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Research article

Some conditions for the existence and uniqueness of monotonic and positive solutions for nonlinear systems of ordinary differential equations

Adrian Nicolae Branga*

Department of Mathematics and Informatics, Lucian Blaga University of Sibiu, 5–7 Dr. I. Raţiu, Sibiu 550012, Romania

* **Correspondence:** Email: adrian.branga@gmail.com.

Abstract: In this paper, applying the theory of fixed points in complete gauge spaces, we establish some conditions for the existence and uniqueness of monotonic and positive solutions for nonlinear systems of ordinary differential equations. Moreover, the paper contains an application of the theoretical results to the study of a class of systems of nonlinear ordinary differential equations.

Keywords: nonlinear systems of ordinary differential equations; monotonic and positive solutions; complete gauge spaces; fixed point theory; existence and uniqueness theorems

1. Introduction

Regarding the existence of monotonic solutions of differential equations, important results were obtained by Chu [1], Aslanov [2], Ertem and Zafer [3], Yin [4], Li and Fan [5], Rovder [6], Tóthová and Palumbíny [7], Rovderová [8], Iseki [9], Demidovich [10], Sanhan et al. [11], Branga and Olaru [12].

The purpose of this article is to study the existence and uniqueness of monotonic and positive solutions for systems of first order nonlinear ordinary differential equations on the positive real axis (defined over an unbounded interval). To achieve this goal we will base our research on the theory of gauge spaces introduced by Dugundji [13], which proved that any separating family of pseudometrics on a nonempty set induces a Hausdorff uniform structure on that set and conversely, any Hausdorff uniform structure on a nonempty set is generated by a separating family of pseudometrics, this allowing us to identify the gauge spaces with Hausdorff uniform spaces. We costruct three families of pseudometrics and the related gauge spaces, showing that the problem of solving systems of first order nonlinear ordinary differential equations on the positive real axis can be reduced to finding the fixed points of an operator which acts on these gauge spaces. Further, using the fixed point results obtained by Colojoara [14] and Gheorghiu [15] in the case of complete gauge spaces, respectively by Knill [16] and Tarafdar [17] for Hausdorff uniform spaces, we proved that under certain conditions the considered operator has a unique fixed point and consequently the system of first order nonlinear ordinary differential equations, with initial condition, has a unique solution on the positive real axis and this solution is monotonic and positive. Also, we explored some sufficient conditions for fulfilling the hypotheses of the main results obtained in the paper. Moreover, an example was given to illustrate the theoretical results.

2. Materials and methods

In the following, we denote by $\mathbb{R}_+$ the real interval $[0, \infty)$.

We consider a system of first order nonlinear ordinary differential equations:

$$x'(t) = f(t, x(t)),\ t \in \mathbb{R}_+, \tag{2.1}$$

where $x : \mathbb{R}_+ \to \mathbb{R}^n$, $x = (x_i)_{i=\overline{1,n}}$, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $f = (f_i)_{i=\overline{1,n}}$.

Definition 2.1. *[18] Let us consider $I \subseteq \mathbb{R}_+$ an interval. We say that the vector functions $x, y : \mathbb{R}_+ \to \mathbb{R}^n$, $x = (x_i)_{i=\overline{1,n}}$, $y = (y_i)_{i=\overline{1,n}}$, satisfies the condition $x(t) \leq y(t)$ on I, if $x_i(t) \leq y_i(t)$, $i = \overline{1, n}$, on I.*

Definition 2.2. *[18] A solution function $x : \mathbb{R}_+ \to \mathbb{R}^n$, $x = (x_i)_{i=\overline{1,n}}$, of the system of first order nonlinear ordinary differential equations (2.1), is called monotonic for $t \to \infty$, if there is a number $T \geq 0$ such that each function x_i, $i = \overline{1, n}$, is monotonic on $[T, \infty)$.*

Theorem 2.1. *[18] Let us consider $a, b \in \mathbb{R}$, $a < b$, and $g : [a, b] \to \mathbb{R}$ an integrable function on $[a, b]$. We define the function $G : [a, b] \to \mathbb{R}$, $G(t) = \int_a^t g(s)ds$. Then, the following statements are valid:*

(i) G is continuous on $[a, b]$;
(ii) If g is continuous on $[a, b]$, then G is differentiable on $[a, b]$ and $G'(t) = g(t)$ for all $t \in [a, b]$.

Definition 2.3. *[13] Let X be a nonempty set. A pseudometric (or gauge) on X is a mapping $p : X \times X \to \mathbb{R}_+$ which fulfills the following conditions:*

1) if $x \in X$, then $p(x, x) = 0$;
2) if $x, y \in X$, then $p(x, y) = p(y, x)$;
3) if $x, y, z \in X$, then $p(x, z) \leq p(x, y) + p(y, z)$.

Definition 2.4. *[13] Let X be a nonempty set. We say that:*

(i) A gauge structure on X is a family $\mathcal{P} = (p_k)_{k\in I}$ of pseudometrics on X;
(ii) A separating gauge structure $\mathcal{P} = (p_k)_{k\in I}$ on X is a gauge structure on X which satisfies the condition: for every pair of elements $x, y \in X$, with $x \neq y$, there exists $k \in I$ such that $p_k(x, y) \neq 0$;
(iii) A gauge space is a pair $(X, \mathcal{P})$ of a set X and a separating gauge structure $\mathcal{P}$ on X.

Definition 2.5. *[13] Let $(X, \mathcal{P})$ be a gauge space, where $\mathcal{P} = (p_k)_{k\in I}$. We say that:*

(i) A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to an element $x \in X$ if for every $\varepsilon > 0$ and $k \in I$ there exists a number $n(\varepsilon, k) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon, k)$ we have $p_k(x_n, x) < \varepsilon$;
(ii) A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called a Cauchy sequence if for every $\varepsilon > 0$ and $k \in I$ there exists a number $n(\varepsilon, k) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon, k)$ and $p \in \mathbb{N}$ we have $p_k(x_n, x_{n+p}) < \varepsilon$;

(iii) *A gauge space $(X, \mathcal{P})$ is named sequentially complete if it satisfies the condition: any Cauchy sequence of elements in X is convergent in X.*

Theorem 2.2. *[14,15] If $(X, \mathcal{P})$ is a sequentially complete gauge space, where $\mathcal{P} = (p_k)_{k\in I}$, $T : X \to X$ is a self-mapping and for every $k \in I$ there exists $\alpha_k \in (0, 1)$ such that*

$$p_k(T(x), T(y)) \le \alpha_k p_k(x, y), \quad \text{for all } x, y \in X, \quad \text{for all } k \in I,$$

then T has a unique fixed point on X.

Definition 2.6. *[19] Let X be a nonempty set and (Y, ρ) a metric space. We say that:*

(i) *A sequence of functions $f_n : X \to Y$, $n \in \mathbb{N}$, is called uniformly convergent on X if: there exists a function $f : X \to Y$ with the property that for every $\varepsilon > 0$ there is a number $n(\varepsilon) \in \mathbb{N}$ such that for all $n \ge n(\varepsilon)$ we have $\rho(f_n(x), f(x)) < \varepsilon$ for all $x \in X$;*

(ii) *A sequence of functions $f_n : X \to Y$, $n \in \mathbb{N}$, is named a uniformly Cauchy sequence if: for every $\varepsilon > 0$ there is a number $n(\varepsilon) \in \mathbb{N}$ such that for all $n \ge n(\varepsilon)$ and $p \in \mathbb{N}$ we have $\rho(f_n(x), f_{n+p}(x)) < \varepsilon$ for all $x \in X$.*

Theorem 2.3. *[19] Let X be a nonempty set and (Y, ρ) a metric space. Then the following affirmations are valid:*

(i) *If $f_n : X \to Y$, $n \in \mathbb{N}$, is a uniformly convergent sequence on X to a function $f : X \to Y$, then f_n, $n \in \mathbb{N}$, is a uniformly Cauchy sequence;*

(ii) *If $f_n : X \to Y$, $n \in \mathbb{N}$, is a uniformly Cauchy sequence and (Y, ρ) is complete, then f_n, $n \in \mathbb{N}$, is a uniformly convergent sequence on X to a function $f : X \to Y$.*

Theorem 2.4. *[19] Let (X, τ) be a topological space, (Y, ρ) a complete metric space and $f_n : X \to Y$, $n \in \mathbb{N}$, a uniformly convergent sequence on X to a function $f : X \to Y$. If every function f_n, $n \in \mathbb{N}$, is continuous on X, then f is continuous on X.*

Theorem 2.5. *[19] Let (X, d) be a metric space, A a compact subset of X and $f : A \to \mathbb{R}$ a continuous function. Then f is bounded on A and there exists $\underline{t}, \bar{t} \in A$ such that $f(\underline{t}) = \inf\limits_{t\in A} f(t)$ and $f(\bar{t}) = \sup\limits_{t\in A} f(t)$ (f attains its infimum and supremum in A).*

3. Results

In the following we consider the spaces $\mathbb{R}^n = \{x = (x_1, ..., x_n) \mid x_i \in \mathbb{R},\ i = \overline{1, n}\}$ and $C(\mathbb{R}_+, \mathbb{R}^n) = \{x : \mathbb{R}_+ \to \mathbb{R}^n \mid x \text{ is continuous on } \mathbb{R}_+\}$.

Lemma 3.1. *The following properties are valid:*

(i) *$(\mathbb{R}^n, \|\cdot\|_2)$ is a complete normed linear space, where $\|\cdot\|_2 : \mathbb{R}^n \to \mathbb{R}_+$, $\|x\|_2 = \left(\sum\limits_{i=1}^{n} x_i^2\right)^{1/2}$;*

(ii) *$(\mathbb{R}^n, \rho_2)$ is a complete metric space, where $\rho_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, $\rho_2(x, y) = \|x - y\|_2$;*

(iii) *the maps $p_{k,2} : C(\mathbb{R}_+, \mathbb{R}^n) \times C(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}_+$, $p_{k,2}(x, y) = \sup\limits_{t\in[0,k]} (\|x(t) - y(t)\|_2 e^{-\tau t})$, $k \in \mathbb{N}^*$, are pseudometrics on $C(\mathbb{R}_+, \mathbb{R}^n)$, where $\tau > 0$;*

(iv) $(C(\mathbb{R}_+,\mathbb{R}^n),\mathcal{P}_2)$ *is a sequentially complete gauge space, where* $\mathcal{P}_2=(p_{k,2})_{k\in\mathbb{N}^*}$.

Proof. The demonstration for statements (i), (ii) can be found, for example, in the paper [19].

(iii) We choose $k\in\mathbb{N}^*$ be an arbitrary number.

Let $x,y\in C(\mathbb{R}_+,\mathbb{R}^n)$ be arbitrary elements. It follows that the function $x-y:\mathbb{R}_+\to\mathbb{R}^n$ is continuous on $\mathbb{R}_+$. Also, it is well known that the norm $\|\cdot\|_2:\mathbb{R}^n\to\mathbb{R}_+$ is a continuous map on $\mathbb{R}^n$. Therefore, $\alpha:\mathbb{R}_+\to\mathbb{R}_+$, $\alpha(t)=\|x(t)-y(t)\|_2$ is a continuous function on $\mathbb{R}_+$. Further, $\beta:\mathbb{R}_+\to\mathbb{R}_+$, $\beta(t)=e^{-\tau t}$ is a continuous function on $\mathbb{R}_+$. Consequently, $\gamma:\mathbb{R}_+\to\mathbb{R}_+$, $\gamma(t)=\alpha(t)\beta(t)=\|x(t)-y(t)\|_2e^{-\tau t}$ is a continuous function on $\mathbb{R}_+$. We obtain that the function γ is continuous on $[0,k]$. Using Theorem 2.5 we deduce that the function γ is bounded on $[0,k]$ and there exists $\bar{t}\in[0,k]$ such that $\gamma(\bar{t})=\sup\limits_{t\in[0,k]}\gamma(t)=\sup\limits_{t\in[0,k]}(\|x(t)-y(t)\|_2e^{-\tau t})\in\mathbb{R}_+$ (γ attains its supremum in $[0,k]$). Hence, the map $p_{k,2}$ is well-defined.

Let us show that the function $p_{k,2}$ satisfies the conditions of a pseudometric. We choose $x,y,z\in C(\mathbb{R}_+,\mathbb{R}^n)$ be arbitrary elements. Applying the properties of the norm $\|\cdot\|_2$ we get:

$$1)p_{k,2}(x,x)=\sup_{t\in[0,k]}(\|x(t)-x(t)\|_2e^{-\tau t})=\sup_{t\in[0,k]}(\|0\|_2e^{-\tau t})=0;$$

$$2)p_{k,2}(x,y)=\sup_{t\in[0,k]}(\|x(t)-y(t)\|_2e^{-\tau t})=\sup_{t\in[0,k]}(\|y(t)-x(t)\|_2e^{-\tau t})=p_{k,2}(y,x);$$

$$3)\|x(t)-z(t)\|_2=\|x(t)-y(t)+y(t)-z(t)\|_2\le\|x(t)-y(t)\|_2+\|y(t)-z(t)\|_2,$$

hence

$$\begin{aligned}\|x(t)-z(t)\|_2e^{-\tau t}&\le\|x(t)-y(t)\|_2e^{-\tau t}+\|y(t)-z(t)\|_2e^{-\tau t}\\&\le\sup_{t\in[0,k]}(\|x(t)-y(t)\|_2e^{-\tau t})+\sup_{t\in[0,k]}(\|y(t)-z(t)\|_2e^{-\tau t})\\&=p_{k,2}(x,y)+p_{k,2}(y,z),\ \text{for all } t\in[0,k],\end{aligned}$$

thus

$$\sup_{t\in[0,k]}(\|x(t)-z(t)\|_2e^{-\tau t})\le p_{k,2}(x,y)+p_{k,2}(y,z),$$

so

$$p_{k,2}(x,z)\le p_{k,2}(x,y)+p_{k,2}(y,z).$$

(iv) According to (iii) $\mathcal{P}_2=(p_{k,2})_{k\in\mathbb{N}^*}$ is a family of pseudometrics on $C(\mathbb{R}_+,\mathbb{R}^n)$, hence it forms a gauge structure on $C(\mathbb{R}_+,\mathbb{R}^n)$. Moreover, for every pair of elements $x,y\in C(\mathbb{R}_+,\mathbb{R}^n)$, with $x\neq y$, and every $k\in\mathbb{N}^*$, we get $p_{k,2}(x,y)=\sup\limits_{t\in[0,k]}(\|x(t)-y(t)\|_2e^{-\tau t})\neq0$, thus $\mathcal{P}_2=(p_{k,2})_{k\in\mathbb{N}^*}$ is a separating gauge structure on $C(\mathbb{R}_+,\mathbb{R}^n)$. It follows that $(C(\mathbb{R}_+,\mathbb{R}^n),\mathcal{P}_2)$ is a gauge space.

Let $\varepsilon>0$ and $k\in\mathbb{N}^*$ be arbitrary elements.

We prove that the gauge space $(C(\mathbb{R}_+,\mathbb{R}^n),\mathcal{P}_2)$ is sequentially complete. We choose $(x_m)_{m\in\mathbb{N}}\subset C(\mathbb{R}_+,\mathbb{R}^n)$ an arbitrary Cauchy sequence. Using Definition 2.5 (ii) we deduce that for $\varepsilon e^{-\tau k}>0$ and $k\in\mathbb{N}^*$ there exists a number $m'(\varepsilon,k)\in\mathbb{N}$ such that for all $m\ge m'(\varepsilon,k)$ and $p\in\mathbb{N}$ we have $p_{k,2}(x_m,x_{m+p})<\varepsilon e^{-\tau k}$, so $\sup\limits_{t\in[0,k]}\left(\|x_m(t)-x_{m+p}(t)\|_2e^{-\tau t}\right)<\varepsilon e^{-\tau k}$. Because $e^{-\tau k}\le e^{-\tau t}$ for all $t\in[0,k]$, we find $\|x_m(t)-x_{m+p}(t)\|_2e^{-\tau k}\le\|x_m(t)-x_{m+p}(t)\|_2e^{-\tau t}$ for all $t\in[0,k]$, thus

$\sup_{t\in[0,k]}\left(\|x_m(t)-x_{m+p}(t)\|_2e^{-\tau k}\right)\leq\sup_{t\in[0,k]}\left(\|x_m(t)-x_{m+p}(t)\|_2e^{-\tau t}\right)$. It follows that for all $m\geq m'(\varepsilon,k)$ and $p\in\mathbb{N}$ we have: $\sup_{t\in[0,k]}\left(\|x_m(t)-x_{m+p}(t)\|_2\right)e^{-\tau k}<\varepsilon e^{-\tau k}$, hence $\sup_{t\in[0,k]}\left(\|x_m(t)-x_{m+p}(t)\|_2\right)<\varepsilon$, thus $\|x_m(t)-x_{m+p}(t)\|_2<\varepsilon$ for all $t\in[0,k]$, so $\rho_2(x_m(t),x_{m+p}(t))<\varepsilon$ for all $t\in[0,k]$. Therefore, for every $\varepsilon>0$ there is a number $m(\varepsilon):=m'(\varepsilon,k)\in\mathbb{N}$ such that for all $m\geq m(\varepsilon)$ and $p\in\mathbb{N}$ we have $\rho_2(x_m(t),x_{m+p}(t))<\varepsilon$ for all $t\in[0,k]$, which means that $x_m:[0,k]\to\mathbb{R}^n$, $m\in\mathbb{N}$, is a uniformly Cauchy sequence (according to Definition 2.6 (ii)). Moreover, $(\mathbb{R}^n,\rho_2)$ being a complete metric space (according to the affirmation (ii)), applying Theorem 2.3 (ii) it follows that x_m, $m\in\mathbb{N}$, is a uniformly convergent sequence on $[0,k]$ to a function $x:[0,k]\to\mathbb{R}^n$. As every function x_m, $m\in\mathbb{N}$, is continuous on $[0,k]$, using Theorem 2.4 we obtain that x is continuous on $[0,k]$. Hence, $x_m:[0,k]\to\mathbb{R}^n$, $m\in\mathbb{N}$, is a sequence of continuous functions, which is uniformly convergent on $[0,k]$ to a continuous function $x:[0,k]\to\mathbb{R}^n$, property that is valid for any value $k\in\mathbb{N}^*$. Consequently, the sequence of continuous functions $x_m:\mathbb{R}_+\to\mathbb{R}^n$, $m\in\mathbb{N}$, is uniformly convergent on $\mathbb{R}_+$ to a continuous function $x:\mathbb{R}_+\to\mathbb{R}^n$.

On the other hand, $x_m:[0,k]\to\mathbb{R}^n$, $m\in\mathbb{N}$, being a uniformly convergent sequence on $[0,k]$ to a function $x:[0,k]\to\mathbb{R}^n$, using Definition 2.6 (i) we deduce that for $\frac{\varepsilon}{2}>0$ there is a number $m'(\varepsilon,k)\in\mathbb{N}$ such that for all $m\geq m'(\varepsilon,k)$ we have $\rho_2(x_m(t),x(t))<\frac{\varepsilon}{2}$ for all $t\in[0,k]$, which is equivalent to $\|x_m(t)-x(t)\|_2<\frac{\varepsilon}{2}$ for all $t\in[0,k]$. It follows that for all $m\geq m'(\varepsilon,k)$ we have: $\|x_m(t)-x(t)\|_2e^{-\tau t}<\frac{\varepsilon}{2}e^{-\tau t}\leq\frac{\varepsilon}{2}$ for all $t\in[0,k]$, hence $\sup_{t\in[0,k]}\left(\|x_m(t)-x(t)\|_2e^{-\tau t}\right)\leq\frac{\varepsilon}{2}<\varepsilon$, thus $p_{k,2}(x_m,x)<\varepsilon$. Therefore, for every $\varepsilon>0$ and $k\in\mathbb{N}^*$ there exists a number $m'(\varepsilon,k)\in\mathbb{N}$ such that for all $m\geq m'(\varepsilon,k)$ we have $p_{k,2}(x_m,x)<\varepsilon$, which means that the sequence $(x_m)_{m\in\mathbb{N}}\subset C(\mathbb{R}_+,\mathbb{R}^n)$ converges to an element $x\in C(\mathbb{R}_+,\mathbb{R}^n)$ (according to Definition 2.5 (i)).

Consequently, any Cauchy sequence of elements in $C(\mathbb{R}_+,\mathbb{R}^n)$ is convergent in $C(\mathbb{R}_+,\mathbb{R}^n)$, which means that the gauge space $(C(\mathbb{R}_+,\mathbb{R}^n),\mathcal{P}_2)$ is sequentially complete (according to Definition 2.5 (iii)).

Theorem 3.1. *Let us suppose that the function $f:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous on $\mathbb{R}_+\times\mathbb{R}^n$ and there is a number $L_2(f)>0$ such that $\|f(t,u)-f(t,v)\|_2\leq L_2(f)\|u-v\|_2$, for all $t\in\mathbb{R}_+$, $u,v\in\mathbb{R}^n$. Then, the following statements are true:*

(i) The system of first order nonlinear ordinary differential equations (2.1), with initial condition $x(0)=x^0\in\mathbb{R}^n$, has a unique solution on $C(\mathbb{R}_+,\mathbb{R}^n)$;
(ii) Moreover, if $f(t,u)\geq 0$, for all $t\in\mathbb{R}_+$, $u\in\mathbb{R}^n$, this solution is monotonic for $t\to\infty$;
(iii) Further, if $x^0\geq 0$ and $f(t,u)\geq 0$, for all $t\in\mathbb{R}_+$, $u\in\mathbb{R}^n$, the solution is positive.

Proof. (i) Since the functions $x:\mathbb{R}_+\to\mathbb{R}^n$, $f:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ are continuous, it follows that the function $g:\mathbb{R}_+\to\mathbb{R}^n$, $g(s)=f(s,x(s))$ is continuous. Applying Theorem 2.1 we deduce that the system of first order nonlinear ordinary differential equations (2.1), with initial condition $x(0)=x^0\in\mathbb{R}^n$, is equivalent to the system of integral equations

$$x(t)=x^0+\int_0^t f(s,x(s))ds,\ t\in\mathbb{R}_+. \tag{3.1}$$

Using the continuity of the function g, relation (3.1) and Theorem 2.1, we can define an operator

$T : C(\mathbb{R}_+, \mathbb{R}^n) \to C(\mathbb{R}_+, \mathbb{R}^n)$,

$$T(x)(t) = x^0 + \int_0^t f(s, x(s))ds,\ t \in \mathbb{R}_+. \tag{3.2}$$

For every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, $t \in [0, k]$, $i = \overline{1, n}$, we have successively:

$$|pr_i(T(x)(t)) - pr_i(T(y)(t))|$$

$$= \left| x_i^0 + \int_0^t f_i(s, x(s))ds - x_i^0 - \int_0^t f_i(s, y(s))ds \right|$$

$$= \left| \int_0^t f_i(s, x(s))ds - \int_0^t f_i(s, y(s))ds \right| = \left| \int_0^t (f_i(s, x(s)) - f_i(s, y(s)))ds \right|$$

$$\leq \int_0^t |f_i(s, x(s)) - f_i(s, y(s))|\, ds.$$

On the other hand, for every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, $s, t \in [0, k]$, $i = \overline{1, n}$, considering the hypothesis $\|f(t, u) - f(t, v)\|_2 \leq L_2(f)\|u - v\|_2$, for all $t \in \mathbb{R}_+$, $u, v \in \mathbb{R}^n$, we find:

$$|f_i(s, x(s)) - f_i(s, y(s))| = \left((f_i(s, x(s)) - f_i(s, y(s)))^2\right)^{1/2}$$

$$\leq \left(\sum_{i=1}^n (f_i(s, x(s)) - f_i(s, y(s)))^2 \right)^{1/2} = \|f(s, x(s)) - f(s, y(s))\|_2$$

$$\leq L_2(f)\|x(s) - y(s)\|_2 = L_2(f)\|x(s) - y(s)\|_2 e^{-\tau s} e^{\tau s}$$

$$\leq L_2(f) \sup_{s \in [0,k]} (\|x(s) - y(s)\|_2 e^{-\tau s}) e^{\tau s} = L_2(f) p_{k,2}(x, y) e^{\tau s},$$

which involves

$$\int_0^t |f_i(s, x(s)) - f_i(s, y(s))|\, ds \leq \int_0^t L_2(f) p_{k,2}(x, y) e^{\tau s} ds$$

$$= L_2(f) p_{k,2}(x, y) \int_0^t e^{\tau s} ds = L_2(f) p_{k,2}(x, y) \frac{e^{\tau s}}{\tau}\Big|_0^t = L_2(f) p_{k,2}(x, y) \left(\frac{e^{\tau t}}{\tau} - \frac{1}{\tau} \right)$$

$$= \frac{L_2(f)}{\tau} p_{k,2}(x, y)(e^{\tau t} - 1) \leq \frac{L_2(f)}{\tau} p_{k,2}(x, y) e^{\tau t}.$$

Hence, for every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, $t \in [0, k]$, $i = \overline{1, n}$, we get

$$|pr_i(T(x)(t)) - pr_i(T(y)(t))| \leq \frac{L_2(f)}{\tau} p_{k,2}(x, y) e^{\tau t},$$

which implies that

$$(pr_i(T(x)(t)) - pr_i(T(y)(t)))^2 \leq \left(\frac{L_2(f)}{\tau} p_{k,2}(x, y) e^{\tau t} \right)^2,$$

thus

$$\sum_{i=1}^{n} (pr_i(T(x)(t)) - pr_i(T(y)(t)))^2 \leq \sum_{i=1}^{n} \left(\frac{L_2(f)}{\tau} p_{k,2}(x, y) e^{\tau t} \right)^2 = n \left(\frac{L_2(f)}{\tau} p_{k,2}(x, y) e^{\tau t} \right)^2,$$

so

$$\left(\sum_{i=1}^{n} (pr_i(T(x)(t)) - pr_i(T(y)(t)))^2 \right)^{1/2} \leq \sqrt{n} \frac{L_2(f)}{\tau} p_{k,2}(x, y) e^{\tau t},$$

i.e.,

$$||T(x)(t) - T(y)(t)||_2 \leq \frac{\sqrt{n} L_2(f)}{\tau} p_{k,2}(x, y) e^{\tau t},$$

which is equivalent to

$$||T(x)(t) - T(y)(t)||_2 e^{-\tau t} \leq \frac{\sqrt{n} L_2(f)}{\tau} p_{k,2}(x, y).$$

Therefore, for every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, we obtain

$$\sup_{t \in [0,k]} ||T(x)(t) - T(y)(t)||_2 e^{-\tau t} \leq \frac{\sqrt{n} L_2(f)}{\tau} p_{k,2}(x, y),$$

thus

$$p_{k,2}(T(x), T(y)) \leq \frac{\sqrt{n} L_2(f)}{\tau} p_{k,2}(x, y).$$

Consequently, for $\tau > \sqrt{n} L_2(f)$ and denoting $\alpha_{k,2} := \dfrac{\sqrt{n} L_2(f)}{\tau} \in (0, 1)$, we have

$$p_{k,2}(T(x), T(y)) \leq \alpha_{k,2} p_{k,2}(x, y), \quad \text{for all } x, y \in C(\mathbb{R}_+, \mathbb{R}^n), \quad \text{for all } k \in \mathbb{N}^*.$$

According to Lemma 3.1, $(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{P}_2)$ is a sequentially complete gauge space, where $\mathcal{P}_2 = (p_{k,2})_{k \in \mathbb{N}^*}$, and above we proved that $T : C(\mathbb{R}_+, \mathbb{R}^n) \to C(\mathbb{R}_+, \mathbb{R}^n)$ is an operator having the property: for every $k \in \mathbb{N}^*$ there exists $\alpha_{k,2} \in (0, 1)$ such that

$$p_{k,2}(T(x), T(y)) \leq \alpha_{k,2} p_{k,2}(x, y), \quad \text{for all } x, y \in C(\mathbb{R}_+, \mathbb{R}^n), \quad \text{for all } k \in \mathbb{N}^*.$$

Applying Theorem 2.2 it follows that T has a unique fixed point on $C(\mathbb{R}_+, \mathbb{R}^n)$. Therefore, the system of integral equations (3.1) has a unique solution on $C(\mathbb{R}_+, \mathbb{R}^n)$. Consequently, the system of first order nonlinear ordinary differential equations (2.1), with initial condition $x(0) = x^0 \in \mathbb{R}^n$, has a unique solution on $C(\mathbb{R}_+, \mathbb{R}^n)$.

(ii) Let us denote by $x^* \in C(\mathbb{R}_+, \mathbb{R}^n)$ the unique solution for the system of first order nonlinear ordinary differential equations (2.1), with initial condition $x(0) = x^0 \in \mathbb{R}^n$. It follows that x^* is the unique solution in $C(\mathbb{R}_+, \mathbb{R}^n)$ for the system of integral equations (3.1), therefore

$$x^*(t) = x^0 + \int_0^t f(s, x^*(s))ds,\ t \in \mathbb{R}_+. \tag{3.3}$$

Let $t_1 < t_2$ be arbitrary numbers in $\mathbb{R}_+$. According to relation (3.3) and considering the hypothesis $f(t,u) \geq 0$, for all $t \in \mathbb{R}_+$, $u \in \mathbb{R}^n$, we deduce

$$x^*(t_1) - x^*(t_2) = x^0 + \int_0^{t_1} f(s, x^*(s))ds - x^0 - \int_0^{t_2} f(s, x^*(s))ds$$

$$= \int_0^{t_1} f(s, x^*(s))ds - \int_0^{t_2} f(s, x^*(s))ds = -\int_{t_1}^{t_2} f(s, x^*(s))ds \leq 0,$$

hence $x^*(t_1) \leq x^*(t_2)$. It follows that x^* is a monotonically increasing function on $\mathbb{R}_+$. Consequently, each function x_i^*, $i = \overline{1,n}$, is monotonic on $[0, \infty)$, i.e., x^* is monotonic for $t \to \infty$.

(iii) In accordance with the relation (3.3) and considering the hypotheses $x^0 \geq 0$ and $f(t,u) \geq 0$, for all $t \in \mathbb{R}_+$, $u \in \mathbb{R}^n$, we find $x^*(t) \geq 0$, for all $t \in \mathbb{R}_+$, therefore the solution x^* is positive.

Lemma 3.2. *The following properties are valid:*

(i) $(\mathbb{R}^n, \|\cdot\|_1)$ *is a complete normed linear space, where* $\|\cdot\|_1 : \mathbb{R}^n \to \mathbb{R}_+$, $\|x\|_1 = \sum_{i=1}^{n} |x_i|$;

(ii) $(\mathbb{R}^n, \rho_1)$ *is a complete metric space, where* $\rho_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, $\rho_1(x,y) = \|x-y\|_1$;

(iii) the maps $p_{k,1} : C(\mathbb{R}_+, \mathbb{R}^n) \times C(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}_+$, $p_{k,1}(x,y) = \sup_{t \in [0,k]} \|x(t) - y(t)\|_1 e^{-\tau t}$, $k \in \mathbb{N}^*$, *are pseudometrics on* $C(\mathbb{R}_+, \mathbb{R}^n)$*, where* $\tau > 0$;

(iv) $(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{P}_1)$ *is a sequentially complete gauge space, where* $\mathcal{P}_1 = (p_{k,1})_{k \in \mathbb{N}^*}$.

Proof. The demonstration for statements (i), (ii) can be found, for example, in the paper [19]. To prove affirmations (iii) and (iv) we can proceed similarly to the demonstration of Lemma 3.1.

Theorem 3.2. *Let us suppose that the function* $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous on* $\mathbb{R}_+ \times \mathbb{R}^n$ *and there is a number* $L_1(f) > 0$ *such that* $\|f(t,u) - f(t,v)\|_1 \leq L_1(f)\|u - v\|_1$*, for all* $t \in \mathbb{R}_+$, $u, v \in \mathbb{R}^n$*. Then, the following statements are true:*

(i) The system of first order nonlinear ordinary differential equations (2.1), with initial condition $x(0) = x^0 \in \mathbb{R}^n$*, has a unique solution on* $C(\mathbb{R}_+, \mathbb{R}^n)$;

(ii) Moreover, if $f(t,u) \geq 0$*, for all* $t \in \mathbb{R}_+$, $u \in \mathbb{R}^n$*, this solution is monotonic for* $t \to \infty$;

(iii) Further, if $x^0 \geq 0$ *and* $f(t,u) \geq 0$*, for all* $t \in \mathbb{R}_+$, $u \in \mathbb{R}^n$*, the solution is positive.*

Proof. (i) Since the functions $x : \mathbb{R}_+ \to \mathbb{R}^n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous, it follows that the function $g : \mathbb{R}_+ \to \mathbb{R}^n$, $g(s) = f(s, x(s))$ is continuous. Applying Theorem 2.1 we deduce that the system of first order nonlinear ordinary differential equations (2.1), with initial condition $x(0) = x^0 \in \mathbb{R}^n$, is equivalent to the system of integral equations

$$x(t) = x^0 + \int_0^t f(s, x(s))ds, \ t \in \mathbb{R}_+. \tag{3.4}$$

Using the continuity of the function g, relation (3.4) and Theorem 2.1, we can define an operator $T : C(\mathbb{R}_+, \mathbb{R}^n) \to C(\mathbb{R}_+, \mathbb{R}^n)$,

$$T(x)(t) = x^0 + \int_0^t f(s, x(s))ds, \ t \in \mathbb{R}_+. \tag{3.5}$$

For every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, $t \in [0, k]$, $i = \overline{1, n}$, we have successively:

$$|pr_i(T(x)(t)) - pr_i(T(y)(t))|$$

$$= \left| x_i^0 + \int_0^t f_i(s, x(s))ds - x_i^0 - \int_0^t f_i(s, y(s))ds \right|$$

$$= \left| \int_0^t f_i(s, x(s))ds - \int_0^t f_i(s, y(s))ds \right| = \left| \int_0^t (f_i(s, x(s)) - f_i(s, y(s)))ds \right|$$

$$\le \int_0^t |f_i(s, x(s)) - f_i(s, y(s))| \, ds,$$

which implies that

$$\sum_{i=1}^n |pr_i(T(x)(t)) - pr_i(T(y)(t))|$$

$$\le \sum_{i=1}^n \int_0^t |f_i(s, x(s)) - f_i(s, y(s))| \, ds = \int_0^t \sum_{i=1}^n |f_i(s, x(s)) - f_i(s, y(s))| \, ds,$$

i.e.,

$$\|T(x)(t) - T(y)(t)\|_1 \le \int_0^t \|f(s, x(s)) - f(s, y(s))\|_1 ds.$$

On the other hand, for every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, $s, t \in [0, k]$, considering the hypothesis $\|f(t, u) - f(t, v)\|_1 \le L_1(f)\|u - v\|_1$, for all $t \in \mathbb{R}_+$, $u, v \in \mathbb{R}^n$, we find:

$$\|f(s, x(s)) - f(s, y(s))\|_1 \le L_1(f)\|x(s) - y(s)\|_1$$

$$= L_1(f)\|x(s) - y(s)\|_1 e^{-\tau s} e^{\tau s} \leq L_1(f) \sup_{s\in[0,k]} (\|x(s) - y(s)\|_1 e^{-\tau s}) e^{\tau s}$$

$$= L_1(f) p_{k,1}(x, y) e^{\tau s},$$

which involves

$$\int_0^t \|f(s, x(s)) - f(s, y(s))\|_1 ds \leq \int_0^t L_1(f) p_{k,1}(x, y) e^{\tau s} ds$$

$$= L_1(f) p_{k,1}(x, y) \int_0^t e^{\tau s} ds = L_1(f) p_{k,1}(x, y) \frac{e^{\tau s}}{\tau}\Big|_0^t = L_1(f) p_{k,1}(x, y) \left(\frac{e^{\tau t}}{\tau} - \frac{1}{\tau}\right)$$

$$= \frac{L_1(f)}{\tau} p_{k,1}(x, y)(e^{\tau t} - 1) \leq \frac{L_1(f)}{\tau} p_{k,1}(x, y) e^{\tau t}.$$

Hence, for every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, $t \in [0, k]$, we get

$$\|T(x)(t) - T(y)(t)\|_1 \leq \frac{L_1(f)}{\tau} p_{k,1}(x, y) e^{\tau t},$$

which is equivalent to

$$\|T(x)(t) - T(y)(t)\|_1 e^{-\tau t} \leq \frac{L_1(f)}{\tau} p_{k,1}(x, y).$$

Therefore, for every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, we obtain

$$\sup_{t\in[0,k]} \|T(x)(t) - T(y)(t)\|_1 e^{-\tau t} \leq \frac{L_1(f)}{\tau} p_{k,1}(x, y),$$

thus

$$p_{k,1}(T(x), T(y)) \leq \frac{L_1(f)}{\tau} p_{k,1}(x, y).$$

Consequently, for $\tau > L_1(f)$ and denoting $\alpha_{k,1} := \dfrac{L_1(f)}{\tau} \in (0, 1)$, we have

$$p_{k,1}(T(x), T(y)) \leq \alpha_{k,1} p_{k,1}(x, y), \quad \text{for all } x, y \in C(\mathbb{R}_+, \mathbb{R}^n), \quad \text{for all } k \in \mathbb{N}^*.$$

According to Lemma 3.2, $(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{P}_1)$ is a sequentially complete gauge space, where $\mathcal{P}_1 = (p_{k,1})_{k\in\mathbb{N}^*}$, and above we proved that $T : C(\mathbb{R}_+, \mathbb{R}^n) \to C(\mathbb{R}_+, \mathbb{R}^n)$ is an operator having the property: for every $k \in \mathbb{N}^*$ there exists $\alpha_{k,1} \in (0, 1)$ such that

$$p_{k,1}(T(x), T(y)) \leq \alpha_{k,1} p_{k,1}(x, y), \quad \text{for all } x, y \in C(\mathbb{R}_+, \mathbb{R}^n), \quad \text{for all } k \in \mathbb{N}^*.$$

Applying Theorem 2.2 it follows that T has a unique fixed point on $C(\mathbb{R}_+, \mathbb{R}^n)$. Therefore, the system of integral equations (3.4) has a unique solution on $C(\mathbb{R}_+, \mathbb{R}^n)$. Consequently, the system of first order nonlinear ordinary differential equations (2.1), with initial condition $x(0) = x^0 \in \mathbb{R}^n$, has a unique solution on $C(\mathbb{R}_+, \mathbb{R}^n)$.

(ii), (iii) The proof is similar to Theorem 3.1.

Lemma 3.3. *The following properties are valid:*

(i) $(\mathbb{R}^n, \|\cdot\|_\infty)$ *is a complete normed linear space, where* $\|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}_+$, $\|x\|_\infty = \max_{i=\overline{1,n}} |x_i|$*;*
(ii) $(\mathbb{R}^n, \rho_\infty)$ *is a complete metric space, where* $\rho_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, $\rho_\infty(x, y) = \|x - y\|_\infty$*;*
(iii) the maps $p_{k,\infty} : C(\mathbb{R}_+, \mathbb{R}^n) \times C(\mathbb{R}_+, \mathbb{R}^n) \to \mathbb{R}_+$, $p_{k,\infty}(x, y) = \sup_{t\in[0,k]} \|x(t) - y(t)\|_\infty e^{-\tau t}$, $k \in \mathbb{N}^*$, *are pseudometrics on* $C(\mathbb{R}_+, \mathbb{R}^n)$*, where* $\tau > 0$*;*
(iv) $(C(\mathbb{R}_+, \mathbb{R}^n), \mathcal{P}_\infty)$ *is a sequentially complete gauge space, where* $\mathcal{P}_\infty = (p_{k,\infty})_{k\in\mathbb{N}^*}$.

Proof. The demonstration for statements (i), (ii) can be found, for example, in the paper [19]. To prove affirmations (iii) and (iv) we can proceed similarly to the demonstration of Lemma 3.1.

Theorem 3.3. *Let us suppose that the function* $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous on* $\mathbb{R}_+ \times \mathbb{R}^n$ *and there is a number* $L_\infty(f) > 0$ *such that* $\|f(t, u) - f(t, v)\|_\infty \le L_\infty(f)\|u - v\|_\infty$*, for all* $t \in \mathbb{R}_+$, $u, v \in \mathbb{R}^n$*. Then, the following statements are true:*

(i) The system of first order nonlinear ordinary differential equations (2.1), with initial condition $x(0) = x^0 \in \mathbb{R}^n$*, has a unique solution on* $C(\mathbb{R}_+, \mathbb{R}^n)$*;*
(ii) Moreover, if $f(t, u) \ge 0$*, for all* $t \in \mathbb{R}_+$, $u \in \mathbb{R}^n$*, this solution is monotonic for* $t \to \infty$*;*
(iii) Further, if $x^0 \ge 0$ *and* $f(t, u) \ge 0$*, for all* $t \in \mathbb{R}_+$, $u \in \mathbb{R}^n$*, the solution is positive.*

Proof. (i) Since the functions $x : \mathbb{R}_+ \to \mathbb{R}^n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous, it follows that the function $g : \mathbb{R}_+ \to \mathbb{R}^n$, $g(s) = f(s, x(s))$ is continuous. Applying Theorem 2.1 we deduce that the system of first order nonlinear ordinary differential equations (2.1), with initial condition $x(0) = x^0 \in \mathbb{R}^n$, is equivalent to the system of integral equations

$$x(t) = x^0 + \int_0^t f(s, x(s))ds,\ t \in \mathbb{R}_+. \tag{3.6}$$

Using the continuity of the function g, relation (3.6) and Theorem 2.1, we can define an operator $T : C(\mathbb{R}_+, \mathbb{R}^n) \to C(\mathbb{R}_+, \mathbb{R}^n)$,

$$T(x)(t) = x^0 + \int_0^t f(s, x(s))ds,\ t \in \mathbb{R}_+. \tag{3.7}$$

For every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, $t \in [0, k]$, $i = \overline{1, n}$, we have successively:

$$|pr_i(T(x)(t)) - pr_i(T(y)(t))|$$

$$= \left| x_i^0 + \int_0^t f_i(s, x(s))ds - x_i^0 - \int_0^t f_i(s, y(s))ds \right|$$

$$= \left| \int_0^t f_i(s, x(s))ds - \int_0^t f_i(s, y(s))ds \right| = \left| \int_0^t (f_i(s, x(s)) - f_i(s, y(s)))ds \right|$$

$$\leq \int_0^t |f_i(s, x(s)) - f_i(s, y(s))|\, ds.$$

On the other hand, for every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, $s, t \in [0, k]$, $i = \overline{1, n}$, considering the hypothesis $\|f(t, u) - f(t, v)\|_\infty \leq L_\infty(f)\|u - v\|_\infty$, for all $t \in \mathbb{R}_+$, $u, v \in \mathbb{R}^n$, we find:

$$|f_i(s, x(s)) - f_i(s, y(s))| \leq \max_{i=\overline{1,n}} |f_i(s, x(s)) - f_i(s, y(s))|$$

$$= \|f(s, x(s)) - f(s, y(s))\|_\infty \leq L_\infty(f)\|x(s) - y(s)\|_\infty$$

$$= L_\infty(f)\|x(s) - y(s)\|_\infty e^{-\tau s} e^{\tau s} \leq L_\infty(f) \sup_{s\in[0,k]} (\|x(s) - y(s)\|_\infty e^{-\tau s}) e^{\tau s}$$

$$= L_\infty(f) p_{k,\infty}(x, y) e^{\tau s},$$

which involves

$$\int_0^t |f_i(s, x(s)) - f_i(s, y(s))|\, ds \leq \int_0^t L_\infty(f) p_{k,\infty}(x, y) e^{\tau s} ds$$

$$= L_\infty(f) p_{k,\infty}(x, y) \int_0^t e^{\tau s} ds = L_\infty(f) p_{k,\infty}(x, y) \frac{e^{\tau s}}{\tau}\Big|_0^t = L_\infty(f) p_{k,\infty}(x, y) \left(\frac{e^{\tau t}}{\tau} - \frac{1}{\tau}\right)$$

$$= \frac{L_\infty(f)}{\tau} p_{k,\infty}(x, y)(e^{\tau t} - 1) \leq \frac{L_\infty(f)}{\tau} p_{k,\infty}(x, y) e^{\tau t}.$$

Hence, for every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, $t \in [0, k]$, $i = \overline{1, n}$, we get

$$|pr_i(T(x)(t)) - pr_i(T(y)(t))| \leq \frac{L_\infty(f)}{\tau} p_{k,\infty}(x, y) e^{\tau t},$$

which implies that

$$\max_{i=\overline{1,n}} |pr_i(T(x)(t)) - pr_i(T(y)(t))| \leq \frac{L_\infty(f)}{\tau} p_{k,\infty}(x, y) e^{\tau t},$$

thus

$$\|T(x)(t) - T(y)(t)\|_\infty \leq \frac{L_\infty(f)}{\tau} p_{k,\infty}(x, y) e^{\tau t},$$

which is equivalent to

$$\|T(x)(t) - T(y)(t)\|_\infty e^{-\tau t} \leq \frac{L_\infty(f)}{\tau} p_{k,\infty}(x, y).$$

Therefore, for every $k \in \mathbb{N}^*$, $x, y \in C(\mathbb{R}_+, \mathbb{R}^n)$, we obtain

$$\sup_{t\in[0,k]} \|T(x)(t) - T(y)(t)\|_\infty e^{-\tau t} \leq \frac{L_\infty(f)}{\tau} p_{k,\infty}(x, y),$$

thus

$$p_{k,\infty}(T(x),T(y)) \leq \frac{L_\infty(f)}{\tau} p_{k,\infty}(x,y).$$

Consequently, for $\tau > L_\infty(f)$ and denoting $\alpha_{k,\infty} := \dfrac{L_\infty(f)}{\tau} \in (0,1)$, we have

$$p_{k,\infty}(T(x),T(y)) \leq \alpha_{k,\infty} p_{k,\infty}(x,y), \quad \text{for all } x,y \in C(\mathbb{R}_+,\mathbb{R}^n), \quad \text{for all } k \in \mathbb{N}^*.$$

According to Lemma 3.3, $(C(\mathbb{R}_+,\mathbb{R}^n),\mathcal{P}_\infty)$ is a sequentially complete gauge space, where $\mathcal{P}_\infty = (p_{k,\infty})_{k\in\mathbb{N}^*}$, and above we proved that $T : C(\mathbb{R}_+,\mathbb{R}^n) \to C(\mathbb{R}_+,\mathbb{R}^n)$ is an operator having the property: for every $k \in \mathbb{N}^*$ there exists $\alpha_{k,\infty} \in (0,1)$ such that

$$p_{k,\infty}(T(x),T(y)) \leq \alpha_{k,\infty} p_{k,\infty}(x,y), \quad \text{for all } x,y \in C(\mathbb{R}_+,\mathbb{R}^n), \quad \text{for all } k \in \mathbb{N}^*.$$

Applying Theorem 2.2 it follows that T has a unique fixed point on $C(\mathbb{R}_+,\mathbb{R}^n)$. Therefore, the system of integral equations (3.6) has a unique solution on $C(\mathbb{R}_+,\mathbb{R}^n)$. Consequently, the system of first order nonlinear ordinary differential equations (2.1), with initial condition $x(0) = x^0 \in \mathbb{R}^n$, has a unique solution on $C(\mathbb{R}_+,\mathbb{R}^n)$.

(ii), (iii) The proof is similar to Theorem 3.1.

In the following, we will investigate some sufficient conditions for fulfilling the hypotheses of Theorem 3.1, Theorem 3.2 and Theorem 3.3. To achieve this goal we need the mean value theorem in several variables, which is presented below.

Lemma 3.4. *([18]) Let $A \subseteq \mathbb{R}^n$ be an open convex set, $g : A \to \mathbb{R}$ a differentiable function on A and u, v points in A. Then there exists a number $\theta \in (0,1)$ such that*

$$g(u) - g(v) = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j}((1-\theta)u + \theta v)(u_j - v_j) = \nabla g((1-\theta)u + \theta v) \cdot (u - v),$$

where ∇ denoted the gradient and $\cdot$ the dot product.

Theorem 3.4. *Let us suppose that the function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous on $\mathbb{R}_+ \times \mathbb{R}^n$, the function $f(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable on $\mathbb{R}^n$ for any $t \in \mathbb{R}_+$ and there exist the numbers $L_{i,j} \geq 0$, $i,j = \overline{1,n}$, not all equal to zero, such that $\left|\dfrac{\partial f_i}{\partial x_j}(t,x)\right| \leq L_{i,j}$ for all $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $i,j = \overline{1,n}$. Then the following affirmations are valid:*

(i) $|f_i(t,u) - f_i(t,v)| \leq \sum_{j=1}^{n} L_{i,j}|u_j - v_j|$, for all $t \in \mathbb{R}_+$, $u,v \in \mathbb{R}^n$, $i = \overline{1,n}$;

(ii) There is a number $L_2(f) := \left(\sum_{i=1}^{n}\sum_{j=1}^{n} L_{i,j}^2\right)^{1/2} > 0$ such that $\|f(t,u) - f(t,v)\|_2 \leq L_2(f)\|u - v\|_2$, for all $t \in \mathbb{R}_+$, $u,v \in \mathbb{R}^n$;

(iii) There is a number $L_1(f) := \max_{j=\overline{1,n}} \left(\sum_{i=1}^{n} L_{i,j}\right) > 0$ such that $\|f(t,u) - f(t,v)\|_1 \leq L_1(f)\|u - v\|_1$, for all $t \in \mathbb{R}_+$, $u,v \in \mathbb{R}^n$;

(iv) There is a number $L_\infty(f) := \max_{i=\overline{1,n}} \left(\sum_{j=1}^{n} L_{i,j} \right) > 0$ *such that* $\|f(t,u) - f(t,v)\|_\infty \le L_\infty(f)\|u - v\|_\infty$*, for all* $t \in \mathbb{R}_+$*,* $u, v \in \mathbb{R}^n$*;*

(v) The system of first order nonlinear ordinary differential equations (2.1), with initial condition $x(0) = x^0 \in \mathbb{R}^n$*, has a unique solution on* $C(\mathbb{R}_+, \mathbb{R}^n)$*;*

(vi) Moreover, if $f(t,u) \ge 0$*, for all* $t \in \mathbb{R}_+$*,* $u \in \mathbb{R}^n$*, this solution is monotonic for* $t \to \infty$*;*

(vii) Further, if $x^0 \ge 0$ *and* $f(t,u) \ge 0$*, for all* $t \in \mathbb{R}_+$*,* $u \in \mathbb{R}^n$*, the solution is positive.*

Proof. (i) Obviously $A := \mathbb{R}^n$ is an open convex set. Let $t \in \mathbb{R}_+$, $i \in \{1, \dots, n\}$ be arbitrary numbers. We define the function $g : \mathbb{R}^n \to \mathbb{R}$, $g(x) = f_i(t,x)$. As the function $f(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable on $\mathbb{R}^n$, it follows that the function $f_i(t,\cdot) : \mathbb{R}^n \to \mathbb{R}$ is differentiable on $\mathbb{R}^n$, hence the function g is differentiable on $\mathbb{R}^n$. Moreover, $\frac{\partial g}{\partial x_j}(x) = \frac{\partial f_i}{\partial x_j}(t,x)$, for all $x \in \mathbb{R}^n$, $j = \overline{1,n}$. We choose $u, v \in \mathbb{R}^n$ be arbitrary points. Applying Lemma 3.4 we deduce that there exists a number $\theta_i \in (0,1)$ such that

$$g(u) - g(v) = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j}((1-\theta_i)u + \theta_i v)(u_j - v_j),$$

thus

$$f_i(t,u) - f_i(t,v) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(t,(1-\theta_i)u + \theta_i v)(u_j - v_j),$$

so

$$|f_i(t,u) - f_i(t,v)| = \left| \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(t,(1-\theta_i)u + \theta_i v)(u_j - v_j) \right|$$

$$\le \sum_{j=1}^{n} \left| \frac{\partial f_i}{\partial x_j}(t,(1-\theta_i)u + \theta_i v) \right| |u_j - v_j| \le \sum_{j=1}^{n} L_{i,j}|u_j - v_j|.$$

As the elements $t \in \mathbb{R}_+$, $u, v \in \mathbb{R}^n$, $i = \overline{1,n}$ were chosen arbitrarily, from the previous relation it follows that $|f_i(t,u) - f_i(t,v)| \le \sum_{j=1}^{n} L_{i,j}|u_j - v_j|$, for all $t \in \mathbb{R}_+$, $u, v \in \mathbb{R}^n$, $i = \overline{1,n}$.

(ii) Using the statement (i) and the Cauchy–Schwarz inequality, for every $t \in \mathbb{R}_+$, $u, v \in \mathbb{R}^n$, $i = \overline{1,n}$ we deduce

$$(f_i(t,u) - f_i(t,v))^2 \le \left(\sum_{j=1}^{n} L_{i,j}|u_j - v_j| \right)^2 \le \sum_{j=1}^{n} L_{i,j}^2 \sum_{j=1}^{n} (u_j - v_j)^2,$$

hence

$$\sum_{i=1}^{n} (f_i(t,u) - f_i(t,v))^2 \le \sum_{i=1}^{n} \left(\sum_{j=1}^{n} L_{i,j}^2 \sum_{j=1}^{n} (u_j - v_j)^2 \right)$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} L_{i,j}^2 \right) \sum_{j=1}^{n} (u_j - v_j)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} L_{i,j}^2 \sum_{j=1}^{n} (u_j - v_j)^2,$$

thus

$$\left(\sum_{i=1}^{n}(f_i(t,u)-f_i(t,v))^2\right)^{1/2} \le \left(\sum_{i=1}^{n}\sum_{j=1}^{n}L_{i,j}^2\right)^{1/2}\left(\sum_{j=1}^{n}(u_j-v_j)^2\right)^{1/2},$$

so

$$\|f(t,u)-f(t,v)\|_2 \le L_2(f)\|u-v\|_2.$$

(iii) Using the statement (i), for every $t\in\mathbb{R}_+$, $u,v\in\mathbb{R}^n$ we find

$$\sum_{i=1}^{n}|f_i(t,u)-f_i(t,v)| \le \sum_{i=1}^{n}\left(\sum_{j=1}^{n}L_{i,j}|u_j-v_j|\right)$$

$$=\sum_{j=1}^{n}\left(\sum_{i=1}^{n}L_{i,j}|u_j-v_j|\right)=\sum_{j=1}^{n}\left(\sum_{i=1}^{n}L_{i,j}\right)|u_j-v_j|$$

$$\le \sum_{j=1}^{n}\max_{j=\overline{1,n}}\left(\sum_{i=1}^{n}L_{i,j}\right)|u_j-v_j| = \max_{j=\overline{1,n}}\left(\sum_{i=1}^{n}L_{i,j}\right)\sum_{j=1}^{n}|u_j-v_j|,$$

hence

$$\|f(t,u)-f(t,v)\|_1 \le L_1(f)\|u-v\|_1.$$

(iv) Using the statement (i), for every $t\in\mathbb{R}_+$, $u,v\in\mathbb{R}^n$, $i=\overline{1,n}$ we deduce

$$|f_i(t,u)-f_i(t,v)| \le \sum_{j=1}^{n}L_{i,j}|u_j-v_j|$$

$$\le \sum_{j=1}^{n}L_{i,j}\max_{j=\overline{1,n}}|u_j-v_j| = \left(\sum_{j=1}^{n}L_{i,j}\right)\max_{j=\overline{1,n}}|u_j-v_j|,$$

hence

$$\max_{i=\overline{1,n}}|f_i(t,u)-f_i(t,v)| \le \max_{i=\overline{1,n}}\left(\sum_{j=1}^{n}L_{i,j}\right)\max_{j=\overline{1,n}}|u_j-v_j|,$$

thus

$$\|f(t,u)-f(t,v)\|_\infty \le L_\infty(f)\|u-v\|_\infty.$$

(v), (vi), (vii) Considering the statements (ii), (iii), (iv) it follows that the hypotheses of Theorem 3.1–3.3 are respectively fulfilled, hence the affirmations (v), (vi), (vii) are valid as a consequence of one of these theorems.

Example 3.1. *Let us consider the following nonlinear system of ordinary differential equations*

$$\begin{cases} x_1'(t)=\alpha_1(t)\sin^2 x_1(t)+\beta_1(t)\ln(1+x_2^2(t))+\gamma_1(t)\\ x_2'(t)=\frac{1}{2}\alpha_2(t)\cos^4 x_1(t)+\beta_2(t)\frac{1}{1+x_2^2(t)}+\gamma_2(t),\end{cases} \tag{3.8}$$

where the functions $\alpha_i,\beta_i,\gamma_i:\mathbb{R}_+\to\mathbb{R}$, $i=\overline{1,n}$, are continuous, positive and bounded on $\mathbb{R}_+$. We denote $\sup_{t\in\mathbb{R}_+}\alpha_i(t)=\alpha_i\in\mathbb{R}_+$, $\sup_{t\in\mathbb{R}_+}\beta_i(t)=\beta_i\in\mathbb{R}_+$, $\sup_{t\in\mathbb{R}_+}\gamma_i(t)=\gamma_i\in\mathbb{R}_+$, $i=\overline{1,2}$, and we suppose that at *least one of the values α_i, β_i, $i = \overline{1,2}$, is non-zero. According to the relation (2.1) we can rewrite the nonlinear system of ordinary differential equations (3.8) in the form*

$$x'(t) = f(t, x(t)),\ t \in \mathbb{R}_+, \tag{3.9}$$

where $x : \mathbb{R}_+ \to \mathbb{R}^2$, $x = (x_i)_{i=\overline{1,2}}$, $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$, $f = (f_i)_{i=\overline{1,2}}$,

$$\begin{cases} f_1(t, x_1, x_2) = \alpha_1(t)\sin^2 x_1 + \beta_1(t)\ln(1 + x_2^2) + \gamma_1(t) \\ f_2(t, x_1, x_2) = \frac{1}{2}\alpha_2(t)\cos^4 x_1 + \beta_2(t)\frac{1}{1+x_2^2} + \gamma_2(t). \end{cases} \tag{3.10}$$

We remark that the function $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous on $\mathbb{R}_+ \times \mathbb{R}^2$, the function $f(t,\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable on $\mathbb{R}^2$ for any $t \in \mathbb{R}_+$. Also, for every $t \in \mathbb{R}_+$, $x \in \mathbb{R}^2$, we have

$$\begin{aligned}
&\tfrac{\partial f_1}{\partial x_1}(t, x_1, x_2) = \alpha_1(t) 2\sin x_1 \cos x_1 = \alpha_1(t)\sin(2x_1),\\
&\tfrac{\partial f_1}{\partial x_2}(t, x_1, x_2) = \beta_1(t)\tfrac{2x_2}{1+x_2^2},\\
&\tfrac{\partial f_2}{\partial x_1}(t, x_1, x_2) = \tfrac{1}{2}\alpha_2(t) 4\cos^3 x_1(-\sin x_1) = -\alpha_2(t)\cos^2 x_1 \sin(2x_1),\\
&\tfrac{\partial f_2}{\partial x_2}(t, x_1, x_2) = \beta_2(t)(-1)(1 + x_2^2)^{-2} 2x_2 = -\beta_2(t)\tfrac{2x_2}{(1+x_2^2)^2},
\end{aligned}$$

hence

$$\begin{aligned}
&\left|\tfrac{\partial f_1}{\partial x_1}(t, x_1, x_2)\right| = |\alpha_1(t)\sin(2x_1)| = \alpha_1(t)\,|\sin(2x_1)| \le \alpha_1(t) \le \alpha_1,\\
&\left|\tfrac{\partial f_1}{\partial x_2}(t, x_1, x_2)\right| = \left|\beta_1(t)\tfrac{2x_2}{1+x_2^2}\right| = \beta_1(t)\tfrac{2|x_2|}{1+|x_2|^2} \le \beta_1(t)\tfrac{1+|x_2|^2}{1+|x_2|^2} = \beta_1(t) \le \beta_1,\\
&\left|\tfrac{\partial f_2}{\partial x_1}(t, x_1, x_2)\right| = \left|-\alpha_2(t)\cos^2 x_1 \sin(2x_1)\right| = \alpha_2(t)\cos^2 x_1\,|\sin(2x_1)| \le \alpha_2(t) \le \alpha_2,\\
&\left|\tfrac{\partial f_2}{\partial x_2}(t, x_1, x_2)\right| = \left|-\beta_2(t)\tfrac{2x_2}{(1+x_2^2)^2}\right| = \beta_2(t)\tfrac{2|x_2|}{(1+|x_2|^2)^2} \le \beta_2(t)\tfrac{(1+|x_2|^2)}{(1+|x_2|^2)^2} = \beta_2(t)\tfrac{1}{1+|x_2|^2} \le \beta_2(t) \le \beta_2,
\end{aligned}$$

thus there exist the numbers $L_{i,j} \ge 0$, $i, j = \overline{1,2}$, not all equal to zero, such that

$$\left|\frac{\partial f_i}{\partial x_j}(t, x)\right| \le L_{i,j} \text{ for all } t \in \mathbb{R}_+,\ x \in \mathbb{R}^2,\ i, j = \overline{1,2},$$

where $L_{1,1} = \alpha_1$, $L_{1,2} = \beta_1$, $L_{2,1} = \alpha_2$, $L_{2,2} = \beta_2$. It follows that the hypotheses of Theorem 3.4 are fulfilled, therefore the following affirmations are valid:

(i) $|f_i(t, u) - f_i(t, v)| \le \alpha_i|u_1 - v_1| + \beta_i|u_2 - v_2|$, *for all $t \in \mathbb{R}_+$, $u, v \in \mathbb{R}^2$, $i = \overline{1,2}$;*

(ii) There is a number $L_2(f) := \left(\alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2\right)^{1/2} > 0$ such that $\|f(t, u) - f(t, v)\|_2 \le L_2(f)\|u - v\|_2$, for all $t \in \mathbb{R}_+$, $u, v \in \mathbb{R}^2$;

(iii) There is a number $L_1(f) := \max\{\alpha_1 + \alpha_2, \beta_1 + \beta_2\} > 0$ such that $\|f(t, u) - f(t, v)\|_1 \le L_1(f)\|u - v\|_1$, for all $t \in \mathbb{R}_+$, $u, v \in \mathbb{R}^2$;

(iv) There is a number $L_\infty(f) := \max\{\alpha_1 + \beta_1, \alpha_2 + \beta_2\} > 0$ such that $\|f(t, u) - f(t, v)\|_\infty \le L_\infty(f)\|u - v\|_\infty$, for all $t \in \mathbb{R}_+$, $u, v \in \mathbb{R}^2$;

(v) The system of first order nonlinear ordinary differential equations (3.8), with initial condition $x(0) = x^0 = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \in \mathbb{R}^2$, has a unique solution on $C(\mathbb{R}_+, \mathbb{R}^2)$;

(vi) This solution is monotonic for $t \to \infty$ (as $f(t, u) \ge 0$, for all $t \in \mathbb{R}_+$, $u \in \mathbb{R}^2$);

(vii) If $\delta_i \ge 0$, $i = \overline{1,2}$, the solution is positive (since $x^0 \ge 0$ and $f(t, u) \ge 0$, for all $t \in \mathbb{R}_+$, $u \in \mathbb{R}^2$).

4. Conclusions and future research

In this article we study the existence and uniqueness of monotonic and positive solutions for systems of first order nonlinear ordinary differential equations on the positive real axis (defined over an unbounded interval). For this approach we costruct three families of pseudometrics and the related gauge spaces and we showed that the problem of solving systems of first order nonlinear ordinary differential equations on the positive real axis can be reduced to finding the fixed points of an operator which acts on these gauge spaces. Applying the theory of fixed points for complete gauge spaces, we proved that under certain conditions the considered operator has a unique fixed point and consequently the system of first order nonlinear ordinary differential equations, with initial condition, has a unique solution on the positive real axis and this solution is monotonic and positive. Also, we investigated some sufficient conditions for fulfilling the hypotheses of the main results obtained in the paper. In the last part of the article, an application of the theoretical results to the study of a class of systems of nonlinear ordinary differential equations was presented.

In a more general context and in connection with the context of this article, it is noteworthy the paper of de Cabral-García et al. [20], in which the authors provide sufficient conditions to ensure when a self-mapping of a non-empty set has a unique fixed point. Precisely, the authors consider a set of measurable functions that is closed, convex, and uniformly integrable. Thus, the proof of the existence result depends on a certain type of sequential compactness for uniformly integrable functions, that is also studied. Moreover, the fixed point theorem is applied in the study of the existence and uniqueness of solutions for some Fredholm and Caputo equations. Since the fractional calculus is an actual topic of interest, this will be a further direction of research to extend the results obtained in the present study.

Furthermore, this paper focuses on first order differential equations, and we know that the theory of first order differential equations is also relevant to approach high order differential equations, via comparative techniques. The fixed point theory play a key role in the development of iteration methods for numerical approximation of solutions. In this sense, it is worth mentioning the paper of Akgun and Rasulov [21], which opens a window on this topic, by proposing a new iterative method for third-order boundary value problems based on embedding Green's function. The results of the mentioned paper extend and generalize the corresponding results in the literature. Moreover, the existence and uniqueness theorems are established, and necessary conditions are derived for convergence. Considering this paper as the basis of research, another direction of extension of the present study could be its application to the construction of new fixed point iterative methods of approximation of solutions for high order differential equations.

Finally, we would like to mention a recently published survey of Debnath et al. [22], which contains a systematic knowledge and the current state-of-the-art development of the fixed point theory and its applications in several areas. This book could serve to discover new directions to extend the results established in the present paper.

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Conflict of interest

The authors declare there is no conflict of interest.

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