + \mathscr{A}\tilde{y}^{r-1})^{[\frac{1}{r-1}]}\}. \tag{3.5}$$

Proof. First show that for any $v, w \in \mathbb{R}^n$, the following inequality holds,

$$\|\tilde{v}_{\tilde{y}} - \tilde{w}_{\tilde{y}}\|_\infty \leq max\{1, \| \mathscr{A} \|_\infty^{\frac{1}{r-1}}\}(\| v - \tilde{y} \|_\infty + \| w - \tilde{y} \|_\infty), \tag{3.6}$$

where $\tilde{v}_{\tilde{y}}$ is defined in (3.5) and $\tilde{w}_{\tilde{y}}$ is determined by (3.5) with v being replaced by w. Denote $y = \tilde{v}_{\tilde{y}}$ and $x = \tilde{w}_{\tilde{y}}$, suppose that $| y_j - x_j |= y_j - x_j, \forall j \in J_n$, then we have

$$y = \tilde{v}_{\tilde{y}} = min\{v, (\mathscr{A}(v - \tilde{y})^{r-1})^{[\frac{1}{r-1}]} + (q + \mathscr{A}\tilde{y}^{r-1})^{[\frac{1}{r-1}]}\},$$

$$x = \tilde{w}_{\tilde{y}} = min\{w, (\mathscr{A}(w - \tilde{y})^{r-1})^{[\frac{1}{r-1}]} + (q + \mathscr{A}\tilde{y}^{r-1})^{[\frac{1}{r-1}]}\}.$$

Thus,

$$y_j \leq v_j \quad and \quad y_j \leq (\mathscr{A}(v - \tilde{y})^{r-1})_j^{\frac{1}{r-1}} + (q + \mathscr{A}\tilde{y}^{r-1})_j^{\frac{1}{r-1}},$$

$$x_j = w_j \quad or \quad x_j = (\mathscr{A}(w - \tilde{y})^{r-1})_j^{\frac{1}{r-1}} + (q + \mathscr{A}\tilde{y}^{r-1})_j^{\frac{1}{r-1}}.$$

In the following, we prove this result in two cases. If $x_j = w_j$, then

$$\begin{aligned} | y_j - x_j |= y_j - x_j &\leq v_j - w_j \\ &\leq \| v - w \|_\infty \\ &= \| (v - \tilde{y}) - (w - \tilde{y}) \|_\infty \\ &\leq \| v - \tilde{y} \|_\infty + \| w - \tilde{y} \|_\infty . \end{aligned}$$

Therefore

$$\| y - x \|_\infty \leq \| v - \tilde{y} \|_\infty + \| w - \tilde{y} \|_\infty . \tag{3.7}$$

If $x_j = (\mathscr{A}(w - \tilde{y})^{r-1})_j^{\frac{1}{r-1}} + (q + \mathscr{A}\tilde{y}^{r-1})_j^{\frac{1}{r-1}}$, then

$$\begin{aligned} | y_j - x_j |= y_j - x_j &\leq (\mathscr{A}(v - \tilde{y})^{r-1})_j^{\frac{1}{r-1}} - (\mathscr{A}(w - \tilde{y})^{r-1})_j^{\frac{1}{r-1}} \\ &\leq \| \mathscr{A}(v - \tilde{y})^{r-1} \|_\infty^{\frac{1}{r-1}} + \| \mathscr{A}(w - \tilde{y})^{r-1} \|_\infty^{\frac{1}{r-1}} . \end{aligned}$$

It follows from Lemma 3.1, that

$$\| \mathscr{A}(v-\tilde{y})^{r-1} \|_{\infty} \leq \| v-\tilde{y} \|_{\infty}^{r-1} \| \mathscr{A} \|_{\infty},$$
$$\| \mathscr{A}(w-\tilde{y})^{r-1} \|_{\infty} \leq \| w-\tilde{y} \|_{\infty}^{r-1} \| \mathscr{A} \|_{\infty} .$$

Thus it holds that

$$\| y-x \|_{\infty} \leq \| \mathscr{A} \|_{\infty}^{\frac{1}{r-1}} (\| v-\tilde{y} \|_{\infty} + \| w-\tilde{y} \|_{\infty}). \tag{3.8}$$

Combining (3.7) and (3.8) results in (3.6). Let $w = \tilde{y}$ in (3.6), then

$$\|\tilde{v}_{\tilde{y}} - min\{\tilde{y}, (q+\mathscr{A}\tilde{y}^{r-1})^{[\frac{1}{r-1}]}\}\|_{\infty} \leq max\{1, \| \mathscr{A} \|_{\infty}^{\frac{1}{r-1}}\} \| v-\tilde{y} \|_{\infty} . \tag{3.9}$$

Given that $\tilde{y}$ solves the TCP($\mathscr{A}, q$), then we have

$$min\{\tilde{y}, (q+\mathscr{A}\tilde{y}^{r-1})^{[\frac{1}{r-1}]}\} = 0.$$

It follows from (3.9) that

$$\| min\{v, (\mathscr{A}(v-\tilde{y})^{r-1})^{[\frac{1}{r-1}]} + (q+\mathscr{A}\tilde{y}^{r-1})^{[\frac{1}{r-1}]}\} \|_{\infty} \leq max\{1, \| \mathscr{A} \|_{\infty}^{\frac{1}{r-1}}\} \| v-\tilde{y} \|_{\infty},$$

which implies (3.4). This completes the proof.

We remark that a lower error bound (Theorem 3.1) for the TCP($\mathscr{A}, q$) with a P-tensor in this paper is sharper than the result (Theorem 3.2) in [15]. Because

$$\frac{\| \tilde{v}_{\tilde{y}} \|_{\infty}}{1+ \| \mathscr{A} \|_{\infty}^{\frac{1}{r-1}}} \leq \frac{\| \tilde{v}_{\tilde{y}} \|_{\infty}}{max\{1, \| \mathscr{A} \|_{\infty}^{\frac{1}{r-1}}\}}, \tilde{v}_{\tilde{y}} := min\{v, (\mathscr{A}(v-\tilde{y})^{r-1})^{[\frac{1}{r-1}]} + (q+\mathscr{A}\tilde{y}^{r-1})^{[\frac{1}{r-1}]}\}.$$

The following result gives a global error bound on the approximate solution of (1.1).

Theorem 3.2. *Suppose $\mathscr{A} \in \mathbb{T}_{r,n}$ is a P-tensor. Let $\tilde{y}$ be the unique solution of the TCP($\mathscr{A}, q$). We have*

$$\frac{(1+ \| \mathscr{A} \|_{\infty}^{\frac{1}{r-1}}) \| \tilde{v}_{\tilde{y}} \|_{\infty} - \sqrt{\triangle_1}}{2\alpha(F_{\mathscr{A}})} \leq \| v-\tilde{y} \|_{\infty} \leq \frac{(1+ \| \mathscr{A} \|_{\infty}^{\frac{1}{r-1}}) \| \tilde{v}_{\tilde{y}} \|_{\infty} + \sqrt{\triangle_1}}{2\alpha(F_{\mathscr{A}})}, \forall v \in \mathbb{R}^n, \tag{3.10}$$

where

$$\triangle_1 = (1+ \| \mathscr{A} \|_{\infty}^{\frac{1}{r-1}})^2 \| \tilde{v}_{\tilde{y}} \|_{\infty}^2 - 4\alpha(F_{\mathscr{A}})(\tilde{v}_{\tilde{y}})_{k_1}^2 \geq 0$$

with k_1 satisfying $(\tilde{v}_{\tilde{y}})_{k_1} \neq 0$,

$$(\mathscr{A}(v-\tilde{y})^{r-1})_{k_1}^{\frac{1}{r-1}}(v-\tilde{y})_{k_1} = \max_{j\in J_n}\{(\mathscr{A}(v-\tilde{y})^{r-1})_j^{\frac{1}{r-1}}(v-\tilde{y})_j\},$$

and $\alpha(F_{\mathscr{A}})$ is defined in (2.2).

Proof. We first show that the following inequality holds

$$\alpha(F_{\mathscr{A}}) \| v-\tilde{y} \|_{\infty}^2 - (1+ \| \mathscr{A} \|_{\infty}^{\frac{1}{r-1}}) \| \tilde{v}_{\tilde{y}} \|_{\infty} \| v-\tilde{y} \|_{\infty} + (\tilde{v}_{\tilde{y}})_{k_1}^2 \leq 0. \tag{3.11}$$

Suppose $w = (q+\mathscr{A}\tilde{y}^{r-1})^{[\frac{1}{r-1}]}$. Since $\tilde{y}$ solves (1.1), we can prove that

$$\tilde{y} \geq 0, w \geq 0, \tilde{y}^T w = 0.$$

Let $x = v - \tilde{v}_{\tilde{y}}$ and $z = (\mathscr{A}(v-\tilde{y})^{r-1})^{[\frac{1}{r-1}]} + (q + \mathscr{A}\tilde{y}^{r-1})^{[\frac{1}{r-1}]} - \tilde{v}_{\tilde{y}}$, where $\tilde{v}_{\tilde{y}}$ is defined as that in (3.5), then we have that

$$x \geq 0, z \geq 0, x^T z = 0.$$

Therefore, for any $j \in J_n$,

$$\begin{aligned}
0 \geq x_j z_j - x_j w_j - \tilde{y}_j z_j + \tilde{y}_j w_j &= (z-w)_j (x-\tilde{y})_j \\
&= [(\mathscr{A}(v-\tilde{y})^{r-1})^{[\frac{1}{r-1}]} - \tilde{v}_{\tilde{y}}]_j (v - \tilde{v}_{\tilde{y}} - \tilde{y})_j \\
&= (\mathscr{A}(v-\tilde{y})^{r-1})_j^{\frac{1}{r-1}} (v-\tilde{y})_j - (\tilde{v}_{\tilde{y}})_j (v-\tilde{y})_j \\
&\quad - (\mathscr{A}(v-\tilde{y})^{r-1})_j^{\frac{1}{r-1}} (\tilde{v}_{\tilde{y}})_j + (\tilde{v}_{\tilde{y}})_j^2.
\end{aligned}$$

Thus

$$(\tilde{v}_{\tilde{y}})_j (v-\tilde{y})_j + (\mathscr{A}(v-\tilde{y})^{r-1})_j^{\frac{1}{r-1}} (\tilde{v}_{\tilde{y}})_j - (\tilde{v}_{\tilde{y}})_j^2 \geq (\mathscr{A}(v-\tilde{y})^{r-1})_j^{\frac{1}{r-1}} (v-\tilde{y})_j.$$

Let

$$(\mathscr{A}(v-\tilde{y})^{r-1})_{k_1}^{\frac{1}{r-1}} (v-\tilde{y})_{k_1} = \max_{j \in J_n} \{(\mathscr{A}(v-\tilde{y})^{r-1})_j^{\frac{1}{r-1}} (v-\tilde{y})_j\},$$

then

$$\begin{aligned}
&\max_{j \in J_n} \{(\mathscr{A}(v-\tilde{y})^{r-1})_j^{\frac{1}{r-1}} (v-\tilde{y})_j\} \\
&\leq (\tilde{v}_{\tilde{y}})_{k_1} (v-\tilde{y})_{k_1} + (\mathscr{A}(v-\tilde{y})^{r-1})_{k_1}^{\frac{1}{r-1}} (\tilde{v}_{\tilde{y}})_{k_1} - (\tilde{v}_{\tilde{y}})_{k_1}^2 \\
&\leq \| \tilde{v}_{\tilde{y}} \|_\infty \| v - \tilde{y} \|_\infty + \| (\mathscr{A}(v-\tilde{y})^{r-1})^{\frac{1}{r-1}} \|_\infty \| \tilde{v}_{\tilde{y}} \|_\infty - (\tilde{v}_{\tilde{y}})_{k_1}^2.
\end{aligned} \tag{3.12}$$

From Lemma 3.1, we have that

$$\| (\mathscr{A}(v-\tilde{y})^{r-1})^{\frac{1}{r-1}} \|_\infty \leq \| v - \tilde{y} \|_\infty \| \mathscr{A} \|_\infty^{\frac{1}{r-1}}. \tag{3.13}$$

Furthermore, we can deduce from (2.5) that

$$\alpha(F_{\mathscr{A}}) \| v - \tilde{y} \|_\infty^2 \leq \max_{j \in J_n} \{(\mathscr{A}(v-\tilde{y})^{r-1})_j^{\frac{1}{r-1}} (v-\tilde{y})_j\}. \tag{3.14}$$

Therefore, combining (3.12)–(3.14) results in

$$\begin{aligned}
\alpha(F_{\mathscr{A}}) \| v - \tilde{y} \|_\infty^2 &\leq (v-\tilde{y})_{k_1} (\tilde{v}_{\tilde{y}})_{k_1} + (\mathscr{A}(v-\tilde{y})^{r-1})_{k_1}^{\frac{1}{r-1}} (\tilde{v}_{\tilde{y}})_{k_1} - (\tilde{v}_{\tilde{y}})_{k_1}^2 \\
&\leq \| v - \tilde{y} \|_\infty \| \tilde{v}_{\tilde{y}} \|_\infty + \| (\mathscr{A}(v-\tilde{y})^{r-1})^{\frac{1}{r-1}} \|_\infty \| \tilde{v}_{\tilde{y}} \|_\infty - (\tilde{v}_{\tilde{y}})_{k_1}^2 \\
&= \| v - \tilde{y} \|_\infty (1 + \| \mathscr{A} \|_\infty^{\frac{1}{r-1}}) \| \tilde{v}_{\tilde{y}} \|_\infty - (\tilde{v}_{\tilde{y}})_{k_1}^2.
\end{aligned}$$

If $(\tilde{v}_{\tilde{y}})_{k_1} = 0$, then $v = \tilde{y}$; otherwise, (3.11) holds. According to Lemma 2.3, we have $\alpha(F_{\mathscr{A}}) > 0$, here $\mathscr{A}$ is a P-tensor. We can derive (3.10) by solving (3.11). The proof is completed.

If we set $v = 0$ in Theorem 3.2, we obtain a new bound for the norm of the exact solution of (1.1).

Corollary 1. *Let $v = 0$, then under the conditions of Theorem 3.2, we have that*

$$\frac{|((-q)_+)_{k_2}|(1+\|\mathscr{A}\|_\infty^{\frac{1}{r-1}}) - \sqrt{\triangle_2}}{2\alpha(F_{\mathscr{A}})} \le \|\tilde{y}\|_\infty \le \frac{|((-q)_+)_{k_2}|(1+\|\mathscr{A}\|_\infty^{\frac{1}{r-1}}) + \sqrt{\triangle_2}}{2\alpha(F_{\mathscr{A}})}, \tag{3.15}$$

where

$$(-q)_+ = min\{-q, 0\},$$

$$\triangle_2 = ((-q)_+)_{k_2}^2(1+\|\mathscr{A}\|_\infty^{\frac{1}{r-1}})^2 - 4\alpha(F_{\mathscr{A}})((-q)_+)_{k_2}^2 \ge 0$$

with k_2 satisfying

$$\tilde{y}_{k_2}(\mathscr{A}\tilde{y}^{r-1})_{k_2}^{\frac{1}{r-1}} = \max_{j\in J_n}\{\tilde{y}_j(\mathscr{A}\tilde{y}^{r-1})_j^{\frac{1}{r-1}}\}, and \quad ((-q)_+)_{k_2} \ne 0,$$

and $\alpha(F_{\mathscr{A}})$ is defined in (2.2).

Proof. The proof is similar to that of Theorem 3.2 and is omitted.

We have obtained a new lower error bound in Theorem 3.1 and a global error bound in Theorem 3.2 to the TCP($\mathscr{A}, q; p$).

We remark that when $r = 2$, Theorem 3.2 and Corollary 1 reduce to Theorem 3.1 in [7] and Corollary 3.1 in [7], respectively.

When $r = 2$, the tensor $\mathscr{A} \in \mathbb{T}_{r,n}$ reduces to a matrix M. Thus, $F_{\mathscr{A}}y := (\mathscr{A}y^{r-1})^{[\frac{1}{r-1}]} = My$, we have

$$\alpha(F_{\mathscr{A}}) := \min_{\|y\|_\infty=1}\max_{j\in J_n} y_j(F_{\mathscr{A}}y)_j = \min_{\|y\|_\infty=1}\max_{j\in J_n} y_j(My)_j = \alpha(M).$$

Furthermore, we can deduce from (3.5) that

$$\begin{aligned}\tilde{v}_{\tilde{y}} :&= min\{v, (\mathscr{A}(v-\tilde{y})^{r-1})^{[\frac{1}{r-1}]} + (q+\mathscr{A}\tilde{y}^{r-1})^{[\frac{1}{r-1}]}\}\\ &= min\{v, M(v-\tilde{y}) + q + M\tilde{y}\}\\ &= min\{v, q+Mv\} = u,\end{aligned}$$

which implies that

$$\begin{aligned}\triangle_1 &= (1+\|\mathscr{A}\|_\infty^{\frac{1}{r-1}})^2\|\tilde{v}_{\tilde{y}}\|_\infty^2 - 4\alpha(F_{\mathscr{A}})(\tilde{v}_{\tilde{y}})_{k_1}^2\\ &= (1+\|M\|_\infty)^2\|min\{v, q+Mv\}\|_\infty^2 - 4\alpha(M)(min\{v, q+Mv\})_{k_1}^2\\ &= (1+\|M\|_\infty)^2\|u\|_\infty^2 - 4\alpha(M)u_{k_1}^2 = \triangle\end{aligned}$$

with k_1 satisfying $u_{k_1} \ne 0$ and

$$(v-\tilde{y})_{k_1}(M(v-\tilde{y}))_{k_1} = \max_{j\in J_n}\{(v-\tilde{y})_j(M(v-\tilde{y}))_j\},$$

and that

$$\begin{aligned}\triangle_2 &= ((-q)_+)_{k_2}^2(1+\|\mathscr{A}\|_\infty^{\frac{1}{r-1}})^2 - 4\alpha(F_{\mathscr{A}})((-q)_+)_{k_2}^2\\ &= ((-q)_+)_{k_2}^2(1+\|M\|_\infty)^2 - 4\alpha(M)((-q)_+)_{k_2}^2\\ &= \triangle'\end{aligned}$$

with k_2 satisfying $((-q)_+)_{k_2} \neq 0$ and

$$\tilde{y}_{k_2}(M\tilde{y})_{k_2} = \max_{j \in J_n}\{\tilde{y}_j(M\tilde{y})_j\}.$$

Therefore, Theorem 3.1 reduces to Theorem 5.10.6 in [2].

Similarly, when $r = 2$, Theorem 3.2 is an extension of Theorem 3.1 in [7], and Corollary 1 is an extension of Corollary 3.1 in [7].

4. Conclusions

In this paper, we present a new lower error bound and a result of the global error bound on the approximate solution to the TCP($\mathscr{A}, q$) with a P-tensor. The new bound on the norm of the exact solution to the TCP($\mathscr{A}, q$) with a P-tensor is derived.

As we all know, error bounds have important applications in iterative methods for solving related optimization problems. For example, we use the obtained error bounds to analyze the convergence of the iterative method for solving the TCP($\mathscr{A}, q$). We can futher study the numerical algorithm to compute the error bounds of the TCP($\mathscr{A}, q$) with a P-tensor (see [19]).

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Conflict of interest

The authors declare there is no conflicts of interest.

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