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Research article

Boundedness of a predator-prey model with density-dependent motilities and stage structure for the predator

Ailing Xiang and Liangchen Wang*

School of Science, Chongqing University of Posts and Telecommunications, Chongqing 400065, China

* **Correspondence:** Email: wanglc@cqupt.edu.cn.

Abstract: In this paper, we consider a predator-prey model with density-dependent prey-taxis and stage structure for the predator. We establish the existence of classical solutions with uniform-in-time bound in a one-dimensional case. In addition, we prove that the solution stabilizes to the prey-only steady state under some conditions.

Keywords: biological predator-prey model; boundedness; density-dependent motilities; stage structure; prey-taxis

1. Introduction

This paper deals with the predator-prey model with density-dependent motilities and stage structure for the predator

$$\begin{cases} u_t = (d_1(w)u)_{xx} + bv - cu, & x \in \Omega, \quad t > 0, \\ v_t = (d_2(u)v)_{xx} + kuw - v, & x \in \Omega, \quad t > 0, \\ w_t = d_3 w_{xx} + aw - w^2 - uw - rvw, & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \ \ v(x,0) = v_0(x), \ \ w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}$ and $\partial/\partial\nu$ represents the outer unit normal vector of $\partial\Omega$, where $u = u(x,t), v = v(x,t)$ and $w = w(x,t)$ are the densities of the mature predator, immature predator and prey at position x and time t, respectively. d_3, a, b, c, k, r are positive constants and more details of the parameters can be found in [1, 2]. The terms $(d_1(w)u)_{xx}$ and $(d_2(u)v)_{xx}$ state that the motility functions $d_1(w)$ and $d_2(u)$ have some influence on the diffusion of mature predator and immature predator.

Biological predator-prey model plays a critical role in survival and reproduction of organisms, especially the predator-prey system with stage structure of predator describes the biological predator-prey phenomenon and its irregular movement more vividly(see [3–7] and reference therein). Recently, the following stage structure of predator with taxis mechanisms model has been studied by Wang and Wang [2]:

$$\begin{cases} u_t = d_1\Delta u - \chi\nabla\cdot(u\nabla w) + bv - cu, & x\in\Omega, \quad t>0,\\ v_t = d_2\Delta v - \rho\nabla\cdot(v\nabla u) + kuw - v, & x\in\Omega, \quad t>0,\\ w_t = d_3\Delta w + aw - w^2 - uw - rvw, & x\in\Omega, \quad t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, & x\in\partial\Omega, \quad t>0,\\ u(x,0)=u_0(x), \ \ v(x,0)=v_0(x), \ \ w(x,0)=w_0(x), & x\in\Omega. \end{cases} \tag{1.2}$$

In the case $n = 1$ with $\rho > 0$ and $n = 2$ with $\rho = 0$, the authors [2] first established that the solutions of problem (1.2) are global existence and boundedness. Secondly, the linearized stability of normal steady state and predator-free steady state are obtained by using local bifurcation and Hopf bifurcation theory. Moreover, they proved the global stability of predator-free steady state. On the other hand, many scholars have also studied the stage state for prey [8, 9] and the different state of the predator [10].

In order to describe the movement of species more meaningfully, we illustrate a chemotaxis system with density-dependent motility to describe the motility law of predators. At present, this kind of model is mostly used in the field of chemical signal substances. The classic model is proposed in [11]

$$\begin{cases} u_t = \Delta(\gamma(v)u) + \mu u(1-u), & x\in\Omega, \quad t>0,\\ v_t = \Delta v - v + u, & x\in\Omega, \quad t>0, \end{cases} \tag{1.3}$$

where $u(x,t)$ is the densities of bacteria and $v(x,t)$ is the concentration of AHL at position x and time t. This system describes bacteria with logistic sources whose diffusion rate depends on the motion function $\gamma(v)$, which considers the repressive effect of AHL concentration on bacteria motility by supposing $\gamma'(v) < 0$. This diffusion mechanism is called "density suppression motility" in [12, 13]. Therefore, it is a very interesting phenomenon and has been widely studied. If $\mu > 0$, Jin et.al [14] proved that the problem in two dimensions possesses a global classical solution and coexistence steady state is globally asymptotically stable. Yoon and Kim [15] obtained a global classical solution with $\mu = 0$ and a particular form of $\gamma(v) = \frac{c_0}{v^k}, c_0 > 0, k > 0$ in any dimensions provided c_0 is small. Moreover, Tao and Winkler [16] proved that some weak solutions exist globally under high dimensional conditions and in a specific three-dimensional case, this solution is bounded and classical with $\mu = 0$. We refer the readers to [17–24] for other interesting results on density-suppressed model.

Recently, this kind of model is also studied in the predator-prey mode [25,26]. In [26], the following density-dependent model with homogeneous Neumann boundary conditions is proposed

$$\begin{cases} u_t = \Delta(d_1(w)u) + u(a_1w - b_1u - c_1v), & x\in\Omega, \quad t>0,\\ v_t = \Delta(d_2(w)v) + v(a_2w - b_2u - c_2v), & x\in\Omega, \quad t>0,\\ w_t = \Delta w - w(u+v) + \mu w(m(x)-w), & x\in\Omega, \quad t>0, \end{cases} \tag{1.4}$$

when $b_1 = c_2, c_1 = c, b_2 = b$ and $m(x) = 1$, the model (1.4) exists the global bounded classical solution, and asymptotic behavior is derived in different parameter regimes. $d_i(w)(i = 1, 2)$ indicates the resource dependent diffusion rate of species with monotonic properties: $d_i'(w) < 0(i = 1, 2)$, which is consistent with the fact that predators reduce their random diffusion when encountering the prey observed by kareiva and odell [27]. The major difference between (1.1) and (1.4) is that the motility of immature predators are influenced by mature predators rather than prey and mature predators grow from immature predators. Hence, due to its biological significance, the density-dependent model has attracted the interest of many scholars.

The goal of this paper is to establish global existence and large time behavior of classical solutions to the model (1.1). We shall suppose that there exist $\eta_2 > \eta_1 > 0$ such that $d_1(w)$ and $d_2(u)$ satisfy

(H_1) $d_1(w) \in C^3([0,\infty))$, $d_1(w) > 0$ and $d_1'(w) \le 0$ for all $w \ge 0$,
(H_2) $d_2(u) \in C^3([0,\infty))$, $\eta_1 \le d_2(u) \le \eta_2$ for all $u \ge 0$.

In this paper, the main results are stated as below. Our first result derives global boundedness of classical solution to (1.1).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}$ be a bounded domain with smooth boundary and the assumptions $(H_1)-(H_2)$ hold. Suppose that the parameters $a, b, c, k, r > 0$ and $(u_0, v_0, w_0) \in \left[W^{1,\infty}(\Omega)\right]^3$ with $u_0, v_0, w_0 \ge 0$. Then the model (1.1) has a unique nonnegative classical solution (u, v, w) satisfying*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \quad \text{for all } t > 0, \tag{1.5}$$

where $C > 0$ is a constant. Particularly, we have $0 \le w \le M$, where

$$M := \max\{a, \|w_0\|_{L^\infty}\}.$$

The second result is that we consider the global stability of the classical solution obtained in Theorem 1.1.

Theorem 1.2. *Let $(u_0, v_0, w_0) \in \left[W^{1,\infty}(\Omega)\right]^3$ with $u_0, v_0 \ge 0 (\not\equiv 0)$ and $w_0 > 0$ in $\bar{\Omega}$. The solution (u, v, w) of (1.1) obtained in Theorem 1.1 has the following properties: If the positive parameters a, b, c, k and r satisfy $\frac{c-abk}{kcar} > 1$, then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t) - a\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \tag{1.6}$$

In the paper, for simplicity, we abbreviate $\int_0^t \int_\Omega f(\cdot,s)dxds$, $\int_\Omega f(\cdot,s)ds$, $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{1,p}(\Omega)}$ as $\int_0^t \int_\Omega f$, $\int_\Omega f$, $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$, respectively. Moreover, C stands for a generic positive constant which may alter from line to line and is independent of time.

The organizational structure of this paper is as below. In Section 2, we show the local existence of a solution to (1.1) and some preliminary results are given. In Section 3, we establish global existence and boundedness for the model (1.1) and proof of Theorem 1.1. Section 4, we obtain the prey-only global stability to achieve Theorem 1.2.

2. Preliminaries

We first give the existence of local solutions of (1.1) by using Amann's theorem [28, 29](cf. also [30, Lemma 1.1] or [31, Lemma 2.6]).

Lemma 2.1. *(Local existence). Let $\Omega \subset \mathbb{R}$ be a bounded domain with smooth boundary. Suppose that the parameters $a, b, c, k, r > 0$ and the assumptions $(H_1) - (H_2)$ hold. Assume that $(u_0, v_0, w_0) \in \left[W^{1,\infty}(\Omega)\right]^3$ with $u_0, v_0, w_0 \geq 0$. Then there exists a constant $T_{\max} \in (0, \infty]$ such that the problem (1.1) has a unique nonnegative classical solution (u, v, w) and satisfies*

$$(u, v, w) \in \left[C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}) \cap L^{\infty}_{loc}([0, T_{\max}); W^{1,\infty}(\Omega))\right]^3,$$

and which is such that if $T_{max} < \infty$,

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,\infty}(\Omega)} \to \infty \quad as\ \ t \nearrow T_{\max}.$$

Moreover, if the initial data $(u_0, v_0, w_0) \in \left[W^{1,\infty}(\Omega)\right]^3$ with $u_0, v_0 \geq 0 (\not\equiv 0)$ and $w_0 > 0$ in $\bar{\Omega}$, then the solution of (1.1) satisfies $u, v, w > 0$ in $\overline{\Omega} \times (0, T_{max})$.

Lemma 2.2. *([32, Lemma 2.2]) Let the assumptions in Lemma 2.1 hold. Then the solution (u, v, w) of system (1.1) fulfills that*

$$0 \leq w(x, t) \leq M \quad for\ all\ x \in \Omega, t > 0, \tag{2.1}$$

where $M := \max\{a, \|w_0\|_{L^\infty}\}$, and it also founds that

$$\limsup_{t\to\infty} w(x, t) \leq a \quad for\ all\ x \in \bar{\Omega}. \tag{2.2}$$

In order to prove our results, we will quote the following lemma.

Lemma 2.3. *([33, Lemma 2.3]) Let $T > 0$ and $\tau \in (0, T)$, assume that $a, b > 0$, and $y : [0, T) \to [0, \infty)$ is absolutely continuous and satisfies*

$$y'(t) + ay(t) \leq b(t)$$

with some nonnegative function $b(t) \in L^1_{loc}([0, T))$ fulfilling

$$\int_t^{t+\tau} b(s)ds \leq b \quad for\ all\ \ t \in [0, T - \tau).$$

Then

$$y(t) \leq \max\left\{y(0) + b, \frac{b}{a\tau} + 2b\right\} \quad for\ all\ \ t \in (0, T).$$

Lemma 2.4. *([34, Lemma 2.4]) Let $T > 0$ and $\tau \in (0, T)$, assume that $\alpha, \beta > 0$, and $y : [0, T) \to [0, \infty)$ is absolutely continuous and satisfies*

$$y'(t) + a(t)y(t) \leq b(t)y(t) + c(t)$$

with the nonnegative functions $a(t), b(t), c(t) \in L^1_{loc}([0, T))$ fulfilling

$$\sup_{0\leq t\leq T} \int_t^{t+\tau} b(s)ds \leq \alpha \quad for\ all\ \ t \in [0, T - \tau)$$

and

$$\sup_{0\le t\le T}\int_t^{t+\tau} c(s)ds \le \beta \quad \text{for all } t\in[0,T-\tau).$$

Moreover, there also exists a positive constant ρ satisfies

$$\int_t^{t+\tau} a(s)ds - \int_t^{t+\tau} b(s)ds > \rho \quad \text{for all } t\in[0,T-\tau).$$

Then

$$y(t) \le e^{\alpha}\left(y(0) + \frac{\beta e^{\alpha}}{1-e^{\rho}} + \beta\right) \quad \text{for all } t\in(0,T).$$

Lemma 2.5. *Under the assumptions in Theorem 1.1, the solution (u, v, w) of (1.1) fulfills*

$$\int_\Omega u \le C \quad \text{and} \quad \int_\Omega v \le C \quad \text{for all } t\in(0,T_{max}), \tag{2.3}$$

where $C > 0$ is a constant.

Proof. The first equation of (1.1) adds the second equation of (1.1) multiplied by $b + 1$ and adds the third equation of (1.1) multiplied by $k(b + 1)$, then integrating we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (u+(b+1)v+k(b+1)w) + \int_\Omega (cu+v+w)\\
&= (ka(b+1)+1)\int_\Omega w - k(b+1)\int_\Omega w^2 - k(b+1)\int_\Omega rvw\\
&\le (ka(b+1)+1)M|\Omega|.
\end{aligned} \tag{2.4}$$

Using Gronwall's inequality to (2.4), we obtain (2.3) immediately. □

Next, we shall obtain $W^{1,p}$ bound for the prey $w(\cdot,t)$.

Lemma 2.6. *Under the assumptions in Theorem 1.1 and (u, v, w) is a solution of (1.1), for any $p > 1$, there exists a constant $C > 0$ such that*

$$\|w_x(\cdot,t)\|_p \le C \quad \text{for all } t\in(0,T_{max}). \tag{2.5}$$

Proof. By the variation-of-constants method, w can be written as

$$w(\cdot,t) = e^{d_3 t\Delta}w_0 + \int_0^t e^{d_3(t-s)\Delta}\left(aw - w^2 - uw - rvw\right),$$

using (2.1) and (2.3), then there exists a constant $c_1 > 0$ satisfies

$$\left\|aw - w^2 - uw - rvw\right\|_1 \le \|aw\|_1 + \left\|w^2\right\|_1 + \|uw\|_1 + \|rvw\|_1 \le c_1. \tag{2.6}$$

According to standard $L^p - L^q$ estimates in [35, Lemma 1.3], there exist $\lambda > 0$ and some constants $c_i > 0 (i = 2, 3)$ such that

$$\begin{aligned}
\|w_x(\cdot,t)\|_p &\le c_2\|w_0\|_{1,\infty} + c_2\int_0^t e^{-\lambda(t-s)}\left(1+(t-s)^{-1+\frac{1}{2p}}\right)\left\|aw - w^2 - uw - rvw\right\|_1\\
&\le c_2\|w_0\|_{1,\infty} + c_1c_2\int_0^t e^{-\lambda(t-s)}\left(1+(t-s)^{-1+\frac{1}{2p}}\right)\\
&\le c_3
\end{aligned}$$

for all $t \in (0, T_{max})$. Hence, the proof of (2.5) is completed. □

Next, we apply the method of [25, Lemma 2.3] to obtain the following estimates.

Lemma 2.7. *Under the conditions in Theorem 1.1 and (u, v, w) is a solution of (1.1). Then there exists a constant $C > 0$ such that*

$$\int_t^{t+\tau}\int_\Omega u^2 \le C \quad \text{and} \quad \int_t^{t+\tau}\int_\Omega v^2 \le C \quad \text{for all } t \in (0, T_{max} - \tau), \tag{2.7}$$

where $\tau = \min\left\{1, \frac{T_{max}}{2}\right\}$.

Proof. Let $\mathcal{A}$ represents the self-adjoint realization of $-\Delta + \delta$ ([36, Lemma 3.1]) under homogeneous Neumann boundary conditions in $L^2(\Omega)$ and

$$0 < \delta < \min\left\{\frac{c}{d_1(0)}, \frac{1}{(b+1)\eta_2}\right\}, \tag{2.8}$$

where $\eta_2 > 0$ is from (H_2) and

$$d_1(0) = \max_{0 \le w \le M} d_1(w)$$

due to (H_1) and Lemma 2.2. Since $\delta > 0$, $\mathcal{A}$ has an order-preserving bounded inverse $\mathcal{A}^{-1}$ on $L^2(\Omega)$, then there exists a constant $c_1 > 0$ such that

$$\left\|\mathcal{A}^{-1}\psi\right\|_2 \le c_1\|\psi\|_2 \quad \text{for all } \psi \in L^2(\Omega) \tag{2.9}$$

and

$$\left\|\mathcal{A}^{-\frac{1}{2}}\psi\right\|_2^2 = \int_\Omega \psi \cdot \mathcal{A}^{-1}\psi \le c_1\|\psi\|_2^2 \quad \text{for all } \psi \in L^2(\Omega). \tag{2.10}$$

From (1.1), we have

$$\begin{aligned}
&(u + (b+1)v + k(b+1)w)_t \\
&= \Delta(d_1(w)u + (b+1)d_2(u)v + k(b+1)d_3w) - cu - v + k(b+1)\left(aw - w^2 - rvw\right),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&(u + (b+1)v + k(b+1)w)_t + \mathcal{A}(d_1(w)u + (b+1)d_2(u)v + k(b+1)d_3w) \\
&= \delta(d_1(w)u + (b+1)d_2(u)v + k(b+1)d_3w) - cu - v + k(b+1)\left(aw - w^2 - rvw\right) \\
&= (\delta d_1(w) - c)u + (\delta(b+1)d_2(u) - 1)v + k(b+1)\left(\delta d_3w + aw - w^2 - rvw\right).
\end{aligned} \tag{2.11}$$

Noting the facts (2.1), (2.8) and $(H_1) - (H_2)$, one can find $c_2 := kM(b+1)(\delta d_3 + a) > 0$ such that

$$\begin{aligned}
&(\delta d_1(w) - c)u + (\delta(b+1)d_2(u) - 1)v + k(b+1)\left(\delta d_3w + aw - w^2 - rvw\right) \\
&\le (\delta d_1(0) - c)u + (\delta(b+1)\eta_2 - 1)v + c_2 \\
&\le c_2.
\end{aligned} \tag{2.12}$$

Substituting (2.12) into (2.11), one has

$$(u + (b+1)v + k(b+1)w)_t + \mathcal{A}(d_1(w)u + (b+1)d_2(u)v + k(b+1)d_3w) \le c_2,$$

hence, multiplying the above inequality by $\mathcal{A}^{-1}(u+(b+1)v+k(b+1)w) \geq 0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(u+(b+1)v+k(b+1)w)|^2 \\
&\quad + \int_\Omega (u+(b+1)v+k(b+1)w)(d_1(w)u+(b+1)d_2(u)v+k(b+1)d_3w) \\
&\leq c_2 \int_\Omega \mathcal{A}^{-1}(u+(b+1)v+k(b+1)w),
\end{aligned}$$

which together with the fact $(H_1)-(H_2)$, we can find $d_1(M) = \min_{0\leq w\leq M} d_1(w)$ and $c_3 := \min\{d_1(M), \eta_1, d_3\}$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(u+(b+1)v+k(b+1)w)|^2 + c_3\int_\Omega (u+(b+1)v+k(b+1)w)^2 \\
&\leq c_2\int_\Omega \mathcal{A}^{-1}(u+(b+1)v+k(b+1)w).
\end{aligned} \tag{2.13}$$

By (2.9) and (2.10), we can obtain that

$$\begin{aligned}
&\frac{c_3}{4c_1}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}(u+(b+1)v+k(b+1)w)|^2 + c_2\int_\Omega \mathcal{A}^{-1}(u+(b+1)v+k(b+1)w) \\
&\leq \frac{c_3}{4}\int_\Omega (u+(b+1)v+k(b+1)w)^2 + c_1c_2|\Omega|^{\frac{1}{2}}\|u+(b+1)v+k(b+1)w\|_2 \\
&\leq \frac{c_3}{2}\int_\Omega (u+(b+1)v+k(b+1)w)^2 + \frac{c_1^2c_2^2|\Omega|}{c_3}.
\end{aligned}$$

Therefore, combining with (2.13), and denoting $y_1(t) := \int_\Omega |\mathcal{A}^{-\frac{1}{2}}(u+(b+1)v+k(b+1)w)|^2$, one has

$$y_1'(t) + \frac{c_3}{2c_1}y_1(t) + c_3\int_\Omega (u+(b+1)v+k(b+1)w)^2 \leq \frac{2c_1^2c_2^2|\Omega|}{c_3}.$$

Then using Gronwall's inequality implies $y_1(t) \leq c_4$ with some constant $c_4 > 0$, thus

$$\begin{aligned}
\int_t^{t+\tau}\int_\Omega u^2 &\leq \int_t^{t+\tau}\int_\Omega (u+(b+1)v+k(b+1)w)^2 \\
&\leq \frac{y_1(t)}{c_3} + \frac{2c_1^2c_2^2|\Omega|\tau}{c_3^2} \\
&\leq \frac{c_4}{c_3} + \frac{2c_1^2c_2^2|\Omega|}{c_3^2} \quad \text{for all } t \in (0, T_{max}-\tau),
\end{aligned}$$

because $\tau \leq 1$. Similarly, we have

$$\int_t^{t+\tau}\int_\Omega v^2 \leq \frac{c_4}{c_3} + \frac{2c_1^2c_2^2|\Omega|}{c_3^2} \quad \text{for all } t \in (0, T_{max}-\tau).$$

Hence, we can obtain (2.7). □

In addition, as the result of Lemma 2.7, we can deduce the following results.

Lemma 2.8. *Under the conditions in Theorem 1.1 and* (u, v, w) *is a solution of (1.1). Then there exists a constant* $C > 0$ *such that*

$$\int_t^{t+\tau}\int_\Omega w_{xx}^2 \le C \quad \text{for all } t \in (0, T_{max} - \tau), \tag{2.14}$$

where $\tau = \min\{1, \frac{T_{max}}{2}\}$.

Proof. Testing the third equation of (1.1) by $-w_{xx}$, using Young's inequality and (2.1), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega w_x^2 &= -d_3\int_\Omega w_{xx}^2 - a\int_\Omega ww_{xx} + \int_\Omega w^2w_{xx} + \int_\Omega uww_{xx} + r\int_\Omega vww_{xx}\\
&\le -d_3\int_\Omega w_{xx}^2 + \frac{d_3}{2}\int_\Omega w_{xx}^2 + \frac{2a^2}{d_3}\int_\Omega w^2\\
&\quad + \frac{2}{d_3}\int_\Omega w^4 + \frac{2}{d_3}\int_\Omega u^2w^2 + \frac{2r^2}{d_3}\int_\Omega v^2w^2\\
&\le -\frac{d_3}{2}\int_\Omega w_{xx}^2 + \frac{2M^2}{d_3}\int_\Omega u^2 + \frac{2r^2M^2}{d_3}\int_\Omega v^2 + c_1,
\end{aligned}$$

where $c_1 := \frac{2M^2|\Omega|(a^2+M^2)}{d_3}$, which yields

$$\frac{d}{dt}\int_\Omega w_x^2 + d_3\int_\Omega w_{xx}^2 \le \frac{4M^2}{d_3}\int_\Omega u^2 + \frac{4r^2M^2}{d_3}\int_\Omega v^2 + 2c_1. \tag{2.15}$$

By the Gagliardo-Nirenberg inequality and the fact $\|w\|_2 \le M|\Omega|^{\frac{1}{2}}$, there exist some constants $c_2, c_3 > 0$ such that

$$\int_\Omega w_x^2 = \|w_x\|_2^2 \le c_2\left(\|w_{xx}\|_2\|w\|_2 + \|w\|_2^2\right) \le \frac{d_3}{2}\|w_{xx}\|_2^2 + c_3. \tag{2.16}$$

Combining (2.15) and (2.16), let $c_4 := 2c_1 + c_3$, then we have

$$\frac{d}{dt}\int_\Omega w_x^2 + \int_\Omega w_x^2 + \frac{d_3}{2}\int_\Omega w_{xx}^2 \le \frac{4M^2}{d_3}\int_\Omega u^2 + \frac{4r^2M^2}{d_3}\int_\Omega v^2 + c_4. \tag{2.17}$$

Let $y(t) := \int_\Omega w_x^2$ and $b(t) := \frac{4M^2}{d_3}\int_\Omega u^2 + \frac{4r^2M^2}{d_3}\int_\Omega v^2 + c_4$. From (2.17) we have

$$y'(t) + y(t) + \frac{d_3}{2}\int_\Omega w_{xx}^2 \le b(t) \quad \text{for all } t \in (0, T_{max}), \tag{2.18}$$

by Lemma 2.7 implies there exists a constant $c_5 > 0$ such that $\int_t^{t+\tau}\int_\Omega(u^2 + v^2) \le c_5$, therefore, we have

$$\int_t^{t+\tau} b(s) \le c_6 := \frac{4M^2c_5\max\{1, r^2\}}{d_3} + c_4 \quad \text{for all } t \in (0, T_{max} - \tau),$$

because $\tau \le 1$. Using (2.18) and Lemma 2.3 to ensure that

$$y(t) \le c_7 := \max\left\{\int_\Omega (w_0)_x^2 + c_6, \frac{c_6}{\tau} + 2c_6\right\} \quad \text{for all } t \in (0, T_{max}).$$

Therefore, an integration of (2.18) over $(t, t+\tau)$ yields

$$y(t+\tau) + \int_t^{t+\tau} y(s) + \frac{d_3}{2}\int_t^{t+\tau}\int_\Omega w_{xx}^2 \le y(t) + \int_t^{t+\tau} b(s) \le c_7 + c_6$$

for all $t \in (0, T_{max} - \tau)$, which in view of the nonnegativity of y implies (2.14). □

3. Boundedness of solutions

In the first, we will obtain a priori $L^2 - \textit{estimate}$ of the predator u.

Lemma 3.1. *Let the assumptions in Theorem 1.1 hold, then there exists a constant $C > 0$ such that*

$$\|u(\cdot,t)\|_2 \le C \quad \textit{for all } t \in (0, T_{max}). \tag{3.1}$$

Proof. Testing the first equation of (1.1) by u, integrating the result by part and using Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + c\int_\Omega u^2 + \int_\Omega d_1(w)u_x^2 \\
&= -\int_\Omega d_1'(w)uu_xw_x + b\int_\Omega uv \\
&\le \frac{1}{2}\int_\Omega d_1(w)u_x^2 + \frac{1}{2}\int_\Omega \frac{(d_1'(w))^2}{d_1(w)}u^2w_x^2 + \frac{b^2}{2c}\int_\Omega v^2 + \frac{c}{2}\int_\Omega u^2,
\end{aligned}$$

which yields

$$\frac{d}{dt}\int_\Omega u^2 + c\int_\Omega u^2 + \int_\Omega d_1(w)u_x^2 \le \int_\Omega \frac{(d_1'(w))^2}{d_1(w)}u^2w_x^2 + \frac{b^2}{c}\int_\Omega v^2, \tag{3.2}$$

by Lemma 2.6 implies $\|w_x\|_2 \le c_1$ with some $c_1 > 0$, thus using (H_1) and (2.1), we have from (3.2) that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^2 + c\int_\Omega u^2 + d_1(M)\int_\Omega u_x^2 &\le K_1^2\|u\|_\infty^2 \int_\Omega w_x^2 + \frac{b^2}{c}\int_\Omega v^2 \\
&\le K_1^2c_1^2\|u\|_\infty^2 + \frac{b^2}{c}\int_\Omega v^2,
\end{aligned} \tag{3.3}$$

where $K_1 := \frac{\max\limits_{0\le w\le M}|d_1'(w)|}{\sqrt{d_1(M)}}$. By the Gagliardo-Nirenberg inequality, Young's inequality and (2.3), there exist constants $c_i > 0 (i = 2, 3)$ satisfy

$$K_1^2c_1^2\|u\|_\infty^2 \le c_2\left(\|u_x\|_2^{\frac{4}{3}}\|u\|_1^{\frac{2}{3}} + \|u\|_1^2\right) \le \frac{d_1(M)}{2}\|u_x\|_2^2 + c_3.$$

This together with (3.3), one has

$$\frac{d}{dt}\int_\Omega u^2 + c\int_\Omega u^2 + \frac{d_1(M)}{2}\int_\Omega u_x^2 \le \frac{b^2}{c}\int_\Omega v^2 + c_3. \tag{3.4}$$

Using (2.7) and Lemma 2.3, we derive (3.1). □

We are now in the position to derive some estimates for u .

Lemma 3.2. *Let the assumptions in Theorem 1.1 hold and (u, v, w) be a solution of (1.1). Then there exists a constant $C > 0$ such that*

$$\int_\Omega u_x^2(\cdot,t) \le C \quad \textit{for all } t \in (0, T_{max}), \tag{3.5}$$

$$\int_t^{t+\tau}\int_\Omega u_{xx}^2(\cdot,t) \le C \quad \textit{for all } t \in (0, T_{max} - \tau) \tag{3.6}$$

and

$$\|u(\cdot,t)\|_\infty \le C \quad \text{for all } t \in (0, T_{max}), \tag{3.7}$$

where $\tau = \min\{1, \frac{T_{max}}{2}\}$.

Proof. Testing the first equation of (1.1) by $-u_{xx}$ and using Young's inequality, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega u_x^2 &= -\int_\Omega u_{xx}\Big(d_1(w)u_{xx} + d_1'(w)uw_{xx} + 2d_1'(w)u_xw_x + d_1''(w)uw_x^2 + bv - cu\Big)\\
&\le -\int_\Omega d_1(w)u_{xx}^2 + \frac{5}{8}\int_\Omega d_1(w)u_{xx}^2 + 2\int_\Omega \frac{(d_1'(w))^2}{d_1(w)}u^2w_{xx}^2 + 2\int_\Omega \frac{c^2}{d_1(w)}u^2\\
&\quad + 8\int_\Omega \frac{(d_1'(w))^2}{d_1(w)}u_x^2w_x^2 + 2\int_\Omega \frac{(d_1''(w))^2}{d_1(w)}u^2w_x^4 + 2\int_\Omega \frac{b^2}{d_1(w)}v^2\\
&\le -\frac{3}{8}\int_\Omega d_1(w)u_{xx}^2 + 2\|u\|_\infty^2\int_\Omega \frac{(d_1'(w))^2}{d_1(w)}w_{xx}^2 + 2\int_\Omega \frac{c^2}{d_1(w)}u^2\\
&\quad + 8\|u_x\|_\infty^2\int_\Omega \frac{(d_1'(w))^2}{d_1(w)}w_x^2 + 2\|u\|_\infty^2\int_\Omega \frac{(d_1''(w))^2}{d_1(w)}w_x^4 + 2\int_\Omega \frac{b^2}{d_1(w)}v^2.
\end{aligned}$$

From Lemma 2.6, we choose $p = 2, 4$, then there exist $c_1, c_2 > 0$ such that $\|w_x\|_2 \le c_1, \|w_x\|_4 \le c_2$, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_x^2 + \frac{3d_1(M)}{4}\int_\Omega u_{xx}^2 &\le 4K_1^2\|u\|_\infty^2\int_\Omega w_{xx}^2 + 16K_1^2c_1^2\|u_x\|_\infty^2 + 4K_2^2c_2^4\|u\|_\infty^2\\
&\quad + \frac{4b^2}{d_1(M)}\int_\Omega v^2 + \frac{4c^2}{d_1(M)}\int_\Omega u^2.
\end{aligned} \tag{3.8}$$

where $K_2 := \frac{\max_{0\le w\le M}|d_1''(w)|}{\sqrt{d_1(M)}}$. Using the Gagliardo-Nirenberg inequality and Lemma 3.1, for each $\varepsilon > 0$ one can find some $c_\varepsilon > 0$ and $c_i > 0 (i = 3, 4, 5)$ such that

$$\|u\|_\infty^2 \le c_3\left(\|u_x\|_2^{\frac{4}{3}}\|u\|_1^{\frac{2}{3}} + \|u\|_1^2\right) \le \varepsilon\|u_x\|_2^2 + c_\varepsilon \tag{3.9}$$

and

$$16K_1^2c_1^2\|u_x\|_\infty^2 \le c_4\left(\|u_{xx}\|_2^{\frac{3}{2}}\|u\|_2^{\frac{1}{2}} + \|u\|_2^2\right) \le \frac{d_1(M)}{4}\|u_{xx}\|_2^2 + c_5. \tag{3.10}$$

Using Lemma 3.1 again, for some $c_6 > 0$, we have $\|u(\cdot,t)\|_2 \le c_6$. Substituting (3.9)-(3.10) into (3.8), we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_x^2 + \frac{d_1(M)}{2}\int_\Omega u_{xx}^2 &\le 4K_1^2\varepsilon\|u_x\|_2^2\|w_{xx}\|_2^2 + 4K_1^2c_\varepsilon\|w_{xx}\|_2^2\\
&\quad + 4K_2^2c_2^4\varepsilon\|u_x\|_2^2 + \frac{4b^2}{d_1(M)}\int_\Omega v^2 + c_7,
\end{aligned} \tag{3.11}$$

where $c_7 := c_5 + 4K_2^2c_2^4c_\varepsilon + \frac{4c^2c_6^2}{d_1(M)}$. Using (3.1), for some $c_8, c_9 > 0$, we obtain

$$\left(4K_2^2c_2^4\varepsilon + 1\right)\|u_x\|_2^2 \le c_8\left(\|u_{xx}\|_2\|u\|_2 + \|u\|_2^2\right) \le \frac{d_1(M)}{4}\|u_{xx}\|_2^2 + c_9.$$

Combining it with (3.11), there exists $c_{10} > 0$ satisfying

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega u_x^2 + \int_\Omega u_x^2 + \frac{d_1(M)}{4}\int_\Omega u_{xx}^2 \\
&\leq 4K_1^2\varepsilon\|u_x\|_2^2\|w_{xx}\|_2^2 + 4K_1^2c_\varepsilon\|w_{xx}\|_2^2 + \frac{4b^2}{d_1(M)}\int_\Omega v^2 + c_{10}.
\end{aligned}
\tag{3.12}
$$

From Lemma 2.8, one has $\int_t^{t+\tau}\int_\Omega w_{xx}^2 \leq c_{11}$ with some $c_{11} > 0$. Let $a(t) := 1, b(t) := 4K_1^2\varepsilon\|w_{xx}\|_2^2$ and $c(t) := 4K_1^2c_\varepsilon\|w_{xx}\|_2^2 + \frac{4b^2}{d_1(M)}\int_\Omega v^2 + c_9$, choosing $\varepsilon = \frac{\tau}{8K_1^2c_{11}} > 0$ such that $\int_t^{t+\tau}a(s)ds - \int_t^{t+\tau}b(s)ds = \frac{\tau}{2} > 0$. Hence, using Lemma 2.4, we can derive the boundedness of $\int_\Omega u_x^2(\cdot,t)$ for all $t \in (0, T_{max})$. Furthermore, (3.6) can be obtained upon an integration in time for (3.12). Finally, using the boundedness of $\int_\Omega u_x^2(\cdot,t)$ and (3.9), which implies (3.7). □

Now we establish some estimates of v.

Lemma 3.3. *Let the assumptions in Theorem 1.1 hold, then there exists a constant $C > 0$ such that*

$$\int_\Omega v^2(\cdot,t) \leq C \quad \textit{for all } t \in (0, T_{max}) \tag{3.13}$$

and

$$\int_t^{t+\tau}\int_\Omega v_x^2(\cdot,t) \leq C \quad \textit{for all } t \in (0, T_{max}-\tau), \tag{3.14}$$

where $\tau = \min\left\{1, \frac{T_{max}}{2}\right\}$.

Proof. Testing the second equation of (1.1) by v, integrating and using Young's inequality, we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 &= -\int_\Omega v_x(d_2(u)v_x + d_2'(u)vu_x) + k\int_\Omega uvw - \int_\Omega v^2 \\
&\leq -\int_\Omega d_2(u)v_x^2 + \frac{1}{4}\int_\Omega d_2(u)v_x^2 + \int_\Omega \frac{(d_2'(u))^2}{d_2(u)}v^2u_x^2 \\
&\quad + \frac{k^2M^2}{2}\int_\Omega u^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega v^2 \\
&\leq -\frac{3}{4}\int_\Omega d_2(u)v_x^2 + \int_\Omega \frac{(d_2'(u))^2}{d_2(u)}v^2u_x^2 + \frac{k^2M^2}{2}\int_\Omega u^2 - \frac{1}{2}\int_\Omega v^2.
\end{aligned}
$$

From (3.7), we can find a constant $u^* > 0$ such that $0 \leq u \leq \text{ess sup}_\Omega u = \|u\|_\infty \leq u^*$. Using (H_2), which yields

$$\frac{d}{dt}\int_\Omega v^2 + \int_\Omega v^2 + \frac{3\eta_1}{2}\int_\Omega v_x^2 \leq 2K_3^2\|v\|_4^2\|u_x\|_4^2 + k^2M^2\int_\Omega u^2, \tag{3.15}$$

where $K_3 := \frac{\max_{0\leq u\leq u^*}|d_2'(u)|}{\sqrt{\eta_1}}$. Using the Gagliardo-Nirenberg inequality, there exist some constants $c_i > 0 (i = 1, 2, 3)$ such that

$$\|v\|_4^2 \leq c_1\left(\|v_x\|_2\|v\|_1 + \|v\|_1^2\right) \leq c_2(\|v_x\|_2 + 1) \tag{3.16}$$

and

$$
\begin{aligned}
\|u_x\|_4^2 &\leq c_3\left(\|u_{xx}\|_2\|u_x\|_1 + \|u_x\|_1^2\right) \\
&\leq c_3\left(\frac{1}{2}\|u_{xx}\|_2\|u_x\|_2^2 + \frac{|\Omega|}{2}\|u_{xx}\|_2 + |\Omega|\|u_x\|_2^2\right),
\end{aligned}
\tag{3.17}
$$

where we use Young's inequality and the Cauchy-Schwarz inequality. Using (3.5), there exists a constant $c_4 > 0$ such that

$$\|u_x\|_4^2 \le c_4(\|u_{xx}\|_2 + 1). \tag{3.18}$$

By Lemma 3.1, there exists $c_5 > 0$ such that $\|u(\cdot,t)\|_2 \le c_5$. Substituting (3.16) and (3.18) into (3.15) and using Young's inequality, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega v^2 + \int_\Omega v^2 + \frac{3\eta_1}{2}\int_\Omega v_x^2 &\le 2K_3^2 c_2 c_4(\|v_x\|_2 + 1)(\|u_{xx}\|_2 + 1) + k^2M^2\int_\Omega u^2 \\ &\le \frac{\eta_1}{2}\int_\Omega v_x^2 + \frac{4K_3^4c_2^2c_4^2 + \eta_1}{\eta_1}\int_\Omega u_{xx}^2 + c_6,\end{aligned} \tag{3.19}$$

where $c_6 := k^2M^2c_5^2 + \frac{2K_3^2c_2c_4\eta_1 + 4K_3^4c_2^2c_4^2 + K_3^4c_2^2c_4^2\eta_1}{\eta_1}$, which yields

$$\frac{d}{dt}\int_\Omega v^2 + \int_\Omega v^2 + \eta_1\int_\Omega v_x^2 \le \frac{4K_3^4c_2^2c_4^2 + \eta_1}{\eta_1}\int_\Omega u_{xx}^2 + c_6.$$

Using (3.6) and Lemma 2.3, we derive (3.13) and (3.14). □

Finally, we shall establish the estimate of $\|v(\cdot,t)\|_\infty$.

Lemma 3.4. *Let the assumptions in Theorem 1.1 hold, then there exists a constant $C > 0$ such that*

$$\|v(\cdot,t)\|_\infty \le C \quad \textit{for all } t \in (0, T_{max}). \tag{3.20}$$

Proof. Testing the second equation of (1.1) by $-v_{xx}$ and using Young's inequality yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega v_x^2 &= -\int_\Omega v_{xx}\left(d_2(u)v_{xx} + d_2'(u)vu_{xx} + 2d_2'(u)u_xv_x + d_2''(u)vu_x^2 + kuw - v\right) \\ &\le -\int_\Omega d_2(u)v_{xx}^2 + \frac{4}{8}\int_\Omega d_2(u)v_{xx}^2 + 2\int_\Omega \frac{(d_2'(u))^2}{d_2(u)}v^2u_{xx}^2 \\ &\quad + 8\int_\Omega \frac{(d_2'(u))^2}{d_2(u)}u_x^2v_x^2 + 2\int_\Omega \frac{(d_2''(u))^2}{d_2(u)}v^2u_x^4 + 2\int_\Omega \frac{k^2M^2}{d_2(u)}u^2 - \int_\Omega v_x^2 \\ &\le -\frac{1}{2}\int_\Omega d_2(u)v_{xx}^2 + 2\|v\|_\infty^2\int_\Omega \frac{(d_2'(u))^2}{d_2(u)}u_{xx}^2 + 8\|v_x\|_\infty^2\int_\Omega \frac{(d_2'(u))^2}{d_2(u)}u_x^2 \\ &\quad + 2\|u_x\|_\infty^4\int_\Omega \frac{(d_2''(u))^2}{d_2(u)}v^2 + 2\int_\Omega \frac{k^2M^2}{d_2(u)}u^2 - \int_\Omega v_x^2.\end{aligned}$$

From Lemmata 2.5, 3.2 and 3.3, there exists $c_i > 0 (i = 1, 2, 3)$ satisfies $\|v(\cdot,t)\|_1 \le c_1$, $\|u_x(\cdot,t)\|_2^2 \le c_2$, $\|v(\cdot,t)\|_2^2 \le c_3$. Using the Gagliardo-Nirenberg inequality, for each $\varepsilon > 0$ one can find some $c_\varepsilon > 0$ and $c_i > 0 (i = 4, 5, 6, 7, 8)$ such that

$$\|v\|_\infty^2 \le c_4\left(\|v_x\|_2^{\frac{4}{3}}\|v\|_1^{\frac{2}{3}} + \|v\|_1^2\right) \le \varepsilon\|v_x\|_2^2 + c_\varepsilon$$

and

$$\|v_x\|_\infty^2 \le c_5\left(\|v_{xx}\|_2^{\frac{3}{2}}\|v\|_2^{\frac{1}{2}} + \|v\|_2^2\right) \le \frac{\eta_1}{16K_3^2c_2}\|v_{xx}\|_2^2 + c_6$$

as well as

$$\|u_x\|_\infty^4 \le c_7(\|u_{xx}\|_2^2\|u_x\|_2^2 + \|u_x\|_2^4) \le c_8(\|u_{xx}\|_2^2 + 1).$$

From Lemma 3.1, for some $c_9 > 0$, one has $\|u(\cdot,t)\|_2 \le c_9$. Combining with the above inequalities and using (H_2), we conclude

$$\frac{d}{dt}\int_\Omega v_x^2 + 2\int_\Omega v_x^2 \le 4K_3^2\varepsilon\|v_x\|_2^2\|u_{xx}\|_2^2 + \left(4K_3^2c_\varepsilon + 4K_4^2c_3c_8\right)\|u_{xx}\|_2^2 + c_{10}.$$

where $K_4 := \frac{\max_{0\le u\le u^*}|d_2''(u)|}{\sqrt{\eta_1}}$ and $c_{10} := 16K_3^2c_2c_6 + 4K_4^2c_3c_8 + \frac{4k^2M^2c_9^2}{\eta_1}$. From (3.6), one has $\int_t^{t+\tau}\int_\Omega u_{xx}^2 \le c_{11}$ with some $c_{11} > 0$. Using Lemma 2.4, denoting $a(t) := 2$, $b(t) := 4K_3^2\varepsilon\|u_{xx}\|_2^2$ and $c(t) := \left(4K_3^2c_\varepsilon + 4K_4^2c_3c_8\right)\|u_{xx}\|_2^2 + c_{10}$, choosing $\varepsilon = \frac{\tau}{4K_3^2c_{11}}$ such that $\int_t^{t+\tau}a(s)ds - \int_t^{t+\tau}b(s)ds = \tau > 0$, therefore, we derive the boundedness of $\int_\Omega v_x^2(\cdot,t)$. Finally, using the boundedness of $\int_\Omega v_x^2(\cdot,t)$ and the Gagliardo-Nirenberg inequality, we obtain (3.20). □

We can now easily prove Theorem 1.1.

Proof of Theorem 1.1. From Lemmata 3.1 and 3.3, there exists a constant $C > 0$ satisfies $\|u(\cdot,t)\|_2 + \|v(\cdot,t)\|_2 \le C$ for all $t \in (0, T_{max})$, then we have $\|w(\cdot,t)\|_{1,\infty} \le C$ ([32, Lemma 3.1]), combining Lemmata 3.2, 3.4 and 2.1, we can obtain Theorem 1.1 immediately. □

4. Global stability

In this section, we shall construct appropriate Lyapunov functional to derive the global stability in Theorem 1.2.

Lemma 4.1. *([26, Lemma 3.6]) Let the assumptions in Theorem 1.2 hold, then there exist $\theta \in (0,1)$ and $C > 0$ such that*

$$\|u\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} + \|v\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} + \|w\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+1])} \le C \quad \textit{for all } t \ge 1. \tag{4.1}$$

Lemma 4.2. *Let $a, b, c, k > 0$. Then*

$$\frac{d}{dt}\int_\Omega u = b\int_\Omega v - c\int_\Omega u, \tag{4.2}$$

$$\frac{d}{dt}\int_\Omega v = k\int_\Omega uw - \int_\Omega v \tag{4.3}$$

and

$$\frac{d}{dt}\int_\Omega w + \int_\Omega uw + r\int_\Omega vw + \int_\Omega (w-a)^2 = -a\int_\Omega (w-a) \tag{4.4}$$

for all $t > 0$.

Proof. Integrating the three equations of (1.1), respectively, we obtain (4.2)-(4.4). □

Lemma 4.3. *Let $a, r, d_3 \ge 0$. Then*

$$-\frac{d}{dt}\int_\Omega \ln w + d_3\int_\Omega \frac{w_x^2}{w^2} = \int_\Omega u + r\int_\Omega v + \int_\Omega (w-a) \tag{4.5}$$

for all $t > 0$.

Proof. By the third equation in (1.1), we make use of the positivity of w in $\bar{\Omega} \times (0, \infty)$ to see that

$$
\begin{aligned}
-\frac{d}{dt}\int_\Omega \ln w &= -\int_\Omega \frac{d_3 w_{xx} - w^2 - uw - rvw + aw}{w} \\
&= -d_3 \int_\Omega \frac{w_x^2}{w^2} + \int_\Omega u + r\int_\Omega v + \int_\Omega (w-a)
\end{aligned}
\tag{4.6}
$$

for all $t > 0$. □

Combining Lemmata 4.2 and 4.3, when $\frac{c-abk}{kcar} > 1$, we have the following lemma.

Lemma 4.4. *Under the assumptions in Theorem 1.2, there exists $C > 0$ such that*

$$
\int_0^\infty \int_\Omega u \le C \tag{4.7}
$$

and

$$
\int_0^\infty \int_\Omega v \le C \tag{4.8}
$$

as well as

$$
\int_0^\infty \int_\Omega w_x^2 \le C, \qquad \int_0^\infty \int_\Omega (w-a)^2 \le C \tag{4.9}
$$

for all $t > 0$.

Proof. Using Lemmata 4.2 and 4.3, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \left\{au + \frac{c}{k}v + cw - ca\ln w\right\} + rc\int_\Omega vw + c\int_\Omega (w-a)^2 + cad_3\int_\Omega \frac{w_x^2}{w^2} \\
&= -\left(\frac{c}{k} - rac - ab\right)\int_\Omega v
\end{aligned}
\tag{4.10}
$$

for all $t > 0$. Since $\frac{c-abk}{kcar} > 1$, integrating (4.10) on $[0, t)$ to obtain

$$
\begin{aligned}
&a\int_\Omega u + \frac{c}{k}\int_\Omega v + c\int_\Omega w + rc\int_0^t\int_\Omega vw + c\int_0^t\int_\Omega (w-a)^2 \\
&\quad + cad_3\int_0^t\int_\Omega \frac{w_x^2}{w^2} + \left(\frac{c}{k} - rac - ab\right)\int_0^t\int_\Omega v \\
&\le a\int_\Omega u_0 + \frac{c}{k}\int_\Omega v_0 + c\int_\Omega w_0 - ac\int_\Omega \ln w_0 + ac\int_\Omega \ln w
\end{aligned}
\tag{4.11}
$$

for all $t > 0$. Due to $\ln w \le w$ for all $w > 0$, one has

$$
\begin{aligned}
&a\int_\Omega u + \frac{c}{k}\int_\Omega v + c\int_\Omega w + rc\int_0^t\int_\Omega vw + c\int_0^t\int_\Omega (w-a)^2 \\
&\quad + cad_3\int_0^t\int_\Omega \frac{w_x^2}{w^2} + \left(\frac{c}{k} - rac - ab\right)\int_0^t\int_\Omega v \\
&\le a\int_\Omega u_0 + \frac{c}{k}\int_\Omega v_0 + c\int_\Omega w_0 - ac\int_\Omega \ln w_0 + ac\int_\Omega w \\
&\le a\int_\Omega u_0 + \frac{c}{k}\int_\Omega v_0 + c\int_\Omega w_0 - ac\int_\Omega \ln w_0 + acM|\Omega|
\end{aligned}
\tag{4.12}
$$

for all $t > 0$, which implies (4.8) and (4.9) hold. Integrating (4.2) on $[0, t)$ to obtain

$$\int_\Omega u + c\int_0^t \int_\Omega u = b\int_0^t \int_\Omega v + \int_\Omega u_0.$$

Using (4.8), we can obtain (4.7). The proof is completed. □

Lemma 4.5. *Under the assumptions in Theorem 1.2, there exists* $C > 0$ *satisfies*

$$\int_0^\infty \int_\Omega u_x^2 \le C, \qquad \int_0^\infty \int_\Omega u^2 \le C \tag{4.13}$$

and

$$\int_0^\infty \int_\Omega v^2 \le C \tag{4.14}$$

for all $t > 0$.

Proof. By (3.7) and (3.20), for some $c_1, c_2 > 0$, we have $\|u(\cdot, t)\|_\infty \le c_1, \|v(\cdot, t)\|_\infty \le c_2$. Testing the first and second equations in (1.1) by u and v, respectively, using Young's inequality and integrating to see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + c\int_\Omega u^2 + \int_\Omega d_1(w)u_x^2 \\
&= -\int_\Omega d_1'(w)uu_xw_x + b\int_\Omega uv \\
&\le \frac{1}{2}\int_\Omega d_1(w)u_x^2 + \frac{c_1^2}{2}\int_\Omega \frac{(d_1'(w))^2}{d_1(w)}w_x^2 + b\int_\Omega uv \\
&\le \frac{1}{2}\int_\Omega d_1(w)u_x^2 + \frac{c_1^2}{2}\int_\Omega \frac{(d_1'(w))^2}{d_1(w)}w_x^2 + b\|v\|_\infty\int_\Omega u
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 + \int_\Omega v^2 + \int_\Omega d_2(u)v_x^2 \\
&= -\int_\Omega d_2'(u)vv_xu_x + k\int_\Omega uvw \\
&\le \frac{1}{2}\int_\Omega d_2(u)v_x^2 + \frac{c_2^2}{2}\int_\Omega \frac{(d_2'(u))^2}{d_2(u)}u_x^2 + k\int_\Omega uvw \\
&\le \frac{1}{2}\int_\Omega d_2(u)v_x^2 + \frac{c_2^2}{2}\int_\Omega \frac{(d_2'(u))^2}{d_2(u)}u_x^2 + kM\|v\|_\infty\int_\Omega u.
\end{aligned}$$

Since $(H_1) - (H_2)$, (2.1) and (3.7), which yields

$$\frac{d}{dt}\int_\Omega u^2 + 2c\int_\Omega u^2 + d_1(M)\int_\Omega u_x^2 \le c_1^2K_1^2\int_\Omega w_x^2 + 2bc_2\int_\Omega u, \tag{4.15}$$

$$\frac{d}{dt}\int_\Omega v^2 + 2\int_\Omega v^2 + \eta_1\int_\Omega v_x^2 \le c_2^2K_3^2\int_\Omega u_x^2 + 2kMc_2\int_\Omega u. \tag{4.16}$$

Then using (4.7) and (4.9) imply (4.13) and (4.14). □

Lemma 4.6. *Let the assumptions in Theorem 1.2 hold, the solution of (1.1) satisfies*

$$\|u(\cdot,t)\|_\infty \to 0 \quad as \ \ t \to \infty, \tag{4.17}$$

$$\|v(\cdot,t)\|_\infty \to 0 \quad as \ \ t \to \infty \tag{4.18}$$

and

$$\|w(\cdot,t) - a\|_\infty \to 0 \quad as \ \ t \to \infty. \tag{4.19}$$

Proof. Suppose that (4.17) is false, for some $c_1 > 0$, there exist $(x_i)_{i\in N} \subset \Omega$ and $(t_i)_{i\in N} \subset (1,\infty)$ satisfying $t_i \to \infty$ as $i \to \infty$ such that

$$|u(x_i,t_i)| \geq c_1 \quad \text{for all} \ \ i \in N.$$

From Lemma 4.1, we know that u is uniformly continuous in $\Omega \times (1,\infty)$, therefore, for any $i \in N$, we can find some $r_1 > 0$ and $\tau_1 > 0$ such that

$$|u(x,t)| \geq \frac{c_1}{2} \quad \text{for all} \ \ x \in L_{r_1}(x_i) \cap \Omega \ \ \text{and} \ \ t \in (t_i, t_i + \tau_1),$$

where $L_{r_1}(x_i)$ denotes a line segment with x_i as the center, r_1 as the radius and $2r_1$ in total length and hence

$$\int_{t_i}^{t_i+\tau_1} \int_\Omega |u(x,t)|^2 \geq \frac{c_1^2 c_2 \tau_1}{4} \quad \text{for all} \ \ i \in N, \tag{4.20}$$

where $c_2 := \inf_{i\in N} |L_{r_1}(x_i) \cap \Omega|$ is positive due to smoothness of $\partial\Omega$. By Lemma 4.5, we have

$$\int_{t_i}^{t_i+\tau_1} \int_\Omega |u(x,t)|^2 \to 0 \quad \text{for all} \ \ i \to \infty.$$

Together with (4.20), this leads to a contradiction, thus (4.17) is established. Similarly, we can obtain (4.18) and (4.19) immediately. □

Proof of Theorem 1.2. Lemma 4.6 derives the conclusions of Theorem 1.2.

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Conflict of interest

The authors declare there is no conflicts of interest.

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