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Research article

Existence and asymptotic behaviour of positive ground state solution for critical Schrödinger-Bopp-Podolsky system

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Abstract: In this paper, we consider a class of critical Schrödinger-Bopp-Podolsky system. By virtue of the Nehari manifold and variational methods, we study the existence, nonexistence and asymptotic behavior of ground state solutions for this problem.

Keywords: Schrödinger-Bopp-Podolsky system; variational methods; ground state solution; asymptotic behavior; critical exponent

1. Introduction and main results

In this paper, we deal with the following system:

$$\begin{cases} -\Delta u + V(x)u + \phi u = \lambda K(x) f(u) + |u|^4 u, & x \in \mathbb{R}^3, \\ -\Delta\phi + \varepsilon^2 \Delta^2 \phi = 4\pi u^2, & x \in \mathbb{R}^3, \end{cases} \tag{$P_{\lambda,\varepsilon}$}$$

where $\lambda \geqslant 0$, $\varepsilon > 0$, f is a continuous, superlinear and subcritical nonlinearity. $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function satisfying the following conditions:
(V_1) $0 < V(x) < V_\infty := \liminf\limits_{|x|\to+\infty} V(x) < +\infty$.
(V_2) There exists a constant $\alpha > 0$ such that

$$\alpha = \inf_{u \in H^1(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u|^2 \mathrm{d}x}{\int_{\mathbb{R}^3} |u|^2 \mathrm{d}x} > 0.$$

Furthermore, for the potential function K, we assume:
(K) $K \in C(\mathbb{R}^3, \mathbb{R})$ and $K_\infty := \limsup\limits_{|x|\to+\infty} K(x) \in (0, +\infty)$ and $K(x) \geqslant K_\infty$ for $x \in \mathbb{R}^3$.

The system $(P_{\lambda,\varepsilon})$ is a version of the so called Schrödinger-Bopp-Podolsky system, which is a Schrödinger equation coupled with a Bopp-Podolsky equation. Podolsky's theory has been proposed by Bopp [1] and independently by Podolsky-Schwed [2] as a second order gauge theory for the electromagnetic field. It appears when one look for standing waves solutions $\psi(x,t) = u(x)e^{i\omega t}$ of the Schrödinger equation coupled with the Bopp-Podolsky Lagrangian of the electromagnetic field, in the purely electrostatic situation. In the physical point of view, ε is the parameter of the Bopp-Podolsky term, u and ϕ represent the modulus of the wave function and the electrostatic potential, respectively. As for more details and physical applications of the Bopp-Podolsky equation, we refer to [3–5] and the references therein.

From a mathematical point of view, the study of system $(P_{\lambda,\varepsilon})$ can be divided into two cases: (1) $\varepsilon = 0$; (2) $\varepsilon \neq 0$.

If $\varepsilon = 0$, then system $(P_{\lambda,\varepsilon})$ gives back the classical Schrödinger-Poisson system as follows:

$$\begin{cases} -\Delta u + V(x)u + \phi u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi = 4\pi u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

which has been introduced by Benci-Fortunato [6] in quantum mechanics as a model describing the interaction of a charged particle with the electrostatic field. In such system, the potential function V is regarded as an external potential, u and ϕ represent the wave functions associated with the particle and electric potential, respectively. For more details on the physical aspects of this system, we refer the readers to [7–9] and the references therein.

In last decades, system (1.1) has been widely studied under variant assumptions on V and f, by variational methods, and existence, nonexistence and multiplicity results are obtained in many papers. For further details, we refer the readers to previous studies [10–15] and the references therein.

In particular, Azzollini-Pomponio [16] proved the existence of ground state solutions to system (1.1) with $f(x,u) = |u|^{p-1}$ and $3 < p < 5$. Ambrosetti-Ruiz [17] obtained multiple solutions to system (1.1) by the monotonicity skills combined with minimax methods. Ruiz [9] dealt with the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + u + \lambda\phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi = u^2, \ \lim\limits_{|x|\to+\infty} \phi(x) = 0, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $2 < p < 6$ and $\lambda > 0$. Via a constraint variational method combining the Nehari-Pohožaev manifold, the existence and nonexistence results were obtained.

If $\varepsilon \neq 0$, then system $(P_{\lambda,\varepsilon})$ is a Schrödinger-Bopp-Podolsky system. D'Avenia-Siciliano [18] first studied the following system from a mathematical point of view:

$$\begin{cases} -\Delta u + \omega u + q^2\phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta\phi + \varepsilon^2\Delta^2\phi = 4\pi u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.3}$$

where $\omega > 0$, $\varepsilon \geqslant 0$ and $q \neq 0$. Based on the variational methods, D'Avenia-Siciliano [18] proved the existence and nonexistence results to system (1.3) depending on the parameters p and q.

Later, for $p \in (2,3]$, Siciliano-Silva [19] obtained the existence and nonexistence of solutions to system (1.3) by means of the fibering map approach and the implicit function theorem.

Motivated by all results mentioned above, a series of interesting questions naturally arises such as:

(I) As we can see, the authors in [18] and [19] merely considered system (1.3) with subcritical growth, so we would much like to know whether similar results hold for system $(P_{\lambda,\varepsilon})$ if its nonlinearity is at critical growth.
(II) Note that in [18] and [19], the authors studied the existence and nonexistence results to system (1.3), but it has not been considered the asymptotic behavior of solutions. Therefore, it is natural to ask a question. Can we obtain the asymptotic behavior of solutions to system $(P_{\lambda,\varepsilon})$?

Compared to [18] and [19], the main purpose of this paper is to fill the gaps. More specifically, we will study the existence, nonexistence and asymptotic behavior of ground state solutions to system $(P_{\lambda,\varepsilon})$ involving a critical nonlinearity.

Now we state our conditions on f. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that
(F_1) $f(t) = o(t^3)$ as $t \to 0$ and $f(t) = 0$ for all $t \leqslant 0$.
(F_2) $\frac{f(t)}{t^3}$ is strictly increasing on interval $(0, +\infty)$.
(F_3) $|f(t)| \leqslant C(1 + |t|^{p-1})$ and $f(t) \geqslant \gamma t^{m-1}$ for some $C > 0$ and $\gamma > 0$, where $4 < p, m < 6$.

1.1. Main Results

We divide the study of system $(P_{\lambda,\varepsilon})$ into three parts: (I) $V(x) \equiv V_\infty$ and $K(x) \equiv K_\infty$; (II) $V(x) < V_\infty$ and $K(x) \geqslant K_\infty$; (III) $V(x) \geqslant V_\infty$ and $K(x) \leqslant K_\infty$, where one of the strictly inequality holds on a positive measure subset.
(I) For $V(x) \equiv V_\infty$ and $K(x) \equiv K_\infty$, system $(P_{\lambda,\varepsilon})$ goes back to its limit system:

$$\begin{cases} -\Delta u + V_\infty u + \phi u = \lambda K_\infty f(u) + |u|^4 u, & x \in \mathbb{R}^3, \\ -\Delta\phi + \varepsilon^2 \Delta^2 \phi = 4\pi u^2, & x \in \mathbb{R}^3. \end{cases} \tag{P_∞}$$

Our first result is as follows:

Theorem 1.1. *Suppose that $\lambda > 0$ and conditions (F_1)-(F_3) hold, then system (P_∞) possesses a positive ground state solution $(u_\infty, \phi_\infty) \in H^1_V(\mathbb{R}^3)\times\mathcal{D}$, where spaces $H^1_V(\mathbb{R}^3)$ and $\mathcal{D}$ are given in section 2 below.*

(II) System $(P_{\lambda,\varepsilon})$ with $V(x) < V_\infty$ and $K(x) \geqslant K_\infty$. Our second result is as follows:

Theorem 1.2. *Suppose that $\lambda > 0$, conditions (V_1)-(V_2), (K) and (F_1)-(F_3) hold. Then the following statements are true.*

(i) System $(P_{\lambda,\varepsilon})$ possesses a positive ground state solution $(u_{\lambda,\varepsilon}, \phi^\varepsilon_u) \in H^1_V(\mathbb{R}^3) \times \mathcal{D}$.

(ii) For every fixed $\varepsilon > 0$, we have

$$\lim_{\lambda\to+\infty} \|u_{\lambda,\varepsilon}\|_{H^1_V(\mathbb{R}^3)} = 0, \qquad \lim_{\lambda\to+\infty} \|\phi^\varepsilon_u\|_{\mathcal{D}} = 0 \qquad \textit{and} \qquad \lim_{\lambda\to+\infty} \|\phi^\varepsilon_u\|_{L^\infty(\mathbb{R}^3)} = 0.$$

(iii) There exist $\lambda^* > 0$ and $\widetilde{\lambda} > \lambda^*$ be fixed. Let $(u_{\widetilde{\lambda},\varepsilon}, \phi^\varepsilon_u)$ be a solution of system $(P_{\lambda,\varepsilon})$ in correspondence of $\widetilde{\lambda}$. Then we have

$$\lim_{\varepsilon\to0} u_{\widetilde{\lambda},\varepsilon} = u_{\widetilde{\lambda},0} \qquad \textit{and} \qquad \lim_{\varepsilon\to0} \phi^\varepsilon_u = \phi^0_u,$$

where $(u_{\widetilde{\lambda},0}, \phi^0_u) \in H^1_V(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a positive ground state solution of

$$\begin{cases} -\Delta u + V(x)u + \phi u = \widetilde{\lambda} K(x) f(u) + |u|^4 u, & x \in \mathbb{R}^3, \\ -\Delta\phi = 4\pi u^2, & x \in \mathbb{R}^3. \end{cases} \tag{$P_{\widetilde{\lambda},0}$}$$

By virtue of the symmetric mountain pass theorem, we also obtain a supplementary result of the infinity many positive solutions for system $(P_{\lambda,\varepsilon})$. Our third result is as follows:

Theorem 1.3. *Suppose that conditions* (V_1)-(V_2), (K) *and* (F_1)-(F_3) *hold, and suppose that* $f(u)$ *is odd. Then system* $(P_{\lambda,\varepsilon})$ *possesses infinitely many positive solutions.*

(III) System $(P_{\lambda,\varepsilon})$ with $V(x) \geqslant V_\infty$ and $K(x) \leqslant K_\infty$, which one of the strictly inequality holds on a positive measure subset. Our last result is as follows:

Theorem 1.4. *Suppose that conditions* (F_1)-(F_3) *hold, then for any* $\lambda > 0$, $\varepsilon > 0$, *system* $(P_{\lambda,\varepsilon})$ *has no ground state solution.*

Remark 1.1. To our best knowledge, there is still no results concerning the existence and asymptotic behavior of solutions for Schrödinger-Bopp-Podolsky system with critical exponent. Hence our results are new. By comparing with [18] and [19], we have to face three major difficulties. First, the existence of critical term and noncompact potential function $V(x)$ set an obstacle that the bounded (PS) sequences may not converge. Second, the presence of the potential functions $V(x)$ and $K(x)$ cause the splitting lemma for recovering the compactness developed in [18] cannot be applied to system $(P_{\lambda,\varepsilon})$. Third, the Podolsky's term in system $(P_{\lambda,\varepsilon})$ makes the corresponding Brézis-Lieb type convergence lemma invalid. As we will see later, these difficulties prevent us from using the way as in [18] and [19]. So we need some new tricks to deal with these essential problems.

Remark 1.2. The proof of Theorems 1.1 and 1.2 is mainly based on the methods of the Nehari manifold and the concentration compactness principle [20]. However, since the nonlinearity f is only continuous, we cannot use standard arguments on the Nehari manifold. To overcome the non-differentiability of the Nehari manifold, we shall use some variants of critical point theorems from Szulkin-Weth [21]. At the same time, because of the presence of the potential functions $V(x)$ and $K(x)$, it is difficult to study the minimization problem of system $(P_{\lambda,\varepsilon})$ directly. Therefore we first study its limit system (P_∞), which is given in section 3. Then by comparing the ground state energy between system $(P_{\lambda,\varepsilon})$ and (P_∞), the existence results is obtained.

In Theorems 1.1 and 1.2, we just consider the following two cases: **(i)** $V(x) \equiv V_\infty$ and $K(x) \equiv K_\infty$; **(ii)** $V(x) < V_\infty$ and $K(x) \geqslant K_\infty$. This motivates an interesting open problem: **Does the existence of ground state solutions for system $(P_{\lambda,\varepsilon})$ hold for $V(x) < V_\infty$ or $K(x) \geqslant K_\infty$?**

The remainder of this paper is as follows. In section 2, variational setting and some preliminaries are presented. In sections 3 to 6, the proof of Theorems 1.1 to 1.4 is given, respectively.

2. Preliminaries and variational settings

Throughout this paper, the letters C, C_i $(i = 1, 2...)$ will denote possibly different positive constants which may change from line to line.

Let

$$H_V^1(\mathbb{R}^3) = \left\{u \in H^1(\mathbb{R}^3) \Big| \int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x < +\infty\right\}$$

endowed with the inner product

$$(u, v)_{H_V^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv)\mathrm{d}x$$

and the related norm

$$\|u\|_{H^1_V(\mathbb{R}^3)} = \left[\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x\right]^{\frac{1}{2}}.$$

Under conditions (V_1)-(V_2), it is easy to see that the norms $\|u\|_{H^1_V(\mathbb{R}^3)}$ and $\|u\|_{H^1(\mathbb{R}^3)}$ are equivalent and the embedding $H^1_V(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for each $s \in [2,6]$.

Next we outline the variational framework for system $(P_{\lambda,\varepsilon})$ and give some preliminary lemmas. In particular, we give some fundamental properties on the operator $-\Delta + \varepsilon^2\Delta^2$.

2.1. The variational settings

We define $\mathcal{D}$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by the scalar product

$$(u,v)_{\mathcal{D}} = \int_{\mathbb{R}^3} \left(\nabla u \nabla v + \varepsilon^2 \Delta u \Delta v\right)\mathrm{d}x.$$

Then $\mathcal{D}$ is a Hilbert space, which is continuously embedded into $D^{1,2}(\mathbb{R}^3)$ and consequently in $L^6(\mathbb{R}^3)$.

Lemma 2.1. ([18]) *The space $\mathcal{D}$ is continuously embedded into $L^\infty(\mathbb{R}^3)$.*

We recall that by the Lax-Milgram theorem, for every fixed $u \in H^1_V(\mathbb{R}^3)$, there exists a unique solution $\phi_u^\varepsilon \in \mathcal{D}$ of the second equation in system $(P_{\lambda,\varepsilon})$. To write explicitly such a solution (see [5]), we consider

$$\mathcal{K}(x) = \frac{1 - e^{\frac{-|x|}{\varepsilon}}}{|x|}.$$

For $\mathcal{K}$, we have the following fundamental properties.

Lemma 2.2. ([18]) *For all $y \in \mathbb{R}^3$, $\mathcal{K}(\cdot - y)$ solves in the sense of distributions*

$$-\Delta\phi + \varepsilon^2\Delta^2\phi = 4\pi\delta_y.$$

Moreover,

(i) if $f \in L^1_{loc}(\mathbb{R}^3)$ and for a.e. $x \in \mathbb{R}^3$, the map $y \in \mathbb{R}^3 \to \frac{f(y)}{|x-y|}$ is summable, then $\mathcal{K} * f \in L^1_{loc}(\mathbb{R}^3)$;

(ii) if $f \in L^p(\mathbb{R}^3)$ with $1 \leqslant p < \frac{3}{2}$, then $\mathcal{K} * f \in L^q(\mathbb{R}^3)$ for $q \in (\frac{3p}{3-2p}, +\infty]$.

In both cases $\mathcal{K} * f$ solves

$$-\Delta\phi + \varepsilon^2\Delta^2\phi = 4\pi f.$$

Then if we fix $u \in H^1_V(\mathbb{R}^3)$, the unique solution in $\mathcal{D}$ of the second equation in system $(P_{\lambda,\varepsilon})$ can be expressed by

$$\phi_u^\varepsilon = \mathcal{K} * u^2 = \int_{\mathbb{R}^3} \frac{1 - e^{\frac{-|x-y|}{\varepsilon}}}{|x-y|} u^2(y)\mathrm{d}y.$$

Now, let us summarize some properties of ϕ_u^ε.

Lemma 2.3. ([18]) *For every $u, v \in H^1_V(\mathbb{R}^3)$, the following statements are true.*

(i) $\phi_u^\varepsilon \geqslant 0$.

(ii) For each $t > 0$, $\phi_{tu}^\varepsilon = t^2\phi_u^\varepsilon$.

(iii) If $u_n \rightharpoonup u$ in $H^1_V(\mathbb{R}^3)$, then $\phi_{u_n}^\varepsilon \rightharpoonup \phi_u^\varepsilon$ in $\mathcal{D}$.

(iv) $\|\phi_u^\varepsilon\|_{\mathcal{D}} \leqslant C\|u\|^2_{L^{\frac{12}{5}}(\mathbb{R}^3)} \leqslant C\|u\|^2_{H^1_V(\mathbb{R}^3)}$ and $\int_{\mathbb{R}^3} \phi_u^\varepsilon |u|^2\mathrm{d}x \leqslant C\|u\|^4_{L^{\frac{12}{5}}(\mathbb{R}^3)} \leqslant C\|u\|^4_{H^1_V(\mathbb{R}^3)}$.

Lemma 2.4. ([18]) *Consider* $f \in L^{\frac{6}{5}}(\mathbb{R}^3)$, $\{f_\varepsilon\}_{\varepsilon\in(0,1)} \subset L^{\frac{6}{5}}(\mathbb{R}^3)$ *and let*

$$\phi^0_{u_f} \in D^{1,2}(\mathbb{R}^3) \text{ be the unique solution of } -\Delta\phi = f \text{ in } \mathbb{R}^3,$$

and

$$\phi^\varepsilon_{u_f} \in \mathcal{D} \text{ be the unique solution of } -\Delta\phi + \varepsilon^2\Delta^2\phi = f_\varepsilon \text{ in } \mathbb{R}^3.$$

As $\varepsilon \to 0$*, we have:*

(i) *If* $f_\varepsilon \rightharpoonup f$ *in* $L^{\frac{6}{5}}(\mathbb{R}^3)$*, then* $\phi^\varepsilon_{u_f} \rightharpoonup \phi^0_{u_f}$ *in* $D^{1,2}(\mathbb{R}^3)$.

(ii) *If* $f_\varepsilon \to f$ *in* $L^{\frac{6}{5}}(\mathbb{R}^3)$*, then* $\phi^\varepsilon_{u_f} \to \phi^0_{u_f}$ *in* $D^{1,2}(\mathbb{R}^3)$ *and* $\varepsilon\Delta\phi^\varepsilon_{u_f} \to 0$ *in* $L^2(\mathbb{R}^3)$.

By using the classical reduction argument, system $(P_{\lambda,\varepsilon})$ can be reduced to a single equation:

$$-\Delta u + V(x)u + \phi^\varepsilon_u u = \lambda K(x) f(u) + |u|^4 u, \quad x \in \mathbb{R}^3. \tag{2.1}$$

Then from now on we speak of solutions of system $(P_{\lambda,\varepsilon})$ is equal to the solutions of equation (2.1). It is easy to see that the solutions of equation (2.1) can be regarded as critical points of the energy functional $I_{\lambda,\varepsilon}$: $H^1_V(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$I_{\lambda,\varepsilon}(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\phi^\varepsilon_u |u|^2 \mathrm{d}x - \lambda\int_{\mathbb{R}^3} K(x)F(u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x.$$

From (F_1) and (F_3), it is easy to check that $I_{\lambda,\varepsilon}$ is a well defined C^1 functional in $H^1_V(\mathbb{R}^3)$. Moreover, $\forall \varphi \in H^1_V(\mathbb{R}^3)$,

$$\left\langle I'_{\lambda,\varepsilon}(u), \varphi\right\rangle = \int_{\mathbb{R}^3}(\nabla u\nabla\varphi + V(x)u\varphi)\mathrm{d}x + \int_{\mathbb{R}^3}\phi^\varepsilon_u u\varphi\mathrm{d}x - \lambda\int_{\mathbb{R}^3}K(x)f(u)\varphi\mathrm{d}x - \int_{\mathbb{R}^3}|u|^4 u\varphi\mathrm{d}x.$$

2.2. The Key Lemmas

The following lemma is the Young convolution inequality, which is a fundamental tool in our analysis.

Lemma 2.5. ([27]) *If* $G \in L^q(\mathbb{R}^3)$ *and* $H \in L^r(\mathbb{R}^3)$ *with* $1 < \frac{1}{q} + \frac{1}{r} \leqslant 2$*, then* $G * H \in L^s(\mathbb{R}^3)$ *with* $\frac{1}{s} = \frac{1}{q} + \frac{1}{r} - 1$ *and*

$$\int_{\mathbb{R}^3}|G * H|^s\mathrm{d}x \leqslant \left(\int_{\mathbb{R}^3}|G|^q\mathrm{d}x\right)^{\frac{s}{q}}\left(\int_{\mathbb{R}^3}|H|^r\mathrm{d}x\right)^{\frac{s}{r}}.$$

We will apply the concentration compactness principle [20] and vanishing lemma [22] to prove the compactness of (PS) sequence of $I_{\lambda,\varepsilon}$. Now, we recall them as follows.

Proposition 2.1. ([20]) *Let* $\rho_n(x) \in L^1(\mathbb{R}^3)$ *be a nonnegative sequence satisfying*

$$\int_{\mathbb{R}^3}\rho_n(x)\mathrm{d}x = l > 0.$$

Then there exists a subsequence, still denoted by $\{\rho_n(x)\}$*, such that one of the following cases occurs.*

(i) Compactness: There exists $\{y_n\} \in \mathbb{R}^3$*, such that for each* $\epsilon > 0$*, there exists* $R > 0$ *such that*

$$\int_{B_R(y_n)}\rho_n(x)\mathrm{d}x \geqslant l - \epsilon.$$

(ii) Vanishing: For every fixed $R > 0$, there holds

$$\lim_{n\to+\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}\rho_n(x)\mathrm{d}x = 0.$$

(iii) Dichotomy: There exist $\beta > 0$ with $0 < \beta < l$, sequence $\{R_n\}$ with $R_n \to +\infty$ and two functions $\rho_n^1(x), \rho_n^2(x) \in L^1(\mathbb{R}^3)$, $\{y_n\} \subset \mathbb{R}^3$ such that for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}^*$, for $n \geqslant n_0$, there holds

$$\left\|\rho_n - (\rho_n^1 + \rho_n^2)\right\|_{L^1(\mathbb{R}^3)} < \epsilon, \qquad \left|\int_{\mathbb{R}^3}\rho_n^1(x)\mathrm{d}x - \beta\right| < \epsilon, \qquad \left|\int_{\mathbb{R}^3}\rho_n^2(x)\mathrm{d}x - (l-\beta)\right| < \epsilon,$$

and

$$\mathrm{supp}\rho_n^1 \subset B_{R_n}(y_n), \quad \mathrm{supp}\rho_n^2 \subset B_{2R_n}^c(y_n).$$

Proposition 2.2. ([22]) *Suppose that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and it satisfies*

$$\lim_{n\to+\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}|u_n|^2\mathrm{d}x = 0,$$

where $R > 0$. Then $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $s \in (2,6)$.

3. The Proof of Theorem 1.1

In this section, we shall prove the existence of positive ground state solutions to system (P_∞).
Set

$$H^1_{V_\infty}(\mathbb{R}^3) = \left\{u \in H^1(\mathbb{R}^3)\Big| \int_{\mathbb{R}^3} V_\infty u^2\mathrm{d}x < +\infty\right\},$$

endowed with the inner product

$$(u,v)_{H^1_{V_\infty}(\mathbb{R}^3)} = \int_{\mathbb{R}^3}(\nabla u\nabla v + V_\infty uv)\mathrm{d}x,$$

and the related norm

$$\|u\|_{H^1_{V_\infty}(\mathbb{R}^3)} = \left[\int_{\mathbb{R}^3}(|\nabla u|^2 + V_\infty u^2)\mathrm{d}x\right]^{\frac{1}{2}}.$$

By the Lax-Milgram theorem and Lemma 2.2, we can define the energy functional corresponding to system (P_∞) by

$$I_\infty(u) = \frac{1}{2}\|u\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u^\varepsilon|u|^2\mathrm{d}x - \lambda K_\infty\int_{\mathbb{R}^3}F(u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x, \quad \forall u \in H^1_{V_\infty}(\mathbb{R}^3).$$

3.1. Mountain Pass Geometry and Nehari Manifold

The Nehari manifold corresponding to I_∞ is defined by

$$\mathcal{N}_\infty = \left\{u \in H^1_{V_\infty}(\mathbb{R}^3)\backslash\{0\}\Big|\langle I'_\infty(u), u\rangle = 0\right\}.$$

We can conclude $\mathcal{N}_\infty$ has the following elementary properties.

Lemma 3.1. (See Appendix) *Suppose that $\varepsilon > 0$ be fixed and conditions (F_1)-(F_3) hold. Then the following statements are true.*

(i) The functional I_∞ possesses the mountain pass geometry.

(ii) For each $u \in H^1_{V_\infty}(\mathbb{R}^3)\backslash\{0\}$, there exists a unique $t_u > 0$ such that $I_\infty(t_u u) = \max_{t\geqslant 0} I_\infty(tu)$. Moreover, $tu \in \mathcal{N}_\infty$ if and only if $t = t_u$ and

$$\lim_{\lambda\to+\infty} t_u = 0.$$

(iii) $c_\infty = \bar{c}_\infty = \bar{\bar{c}}_\infty > 0$, where

$$c_\infty = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\infty(\gamma(t)), \qquad \bar{c}_\infty = \inf_{u\in\mathcal{N}_\infty} I_\infty(u) \qquad \text{and} \qquad \bar{\bar{c}}_\infty = \inf_{u\in H^1_{V_\infty}(\mathbb{R}^3)\backslash\{0\}} \max_{t>0} I_\infty(tu),$$

and $\Gamma = \{\gamma \in C([0,1], H^1_{V_\infty}(\mathbb{R}^3)) \big| \gamma(0) = 0, I_\infty(\gamma(1)) < 0\}$.

According to Lemma 3.1 (*i*), it follows that for any $u \in H^1_{V_\infty}(\mathbb{R}^3)\backslash\{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\infty$. We define a mapping $\widehat{m}_\infty : H^1_{V_\infty}(\mathbb{R}^3)\backslash\{0\} \to \mathcal{N}_\infty$ by

$$\widehat{m}_\infty = t_u u \qquad \text{and} \qquad m_\infty = \widehat{m}_\infty|_{S_\infty}, \qquad S_\infty = \{u \in H^1_{V_\infty}(\mathbb{R}^3) \big| \|u\|_{H^1_{V_\infty}(\mathbb{R}^3)} = 1\}.$$

Moreover, the inverse of m_∞ can be given by

$$m_\infty^{-1}(u) = \frac{u}{\|u\|_{H^1_{V_\infty}(\mathbb{R}^3)}}.$$

Considering the functionals $\widehat{\Upsilon}_\infty : H^1_{V_\infty}(\mathbb{R}^3)\backslash\{0\} \to \mathbb{R}$ and $\Upsilon_\infty : S_\infty \to \mathbb{R}$ given by

$$\widehat{\Upsilon}_\infty(\omega) = I_\infty(\widehat{m}_\infty(u)) \qquad \text{and} \qquad \Upsilon_\infty = \widehat{\Upsilon}_\infty|_{S_\infty}.$$

Then we have the following lemma.

Lemma 3.2. ([21]) *Suppose that all conditions described in Lemma 3.1 hold. Then the following statements are true.*

(*i*) $\Upsilon_\infty \in C^1(S_\infty, \mathbb{R})$ *and*

$$\langle \Upsilon'_\infty(\omega), z\rangle = \|m_\infty(\omega)\|_{H^1_{V_\infty}(\mathbb{R}^3)} \langle I'(m_\infty(\omega)), z\rangle,$$

for all $z \in T_\omega(S_\infty) := \{v \in H^1_{V_\infty}(\mathbb{R}^3) \big| \langle\omega, v\rangle = 0\}$.

(*ii*) $\{\omega_n\}$ *is a* (PS) *sequence for* Υ_∞*, if and only if* $\{m_\infty(\omega_n)\}$ *is a* (PS) *sequence for* I_∞*. If* $\{u_n\} \subset \mathcal{N}_\infty$ *is a bounded* (PS) *sequence for* I_∞*, then* $\{m_\infty^{-1}(u_n)\}$ *is a* (PS) *sequence for* Υ_∞.

(*iii*) $\omega \in S_\infty$ *is a critical point of* Υ_∞*, if and only if* $m_\infty(\omega)$ *is a critical point of* I_∞*. Moreover, the corresponding values of* I_∞ *and* Υ_∞ *coincide and*

$$\inf_{u\in\mathcal{N}_\infty} I_\infty(u) = \inf_{\omega\in S_\infty} \Upsilon_\infty(\omega) = c_\infty.$$

3.2. Estimates of c_∞

The main feature of the functional I_∞ is that it satisfies the local compactness condition, as we can see in the following result.

Lemma 3.3. *For all $\lambda, \varepsilon > 0$, there exists some $v \in H^1_{V_\infty}(\mathbb{R}^3)\backslash\{0\}$ such that*

$$\max_{t\geqslant 0} I_\infty(tv) < \frac{1}{3}\mathcal{S}^{\frac{3}{2}},$$

where $\mathcal{S} = \inf\limits_{u\in D^{1,2}(\mathbb{R}^3)\backslash\{0\}} \frac{\|u\|^2_{D^{1,2}(\mathbb{R}^3)}}{\|u\|^2_{L^6(\mathbb{R}^3)}}$.

Proof. For each $\epsilon > 0$, consider the function

$$U_\epsilon = \frac{C\epsilon^{\frac{1}{4}}}{(\epsilon + |x|^2)^{\frac{1}{2}}},$$

where C is a normalized constant. We recall that U_ϵ satisfies

$$-\Delta u = u^5, \ \ u \in D^{1,2}(\mathbb{R}^3),$$

and

$$\int_{\mathbb{R}^3} |\nabla U_\epsilon|^2 \mathrm{d}x = \int_{\mathbb{R}^3} |U_\epsilon|^6 \mathrm{d}x = \mathcal{S}^{\frac{3}{2}}.$$

Let $\eta \in C_0^\infty(\mathbb{R}^3, [0,1])$ be such that $0 \leqslant \eta \leqslant 1$, if $|x| < 1$ and $\eta = 0$ if $|x| \geqslant 2$. Now, consider $v_\epsilon(x) = \eta U_\epsilon / \|\eta U_\epsilon\|_{L^6(\mathbb{R}^3)}$ then we have the following estimates, if $\epsilon > 0$ small enough:

$$\|\nabla v_\epsilon\|^2_{L^2(\mathbb{R}^3)} = \mathcal{S} + O(\epsilon^{\frac{1}{2}}), \tag{3.1}$$

$$\|v_\epsilon\|^s_{L^s(\mathbb{R}^3)} = \begin{cases} O(\epsilon^{\frac{s}{4}}), & s \in [2,3), \\ O(\epsilon^{\frac{s}{4}} \ln|\epsilon|), & s = 3, \\ O(\epsilon^{\frac{6-s}{4}}), & s \in (3,6). \end{cases} \tag{3.2}$$

By (F_3), we obtain

$$I_\infty(tv_\epsilon) \leqslant \frac{t^2}{2}\|v_\epsilon\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi^\varepsilon_{v_\epsilon}|v_\epsilon|^2 \mathrm{d}x - \frac{C_1 t^m}{m}\int_{\mathbb{R}^3} |v_\epsilon|^m \mathrm{d}x - \frac{t^6}{6} := J_\infty(t).$$

Note that $\lim\limits_{t\to+\infty} J_\infty(t) = -\infty$ and $J_\infty(t) > 0$ as $t > 0$ small enough. So $\sup\limits_{t\geqslant 0} J_\infty(t)$ is attained at some $t_\epsilon > 0$.

From

$$J'_\infty(t_\epsilon) = t_\epsilon\|v_\epsilon\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} + t_\epsilon^3\int_{\mathbb{R}^3} \phi^\varepsilon_{v_\epsilon}|v_\epsilon|^2 \mathrm{d}x - C_1 t_\epsilon^{m-1}\int_{\mathbb{R}^3} |v_\epsilon|^m \mathrm{d}x - t_\epsilon^5 = 0, \tag{3.3}$$

we have

$$t_\epsilon^5 \leqslant t_\epsilon\|v_\epsilon\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} + t_\epsilon^3\int_{\mathbb{R}^3} \phi^\varepsilon_{v_\epsilon}|v_\epsilon|^2 \mathrm{d}x,$$

which implies that t_ϵ is bounded from above by some $t^* > 0$. In view of (3.3), we get

$$\int_{\mathbb{R}^3} |\nabla v_\epsilon|^2 \mathrm{d}x \leqslant t_\epsilon^4 + C_1 t_\epsilon^{m-2} \int_{\mathbb{R}^3} |v_\epsilon|^m \mathrm{d}x.$$

Choosing $\epsilon > 0$ small enough, by (3.1), we obtain

$$t_\epsilon^4 \geqslant \frac{\mathcal{S}}{2}.$$

Thus, we have t_ϵ is bounded from above and below for $\epsilon > 0$ small enough.

Next, we estimate $J_\infty(t)$. Set

$$g(t) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla v_\epsilon|^2 \mathrm{d}x - \frac{t^6}{6}.$$

Then $g(t)$ attains its maximum at $\bar{t} = \left(\int_{\mathbb{R}^3} |\nabla v_\epsilon|^2 \mathrm{d}x \right)^{\frac{1}{4}}$. Consequently, by (3.2) and Lemma 2.3, there holds

$$\begin{aligned}
J_\infty(t_\epsilon) &= g(t_\epsilon) + \frac{t_\epsilon^2}{2} \int_{\mathbb{R}^3} V_\infty |v_\epsilon|^2 \mathrm{d}x + \frac{t_\epsilon^4}{4} \int_{\mathbb{R}^3} \phi_{v_\epsilon}^\varepsilon |v_\epsilon|^2 \mathrm{d}x - \frac{C_1 t_\epsilon^m}{m} \int_{\mathbb{R}^3} |v_\epsilon|^m \mathrm{d}x \\
&\leqslant g(\bar{t}) + \frac{t_\epsilon^2}{2} \int_{\mathbb{R}^3} V_\infty |v_\epsilon|^2 \mathrm{d}x + \frac{t_\epsilon^4}{4} \int_{\mathbb{R}^3} \phi_{v_\epsilon}^\varepsilon |v_\epsilon|^2 \mathrm{d}x - \frac{C_1 t_\epsilon^m}{m} \int_{\mathbb{R}^3} |v_\epsilon|^m \mathrm{d}x \\
&\leqslant \frac{1}{3} \mathcal{S}^{\frac{3}{2}} + O(\epsilon^{\frac{3}{4}}) + C_2 \|v_\epsilon\|_{L^2(\mathbb{R}^3)}^2 + C_3 \|v_\epsilon\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}^4 - C_4 \|v_\epsilon\|_{L^m(\mathbb{R}^3)}^m \\
&\leqslant \frac{1}{3} \mathcal{S}^{\frac{3}{2}} + O(\epsilon^{\frac{3}{4}}) + C_2 O(\epsilon^{\frac{1}{2}}) + C_3 O(\epsilon) - C_4 O(\epsilon^{\frac{6-m}{4}}) \\
&< \frac{1}{3} \mathcal{S}^{\frac{3}{2}},
\end{aligned} \tag{3.4}$$

for $\epsilon > 0$ small enough. Thus, $\max_{t \geqslant 0} I_\infty(t v_\epsilon) < \frac{1}{3} \mathcal{S}^{\frac{3}{2}}$ is obtained. The proof is completed. □

Lemma 3.4. *The following statement holds:*

$$\lim_{\lambda \to +\infty} \sup_{\varepsilon > 0} c_\infty = 0.$$

Proof. We need to prove that for every $\epsilon > 0$, there exists $\overline{\lambda} > 0$ such that

$$0 < \inf_{u \in H_{V_\infty}^1(\mathbb{R}^3) \setminus \{0\}} \max_{t > 0} I_\infty(tu) < \epsilon, \quad \forall \lambda > \overline{\lambda}.$$

Let $v \in C_0^\infty(\mathbb{R}^3)$, with $\|v\|_{H_{V_\infty}^1(\mathbb{R}^3)} = 1$. In view of Lemma 3.1, we know that there exists $t_v > 0$ such that $I_\infty(t_v v) = \max_{t \geqslant 0} I_\infty(tv)$ and $\lim_{\lambda \to +\infty} \sup_{\varepsilon > 0} t_v = 0$. By virtue of Lemmas 2.3 and 3.1 for $\lambda > \overline{\lambda}$, we have the following estimates:

$$0 < c_\infty \leqslant I_\infty(t_v v) \leqslant \frac{t_v^2}{2} \|v\|_{H_{V_\infty}^1(\mathbb{R}^3)}^2 + \frac{t_v^4}{4} \int_{\mathbb{R}^3} \phi_v^\varepsilon |v|^2 \mathrm{d}x \leqslant \frac{t_v^2}{2} + \frac{C t_v^4}{4} < \epsilon.$$

The proof is completed. □

To prove the compactness of the minimizing sequence for I_∞, we need the following result.

Lemma 3.5. *Let $\{u_n\} \subset \mathcal{N}_\infty$ be a minimizing sequence for I_∞. Then $\{u_n\}$ is bounded. Moreover, there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that*

$$\liminf_{n\to+\infty} \int_{B_r(y_n)} |u_n|^2 \mathrm{d}x \geqslant \delta > 0,$$

where $B_r(y_n) = \{y \in \mathbb{R}^3 \big| |y - y_n| \leqslant r\}$.

Proof. For any $\epsilon > 0$, it follows from (F_1), (F_3) that there exists $C_\epsilon > 0$ such that

$$|f(u)| \leqslant \epsilon|u|^3 + C_\epsilon|u|^{p-1} \qquad \text{and} \qquad |F(u)| \leqslant \frac{\epsilon}{4}|u|^4 + \frac{C_\epsilon}{p}|u|^p, \ \ \forall u \in H^1_{V_\infty}(\mathbb{R}^3). \tag{3.5}$$

In view of (F_2), one can see that

$$F(u) \geqslant 0 \qquad \text{and} \qquad 4F(u) - f(u)u \leqslant 0, \ \ \forall u \in H^1_{V_\infty}(\mathbb{R}^3). \tag{3.6}$$

By $\{u_n\} \subset \mathcal{N}_\infty$, we have

$$\begin{aligned} I_\infty(u_n) &= I_\infty(u_n) - \frac{1}{4}\langle I'_\infty(u_n), u\rangle \\ &= \frac{1}{4}\|u_n\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} + \frac{\lambda K_\infty}{4}\int_{\mathbb{R}^3}\Big[f(u_n)u_n - 4F(u_n)\Big]\mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x \\ &\geqslant \frac{1}{4}\|u_n\|^2_{H^1_{V_\infty}(\mathbb{R}^3)}. \end{aligned}$$

Hence, I_∞ is coercive on $\mathcal{N}_\infty$, i.e., $I_\infty(u) \to +\infty$ as $\|u\|_{H^1_{V_\infty}(\mathbb{R}^3)} \to +\infty$, for $u \in \mathcal{N}_\infty$. Thus, we can easily get the boundedness of $\{u_n\}$.

Next we prove the latter conclusion of this lemma. Arguing by contradiction, we assume

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_r(y)} |u_n|^2 \mathrm{d}x = 0,$$

then by Proposition 2.2, there holds $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $s \in (2, 6)$. Taking into account (3.5) and Lemma 2.3, we can deduce

$$\int_{\mathbb{R}^3} F(u_n)\mathrm{d}x \to 0, \qquad \int_{\mathbb{R}^3} f(u_n)u_n\mathrm{d}x \to 0 \qquad \text{and} \qquad \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^2\mathrm{d}x \to 0, \ \ \text{as } n \to +\infty. \tag{3.7}$$

So, combined $\langle I'_\infty(u_n), u_n\rangle = 0$ with (3.7), we have

$$\|u_n\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |u_n|^6\mathrm{d}x + o_n(1).$$

We assume $\|u_n\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} \to l \geqslant 0$. If $l > 0$, by $\{u_n\}$ is a minimizing sequence of I_∞ and (3.7), we get

$$\frac{1}{2}\|u_n\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} - \frac{1}{6}\int_{\mathbb{R}^3} |u_n|^6\mathrm{d}x \to c_\infty.$$

Thus, we obtain $c_\infty = \frac{1}{3}l$. On the other hand, by the definition of $\mathcal{S}$, we know that $l \geqslant \mathcal{S}l^{\frac{1}{3}}$. Namely, $l \geqslant \mathcal{S}^{\frac{3}{2}}$. So $c_\infty = \frac{1}{3}l \geqslant \frac{1}{3}\mathcal{S}^{\frac{3}{2}}$. This contradicts with Lemma 3.3. Hence $l = 0$. However, this contradicts with Lemma 3.1. The proof is completed. □

Now we are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\{\omega_n\} \subset S_{V_\infty}$ be a minimizing sequence of Υ_∞. By the Ekeland variational principle [23], we assume

$$\Upsilon_\infty(\omega_n) \to c_\infty \qquad \text{and} \qquad \Upsilon'_\infty(\omega_n) \to 0, \ \text{ as } n \to +\infty.$$

Set $u_n = m_\infty(\omega_n) \in \mathcal{N}_\infty$ for all $n \in \mathbb{N}^*$. Then

$$I_\infty(u_n) \to c_\infty \qquad \text{and} \qquad I'_\infty(u_n) \to 0, \ \text{ as } n \to +\infty.$$

By Lemma 3.5, we know that $\{u_n\}$ is bounded and there exist $r, \delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\lim_{n\to+\infty} \inf_{y\in\mathbb{R}^3} \int_{B_r(y_n)} |u_n|^2 \mathrm{d}x \geqslant \delta > 0.$$

So we can choose $r_1 > r > 0$ and a sequence $\{y_n^1\} \subset \mathbb{R}^3$ such that

$$\lim_{n\to+\infty} \inf_{y\in\mathbb{R}^3} \int_{B_{r_1}(y_n^1)} |u_n|^2 \mathrm{d}x \geqslant \frac{\delta}{2} > 0.$$

Since I_∞ and $\mathcal{N}_\infty$ are invariant under translations in our case, so we can assume $\{y_n\} \subset \mathbb{Z}^3$ is bounded. Moreover we assume, up to a subsequence, there exists $u_\infty \in H^1_{V_\infty}(\mathbb{R}^3)$ such that $u_n \rightharpoonup u_\infty$ and $u_n \to u_\infty$ a.e. in $\mathbb{R}^3$. Then the weak convergence of $\{u_n\}$ implies $I'_\infty(u_\infty) = 0$.

According to the Fatou lemma, we can obtain

$$\begin{aligned}
c_\infty &\leqslant I_\infty(u_\infty) \\
&= I_\infty(u_\infty) - \frac{1}{4}\langle I'_\infty(u_\infty), u_\infty\rangle \\
&= \frac{1}{4}\|u_\infty\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} + \frac{\lambda K_\infty}{4}\int_{\mathbb{R}^3}\Big[f(u_\infty)u_\infty - 4F(u_\infty)\Big]\mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3}|u_\infty|^6\mathrm{d}x \\
&\leqslant \liminf_{n\to+\infty}\left\{\frac{1}{4}\|u_n\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} + \frac{\lambda K_\infty}{4}\int_{\mathbb{R}^3}\Big[f(u_n)u_n - 4F(u_n)\Big]\mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x\right\} \\
&= \liminf_{n\to+\infty}\left[I_\infty(u_n) - \frac{1}{4}\langle I'_\infty(u_n), u_n\rangle\right] \\
&= c_\infty,
\end{aligned}$$

which implies $I_\infty(u_\infty) = c_\infty$. Next, we need to show the ground state solution u_∞ is positive. In fact, for $|u_\infty| \in H^1_{V_\infty}(\mathbb{R}^3)$, there exists $t_\infty > 0$ such that $t_\infty|u_\infty| \in \mathcal{N}_\infty$. From (F_1) and the form of I_∞, we can infer $I_\infty(t_\infty|u_\infty|) \leqslant I_\infty(t_\infty u_\infty)$. Furthermore, it follows from $u_\infty \in \mathcal{N}_\infty$ that $I_\infty(t_\infty u_\infty) \leqslant I_\infty(u_\infty)$. So, we obtain $I_\infty(t_\infty|u_\infty|) \leqslant I_\infty(u_\infty)$, which implies $t_\infty|u_\infty|$ is a nonnegative ground state solution. It follows from the Harnack inequality [24] that $t_\infty|u_\infty| > 0$, for all $x \in \mathbb{R}^3$. The proof is completed. □

4. Proof of Theorem 1.2

In this section, we investigate the existence of positive ground state solutions to system $(P_{\lambda,\varepsilon})$.

4.1. Mountain pass geometry and Nehari Manifold

Define the Nehari manifold of system $(P_{\lambda,\varepsilon})$ as follows:

$$\mathcal{N}_{\lambda,\varepsilon} = \left\{u \in H_V^1(\mathbb{R}^3)\backslash\{0\} \middle| \langle I'_{\lambda,\varepsilon}(u), u\rangle = 0\right\}.$$

We can conclude $\mathcal{N}_{\lambda,\varepsilon}$ has the following elementary properties without proof.

Lemma 4.1. *Suppose that all conditions described in Theorem 1.2 hold. Then the following statements are true.*

(i) The functional $I_{\lambda,\varepsilon}$ possesses the mountain pass geometry.

(ii) For every $u \in H_V^1(\mathbb{R}^3)\backslash\{0\}$ and a fixed $\varepsilon > 0$, there exists a unique $t_u > 0$ such that $I_{\lambda,\varepsilon}(t_u u) = \max_{t\geqslant 0} I_{\lambda,\varepsilon}(tu)$. Moreover, $tu \in \mathcal{N}_{\lambda,\varepsilon}$ if and only if $t = t_u$ and

$$\lim_{\lambda\to+\infty} t_u = 0.$$

(iii) $c_{\lambda,\varepsilon} = \bar{c}_{\lambda,\varepsilon} = \bar{\bar{c}}_{\lambda,\varepsilon} > 0$, where

$$c_{\lambda,\varepsilon} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_{\lambda,\varepsilon}(\gamma(t)), \qquad \bar{c}_{\lambda,\varepsilon} = \inf_{u\in\mathcal{N}_{\lambda,\varepsilon}} J_{\lambda,\varepsilon}(u) \qquad \text{and} \qquad \bar{\bar{c}}_{\lambda,\varepsilon} = \inf_{u\in H_V^1(\mathbb{R}^3)\backslash\{0\}}\max_{t>0} I_{\lambda,\varepsilon}(tu),$$

and $\Gamma = \{\gamma \in C([0,1], H_V^1(\mathbb{R}^3)) \big| \gamma(0) = 0, I_{\lambda,\varepsilon}(\gamma(1)) < 0\}$.

Proof. The proof is similar to Lemma 3.1, so we omit it for details. □

Similar to section 2, we define the mappings $\widehat{m}_{\lambda,\varepsilon} : H_V^1(\mathbb{R}^3)\backslash\{0\} \to \mathcal{N}_{\lambda,\varepsilon}$ by

$$\widehat{m}_{\lambda,\varepsilon} = t_u u \qquad \text{and} \qquad m_{\lambda,\varepsilon} = \widehat{m}_{\lambda,\varepsilon}|_S, \qquad S = \{u \in H_V^1(\mathbb{R}^3) \big| \|u\|_{H_V^1(\mathbb{R}^3)} = 1\}.$$

Moreover, the inverse of $m_{\lambda,\varepsilon}$ can be given by

$$m_{\lambda,\varepsilon}^{-1}(u) = \frac{u}{\|u\|_{H_V^1(\mathbb{R}^3)}}.$$

Considering the functionals $\widehat{\Upsilon}_{\lambda,\varepsilon} : H_V^1(\mathbb{R}^3)\backslash\{0\} \to \mathbb{R}$ and $\Upsilon_{\lambda,\varepsilon} : S \to \mathbb{R}$ given by

$$\widehat{\Upsilon}_{\lambda,\varepsilon} = I_{\lambda,\varepsilon}(\widehat{m}_{\lambda,\varepsilon}(u)) \qquad \text{and} \qquad \Upsilon_{\lambda,\varepsilon} = \widehat{\Upsilon}_{\lambda,\varepsilon}|_S.$$

Then we have the following lemma.

Lemma 4.2. ([21]) *Suppose that all conditions described in Lemma 4.1 hold. Then the following statements are true.*

(i) $\Upsilon_{\lambda,\varepsilon} \in C^1(S,\mathbb{R})$ and

$$\left\langle \Upsilon'_{\lambda,\varepsilon}(\omega), z\right\rangle = \|m_{\lambda,\varepsilon}(\omega)\|_{H_V^1(\mathbb{R}^3)} \left\langle I'_{\lambda,\varepsilon}(m_{\lambda,\varepsilon}(\omega)), z\right\rangle,$$

for all $z \in T_\omega(S) := \{v \in H_V^1(\mathbb{R}^3) \big| \langle \omega, v\rangle = 0\}$.

(ii) $\{\omega_n\}$ is a (PS) sequence for $\Upsilon_{\lambda,\varepsilon}$, if and only if $\{m_{\lambda,\varepsilon}(\omega_n)\}$ is a (PS) sequence for $I_{\lambda,\varepsilon}$. If $\{u_n\} \subset \mathcal{N}_{\lambda,\varepsilon}$ is a bounded (PS) sequence for $I_{\lambda,\varepsilon}$, then $\{m_{\lambda,\varepsilon}^{-1}(u_n)\}$ is a (PS) sequence for $\Upsilon_{\lambda,\varepsilon}$.

(iii) $\omega \in S$ is a critical point of $\Upsilon_{\lambda,\varepsilon}$, if and only if $m_{\lambda,\varepsilon}(\omega)$ is a critical point of $I_{\lambda,\varepsilon}$. Moreover, the corresponding values of $I_{\lambda,\varepsilon}$ and $\Upsilon_{\lambda,\varepsilon}$ coincide and

$$\inf_{u\in\mathcal{N}_{\lambda,\varepsilon}} I_{\lambda,\varepsilon}(u) = \inf_{\omega\in S} \Upsilon_{\lambda,\varepsilon}(\omega) = c_{\lambda,\varepsilon}.$$

In order to prove that the minimizer of $I_{\lambda,\varepsilon}$ constrained on $\mathcal{N}_{\lambda,\varepsilon}$ is a critical point of $I_{\lambda,\varepsilon}$, we need the following lemmas.

4.2. The behaviors of $(PS)_c$ sequence

In this subsection, we study the behaviors of $(PS)_c$ sequence, which play key roles in the proof of Theorem 1.2.

Lemma 4.3. *If $u_n \rightharpoonup u$ in $H^1_V(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then*

$$\lim_{n\to+\infty}\left[\int_{\mathbb{R}^3}\phi^{\varepsilon}_{u_n}|u_n|^2\mathrm{d}x-\int_{\mathbb{R}^3}\phi^{\varepsilon}_{u_n-u}|u_n-u|^2\mathrm{d}x\right]\to\int_{\mathbb{R}^3}\phi^{\varepsilon}_{u}|u|^2\mathrm{d}x.$$

Proof. Since $\mathcal{K}\in L^{\tau}(\mathbb{R}^3)$ for $\tau\in(3,+\infty]$. As a result of $\{u_n\}$ is bounded in $H^1_V(\mathbb{R}^3)$ and converges almost everywhere to u, the sequence $\{|u_n-u|^2\}$ converges weakly to 0 in $L^{\frac{8}{7}}(\mathbb{R}^3)$ and by the Brézis-Lieb lemma [25], the sequence $\{|u_n|^2-|u_n-u|^2\}$ converges strongly to the function $|u|^2$ in $L^{\frac{8}{7}}(\mathbb{R}^3)$. Putting together Lemma 2.5 with the definition of ϕ^{ε}_u and letting $n\to+\infty$, we get

$$\begin{aligned}&\lim_{n\to+\infty}\int_{\mathbb{R}^3}\left|\phi^{\varepsilon}_{u_n}-\phi^{\varepsilon}_{u_n-u}-\phi^{\varepsilon}_{u}\right|^8\mathrm{d}x\\&\leqslant\left[\int_{\mathbb{R}^3}|\mathcal{K}|^4\mathrm{d}x\right]^2\left[\int_{\mathbb{R}^3}(|u_n|^2-|u_n-u|^2-|u|^2)^{\frac{8}{7}}\mathrm{d}x\right]^7\\&\to 0.\end{aligned}$$

Therefore, we can deduce

$$\begin{aligned}&\lim_{n\to+\infty}\left[\int_{\mathbb{R}^3}\phi^{\varepsilon}_{u_n}|u_n|^2\mathrm{d}x-\int_{\mathbb{R}^3}\phi^{\varepsilon}_{u_n-u}|u_n-u|^2\mathrm{d}x\right]\\&=\lim_{n\to+\infty}\int_{\mathbb{R}^3}(\phi^{\varepsilon}_{u_n}-\phi^{\varepsilon}_{u_n-u})\left[(|u_n|^2-|u_n-u|^2)+2|u_n-u|^2\right]\mathrm{d}x\\&=\int_{\mathbb{R}^3}\phi^{\varepsilon}_{u}|u|^2\mathrm{d}x.\end{aligned}$$

The proof is completed. □

Lemma 4.4. *If $u_n \rightharpoonup u$ in $H^1_V(\mathbb{R}^3)$ and $u_n \to u$ a.e. in $\mathbb{R}^3$, then*

$$\lim_{n\to+\infty}\left[\int_{\mathbb{R}^3}F(u_n)\mathrm{d}x-\int_{\mathbb{R}^3}F(u_n-u)\mathrm{d}x\right]\to\int_{\mathbb{R}^3}F(u)\mathrm{d}x.$$

Proof. The proof is similar to [26, Lemma 3.2], so we omit it here. □

Lemma 4.5. *Let $\{u_n\}\subset H^1_V(\mathbb{R}^3)$ be a $(PS)_c$ sequence of $I_{\lambda,\varepsilon}$ with $0<c\leqslant c_\infty$. If $u_n\rightharpoonup 0$ in $H^1_V(\mathbb{R}^3)$, then one of the following statements is true.*

(i) $u_n\to 0$ in $H^1_V(\mathbb{R}^3)$.

(ii) There exist a sequence $\{y_n\}\subset\mathbb{R}^3$ and constants $r,\delta>0$ such that

$$\liminf_{n\to+\infty}\int_{B_r(y_n)}|u_n|^2\mathrm{d}x\geqslant\delta>0.$$

Proof. Suppose that (*ii*) does not occur, then there exists $r > 0$ such that

$$\lim_{n\to+\infty}\sup_{y\in\mathbb{R}^3}\int_{B_r(y)}|u_n|^2\mathrm{d}x = 0.$$

In view of Proposition 2.2, we get $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $s \in (2, 6)$. So from (3.7) and $\langle I'_{\lambda,\varepsilon}(u_n), u_n\rangle = 0$, it follows that

$$\|u_n\|^2_{H^1_V(\mathbb{R}^3)} = \int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x.$$

Assume that $\|u_n\|^2_{H^1_V(\mathbb{R}^3)} \to l \geqslant 0$. So, we get $c = \frac{1}{3}l$. Moreover, we have

$$\|u_n\|^2_{H^1_V(\mathbb{R}^3)} \geqslant \int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x \geqslant \mathcal{S}\Big(\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x\Big)^{\frac{1}{3}}.$$

Taking the limit as $n \to +\infty$ in the above inequality, we obtain

$$c \geqslant \frac{1}{3}\mathcal{S}^{\frac{3}{2}},$$

which contradicts with our assumption. Thus, $l = 0$. The proof is completed. □

Lemma 4.6. *Suppose the all conditions described in Theorem 1.2 hold. Let $\{u_n\} \subset H^1_V(\mathbb{R}^3)$ be a $(PS)_c$ sequence of $I_{\lambda,\varepsilon}$ with $0 < c \leqslant c_{\lambda,\varepsilon} < c_\infty$. If $u_n \rightharpoonup 0$ in $H^1_V(\mathbb{R}^3)$, then $u_n \to 0$ in $H^1_V(\mathbb{R}^3)$.*

Proof. It is easy to see that $\{u_n\}$ is bounded in $H^1_V(\mathbb{R}^3)$. Therefore, up to a subsequence, we have

$$u_n \rightharpoonup 0 \text{ in } H^1_V(\mathbb{R}^3),\quad u_n \to 0 \text{ in } L^s_{loc}(\mathbb{R}^3) \text{ for } 2 \leqslant s < 6,\quad u_n \to 0 \ a.e. \text{ on } \mathbb{R}^3.$$

Next, we use Proposition 2.1 to prove $u_n \to 0$ in $H^1_V(\mathbb{R}^3)$. For this purpose, we set

$$\rho_n(x) = \frac{1}{4}|(-\Delta)^{\frac{1}{2}}u_n|^2 + \frac{1}{4}V(x)|u_n|^2 + \frac{\lambda}{4}K(x)\Big[f(u_n)u_n - 4F(u_n)\Big] + \frac{1}{12}|u_n|^6.$$

Clearly, one has $\{\rho_n\} \subset L^1(\mathbb{R}^3)$. Thus, passing to a subsequence, we assume that $\Phi(u_n) := \|\rho_n\|_{L^1(\mathbb{R}^3)} \to l$ as $n \to +\infty$. Using the fact that $\Phi(u_n) = I_{\lambda,\varepsilon}(u_n) - \frac{1}{4}\langle I'_{\lambda,\varepsilon}(u_n), u_n\rangle = l$, we get $l = c > 0$. We next claim that neither vanishing nor dichotomy occurs.

Claim 1. Vanishing does not occur.

If $\{\rho_n\}$ vanishing, then $\{u_n^2\}$ also vanishing, i.e., there exists $R > 0$ such that

$$\lim_{n\to+\infty}\sup_{y\in\mathbb{R}^3}\int_{B_R(y)}|u_n|^2\mathrm{d}x = 0.$$

As in the proof of Lemma 4.5, we can prove vanishing does not happen.

Claim 2. Dichotomy does not occur.

Otherwise, there exist $\beta \in (0, l)$ and $\{y_n\} \subset \mathbb{R}^3$ such that for every $\epsilon_n > 0$, we can choose $\{R_n\} \subset \mathbb{R}^+(R_n > \widetilde{R} + R_0/\epsilon$, for any fixed $\epsilon > 0$, $\widetilde{R}, R_0$ are positive constants defined later) with $R_n \to +\infty$ satisfying

$$\limsup_{n\to+\infty}\left(\Big|\beta - \int_{B_{R_n(y_n)}}\rho_n(x)\mathrm{d}x\Big| + \Big|(l-\beta) - \int_{B^c_{2R_n(y_n)}}\rho_n(x)\mathrm{d}x\Big|\right) < \epsilon_n. \tag{4.1}$$

Consider a smooth cut-off function $\psi : [0, +\infty) \to \mathbb{R}^+$ such that

$$\begin{cases} \psi(x) = 1, & x \in B_{R_n}(y_n), \\ 0 \leqslant \psi(x) \leqslant 1, & x \in B_{2R_n}(y_n) \backslash B_{R_n}(y_n), \\ \psi(x) = 0, & x \in B^c_{2R_n}(y_n), \\ |\psi'|_{L^\infty(\mathbb{R}^3)} \leqslant 2. \end{cases}$$

Set

$$u_n = \psi u_n + (1 - \psi)u_n =: \theta_n + \omega_n.$$

Then, one can infer

$$\liminf_{n\to+\infty} \Phi(\theta_n) \geqslant \int_{B_{R_n}(y_n)} \rho_n(x)\mathrm{d}x \to \beta, \tag{4.2}$$

and

$$\liminf_{n\to+\infty} \Phi(\omega_n) \geqslant \int_{B^c_{2R_n}(y_n)} \rho_n(x)\mathrm{d}x \to l - \beta. \tag{4.3}$$

Let $\Omega_n = B_{2R_n}(y_n)\backslash B_{R_n}(y_n)$. Taking the limit as $n \to +\infty$, then we have

$$\int_{\Omega_n} \rho_n(x)\mathrm{d}x = \int_{\mathbb{R}^3} \rho_n(x)\mathrm{d}x - \int_{B_{R_n}(y_n)} \rho_n(x)\mathrm{d}x - \int_{B^c_{2R_n}(y_n)} \rho_n(x)\mathrm{d}x \to 0. \tag{4.4}$$

By (4.4), we can deduce

$$\int_{\Omega_n} (|\nabla u_n|^2 + V(x)|u_n|^2)\mathrm{d}x \to 0 \ \text{ and } \ \int_{\Omega_n} |u_n|^6\mathrm{d}x \to 0. \tag{4.5}$$

According to Lemma 2.3, we get

$$\int_{\Omega_n} \phi^\varepsilon_{u_n}|u_n|^2\mathrm{d}x \to 0. \tag{4.6}$$

Putting (3.5), (4.5), (4.6) together with the definition of θ_n, ω_n, we can easily get

$$\|u_n\|^2_{H^1_V(\mathbb{R}^3)} = \|\theta_n\|^2_{H^1_V(\mathbb{R}^3)} + \|\omega_n\|^2_{H^1_V(\mathbb{R}^3)} + o_n(1), \tag{4.7}$$

$$\int_{\mathbb{R}^3} K(x)F(u_n)\mathrm{d}x = \int_{\mathbb{R}^3} K(x)F(\theta_n)\mathrm{d}x + \int_{\mathbb{R}^3} K(x)F(\omega_n)\mathrm{d}x + o_n(1), \tag{4.8}$$

$$\int_{\mathbb{R}^3} K(x)f(u_n)u_n\mathrm{d}x = \int_{\mathbb{R}^3} K(x)f(\theta_n)\theta_n\mathrm{d}x + \int_{\mathbb{R}^3} K(x)f(\omega_n)\omega_n\mathrm{d}x + o_n(1), \tag{4.9}$$

$$\int_{\mathbb{R}^3} |u_n|^6\mathrm{d}x = \int_{\mathbb{R}^3} |\theta_n|^6\mathrm{d}x + \int_{\mathbb{R}^3} |\omega_n|^6\mathrm{d}x + o_n(1), \tag{4.10}$$

$$\int_{\mathbb{R}^3} \phi^\varepsilon_{u_n}|u_n|^2\mathrm{d}x = \int_{\mathbb{R}^3} \phi^\varepsilon_{\theta_n}|\theta_n|^2\mathrm{d}x + \int_{\mathbb{R}^3} \phi^\varepsilon_{\omega_n}|\omega_n|^2\mathrm{d}x + o_n(1). \tag{4.11}$$

Taking into account (4.7)–(4.11), we get

$$\Phi(u_n) = \Phi(\theta_n) + \Phi(\omega_n) + o_n(1).$$

Combining (4.2) and (4.3), we have

$$l = \lim_{n\to+\infty} \Phi(u_n) = \liminf_{n\to+\infty} \Phi(\theta_n) + \liminf_{n\to+\infty} \Phi(\omega_n) \geqslant \beta + (l-\beta) = l.$$

Therefore, we obtain

$$\liminf_{n\to+\infty} \Phi(\theta_n) = \beta \qquad \text{and} \qquad \liminf_{n\to+\infty} \Phi(\omega_n) = l - \beta. \tag{4.12}$$

Moreover, from (4.7) to (4.11), we get

$$o_n(1) = \left\langle I'_{\lambda,\varepsilon}(u_n), u_n \right\rangle = \left\langle I'_{\lambda,\varepsilon}(\theta_n), \theta_n \right\rangle + \left\langle I'_{\lambda,\varepsilon}(\omega_n), \omega_n \right\rangle + o_n(1). \tag{4.13}$$

In order to finish our proof, it suffices to show (4.13) is not true. We separate the following discussion into three possibilities and show each leads to a contradiction.

Case 1. After passing to a subsequence, we assume $\langle I'_{\lambda,\varepsilon}(\theta_n), \theta_n \rangle \leqslant 0$, then

$$\|\theta_n\|^2_{H^1_V(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \phi^{\varepsilon}_{\theta_n} |\theta_n|^2 \mathrm{d}x - \lambda \int_{\mathbb{R}^3} K(x) f(\theta_n)\theta_n \mathrm{d}x - \int_{\mathbb{R}^3} |\theta_n|^6 \mathrm{d}x \leqslant 0. \tag{4.14}$$

By Lemma 4.1, we know that there exists $t_{\theta_n} > 0$ such that $t_{\theta_n}\theta_n \in \mathcal{N}_{\lambda,\varepsilon}$. Then

$$t^2_{\theta_n} \|\theta_n\|^2_{H^1_V(\mathbb{R}^3)} + t^4_{\theta_n} \int_{\mathbb{R}^3} \phi^{\varepsilon}_{\theta_n} |\theta_n|^2 \mathrm{d}x = \lambda \int_{\mathbb{R}^3} K(x) f(t_{\theta_n}\theta_n) t_{\theta_n}\theta_n \mathrm{d}x + t^6_{\theta_n} \int_{\mathbb{R}^3} |\theta_n|^6 \mathrm{d}x. \tag{4.15}$$

Combined (4.14) with (4.15), one has

$$\left(\frac{1}{t^2_{\theta_n}} - 1\right) \|\theta_n\|^2_{H^1_V(\mathbb{R}^3)} - \lambda \int_{\mathbb{R}^3} K(x) \left[\frac{f(t_{\theta_n}\theta_n)}{(t_{\theta_n}\theta_n)^3} - \frac{f(\theta_n)}{(\theta_n)^3}\right] |\theta_n|^4 \mathrm{d}x - (t^2_{\theta_n} - 1) \int_{\mathbb{R}^3} |\theta_n|^6 \mathrm{d}x \geqslant 0,$$

which implies $t_{\theta_n} \leqslant 1$. From $t_{\theta_n}\theta_n \in \mathcal{N}_{\lambda,\varepsilon}$ and (4.12), we deduce

$$\begin{aligned} c_{\lambda,\varepsilon} \leqslant & I_{\lambda,\varepsilon}(t_{\theta_n}\theta_n) = I_{\lambda,\varepsilon}(t_{\theta_n}\theta_n) - \frac{1}{4}\langle I'_{\lambda,\varepsilon}(t_{\theta_n}\theta_n), t_{\theta_n}\theta_n \rangle \\ = & \frac{t^2_{\theta_n}}{4} \|\theta_n\|^2_{H^1_V(\mathbb{R}^3)} + \frac{\lambda}{4} \int_{\mathbb{R}^3} K(x) \big[f(t_{\theta_n}\theta_n) t_{\theta_n}\theta_n - 4F(t_{\theta_n}\theta_n)\big] \mathrm{d}x + \frac{t^6_{\theta_n}}{12} \int_{\mathbb{R}^3} |\theta_n|^6 \mathrm{d}x \\ \leqslant & \Phi(\theta_n) \to \beta < l = c, \end{aligned}$$

which leads to a contradiction.

Case 2. After passing to a subsequence, we assume $\langle I'_{\lambda,\varepsilon}(\omega_n), \omega_n \rangle \leqslant 0$. This case will lead to a contradiction again as in Case 1.

Case 3. After passing to a subsequence, we assume $\langle I'_{\lambda,\varepsilon}(\theta_n), \theta_n \rangle > 0$ and $\langle I'_{\lambda,\varepsilon}(\omega_n), \omega_n \rangle > 0$. In view of (4.13), we get $\langle I'_{\lambda,\varepsilon}(\theta_n), \theta_n \rangle = o_n(1)$ and $\langle I'_{\lambda,\varepsilon}(\omega_n), \omega_n \rangle = o_n(1)$. Moreover, from (4.7) to (4.11), one has

$$I_{\lambda,\varepsilon}(u_n) = I_{\lambda,\varepsilon}(\theta_n) + I_{\lambda,\varepsilon}(\omega_n) + o_n(1). \tag{4.16}$$

If the sequence $\{y_n\} \subset \mathbb{R}^3$ is bounded, then by conditions (V_1) and (K), we have for every $\epsilon > 0$, there exists $R_0 > 0$, such that

$$V(x) - V_\infty > -\epsilon \qquad \text{and} \qquad |K(x) - K_\infty| \leqslant \epsilon, \ \ \forall\ |x| > R_0/\epsilon. \tag{4.17}$$

By the boundedness of $\{y_n\} \subset \mathbb{R}^3$, there exists $\widetilde{R} > 0$ such that $|y_n| \leqslant \widetilde{R}$. Therefore, we have $\mathbb{R}^3 \backslash B_{R_n}(y_n) \subset \mathbb{R}^3 \backslash B_{R_n - \widetilde{R}}(0) \subset \mathbb{R}^3 \backslash B_{R_0/\epsilon}(0)$ for $n > 0$ large enough. According to (4.17), it follows that

$$\begin{aligned}\int_{\mathbb{R}^3} (V(x) - V_\infty)\, |\omega_n|^2 \mathrm{d}x &= \int_{|x-y_n|>R_n} (V(x) - V_\infty)\, |\omega_n|^2 \mathrm{d}x \\ &> -\epsilon \int_{|x-y_n|>R_n} |\omega_n|^2 \mathrm{d}x \\ &\geqslant -C\epsilon,\end{aligned}$$

which implies

$$\int_{\mathbb{R}^3} (V(x) - V_\infty)\, |\omega_n|^2 \mathrm{d}x \geqslant o_n(1). \tag{4.18}$$

Similarly, it is easy to check

$$\int_{\mathbb{R}^3} (K(x) - K_\infty)\, F(\omega_n) \mathrm{d}x = o_n(1) \qquad \text{and} \qquad \int_{\mathbb{R}^3} (K(x) - K_\infty)\, f(\omega_n)\omega_n \mathrm{d}x = o_n(1). \tag{4.19}$$

Combined (4.18) with (4.19), there holds

$$I_{\lambda,\varepsilon}(\omega_n) \geqslant I_\infty(\omega_n) + o_n(1) \qquad \text{and} \qquad o_n(1) = \langle I'_{\lambda,\varepsilon}(\omega_n), \omega_n\rangle \geqslant \langle I'_\infty(\omega_n), \omega_n\rangle + o(1). \tag{4.20}$$

By the latter conclusion of (4.20), one has $\langle I'_\infty(\omega_n), \omega_n\rangle \leqslant 0$, as $n \to +\infty$. Similar to the proof in Case 1, there exists $t_{\omega_n} \leqslant 1$ such that $t_{\omega_n}\omega_n \in \mathcal{N}_\infty$. Then, we can derive from (4.19) and (4.20) that

$$\begin{aligned} c_\infty \leqslant& I_\infty(t_{\omega_n}\omega_n) = I_\infty(t_{\omega_n}\omega_n) - \frac{1}{4}\langle I'_\infty(t_{\omega_n}\omega_n), t_{\omega_n}\omega_n\rangle \\ =& \frac{t_{\omega_n}^2}{4}\|\omega_n\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} + \frac{\lambda}{4}\int_{\mathbb{R}^3} K_\infty(x)\Big[f(t_{\omega_n}\omega_n)t_{\omega_n}\omega_n - 4F(t_{\omega_n}\omega_n)\Big]\mathrm{d}x + \frac{t_{\omega_n}^6}{12}\int_{\mathbb{R}^3} |\omega_n|^6 \mathrm{d}x \\ \leqslant& \frac{1}{4}\|\omega_n\|^2_{H^1_V(\mathbb{R}^3)} + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\Big[f(\omega_n)\omega_n - 4F(\omega_n)\Big]\mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3} |\omega_n|^6 \mathrm{d}x \\ =& \Phi(\omega_n) \to l - \beta = c - \beta < c, \end{aligned}$$

which leads to a contradiction.

If $\{y_n\} \subset \mathbb{R}^3$ is unbounded, we choose a subsequence, stilled denoted by $\{y_n\}$, such that $|y_n| \geqslant 3R_n$. Then $B_{2R_n}(y_n) \subset \mathbb{R}^3 \backslash B_{R_n}(0) \subset \mathbb{R}^3 \backslash B_{R_0/\epsilon}(0)$. Using the fact of (4.17) and a similar proof of (4.18) and (4.19), one has

$$\int_{\mathbb{R}^3} (V(x) - V_\infty)\, |\theta_n|^2 \mathrm{d}x \geqslant o_n(1),$$

and

$$\int_{\mathbb{R}^3} (K(x) - K_\infty)\, F(\theta_n) \mathrm{d}x = o_n(1) \qquad \text{and} \qquad \int_{\mathbb{R}^3} (K(x) - K_\infty)\, f(\theta_n)\theta_n \mathrm{d}x = o_n(1).$$

Similar to the case $\{y_n\}$ is bounded, we can obtain a contradiction by comparing $I_{\lambda,\varepsilon}(\theta_n)$ and c_∞. Therefore, dichotomy does not occur.

According to the above arguments, by Proposition 2.1, we know that $\{\rho_n\}$ must be compactness; i.e, there exists $\{y_n\} \subset \mathbb{R}^3$ such that for every $\epsilon > 0$, there exists $\widehat{R} > 0$ such that

$$\int_{\mathbb{R}^3 \backslash B_{\widehat{R}}(y_n)} \rho_n(x) \mathrm{d}x < \epsilon.$$

From the Hölder inequality, we obtain

$$\int_{\mathbb{R}^3\backslash B_{\widehat{R}}(y_n)}|u_n|^m\mathrm{d}x \leqslant \left(\int_{\mathbb{R}^3\backslash B_{\widehat{R}}(y_n)}|u_n|^2\mathrm{d}x\right)^{\frac{m\alpha}{2}}\left(\int_{\mathbb{R}^3\backslash B_{\widehat{R}}(y_n)}|u_n|^6\mathrm{d}x\right)^{\frac{m(1-\alpha)}{6}} < C\epsilon, \tag{4.21}$$

where $m \in [2, 6]$, $\alpha \in [0, 1]$ and satisfies $\frac{1}{m} = \frac{\alpha}{2} + \frac{1-\alpha}{6}$. By (4.21), we conclude $\{u_n^m\}$ is also compactness with $m \in [2, 6]$.

Next we prove the sequence $\{y_n\}$ is bounded. Otherwise, up to a subsequence, we can choose $\{R_n\} \subset \mathbb{R}^+$ with $R_n \to +\infty$ satisfying $|y_n| \geqslant R_n \geqslant \widehat{R} + R_0/\epsilon$. Then we have $B_{\widehat{R}}(y_n) \subset \mathbb{R}^3\backslash B_{R_n-\widehat{R}}(0) \subset \mathbb{R}^3\backslash B_{R_0/\epsilon}(0)$. In view of (4.21), there holds

$$\int_{\mathbb{R}^3}(V(x) - V_\infty)|u_n|^2\mathrm{d}x = \int_{B_{\widehat{R}}(y_n)}(V(x) - V_\infty)|u_n|^2\mathrm{d}x + \int_{\mathbb{R}^3\backslash B_{\widehat{R}}(y_n)}(V(x) - V_\infty)|u_n|^2\mathrm{d}x \geqslant o_n(1). \tag{4.22}$$

Similarly, we get

$$\int_{\mathbb{R}^3}(K(x) - K_\infty)F(u_n)\mathrm{d}x = o_n(1) \quad \text{and} \quad \int_{\mathbb{R}^3}(K(x) - K_\infty)f(u_n)u_n\mathrm{d}x = o_n(1). \tag{4.23}$$

It follows from (4.22) and (4.23) that

$$I_{\lambda,\varepsilon}(u_n) \geqslant I_\infty(u_n) + o_n(1) \quad \text{and} \quad o_n(1) = \langle I'_{\lambda,\varepsilon}(u_n), u_n\rangle \geqslant \langle I'_\infty(u_n), u_n\rangle + o_n(1). \tag{4.24}$$

By the latter conclusion of (4.24), one can see $\langle I'_\infty(u_n), u_n\rangle \leqslant 0$, as $n \to +\infty$. Similar to the proof of Case 1, there exists $t_{u_n} \leqslant 1$ such that $t_{u_n}u_n \in \mathcal{N}_\infty$. It follows from (4.23) and (4.24) that

$$\begin{aligned}
c_\infty \leqslant& I_\infty(t_{u_n}u_n) = I_\infty(t_{u_n}u_n) - \frac{1}{4}\langle I'_\infty(t_{u_n}u_n), t_{u_n}u_n\rangle \\
=&\frac{t_{u_n}^2}{4}\|u_n\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} + \frac{\lambda}{4}\int_{\mathbb{R}^3}K_\infty(x)\big[f(t_{u_n}u_n)t_{u_n}u_n - 4F(t_{u_n}u_n)\big]\mathrm{d}x + \frac{t_{u_n}^6}{12}\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x \\
\leqslant&\frac{1}{4}\|u_n\|^2_{H^1_V(\mathbb{R}^3)} + \frac{\lambda}{4}\int_{\mathbb{R}^3}K(x)\big[f(u_n)u_n - 4F(u_n)\big]\mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3}|u_n|^6\mathrm{d}x + o_n(1) \\
=&I_{\lambda,\varepsilon}(u_n) - \frac{1}{4}\langle I'_{\lambda,\varepsilon}(u_n), u_n\rangle + o_n(1) \to c,
\end{aligned}$$

which leads to a contradiction. Hence, $\{y_n\}$ is bounded in $\mathbb{R}^3$.

In view of the boundedness of $\{y_n\}$ and $u_n \to u$ in $L^s_{loc}(\mathbb{R}^3)$ for $2 \leqslant s < 6$, by (4.21) it is easy to check $u_n \to 0$ in $L^s(\mathbb{R}^3)$ for $s \in [2, 6)$. Thus, we can derive from Lemma 4.5 that $u_n \to 0$ in $H^1_V(\mathbb{R}^3)$. The proof is completed. □

Now, we state the proof of Theorem 1.2.

Proof of Theorem 1.2. We divide this proof into five steps.

Step 1. Making use of the Ekeland variational principle [23], there exists a sequence $\{\omega_n\} \subset S$ such that

$$\Upsilon_{\lambda,\varepsilon}(\omega_n) \to c_{\lambda,\varepsilon} \quad \text{and} \quad \Upsilon'_{\lambda,\varepsilon}(\omega_n) \to 0, \ \text{as } n \to +\infty.$$

Set $v_n = m_{\lambda,\varepsilon}(\omega_n)$, we have $v_n \in \mathcal{N}_{\lambda,\varepsilon}$ for all $n \in \mathbb{N}^*$. By Lemma 4.2, we can deduce

$$I_{\lambda,\varepsilon}(v_n) \to c_{\lambda,\varepsilon} \qquad \text{and} \qquad I'_{\lambda,\varepsilon}(v_n) \to 0, \ \text{ as } n \to +\infty.$$

By $\{v_n\}$ is bounded in $H^1_V(\mathbb{R}^3)$, there exists $v \in H^1_V(\mathbb{R}^3)$ such that $v_n \rightharpoonup v$ in $H^1_V(\mathbb{R}^3)$. From Lemma 2.3, by a standard argument, we know that v is a critical point of $I_{\lambda,\varepsilon}$ and $I'_{\lambda,\varepsilon}(v_n) \to I'_{\lambda,\varepsilon}(v) = 0$. Set $u_n = v_n - v$, then $u_n \rightharpoonup 0$ in $H^1_V(\mathbb{R}^3)$. Making use of Lemmas 4.3-4.4 and the Brézis-Lieb lemma [25], it is easy to check

$$I_{\lambda,\varepsilon}(u_n) = I_{\lambda,\varepsilon}(v_n) - I_{\lambda,\varepsilon}(v) + o_n(1), \ \text{ as } n \to +\infty.$$

It follows from $I'_{\lambda,\varepsilon}(v) = 0$ and (3.6) that

$$I_{\lambda,\varepsilon}(v) = \frac{1}{4}\|v\|^2_{H^1_V(\mathbb{R}^3)} + \frac{\lambda}{4}\int_{\mathbb{R}^3} K(x)\Big[f(v)v - 4F(v)\Big]\mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3} |v|^6 \mathrm{d}x \geqslant 0.$$

Thus, we have

$$I_{\lambda,\varepsilon}(u_n) = I_{\lambda,\varepsilon}(v_n) - I_{\lambda,\varepsilon}(v) + o_n(1) \to c_{\lambda,\varepsilon} - d, \ \text{ as } n \to +\infty,$$

where $d := I_{\lambda,\varepsilon}(v) \geqslant 0$.

For any $\varphi \in H^1_V(\mathbb{R}^3)$, according to $u_n \rightharpoonup 0$ in $H^1_V(\mathbb{R}^3)$, one has

$$\langle I'_{\lambda,\varepsilon}(u_n), \varphi\rangle = \langle I'_{\lambda,\varepsilon}(0), \varphi\rangle = 0, \ \text{ as } n \to +\infty.$$

Hence, we know that $\{u_n\}$ is a $(PS)_{c_{\lambda,\varepsilon}-d}$ sequence of $I_{\lambda,\varepsilon}$. In view of $I'_{\lambda,\varepsilon}(u_n) = 0$, it is easy to obtain $d \in [0, c_{\lambda,\varepsilon}]$.

Step 2. In this step, we show $c_{\lambda,\varepsilon} < c_\infty$. Denote by u_∞ be a positive ground state solution of system (P_∞). Then, we have $I_\infty(u_\infty) = c_\infty$. Moreover, by Lemma 4.1, we know that there exists $t_\infty > 0$ such that $t_\infty u_\infty \in \mathcal{N}_{\lambda,\varepsilon}$. We next claim $t_\infty < 1$.

Since $u_\infty \in \mathcal{N}_\infty$, then we have

$$\int_{\mathbb{R}^3} \left(|\nabla u_\infty|^2 + V_\infty |u_\infty|^2\right)\mathrm{d}x + \int_{\mathbb{R}^3} \phi^\varepsilon_{u_\infty}|u_\infty|^2\mathrm{d}x = \lambda\int_{\mathbb{R}^3} K_\infty f(u_\infty)u_\infty \mathrm{d}x + \int_{\mathbb{R}^3} |u_\infty|^6\mathrm{d}x \tag{4.25}$$

Furthermore, it follows from $t_\infty u_\infty \in \mathcal{N}_{\lambda,\varepsilon}$ that

$$\begin{aligned} &\frac{1}{t^2_\infty}\int_{\mathbb{R}^3} \left(|\nabla u_\infty|^2 + V(x)|u_\infty|^2\right)\mathrm{d}x + \int_{\mathbb{R}^3} \phi^\varepsilon_{u_\infty}|u_\infty|^2\mathrm{d}x \\ =&\lambda\int_{\mathbb{R}^3} K(x)\left[\frac{f(t_\infty u_\infty)}{(t_\infty u_\infty)^3}\right]|u_\infty|^4\mathrm{d}x + t^2_\infty\int_{\mathbb{R}^3} |u_\infty|^6\mathrm{d}x. \end{aligned} \tag{4.26}$$

Comparing (4.25) and (4.26), it is easy to get $t_\infty < 1$. Moreover, we have

$$\begin{aligned} I_{\lambda,\varepsilon}(t_\infty u_\infty) =& I_\infty(t_\infty u_\infty) + \frac{t^2_\infty}{2}\int_{\mathbb{R}^3} (V(x) - V_\infty)\,|u_\infty|^2\mathrm{d}x \\ &+ \lambda\int_{\mathbb{R}^3} (K_\infty - K(x))\,F(t_\infty u_\infty)\mathrm{d}x. \end{aligned}$$

Taking into account (V_1), (K) and (F_1), there holds

$$I_{\lambda,\varepsilon}(t_\infty u_\infty) < I_\infty(t_\infty u_\infty).$$

So in general, we get

$$c_{\lambda,\varepsilon} \leqslant I_{\lambda,\varepsilon}(t_\infty u_\infty) < I_\infty(t_\infty u_\infty) < I_\infty(u_\infty) = c_\infty.$$

Step 3. According to $d \in [0, c_{\lambda,\varepsilon}]$ and $c_{\lambda,\varepsilon} < c_\infty$, we have

$$0 \leqslant c_{\lambda,\varepsilon} - d \leqslant c_{\lambda,\varepsilon} < c_\infty.$$

By Lemma 4.6, we derive v is a ground state solution of system $(P_{\lambda,\varepsilon})$. Similar to the arguments in the proof of Theorem 1.1, one can easily prove v is positive. Denote it by $(u_{\lambda,\varepsilon}, \phi_u^\varepsilon)$. So conclusion (i) follows.

Step 4. Similar to the proof of Lemma 3.4, for a fixed $\varepsilon > 0$, it follows that

$$\begin{aligned} 0 = \lim_{\lambda\to+\infty} c_{\lambda,\varepsilon} =& I_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}) - \frac{1}{4}\langle I'_{\lambda,\varepsilon}(u_{\lambda,\varepsilon}), u_{\lambda,\varepsilon}\rangle \\ \geqslant& \frac{1}{4}\|u_{\lambda,\varepsilon}\|^2_{H^1_V(\mathbb{R}^3)}, \end{aligned}$$

which implies $\lim_{\lambda\to+\infty} \|u_{\lambda,\varepsilon}\|_{H^1_V(\mathbb{R}^3)} = 0$. By Lemma 2.3, we get also $\lim_{\lambda\to+\infty} \|\phi_u^\varepsilon\|_{\mathcal{D}} = 0$. At last, using the fact of Lemma 2.1, one can deduce $\lim_{\lambda\to+\infty} \|\phi_u^\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 0$. So conclusion (ii) follows.

Step 5. For fixed $\lambda = \widetilde{\lambda} > 0$, it is easy to get $\{u_{\widetilde{\lambda},\varepsilon}\}_{\varepsilon\geqslant 0}$ is bounded. Therefore, up to a subsequence, there exists $u_{\widetilde{\lambda},0} \in H^1_V(\mathbb{R}^3)$ such that

$$u_{\widetilde{\lambda},\varepsilon} \rightharpoonup u_{\widetilde{\lambda},0}, \quad \text{as } \varepsilon \to 0.$$

Set $\eta_\varepsilon = u_{\widetilde{\lambda},\varepsilon} - u_{\widetilde{\lambda},0}$. Then $\eta_\varepsilon \rightharpoonup 0$ in $H^1_V(\mathbb{R}^3)$. Similar to the proof of Lemma 3.4, we can deduce there exists $\lambda^* > 0$ such that

$$\sup_{\varepsilon>0} c_{\lambda,\varepsilon} = 0, \quad \forall \lambda > \lambda^*.$$

Hence we get $c_{\widetilde{\lambda},\varepsilon} < c_\infty$, for all $\widetilde{\lambda} > \lambda^*$, $\varepsilon > 0$. Note that all the conditions of Lemma 4.6 are satisfied, so by Lemma 4.6 we obtain the strong convergence, more precisely it satisfies

$$\lim_{\varepsilon\to 0} u_{\widetilde{\lambda},\varepsilon} = u_{\widetilde{\lambda},0}.$$

In particular, we have $(u_{\widetilde{\lambda},\varepsilon})^2 \to (u_{\widetilde{\lambda},0})^2$ in $L^{\frac{6}{5}}(\mathbb{R}^3)$.

Let $\varphi \in H^1_V(\mathbb{R}^3)$. Then we have

$$(u_{\widetilde{\lambda},\varepsilon}, \varphi)_{H^1_V(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \phi_u^\varepsilon u_{\widetilde{\lambda},\varepsilon}\varphi \mathrm{d}x = \widetilde{\lambda}\int_{\mathbb{R}^3} K(x) f(u_{\widetilde{\lambda},\varepsilon})\varphi \mathrm{d}x + \int_{\mathbb{R}^3} |u_{\widetilde{\lambda},\varepsilon}|^4 u_{\widetilde{\lambda},\varepsilon}\varphi \mathrm{d}x. \tag{4.27}$$

Pass the limit as $\varepsilon \to 0$ to the above equality. Now we see each term in (4.27), then we have

$$(u_{\widetilde{\lambda},\varepsilon}, \varphi)_{H^1_V(\mathbb{R}^3)} = (u_{\widetilde{\lambda},0}, \varphi)_{H^1_V(\mathbb{R}^3)}, \tag{4.28}$$

and as follows by standard arguments we can deduce

$$\int_{\mathbb{R}^3} K(x) f(u_{\widetilde{\lambda},\varepsilon})\varphi \mathrm{d}x \to \int_{\mathbb{R}^3} K(x) f(u_{\widetilde{\lambda},0})\varphi \mathrm{d}x, \tag{4.29}$$

and

$$\int_{\mathbb{R}^3} |u_{\widetilde{\lambda},\varepsilon}|^4 u_{\widetilde{\lambda},\varepsilon}\varphi \mathrm{d}x \to \int_{\mathbb{R}^3} |u_{\widetilde{\lambda},0}|^4 u_{\widetilde{\lambda},0}\varphi \mathrm{d}x. \tag{4.30}$$

Making use of Lemma 2.4 and taking into account $u_{\widetilde{\lambda},\varepsilon} \to u_{\widetilde{\lambda},0}$ in $L^{\frac{12}{5}}(\mathbb{R}^3)$, $\varphi \in L^{\frac{12}{5}}(\mathbb{R}^3)$ and the Hölder inequality, we get

$$\int_{\mathbb{R}^3} \phi_u^{\varepsilon} u_{\widetilde{\lambda},\varepsilon} \varphi \mathrm{d}x \to \int_{\mathbb{R}^3} \phi_u^0 u_{\widetilde{\lambda},0} \varphi \mathrm{d}x. \tag{4.31}$$

It follows from (4.28)–(4.31) that

$$(u_{\widetilde{\lambda},0}, \varphi)_{H_V^1(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \phi_u^0 u_{\widetilde{\lambda},0} \varphi \mathrm{d}x = \widetilde{\lambda} \int_{\mathbb{R}^3} K(x) f(u_{\widetilde{\lambda},0}) \varphi \mathrm{d}x + \int_{\mathbb{R}^3} |u_{\widetilde{\lambda},0}|^4 u_{\widetilde{\lambda},0} \varphi \mathrm{d}x,$$

which shows $(u_{\widetilde{\lambda},0}, \phi_u^0)$ solves system $(P_{\widetilde{\lambda},0})$. Using the same method in proving Theorem 1.1, we can prove $(u_{\widetilde{\lambda},0}, \phi_u^0)$ is a positive ground state solution of system $(P_{\widetilde{\lambda},0})$. So conclusion (*iii*) follows. The proof is completed. □

5. Proof of Theorem 1.3

In this section, we study the existence of infinitely many solutions to system $(P_{\lambda,\varepsilon})$. To complete this proof, we need the following result.

Lemma 5.1. ([28]) *Let X be an infinite dimensional Banach space and let $I \in C^1(X, \mathbb{R})$ be even, satisfy (PS) condition, and $I(0) = 0$, If $X = Y \bigoplus Z$, where Y is finite dimensional and I satisfies the following conditions.*

(*i*) *There exist constants $\rho, \alpha > 0$ such that $I|_{\{u|\|u\|=\rho\}\cap Z} \geqslant \alpha$;*

(*ii*) *For any finite dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X}) > 0$ such that $I(u) \leqslant 0$ on $\widetilde{X} \backslash B_R$. Then I possesses an unbounded sequence of critical values.*

Now we give the proof of Theorem 1.3.

Proof of Theorem 1.3. To prove Theorem 1.3, it suffices to give the verification of (*i*) and (*ii*).
Verification of (*i*): In view of (3.5) and the Sobolev inequality, we have

$$\begin{aligned}
I_{\lambda,\varepsilon}(u) =& \frac{1}{2}\|u\|^2_{H_V^1(\mathbb{R}^3)} + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^{\varepsilon}|u|^2 \mathrm{d}x - \lambda \int_{\mathbb{R}^3} K(x)F(u)\mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6 \mathrm{d}x \\
\geqslant& \frac{1}{2}\|u\|^2_{H_V^1(\mathbb{R}^3)} - \frac{\lambda\varepsilon C_1}{4}\int_{\mathbb{R}^3}|u|^4\mathrm{d}x - \frac{\lambda C_2 C_\varepsilon}{p}\int_{\mathbb{R}^3}|u|^p \mathrm{d}x - \frac{1}{6}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x \\
\geqslant& \frac{1}{2}\|u\|^2_{H_V^1(\mathbb{R}^3)} - \lambda\varepsilon C_3\|u\|^4_{H_V^1(\mathbb{R}^3)} - \lambda C_4 C_\varepsilon \|u\|^p_{H_V^1(\mathbb{R}^3)} - C_5\|u\|^6_{H_V^1(\mathbb{R}^3)}.
\end{aligned}$$

For $\rho > 0$ small enough, let $\delta = \frac{1}{2}\rho^2 - (\lambda\varepsilon C_3 + \lambda C_4 C_\varepsilon + C_5)\rho^4$, then $I_{\lambda,\varepsilon}(u)|_{\partial B_\rho \cap Z} \geqslant \delta > 0$.
Verification of (*ii*): For any finite dimensional subspace $\widetilde{X} \subset H_V^1(\mathbb{R}^3)$, by the equivalence of norms in the finite dimensional space, there exists constant $C > 0$ such that

$$C\|u\|_{H_V^1(\mathbb{R}^3)} \leqslant \|u\|_{L^s(\mathbb{R}^3)}, \quad s \in [2,6], \quad \forall u \in \widetilde{X}.$$

Putting this together with (3.5) and Lemma 2.3, one can infer

$$\begin{aligned}I_{\lambda,\varepsilon}(u) =&\frac{1}{2}\|u\|^2_{H^1_V(\mathbb{R}^3)}+\frac{1}{4}\int_{\mathbb{R}^3}\phi^\varepsilon_u|u|^2\mathrm{d}x-\lambda\int_{\mathbb{R}^3}K(x)F(u)\mathrm{d}x-\frac{1}{6}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x\\ \leqslant&\frac{1}{2}\|u\|^2_{H^1_V(\mathbb{R}^3)}+C_1\|u\|^4_{H^1_V(\mathbb{R}^3)}+\lambda C_2\int_{\mathbb{R}^3}|F(u)|\mathrm{d}x-\frac{1}{6}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x\\ \leqslant&\frac{1}{2}\|u\|^2_{H^1_V(\mathbb{R}^3)}+C_1\|u\|^4_{H^1_V(\mathbb{R}^3)}+\frac{\lambda\varepsilon C_2}{4}\int_{\mathbb{R}^3}|u|^4\mathrm{d}x+\frac{\lambda C_2C_\varepsilon}{p}\int_{\mathbb{R}^3}|u|^p\mathrm{d}x-\frac{1}{6}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x\\ \leqslant&\frac{1}{2}\|u\|^2_{H^1_V(\mathbb{R}^3)}+C_3\|u\|^4_{H^1_V(\mathbb{R}^3)}+\lambda C_4C_\varepsilon\|u\|^p_{H^1_V(\mathbb{R}^3)}-C_5\|u\|^6_{H^1_V(\mathbb{R}^3)}.\end{aligned}$$

Since $4<p<6$, there exists $R>0$ large enough such that $I_{\lambda,\varepsilon}(u)<0$ on $\widetilde{X}\backslash B_R$. Based on the above facts, all conditions described in Lemma 5.1 are satisfied. Similar to the proof of Theorem 1.1, we can show that the infinitely many solutions are positive. The proof is completed. □

6. Proof of Theorem 1.4

In this section, our goal is to show the nonexistence of ground state solution to system $(P_{\lambda,\varepsilon})$.

Lemma 6.1. *Suppose that all conditions described in Theorem 1.4 hold. Then for any $\lambda,\varepsilon>0$, $c_{\lambda,\varepsilon}=c_\infty$.*

Proof. By the assumptions of $V(x)$ and $K(x)$, one can easily get $I_\infty(u)<I_{\lambda,\varepsilon}(u)$, for all $u\in H^1_V(\mathbb{R}^3)$. In view of Lemma 4.1, we have for each $u\in\mathcal{N}_\infty$, there exists $t_u>0$ such that $t_uu\in\mathcal{N}_{\lambda,\varepsilon}$. So, for each $u\in\mathcal{N}_\infty$, there holds

$$0<c_\infty=\inf_{u\in\mathcal{N}_\infty}I_\infty(u)\leqslant\max_{t\geqslant0}I_\infty(tu)\leqslant\max_{t\geqslant0}I_{\lambda,\varepsilon}(tu)=I_{\lambda,\varepsilon}(t_uu).$$

Moreover, according to Lemma 4.1,

$$0<c_\infty\leqslant\inf_{u\in\mathcal{N}_\infty}I_{\lambda,\varepsilon}(t_uu)=\inf_{v\in\mathcal{N}_{\lambda,\varepsilon}}I_{\lambda,\varepsilon}(v)=c_{\lambda,\varepsilon}.$$

Hence, it remains to show $c_{\lambda,\varepsilon}\leqslant c_\infty$.

By Theorem 1.1, we know that system (P_∞) has a positive ground state solution $u_\infty\in\mathcal{N}_\infty$. Denote by $\omega_n(x)=u_\infty(x-y_n)$, where $\{y_n\}\subset\mathbb{R}^3$ and $|y_n|\to+\infty$ as $n\to+\infty$. Then, there exists a $t_{\omega_n}>0$ such that $t_{\omega_n}\omega_n\in\mathcal{N}_{\lambda,\varepsilon}$, that is,

$$\begin{aligned}&t^2_{\omega_n}\int_{\mathbb{R}^3}\left(|\nabla u_\infty|^2+V(x+y_n)|u_\infty|^2\right)\mathrm{d}x+t^4_{\omega_n}\int_{\mathbb{R}^3}\phi^\varepsilon_{u_\infty}|u_\infty|^2\mathrm{d}x\\ =&\int_{\mathbb{R}^3}K(x+y_n)f(t_{\omega_n}u_\infty)t_{\omega_n}u_\infty\mathrm{d}x+t^6_{\omega_n}\int_{\mathbb{R}^3}|u_\infty|^6\mathrm{d}x.\end{aligned}\tag{6.1}$$

It is easy to see that $\{t_{\omega_n}\}$ cannot converge to zero and infinity. We assume $t_{\omega_n}\to t_0$, as $n\to+\infty$. Passing the limit as $n\to+\infty$ in (6.1), we get

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(|\nabla u_\infty|^2+V_\infty|u_\infty|^2\right)\mathrm{d}x+t^2_{\omega_n}\int_{\mathbb{R}^3}\phi^\varepsilon_{u_\infty}|u_\infty|^2\mathrm{d}x\\ =&\int_{\mathbb{R}^3}K_\infty\frac{f(t_{\omega_n}u_\infty)u_\infty}{t_{\omega_n}}\mathrm{d}x+t^4_{\omega_n}\int_{\mathbb{R}^3}|u_\infty|^6\mathrm{d}x.\end{aligned}$$

By $u_\infty \in \mathcal{N}_\infty$, we can conclude $\lim\limits_{n\to+\infty} t_{\omega_n} = 1$. Since

$$\begin{aligned} c_{\lambda,\varepsilon} \leqslant & I_{\lambda,\varepsilon}(t_{\omega_n}\omega_n) = I_\infty(t_{\omega_n}u_\infty) + \frac{t_{\omega_n}^2}{2}\int_{\mathbb{R}^3}(V(x+y_n)-V_\infty)\,|u_\infty|^2\mathrm{d}x \\ & -\lambda\int_{\mathbb{R}^3}(K(x+y_n)-K_\infty)\,F(t_{\omega_n}u_\infty)\mathrm{d}x. \end{aligned} \tag{6.2}$$

Furthermore, by the assumption of $V(x)$, we can infer for any $\epsilon > 0$, there exists $R > 0$ such that

$$\int_{|x|\geqslant R}(V(x+y_n)-V_\infty)\,|u_\infty|^2\mathrm{d}x \leqslant \epsilon.$$

By $|y_n| \to +\infty$ and the Lebesgue dominated convergence theorem, we have

$$\int_{|x|<R}(V(x+y_n)-V_\infty)\,|u_\infty|^2\mathrm{d}x = 0.$$

Thus, we get

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}(V(x+y_n)-V_\infty)\,|u_\infty|^2\mathrm{d}x = 0.$$

Similarly, we can arrive at

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}(K(x+y_n)-K_\infty)\,F(u_\infty)\mathrm{d}x = 0.$$

Hence, using $t_{\omega_n} \to 1$ and letting $n \to +\infty$ in (6.2), we have $c_{\lambda,\varepsilon} \leqslant c_\infty$. The proof is completed. □

We give the proof of Theorem 1.4.

Proof of Theorem 1.4. By way of contradiction, we assume that there exist $\lambda_0 > 0$ and $u_0 \in \mathcal{N}_{\lambda_0,\varepsilon}$ such that $I_{\lambda_0,\varepsilon}(u_0) = c_{\lambda_0,\varepsilon}$. In view of Lemma 6.1, one has $c_{\lambda_0,\varepsilon} = c_\infty$. According to Lemma 3.1, we know that there exists $t_0 > 0$ such that $t_0u_0 \in \mathcal{N}_\infty$. Thus, we have

$$c_\infty \leqslant I_\infty(t_0u_0) < I_{\lambda_0,\varepsilon}(t_0u_0) \leqslant \max_{t\geqslant 0} I_{\lambda_0,\varepsilon}(tu_0) = I_{\lambda_0,\varepsilon}(u_0) = c_{\lambda_0,\varepsilon} = c_\infty,$$

which yields a contradiction. Moreover, the proof of ε is similar to λ, so we omit it here. The proof is completed. □

Appendix

Proof of Lemma 3.1. (i) It is standard to show that I_∞ satisfies the mountain pass geometry. By the mountain pass theorem, we can obtain a $(PS)_{c_\infty}$ sequence of I_∞.

(ii) For $t > 0$, let

$$h(t) = I_\infty(tu) = \frac{t^2}{2}\|u\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} + \frac{t^4}{4}\int_{\mathbb{R}^3}\phi_u^\varepsilon|u|^2\mathrm{d}x - \lambda K_\infty\int_{\mathbb{R}^3}F(tu)\mathrm{d}x - \frac{t^6}{6}\int_{\mathbb{R}^3}|u|^6\mathrm{d}x.$$

For $t > 0$ small enough, it follows from (3.5) and Sobolev inequality that

$$\begin{aligned} h(t) \geqslant & \frac{t^2}{2}\|u\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} - \frac{\lambda K_\infty \epsilon}{4} t^4 \int_{\mathbb{R}^3} |u|^4 \mathrm{d}x - \frac{\lambda K_\infty C_\epsilon}{p} t^p \int_{\mathbb{R}^3} |u|^p \mathrm{d}x - \frac{t^6}{6}\int_{\mathbb{R}^3} |u|^6 \mathrm{d}x \\ \geqslant & \frac{t^2}{2}\|u\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} - Ct^4\|u\|^4_{H^1_{V_\infty}(\mathbb{R}^3)} - Ct^p\|u\|^p_{H^1_{V_\infty}(\mathbb{R}^3)} - Ct^6\|u\|^6_{H^1_{V_\infty}(\mathbb{R}^3)}. \end{aligned}$$

Hence, we get $h(t) > 0$ for $t > 0$ small enough. Moreover, it is easy to see $I_\infty(tu) \to -\infty$ as $t \to +\infty$. Therefore, $h(t)$ has a maximum at $t = t_u > 0$. So that $h'(t_u) = 0$ and $t_u u \in \mathcal{N}_\infty$. Next, we show that t_u is unique. By the way of contradiction, we assume that there exist $0 < t_u < \tilde{t}_u$ such that $\tilde{t}_u u, t_u u \in \mathcal{N}_\infty$. Then, we have

$$\left(\frac{1}{\tilde{t}_u^2} - \frac{1}{t_u^2}\right)\|u\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} = \lambda K_\infty \int_{\mathbb{R}^3} \left[\frac{f(\tilde{t}_u u)}{(\tilde{t}_u u)^3} - \frac{f(t_u u)}{(t_u u)^3}\right]|u|^4 \mathrm{d}x + \left(\tilde{t}_u^2 - t_u^2\right)\int_{\mathbb{R}^3}|u|^6\mathrm{d}x,$$

which is impossible by $0 < t_u < \tilde{t}_u$. We now show

$$\lim_{\lambda\to+\infty} t_u = 0.$$

By $I'_\infty(t_u u) = 0$, then t_u satisfies

$$t_u^2\|u\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} + t_u^4\int_{\mathbb{R}^3}\phi_u^\varepsilon|u|^2\mathrm{d}x = \lambda K_\infty\int_{\mathbb{R}^3} f(t_u u)t_u u\mathrm{d}x + t_u^6\int_{\mathbb{R}^3}|u|^6\mathrm{d}x. \tag{A.1}$$

If $\lim_{\lambda\to+\infty} t_u = +\infty$, then in view of (F_1), it is easy to lead a contradiction. Thus, $\lim_{\lambda\to+\infty} t_u = \eta \geqslant 0$. If $\eta > 0$, then combined (A.1) with Lemma 2.3, as $\lambda \to +\infty$, we can infer

$$C(\eta^2 + \eta^4) \geqslant \lambda K_\infty \int_{\mathbb{R}^3} f(t_u u)t_u u\mathrm{d}x + t_u^6\int_{\mathbb{R}^3}|u|^6\mathrm{d}x \to +\infty,$$

which yields a contradiction. Hence we conclude $\eta = 0$.

(*iii*) By (*ii*) one has $\bar{c}_\infty = \bar{\bar{c}}_\infty$. Choosing $t_1 > 0$ large enough such that

$$I_\infty(t_1 u) < 0.$$

Define a path $\gamma : [0, 1] \to H^1_{V_\infty}(\mathbb{R}^3)$ by $\gamma(t) = t_1 t u$, then we have $\gamma \in \Gamma$. Thus, we obtain $c_\infty \leqslant \bar{\bar{c}}_\infty$. On the other hand, let $k(t) := \langle I'_\infty(\gamma(t)), \gamma(t)\rangle$, where $\gamma \in \Gamma$. Then, $k(t) > 0$ for $t > 0$ small enough. Set $\gamma(1) = e$, one has

$$\begin{aligned} & I_\infty(e) - \frac{1}{4}\langle I'_\infty(e), e\rangle \\ =& \frac{1}{4}\|e\|^2_{H^1_{V_\infty}(\mathbb{R}^3)} + \lambda K_\infty\int_{\mathbb{R}^3}\left(\frac{1}{4}f(e)e - F(e)\right)\mathrm{d}x + \frac{1}{12}\int_{\mathbb{R}^3}|e|^6\mathrm{d}x \\ >& 0, \end{aligned}$$

from which we obtain

$$\langle I'_\infty(e), e\rangle < 4I_\infty(e) < 0.$$

Then there exists $t_2 \in (0, 1)$ such that $\langle I'_\infty(\gamma(t_2)), \gamma(t_2)\rangle = 0$, which implies $\gamma(t_2) \in \mathcal{N}_\infty$. Therefore, we get $\bar{c}_\infty \leqslant c_\infty$. The proof is completed. □

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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