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Research article

f-Statistical convergence on topological modules

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Abstract: The classical notion of statistical convergence has recently been transported to the scope of real normed spaces by means of the f-statistical convergence for f a modulus function. Here, we go several steps further and extend the f-statistical convergence to the scope of uniform spaces, obtaining particular cases of f-statistical convergence on pseudometric spaces and topological modules.

Keywords: f-statistical convergence; statistical convergence; uniform space; metric space; topological module

1. Introduction

The idea of statistical convergence was given by Zygmund [1] in the first edition of his monograph published in Warsaw in 1935. Later on, Fast [2] introduced the statistical convergence of number sequences in terms of the density of subsets of $\mathbb{N}$. Steinhaus [3] also defined, independently, the notion of statistical convergence. Other primary works on this topic are [4, 5]. Ever since, the concept of statistical convergence has been developed and enriched with deep and beautiful results by many authors [6–12].

Kolk [13] initiated the study of applications of statistical convergence to the scope of Banach spaces. Later in [14], there are important results that relate the statistical convergence to classical properties of Banach spaces. In [15–18], spaces of sequences defined by the statistical convergence are introduced and studied, serving, for instance, to characterize the weakly unconditionally Cauchy series in terms of statistical convergence. Outside the context of normed spaces, we find the works of İlkhan and Kara [9] and Maddock [11], were the statistical convergence is transported to the settings of quasi-metric spaces and locally convex spaces, respectively. In [19–21], statistical convergence was transported to more abstract settings such as topological groups, function spaces, and topological spaces, respectively.

The notion of a modulus function was introduced by Nakano [22]. Maddox [23] and Ruckle [24] have introduced and discussed some properties of sequence spaces defined by using a modulus function. Pehlivan [25] generalized the strong almost convergence with the help of modulus functions. Connor [26] considered strong matrix summability with respect to a modulus and statistical convergence. Finally, in [27–31] the statistical convergence by moduli is defined and studied in a deep manner.

The aim of this manuscript is to go several steps further and extend the statistical convergence by moduli to the scope of uniform spaces, obtaining particular cases of statistical convergence by moduli on pseudometric spaces and topological modules.

2. Methodology

This section will serve to gather all the necessary results and techniques on which we will rely to accomplish our main results.

2.1. Uniform spaces

Uniform spaces were conceived as general spaces where uniform continuity and uniform convergence can be naturally defined.

Definition 2.1 (Uniform space)**.** *Let X be a set. A uniformity on X is a filter $\mathcal{U} \subseteq \mathcal{P}(X \times X)$ satisfying, for every $U \in \mathcal{U}$, the following conditions:*

- *$U \subseteq X \times X$ is a reflexive internal binary relation on X, that is, $\Delta_X \subseteq U$, where $\Delta_X := \{(x, x) : x \in X\}$ is the diagonal of X.*
- *There exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$, where $V \circ V := \{(v, w) \in X \times X : \exists u \in X \ \ (v, u), (u, w) \in V\}$.*
- *$U^{-1} \in \mathcal{U}$, where $U^{-1} := \{(v, u) \in X \times X : (u, v) \in V\}$.*

The pair $(X, \mathcal{U})$ is called a uniform space. The elements of $\mathcal{U}$ are called entourages or vicinities.

Every filter base of $\mathcal{U}$ is called a base of entourages or vicinities. For every $x \in X$ and every $U \in \mathcal{U}$, $U[x] := \{y \in X : (x, y) \in U\}$. An entourage U is said to be symmetric provided that $U = U^{-1}$. If U is an entourage, then $V := U \cap U^{-1}$ is a symmetric entourage. If $\mathcal{B}$ is a base of entourages, then $\mathcal{B}_1 := \left\{U \cap U^{-1} : U \in \mathcal{B}\right\}$ is also a base of entourages.

Every uniform space becomes a topological space by defining the topology by means of the entourages. Let X be a uniform space. Then

$$\tau := \{A \subseteq X : \forall a \in A \ \exists U \text{ entourage } U[a] \subseteq A\} \cup \{\varnothing\}$$

is a topology on X that turns it into a regular topological space. If $\mathcal{B}$ is a base of entourages, then $\mathcal{B}[x_0] := \{U[x_0] : U \in \mathcal{B}\}$ is a base of neighborhoods of x_0.

Definition 2.2 (Complete uniform space)**.** *Let X be a uniform space. A Cauchy prefilter in X is a prefilter $\mathcal{F} \subseteq \mathcal{P}(X)$ such that for every entourage $U \subseteq X \times X$ there exists $B \in \mathcal{F}$ with $B \times B \subseteq U$. We say that X is complete if every Cauchy prefilter in X is convergent, that is, there exists $x_0 \in X$ such that for every entourage $U \subseteq X \times X$ there exists $B \in \mathcal{F}$ with $B \subseteq U[x_0]$.*

Special cases of uniform spaces are the pseudometric spaces and the topological groups.

Example 2.3. *Let X be a pseudometric space. The sets of the form*

$$\mathsf{U}_\delta := \{(x, y) \in X \times X : d(x, y) < \delta\},$$

for every $\delta > 0$, form a base of entourages whose generated filter is called the pseudometric uniformity. For every $x \in X$ and every $\delta > 0$, $\mathsf{U}_\delta[x] = \mathsf{U}(x, \delta)$, that is, the open ball of center x and radius δ.

Example 2.4. *Let G be a topological group. The sets of the form*

$$U_V := \left\{(g, h) \in G \times G : gh^{-1} \in V\right\},$$

for every $V \subseteq G$ neighborhood of 1, constitute a base of entourages whose generated filter is called the group uniformity. For every $g \in G$ and every $V \subseteq G$ neighborhood of 1, $\mathsf{U}_V[g] = V^{-1}g$.

We will work with a special class of topological groups: the topological modules [32–36]. The following characterization of module topology [36, Theorem 3.6] will be very much employed throughout this manuscript.

Theorem 2.5. *If M is a topological module over a topological ring R and $\mathcal{B}$ is a base of neighborhoods of 0 in M, then it is verified that:*

1. *For every $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $V + V \subseteq U$.*
2. *For every $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $-V \subseteq U$.*
3. *For every $U \in \mathcal{B}$ there exist $V \in \mathcal{B}$ and a 0-neighborhood $W \subseteq R$ such that $WV \subseteq U$.*
4. *For every $U \in \mathcal{B}$ and every $r \in R$ there exists $V \in \mathcal{B}$ such that $rV \subseteq U$.*
5. *For every $U \in \mathcal{B}$ and every $m \in M$ there exists a 0-neighborhood $W \subseteq R$ such that $Wm \subseteq U$.*

Conversely, for any filter base on a module over a topological ring verifying all five properties above there exists a unique module topology on the module such that the filter base is a basis of neighborhoods of zero.

2.2. Modulus functions

Modulus functions were introduced in [22].

Definition 2.6 (Modulus function)**.** *A modulus function is a function $f : [0, \infty) \to [0, \infty)$ satisfying the following conditions for all $x, y \in [0, \infty)$:*

- *$f(x) = 0$ if and only if $x = 0$.*
- *$f(x + y) \le f(x) + f(y)$.*
- *f is increasing.*
- *f is continuous from the right at 0.*

It follows that f must be continuous everywhere on $[0, \infty)$, and $f\left(\frac{x}{r}\right) \ge \frac{1}{r} f(x)$ for all $x \in \mathbb{R}^+$ and all $r \in \mathbb{N}$. Notice that a modulus f may be bounded or unbounded. For example, $f(x) = \frac{x}{x+1}$ is bounded, whereas $f(x) = x^p$, for $0 < p < 1$, is unbounded.

Definition 2.7 (Compatible modulus)**.** *A modulus function f is compatible if for any $\varepsilon > 0$ there exist $\tilde{\varepsilon} > 0$ and $n_0 = n_0(\varepsilon)$ such that $\frac{f(n\tilde{\varepsilon})}{f(n)} < \varepsilon$ for all $n \ge n_0$.*

Examples [30] of compatible modulus functions are $f(x) = x + \log(x + 1)$ and $f(x) = x + \dfrac{x}{x+1}$. Examples of noncompatible modulus functions are $f(x) = \log(x + 1)$ and $f(x) = W(x)$, where W is the W-Lambert function restricted to $[0, \infty)$, that is, the inverse of xe^x.

2.3. f-Density of subsets of $\mathbb{N}$

The notion of f-density for subsets of the natural numbers was introduced in [28].

Definition 2.8 (f-Density)**.** *Let f be a modulus function. The f-density of a subset $A \subseteq \mathbb{N}$ is defined as*

$$d_f(A) := \lim_{n\to\infty} \frac{f(\mathrm{card}(A \cap [1, n]))}{f(n)}$$

if the limit exists.

When f is the identity, we obtain the classical version of density [37] of subsets of $\mathbb{N}$, denoted by $d(A)$. Several basic properties of d_f will be employed in the upcoming sections.

Remark 2.9. *Let f be a modulus function. Then:*

1. *d_f is increasing, that is, $d_f(A) \leq d_f(B)$ whenever $A \subseteq B \subseteq \mathbb{N}$ and $d_f(A), d_f(B)$ exist.*
2. *Since $d_f(\emptyset) = 0$ and $d_f(\mathbb{N}) = 1$, we have that $0 \leq d_f(A) \leq 1$ for all $A \subseteq \mathbb{N}$ for which $d_f(A)$ exists.*
3. *d_f is subadditive, that is, $d_f(A \cup B) \leq d_f(A) + d_f(B)$ for all $A, B \subseteq \mathbb{N}$ for which $d_f(A), d_f(B)$ exist.*
4. *An example displayed in* [28] *shows that d_f is not additive even for disjoint pairs of subsets of $\mathbb{N}$.*
5. *If $A \subseteq \mathbb{N}$ and $d_f(A) = 0$, then $d_f(\mathbb{N} \setminus A) = 1$.*
6. *In* [28, Example 2.1]*, it is shown that the converse to the previous proposition does not hold, that is, $d_f(A) = 1$ does not necessarily mean $d_f(\mathbb{N} \setminus A) = 0$.*
7. *$d_f(A) = 0$ implies $d(A) = 0$ for all $A \subseteq \mathbb{N}$.*
8. *If $A \subseteq \mathbb{N}$ is finite and f is unbounded, then $d_f(A) = 0$.*

The following lemma, which can be found in [28, Lemma 3.4], will be very useful in the upcoming section.

Lemma 2.10. *If H is a infinite subset of $\mathbb{N}$, then there exists an unbounded modulus function f such that $d_f(H) = 1$.*

3. Results

We will present in this section our main results of this manuscript. This section will be divided into two subsections. The first subsection is devoted to present basic results on f-statistical convergence on uniform spaces. The second and final subsection contains specific results on f-statistical convergence on topological modules.

3.1. f-Statistical convergence in uniform spaces

Like we mentioned before in Section 2, uniform spaces are abstract generalizations of pseudometric spaces and topological groups. Thus, it makes sense to extend the concept of f-statistical convergence to uniform spaces.

Definition 3.1 (f-Statistical convergence)**.** *Let X be a uniform space. Let f be a modulus function. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to be f-statistically convergent to $x_0 \in X$ if the set $\{n \in \mathbb{N} : (x_n, x_0) \notin U\}$ has f-density 0 for every entourage $U \subseteq X \times X$. We will denote by $f\text{-}\operatorname{st}\lim(x_n)$ to the set of all f-statistical limits of $(x_n)_{n\in\mathbb{N}}$.*

Under the settings of the previous definition,

$$\lim_{n\to\infty} \frac{f\left(\operatorname{card}\left(\{k \le n : (x_k, x_0) \notin U\}\right)\right)}{f(n)} = 0$$

for every entourage $U \subseteq X \times X$. As expected, when the modulus f is the identity, then we call it statistical convergence and denote it by $\operatorname{st}\lim(x_n)$.

Notice that, due to the increasing character of d_f, in order to take the f-statistical limit of a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$, it only suffices to show that $\{n \in \mathbb{N} : (x_n, x_0) \notin U\}$ has f-density 0 for all U in a base of entourages.

Our first basic result is aimed at showing that, in Hausdorff uniform spaces, the f-statistical limit is unique it it exists.

Proposition 3.2. *Let X be a Hausdorff uniform space. Let f be a modulus function. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence. Then $f\text{-}\operatorname{st}\lim(x_n)$ is either empty or a singleton.*

Proof. Suppose on the contrary that there are $x_0 \neq y_0$ in $f\text{-}\operatorname{st}\lim(x_n)$. We can find a symmetric entourage $U \subseteq X \times X$ such that $U[x_0] \cap U[y_0] = \varnothing$. Since $x_0, y_0 \in f\text{-}\operatorname{st}\lim(x_n)$, we have that $d_f(\{n \in \mathbb{N} : (x_n, x_0) \notin U\}) = d_f(\{n \in \mathbb{N} : (x_n, y_0) \notin U\}) = 0$. By Remark 2.9(5), $d_f(\{n \in \mathbb{N} : (x_n, x_0) \in U\}) = d_f(\mathbb{N} \setminus \{n \in \mathbb{N} : (x_n, x_0) \notin U\}) = 1$. However, $\{n \in \mathbb{N} : (x_n, x_0) \in U\} \subseteq \{n \in \mathbb{N} : (x_n, y_0) \notin U\}$ due to the fact that $U[x_0] \cap U[y_0] = \varnothing$, reaching the contradiction that $\{n \in \mathbb{N} : (x_n, y_0) \notin U\}$ has f-density 1 in view of Remark 2.9(1). □

The following results relate the f-statistical convergence with the statistical convergence and the usual convergence.

Proposition 3.3. *Let X be a uniform space. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is convergent to $x_0 \in X$ if and only if $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to x_0 for every unbounded modulus f. In short,*

$$\lim_{n\to\infty} x_n = \bigcap \left\{f\text{-}\operatorname{st}\lim(x_n) : f \text{ unbounded modulus function}\right\}.$$

Proof.

- $\Rightarrow$ Fix an arbitrary unbounded modulus f. For every symmetric entourage $U \subseteq X \times X$ there exists $n_0 \in \mathbb{N}$ with $x_n \in U[x_0]$ for all $n \ge n_0$, which assures that

  $$\lim_{n\to\infty} \frac{f(\operatorname{card}(\{k \le n : (x_k, x_0) \notin U\}))}{f(n)} \le \lim_{n\to\infty} \frac{f(n_0)}{f(n)} = 0.$$

  This assures that $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to x_0.
- $\Leftarrow$ Conversely, if $(x_n)_{n\in\mathbb{N}}$ is not convergent to x_0, then there exists a symmetric entourage $U \subseteq X \times X$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $x_{n_k} \notin U[x_0]$ for each $k \in \mathbb{N}$. As a consequence, $H := \{n \in \mathbb{N} : (x_n, x_0) \notin U\}$ is infinite. By Lemma 2.10, there exists an unbounded modulus function f with $d_f(H) = 1$, meaning that $(x_n)_{n\in\mathbb{N}}$ is not f-statistically convergent to x_0.

□

Proposition 3.4. *Let X be a uniform space. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x_0 \in X$. Then:*

1. *If there exists a modulus f such that $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to x_0, then $(x_n)_{n\in\mathbb{N}}$ is statistically convergent to x_0. In short,*
$$\bigcup \{f\text{-}\mathrm{st}\lim(x_n) : f \text{ modulus function}\} \subseteq \mathrm{st}\lim(x_n).$$
2. *Conversely, if $(x_n)_{n\in\mathbb{N}}$ is statistically convergent to x_0, then $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to x_0 for every compatible modulus function f. In short,*
$$\mathrm{st}\lim(x_n) \subseteq \bigcap \{f\text{-}\mathrm{st}\lim(x_n) : f \text{ compatible modulus function}\}.$$

Proof.

1. For every symmetric entourage $U \subseteq X \times X$ and every $r \in \mathbb{N}$, there exists $n_r \in \mathbb{N}$ such that
$$\frac{f(\mathrm{card}(\{k \le n : (x_k, x_0) \notin U\}))}{f(n)} < \frac{1}{r}$$
for all $n \ge n_r$, that is,
$$f(\mathrm{card}(\{k \le n : (x_k, x_0) \notin U\})) < \frac{f(n)}{r} \le f\left(\frac{n}{r}\right)$$
for all $n \ge n_r$, which implies, in view that f is increasing, that
$$\mathrm{card}(\{k \le n : (x_k, x_0) \notin U\}| < \frac{n}{r}$$
for all $n \ge n_r$, yielding $x_0 \in \mathrm{st}\lim(x_n)$.
2. Take f any compatible modulus functions. Take any symmetric entourage $U \subseteq X \times X$. Fix an arbitrary $\varepsilon > 0$. Since f is compatible, there exists $\tilde{\varepsilon} > 0$ and $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\dfrac{f(n\tilde{\varepsilon})}{f(n)} < \varepsilon$ for all $n \ge n_0$. Since $x_0 \in \mathrm{st}\lim(x_n)$, there exists $r_0 = r_0(\varepsilon) \in \mathbb{N}$ such that if $n \ge r_0$, then $\mathrm{card}\left(\{k \le n : (x_k, x_0) \notin U\}\right) \le n\tilde{\varepsilon}$. Using the increasing monotonicity of f, we obtain
$$\frac{f\left(\mathrm{card}\left(\{k \le n : (x_k, x_0) \notin U\}\right)\right)}{f(n)} \le \frac{f(n\tilde{\varepsilon})}{f(n)} < \varepsilon$$
for all $n \ge \max\{n_0, r_0\}$. Thus, $(x_n)_{n\in\mathbb{N}}$ is f-statistically convergent to x_0.

□

Under the settings of the previous proposition, we conclude that
$$\bigcup \{f\text{-}\mathrm{st}\lim(x_n) : f \text{ modulus}\} \subseteq \mathrm{st}\lim(x_n) \subseteq \bigcap \{f\text{-}\mathrm{st}\lim(x_n) : f \text{ compatible modulus }\}.$$
Since trivially
$$\bigcap \{f\text{-}\mathrm{st}\lim(x_n) : f \text{ compatible modulus }\} \subseteq \bigcup \{f\text{-}\mathrm{st}\lim(x_n) : f \text{ modulus}\},$$
we obtain the following chain of equalities:
$$\bigcup \{f\text{-}\mathrm{st}\lim(x_n) : f \text{ modulus}\} = \mathrm{st}\lim(x_n) = \bigcap \{f\text{-}\mathrm{st}\lim(x_n) : f \text{ compatible modulus }\}.$$

The next result in this subsection is a generalization of [28, Theorem 3.1], which is itself a generalization of a theorem by Fast [2]. First, a technical lemma is needed.

Lemma 3.5. *Let f be a modulus function. Let $(B_j)_{j\in\mathbb{N}}$ be an increasing sequence of subsets of $\mathbb{N}$ with f-density 0. If there exists one B_j which is infinite, then there are strictly increasing sequences $(j_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ of naturals such that:*

1. *For all $k \in \mathbb{N}$, $n_k \in B_{j_k}$ and $\frac{f\left(\operatorname{card}(B_{j_k}\cap[1,i])\right)}{f(i)} \leq \frac{1}{k}$ whenever $i \geq n_k$.*
2. *$A := \bigcup_{k\in\mathbb{N}} B_{j_k} \cap [n_k, n_{k+1})$ has f-density 0.*

Proof. We will follow an inductive process. Let $j_1 := \min\{j \in \mathbb{N} : \operatorname{card}(B_j) = \infty\}$. Choose any $n_1 \in B_{j_1}$. There exist $j_2 \in \mathbb{N}$ with $j_2 > j_1$, which can actually be taken $j_2 := j_1 + 1$, and $n_2 \in B_{j_2}$ such that $n_2 > n_1$ and $\frac{f\left(\operatorname{card}(B_{j_2}\cap[1,i])\right)}{f(i)} \leq \frac{1}{2}$ whenever $i \geq n_2$. Inductively, we find strictly increasing sequences $(j_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ of naturals such that, for all $k \in \mathbb{N}$, $n_k \in B_{j_k}$ and $\frac{f\left(\operatorname{card}(B_{j_k}\cap[1,i])\right)}{f(i)} \leq \frac{1}{k}$ whenever $i \geq n_k$. Finally, we will show that $d_f(A) = 0$. Indeed, fix an arbitrary $\varepsilon > 0$ and take $k \in \mathbb{N}$ with $\frac{1}{k} < \varepsilon$. If $i \geq n_\varepsilon := n_k$, then we can find $l \in \mathbb{N}$ with $l \geq k$ such that $n_l \leq i < n_{l+1}$, meaning that $A\cap[1,i] \subseteq B_{j_l}\cap[1,i]$ and

$$\frac{f\left(\operatorname{card}(A\cap[1,i])\right)}{f(i)} \leq \frac{f\left(\operatorname{card}(B_{j_l}\cap[1,i])\right)}{f(i)} \leq \frac{1}{l} \leq \frac{1}{k} < \varepsilon.$$

As a consequence, $d_f(A) = 0$. □

Before proving the generalization of [28, Theorem 3.1], let us observe that if f is an unbounded modulus function and $A \subseteq \mathbb{N}$ has f-density 0, then $d_f(\mathbb{N} \setminus A) = 1$ so $\mathbb{N} \setminus A$ cannot be finite.

Theorem 3.6. *Let X be a uniform space with a countable base of entourages. Let f be an unbounded modulus function. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x_0 \in X$. Then $x_0 \in f\text{-}\operatorname{st}\lim(x_n)$ if and only if there exists $A \subseteq \mathbb{N}$ with $d_f(A) = 0$ and $x_0 \in \lim_{i\in\mathbb{N}\setminus A} x_i$. In short,*

$$f\text{-}\operatorname{st}\lim(x_n) = \bigcup\left\{\lim_{i\in\mathbb{N}\setminus A} x_i : A \subseteq \mathbb{N},\ d_f(A) = 0\right\}.$$

Proof. Let $\mathcal{B}$ be a countable base of entourages. We may assume without any loss of generality that the entourages of $\mathcal{B}$ are symmetric and nested downward, that is, $\mathcal{B} = \{U_j : j \in \mathbb{N}\}$ with $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$.

$\Rightarrow$ For every $j \in \mathbb{N}$, let $B_j := \{i \in \mathbb{N} : (x_i, x_0) \notin U_j\}$. Notice that $B_j \subseteq B_{j+1}$ and $d_f\left(B_j\right) = 0$ for all $j \in \mathbb{N}$. At this stage, we will distinguish between two cases:

- All the B_j's are finite. In this case, it is trivial that $x_0 \in \lim_{n\to\infty} x_n$, so it only suffices to take $A = \varnothing$.
- There exists one B_j which is infinite. In this case, we will call on Lemma 3.5 to find strictly increasing sequences $(j_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ of naturals such that, for all $k \in \mathbb{N}$, $n_k \in B_{j_k}$ and $\frac{f\left(\operatorname{card}(B_{j_k}\cap[1,i])\right)}{f(i)} \leq \frac{1}{k}$ whenever $i \geq n_k$. Now, let $A := \bigcup_{k\in\mathbb{N}} B_{j_k} \cap [n_k, n_{k+1})$. We know that $d_f(A) = 0$. Let us finally prove that $x_0 \in \lim_{i\in\mathbb{N}\setminus A} x_i$. Indeed, fix an arbitrary symmetric entourage $U \subseteq X \times X$ and take $k \in \mathbb{N}$ such that $U_{j_k} \subseteq U$. Since $\mathbb{N} \setminus A$ is infinite (because it has f-density 1 and f is unbounded), we can take $i_k := \min\{i \in \mathbb{N} \setminus A : i \geq n_k\}$. If $i \in \mathbb{N} \setminus A$ and $i \geq i_k \geq n_k$,

then we can find $l \in \mathbb{N}$ with $l \geq k$ such that $n_l \leq i < n_{l+1}$, meaning that $i \notin B_{j_l}$, which implies that $(x_i, x_0) \in U_{j_l} \subseteq U_{j_k} \subseteq U$. As a consequence, $x_0 \in \lim\limits_{i \in \mathbb{N}\setminus A} x_i$.

$\Leftarrow$ Conversely, assume that $A \subseteq \mathbb{N}$ satisfies that $x_0 \in \lim\limits_{i \in \mathbb{N}\setminus A} x_i$ and $d_f(A) = 0$. Fix an arbitrary symmetric entourage $U \subseteq X \times X$. There exists $i_U \in \mathbb{N} \setminus A$ such that $(x_i, x_0) \in U$ for each $i \in \mathbb{N} \setminus A$ and $i > i_U$. Therefore, $\{i \in \mathbb{N} : (x_i, x_0) \notin U\} \subseteq A \cup \{1, \ldots, i_U\}$, meaning that

$$d_f (\{i \in \mathbb{N} : (x_i, x_0) \notin U\}) \leq d_f (A \cup \{1, \ldots, i_U\}) \leq d_f(A) + d_f (\{1, \ldots, i_U\}) = 0.$$

Observe that right above we have applied Remark 2.9(8) due to the unboundedness of f, and subadditivity of d_f given by Remark 2.9(3). The arbitrariness of U shows that $x_0 \in f\text{-st}\lim(x_n)$.

□

Theorem 3.6 has strong consequences on the f-statistical convergence of f-statistically Cauchy sequences.

Definition 3.7 (f-Statistical Cauchy). *Let X be a uniform space. Let f be a modulus function. A sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is said to be f-statistically Cauchy if for every entourage $U \subseteq X \times X$ there exists $n_U \in \mathbb{N}$ such that the set $\{n \in \mathbb{N} : (x_n, x_{n_U}) \notin U\}$ has f-density* 0.

The following corollary is an abstract generalization of [27, Theorem 3.3].

Corollary 3.8. *Let X be a uniform space with a countable base of entourages. Let f be an unbounded modulus function. If X is complete, then every f-statistically Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ is f-statistically convergent.*

Proof. Let $\mathcal{B}$ be a countable base of entourages. Like in the proof of Theorem 3.6, we may assume without any loss of generality that the entourages of $\mathcal{B}$ are symmetric and nested downward, that is, $\mathcal{B} = \{U_l : l \in \mathbb{N}\}$ with $U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$. For every $l \in \mathbb{N}$, take $m_l := n_{U_l}$ as in Definition 3.7 for the entourage $U_l \in \mathcal{B}$, that is, $d_f (\{i \in \mathbb{N} : (x_i, x_{m_l}) \notin U_l\}) = 0$. For each $j \in \mathbb{N}$, define $V_j := \bigcap_{l\leq j} U_l[x_{m_l}]$ and $B_j := \{i \in \mathbb{N} : x_i \notin V_j\} = \bigcup_{l\leq j} \{i \in \mathbb{N} : (x_i, x_{m_l}) \notin U_l\}$, meaning that $d_f(B_j) = 0$ in view of Remark 2.9(4), that is, subadditivity of d_f, hence $V_j \neq \varnothing$. Notice that $(V_j)_{j\in\mathbb{N}}$ is decreasing, thus it is a prefilter (or filter base) in X. We will show next that $(V_j)_{j\in\mathbb{N}}$ is a Cauchy prefilter in X. Indeed, fix an arbitrary entourage $U \subseteq X \times X$. Take another entourage $V \subseteq X \times X$ such that $V \circ V \subseteq U$. Since $\mathcal{B}$ is base of entourages, there exists $l \in \mathbb{N}$ with $U_l \subseteq V$. Then $V_l \times V_l \subseteq U_l[x_{m_l}] \times U_l[x_{m_l}] \subseteq U_l \circ U_l \subseteq V \circ V \subseteq U$. This shows that $(V_j)_{j\in\mathbb{N}}$ is a Cauchy prefilter in X. Since X is complete, $(V_j)_{j\in\mathbb{N}}$ is convergent to some $x_0 \in X$, meaning that for every entourage $U \subseteq X \times X$, there exists $j \in \mathbb{N}$ such that $V_j \subseteq U[x_0]$. On the other hand, $(B_j)_{j\in\mathbb{N}}$ is increasing. At this stage, we will distinguish between two possibilities:

- All the B_j's are finite. In this case, it is trivial to check that $x_0 \in \lim\limits_{n\to\infty} x_n$. Since f is unbounded, we conclude that $x_0 \in f\text{-st}\lim(x_n)$ in virtue of Proposition 3.3.
- There exists one B_j which is infinite. In this case, we will call on Lemma 3.5 to find strictly increasing sequences $(j_k)_{k\in\mathbb{N}}$ and $(n_k)_{k\in\mathbb{N}}$ of naturals such that, for all $k \in \mathbb{N}$, $n_k \in B_{j_k}$ and $\frac{f\left(\text{card}(B_{j_k} \cap [1,i])\right)}{f(i)} \leq \frac{1}{k}$ whenever $i \geq n_k$. Now, let $A := \bigcup\limits_{k\in\mathbb{N}} B_{j_k} \cap [n_k, n_{k+1})$. We know that $d_f(A) = 0$.

  Let us finally prove that $x_0 \in \lim\limits_{i\in\mathbb{N}\setminus A} x_i$, which will imply that $x_0 \in f\text{-st}\lim(x_n)$ in accordance with

Theorem 3.6. Indeed, fix an arbitrary symmetric entourage $U \subseteq X \times X$. Since $(V_j)_{j\in\mathbb{N}}$ is convergent to $x_0 \in X$, there exists $k \in \mathbb{N}$ such that $V_{j_k} \subseteq U[x_0]$. Since $\mathbb{N} \setminus A$ is infinite (because it has f-density 1 and f is unbounded), we can take $i_k := \min\{i \in \mathbb{N} \setminus A : i \geq n_k\}$. If $i \in \mathbb{N} \setminus A$ and $i \geq i_k \geq n_k$, then we can find $l \in \mathbb{N}$ with $l \geq k$ such that $n_l \leq i < n_{l+1}$, meaning that $i \notin B_{j_l}$, which implies that $x_i \in V_{j_l} \subseteq V_{j_k} \subseteq U[x_0]$. As a consequence, $x_0 \in \lim\limits_{i\in\mathbb{N}\setminus A} x_i$.

□

3.2. f-Statistical convergence on topological modules

Even though topological modules are special cases of topological groups, we decide to study f-statistical convergence on topological modules because in order to prove the most natural results we are in need of commutativity. And it is well known that every topological commutative group, with additive notation, is a topological $\mathbb{Z}$-module when $\mathbb{Z}$ is endowed with the discrete topology.

Let R be a topological ring and M a topological R-module. Let f be a modulus function. Note that a sequence $(x_n)_{n\in\mathbb{N}} \subseteq M$ is f-statistically convergent to $x_0 \in M$ if the set $\{n \in \mathbb{N} : x_n \notin x_0 + U\}$ has f-density 0 for every additively symmetric 0-neighborhood U in M (recall that by additively symmetric we mean $U = -U$).

The following remark, although it is trivial, is extremely useful to perform operations with f-statistical limits.

Remark 3.9. *Let M be a module over a ring R. Let A, B, C be subsets of M. Then:*

1. *If $A + B \subseteq C$ and $C - A \subseteq B$, then $A + B = C$.*
2. *If $C - A \subseteq B$ and $C - B \subseteq A$, then $B = C - A$.*

Theorem 3.10. *Let R be a topological ring and M a topological R-module. Let f be a modulus function. Consider sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq M$ and $r \in R$. Then:*

1. *$f\text{-st}\lim(x_n + y_n) = f\text{-st}\lim(x_n) + f\text{-st}\lim(y_n)$.*
2. *$rf\text{-st}\lim(x_n) \subseteq f\text{-st}\lim(rx_n)$.*
3. *If $r \in R$ is invertible, then $f\text{-st}\lim(rx_n) = rf\text{-st}\lim(x_n)$.*
4. *If $M = R$, then $f\text{-st}\lim(x_n) f\text{-st}\lim(y_n) \subseteq f\text{-st}\lim(x_n y_n)$.*

Proof.

1. Fix arbitrary elements $x_0 \in f\text{-st}\lim(x_n)$ and $y_0 \in f\text{-st}\lim(y_n)$. Take any additively symmetric 0-neighborhood $U \subseteq M$. There exists another addivitely symmetric 0-neighborhood $V \subseteq M$ such that $V + V \subseteq U$. Then

$$\{n \in \mathbb{N} : x_n + y_n \notin (x_0 + y_0) + U\} \subseteq \{n \in \mathbb{N} : x_n \notin x_0 + V\} \cup \{n \in \mathbb{N} : y_n \notin y_0 + V\}.$$

As a consequence,

$$\begin{aligned} d_f(\{n \in \mathbb{N} : x_n + y_n \notin (x_0 + y_0) + U\}) &\leq d_f(\{n \in \mathbb{N} : x_n \notin x_0 + V\}) \\ &+ d_f(\{n \in \mathbb{N} : y_n \notin y_0 + V\}) \\ &= 0. \end{aligned}$$

The arbitrariness of U shows that $x_0 + y_0 \in f\text{-st}\lim(x_n + y_n)$. All of these prove that $f\text{-st}\lim(x_n) + f\text{-st}\lim(y_n) \subseteq f\text{-st}\lim(x_n + y_n)$. Following a similar reasoning, it can be proved that $f\text{-st}\lim(x_n + y_n) - f\text{-st}\lim(x_n) \subseteq f\text{-st}\lim(y_n)$. In view of Remark 3.9, we conclude that $f\text{-st}\lim(x_n + y_n) = f\text{-st}\lim(x_n) + f\text{-st}\lim(y_n)$.

2. Fix an arbitrary element $x_0 \in f\text{-st}\lim(x_n)$. Take any additively symmetric 0-neighborhood $U \subseteq M$. There exists another addivitely symmetric 0-neighborhood $V \subseteq M$ such that $rV \subseteq U$. Then

$$\{n \in \mathbb{N} : rx_n \notin rx_0 + U\} \subseteq \{n \in \mathbb{N} : x_n \notin x_0 + V\}.$$

As a consequence,

$$d_f(\{n \in \mathbb{N} : rx_n \notin rx_0 + U\}) \leq d_f(\{n \in \mathbb{N} : x_n \notin x_0 + V\}) = 0.$$

The arbitrariness of U shows that $rx_0 \in f\text{-st}\lim(rx_n)$.

3. From the previous item, we know that $rf\text{-st}\lim(x_n) \subseteq f\text{-st}\lim(rx_n)$. If we apply the same reasoning with r^{-1}, we obtain that

$$f\text{-st}\lim(x_n) = r^{-1}(rf\text{-st}\lim(x_n)) \subseteq r^{-1}f\text{-st}\lim(rx_n) \subseteq f\text{-st}\lim\left(r^{-1}rx_n\right) = f\text{-st}\lim(x_n).$$

4. Fix arbitrary elements $x_0 \in f\text{-st}\lim(x_n)$ and $y_0 \in f\text{-st}\lim(y_n)$. Take any additively symmetric 0-neighborhood $U \subseteq R$. Let $W \subseteq R$ be an additively symmetric 0-neighborhood such that $W + W + W \subseteq U$. There exists another addivitely symmetric 0-neighborhood $V_1 \subseteq R$ such that $V_1V_1 \subseteq W$. We can also find additively symmetric 0-neighborhoods $V_2, V_3 \subseteq R$ such that $V_2y_0 \subseteq W$ and $x_0V_3 \subseteq W$. If we take $V := V_1 \cap V_2 \cap V_3$, then we obtain that V is an addivitely symmetric 0-neighborhood satisfying that $Vy_0 + x_0V + VV \subseteq W + W + W \subseteq U$. Then

$$\{n \in \mathbb{N} : x_ny_n \notin x_0y_0 + U\} \subseteq \{n \in \mathbb{N} : x_n \notin x_0 + V\} \cup \{n \in \mathbb{N} : y_n \notin y_0 + V\}.$$

As a consequence,

$$\begin{aligned} d_f(\{n \in \mathbb{N} : x_ny_n \notin x_0y_0 + U\}) &\leq d_f(\{n \in \mathbb{N} : x_n \notin x_0 + V\}) \\ &+ d_f(\{n \in \mathbb{N} : y_n \notin y_0 + V\}) \\ &= 0. \end{aligned}$$

The arbitrariness of U shows that $x_0y_0 \in f\text{-st}\lim(x_ny_n)$.

□

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Conflict of interest

The authors declare that there is no conflicts of interest.

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