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Research article

Ordering properties of positive solutions for a class of φ-Laplacian quasilinear Dirichlet problems

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Abstract: We study ordering properties of positive solutions u for the one-dimensional φ-Laplacian quasilinear Dirichlet problem

$$\begin{cases} -(\varphi(u'))' = \lambda f(u), \quad -L < x < L, \\ u(-L) = u(L) = 0, \end{cases}$$

where $\lambda, L > 0$ are two parameters. Assume that $\varphi \in C(-\kappa, \kappa) \cap C^2((-\kappa, 0) \cup (0, \kappa))$ is odd for some positive $\kappa \leq \infty$, and $\varphi'(t) > 0$ for all $t \in (-\kappa, 0) \cup (0, \kappa)$ and $f \in C[0, \eta)$, $f(0) \geq 0$, $f(u) > 0$ on $(0, \eta)$ for some positive $\eta \leq \infty$, where either $\eta = \infty$, or $\eta < \infty$ with $\lim_{u \to \eta^-} f(u) = \infty$ or $\lim_{u \to \eta^-} f(u) = 0$. Some applications are given, including $f(u) = u^p$ $(p > 0)$, $u^p + u^q$ $(0 < p < q < \infty)$, $\frac{1}{(1-u)^p}$ $(p > 0)$, $\exp(u)$, $\exp\left(\frac{au}{a+u}\right)$ $(a > 0)$, and $\frac{1}{(1-u)^2} - \frac{\varepsilon^2}{(1-u)^4}$ $(\varepsilon \in (0, 1))$.

Keywords: prescribed mean curvature problem; m-Laplacian problem; (m, n)-Laplacian problem; positive solution; bifurcation diagram; ordering property

1. Introduction

In this paper we study ordering properties of positive solutions $u \in C^2(-L, L) \cap C[-L, L]$ for the one-dimensional φ-Laplacian quasilinear Dirichlet problem

$$\begin{cases} -(\varphi(u'(x)))' = \lambda f(u), \quad -L < x < L, \\ u(-L) = u(L) = 0, \end{cases} \tag{1.1}$$

where $\lambda, L > 0$ are two parameters, φ and f satisfy the following hypotheses (H1) and (H2) respectively:

(H1) $\varphi \in C(-\kappa, \kappa) \cap C^2((-\kappa, 0) \cup (0, \kappa))$ is odd for some positive $\kappa \leq \infty$, and $\varphi'(t) > 0$ for all $t \in (-\kappa, 0) \cup (0, \kappa)$.

(H2) $f \in C[0, \eta)$, $f(0) \geq 0$, $f(u) > 0$ on $(0, \eta)$ for some positive $\eta \leq \infty$, where either $\eta = \infty$, or $\eta < \infty$ with $\lim_{u\to\eta^-} f(u) = \infty$ or $\lim_{u\to\eta^-} f(u) = 0$.

The main examples of the one-dimensional φ-Laplacian Dirichlet problem are the following (i)–(iv):

(i) $\varphi(t) = \frac{t}{\sqrt{1+t^2}}$ with $\kappa = \infty$, which corresponds to the prescribed mean curvature problem (capillary surface problem) in Euclidean space

$$\begin{cases} -\left(\frac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \lambda f(u), \quad -L < x < L, \\ u(-L) = u(L) = 0. \end{cases} \tag{1.2}$$

Problem (1.2) with general nonlinearity $f(u)$ or with many different types nonlinearities, like u^p $(p > 0)$, $u^p + u^q$ $(0 \leq p < q < \infty)$, $(1 + u)^p$ $(p > 0)$, $\exp(u)$, $\exp(u) - 1$, $\exp\left(\frac{au}{a+u}\right)$ $(a > 0)$, $(1 - u)^{-p}$ $(p > 0)$, and $(1 - u)^{-2} - \varepsilon^2(1 - u)^{-4}$ $(\varepsilon \in (0, 1))$ has been investigated intensively since 1990, see, e.g., [1–8].

(ii) $\varphi(t) = \frac{t}{\sqrt{1-t^2}}$ with $\kappa = 1 < \infty$, which corresponds to the prescribed mean curvature problem (capillary surface problem) in Minkowski space

$$\begin{cases} -\left(\frac{u'(x)}{\sqrt{1-(u'(x))^2}}\right)' = \lambda f(u), \quad -L < x < L, \\ u(-L) = u(L) = 0. \end{cases} \tag{1.3}$$

One-dimensional problem (1.3) and n-dimensional problem of it with Dirichlet or Neumann boundary condition, with general nonlinearity $f(u)$ or with many different types nonlinearities, like u^p $(p > 0)$, $u^p + u^q$ $(0 < p \leq q < \infty)$, $u^p - u^q$ $(p, q > 0$ and $p \neq q)$, $(1 + u)^p$ $(p > 0)$, $\exp(u)$, $\exp(u) - 1$, $\exp\left(\frac{au}{a+u}\right)$ $(a > 0)$ has been investigated intensively in recent years, see, e.g., [9–16].

(iii) $\varphi(t) = |t|^{m-2}t$ $(m > 1)$ with $\kappa = \infty$, which corresponds to the m-Laplacian problem

$$\begin{cases} -\left(|u'(x)|^{m-2}u'(x)\right)' = \lambda f(u), \quad -L < x < L, \\ u(-L) = u(L) = 0. \end{cases} \tag{1.4}$$

In particular, when $m = 2$, then $\varphi(t) = t$, which corresponds to the usual Laplacian problem

$$\begin{cases} -u''(x) = \lambda f(u), \quad -L < x < L, \\ u(-L) = u(L) = 0. \end{cases} \tag{1.5}$$

Problem (1.4) arises in the study of non-Newtonian fluids and nonlinear diffusion problems. The quantity m is a characteristic of the medium. In particular, for $m > 2$ the fluids medium are called dilatant fluids, and those with $1 < m < 2$ are called pseudoplastics. When $m = 2$ they are Newtonian fluids (see, e.g., Díaz [17, 18] and its bibliography).

Problem (1.4) with general $f(u)$ of the types of convex, concave, convex-concave, concave-convex, concave-convex-concave or even concave-convex-concave-convex nonlinearities on $(0, \infty)$ has been extensively and intensively investigated, see, e.g., [19–25].

Lao et al. [20] very recently studied the global bifurcation curve and exact multiplicity of positive solutions of one-dimensional Laplacian regularized MEMS problem (1.5) with $L = 1$ and

$$f_\varepsilon(u) \equiv \frac{1}{(1-u)^2} - \frac{\varepsilon^2}{(1-u)^4} \; (\varepsilon \in (0,1))$$

which is concave on $(0,\infty)$ if $0 < \varepsilon < \frac{\sqrt{30}}{10} \approx 0.548$ and is convex–concave on $(0,\infty)$ if $\frac{\sqrt{30}}{10} \le \varepsilon < 1$.

(iv) $\varphi(t) = |t|^{m-2}t + |t|^{n-2}t$ $(1 < m < n < \infty)$ with $\kappa = \infty$, which corresponds to the (m,n)-Laplacian problem

$$\begin{cases} -\left(|u'(x)|^{m-2}u'(x) + |u'(x)|^{n-2}u'(x)\right)' = \lambda f(u), & -L < x < L, \\ u(-L) = u(L) = 0. \end{cases} \tag{1.6}$$

Problem (1.6) with general nonlinearity $f(u)$ or with different types nonlinearities, like u^p $(p > 0)$ and $u^{m-1} + u^{n-1}$, $(u+1)^\gamma - 2$ $(\gamma \in (0,3)$, $m = 4$, and $n = 2)$ has been studied in recent years, see, e.g., [26–30].

To study ordering properties of solutions of φ-Laplacian problem (1.1), we start with an equivalent quasilinear Dirichlet problem as follows:

$$\begin{cases} u''(x) + \lambda h(u')f(u) = 0, & -L < x < L, \\ u(-L) = u(L) = 0, \end{cases} \tag{1.7}$$

where $h(t) = \frac{1}{\varphi'(t)} > 0$ by (H1), see [4, p. 1199]. For four φ-Laplacian operators

$$\varphi(t) = \frac{t}{\sqrt{1+t^2}}, \; \frac{t}{\sqrt{1-t^2}}, \; |t|^{m-2}t \; (m > 1), \; |t|^{m-2}t + |t|^{n-2}t \; (1 < m < n < \infty), \tag{1.8}$$

we check that $\varphi \in C(-\kappa,\kappa) \cap C^2((-\kappa,0)\cup(0,\kappa))$ with $\kappa = \infty$, 1, ∞, ∞, respectively. In addition, φ is odd on $(-\kappa,\kappa)$,

$$\varphi'(t) = \begin{cases} (1+t^2)^{-3/2} > 0 \text{ for } t \in (-\infty,0)\cup(0,\infty) & \text{if } \varphi(t) = \dfrac{t}{\sqrt{1+t^2}}, \\ (1-t^2)^{-3/2} > 0 \text{ for } t \in (-1,0)\cup(0,1) & \text{if } \varphi(t) = \dfrac{t}{\sqrt{1-t^2}}, \\ (m-1)\,|t|^{m-2} > 0 \text{ for } t \in (-\infty,0)\cup(0,\infty) & \text{if } \varphi(t) = |t|^{m-2}t \text{ with } m > 1, \\ (m-1)\,|t|^{m-2} + (n-1)\,|t|^{n-2} & \text{if } \varphi(t) = |t|^{m-2}t + |t|^{n-2}t \\ > 0 \text{ for } t \in (-\infty,0)\cup(0,\infty) & \text{with } 1 < m < n < \infty, \end{cases} \tag{1.9}$$

and

$$\varphi'(-t) = \varphi'(t) \text{ for all} t \in (-\kappa,0)\cup(0,\kappa). \tag{1.10}$$

So by (1.9) and (1.10), for each $\varphi(t)$ in (1.8), $\varphi(t)$ satisfies (H1). While, it is important to notice that

$$t\varphi''(t) = \begin{cases} -3t^2(1+t^2)^{-5/2} < 0 \;\text{ for } t \in (-\infty,0)\cup(0,\infty) & \text{if } \varphi(t) = \frac{t}{\sqrt{1+t^2}}, \\ 3t^2(1-t^2)^{-5/2} > 0 \;\text{ for } t \in (-1,0)\cup(0,1) & \text{if } \varphi(t) = \frac{t}{\sqrt{1-t^2}}, \end{cases} \tag{1.11}$$

$$t\varphi''(t) = (m-1)(m-2)\,|t|^{m-2} \begin{cases} \le 0 \text{ if } 1 < m \le 2 \\ > 0 \text{ if } 2 < m < \infty \end{cases} \text{ for } t \in (-\infty, 0) \cup (0, \infty)$$
$$\text{if } \varphi(t) = |t|^{m-2} t \text{ with } m > 1 \tag{1.12}$$

and

$$t\varphi''(t) = (m-1)(m-2)\,|t|^{m-2} + (n-1)(n-2)\,|t|^{n-2}$$
$$\begin{cases} < 0 \text{ if } 1 < m < n \le 2 \\ > 0 \text{ if } 2 \le m < n < \infty \end{cases} \text{ for } t \in (-\infty, 0) \cup (0, \infty) \tag{1.13}$$
$$\text{if } \varphi(t) = |t|^{m-2} t + |t|^{n-2} t \text{ with } 1 < m < n < \infty.$$

The sign of $t\varphi''(t)$ plays an important role in the analysis of ordering properties of positive solutions u for φ-Laplacian problem (1.1); see Theorem 2.1 stated behind.

A solution $u \in C^2(-L, L) \cap C[-L, L]$ of φ-Laplacian Dirichlet problem (1.1) with $u' \in C([-L, L], [-\infty, \infty])$ is called classical if $|u'(\pm L)| < \infty$, and it is called non-classical if $u'(-L) = \infty$ or $u'(L) = -\infty$, see [31] and cf. e.g., [5, 8]. In this paper, we always allow that solutions $u \in C^2(-L, L) \cap C[-L, L]$ satisfy $u' \in C([-L, L], [-\infty, \infty])$; that is, we consider classical solutions as well as non-classical solutions.

It can be shown that (see [4, (1.4), (1.5) and Lemma 2.1]), for (1.1) with φ and f satisfying (H1) and (H2) respectively,

(i) Any non-trivial solution $u \in C^2(-L, L) \cap C[-L, L]$ is concave and positive on $(-L, L)$ since the φ-Laplacian equation in (1.1) can be written in the equivalent form $u''(x) = -\lambda h(u') f(u) < 0$ on $(-L, L)$ by (1.7) and (H2).
(ii) A positive solution $u \in C^2(-L, L) \cap C[-L, L]$ must be symmetric on $[-L, L]$. Thus $u'(-L) = -u'(+L)$.

We define the bifurcation diagram C_L of φ-Laplacian Dirichlet problem (1.1) by

$$C_L \equiv \left\{ (\lambda, \|u_\lambda\|_\infty) : \lambda > 0 \text{ and } u_\lambda \in C^2(-L, L) \cap C[-L, L] \text{ is a positive solution of (1.1)} \right\}.$$

For one-dimensional φ-Laplacian Dirichlet problem (1.1), Korman and Li [4] applied the Crandall-Rabinowitz local bifurcation theorem [32] to study the uniqueness and exact multiplicity of positive solutions. The next Theorem 1.1 is due to Korman and Li [4, Theorem3.4].

Theorem 1.1. *([4, Theorem 3.4]) Consider (1.1) where φ satisfies*

$$\varphi \in C^2(\mathbb{R}) \textit{ is odd and } \varphi'(t) > 0 \textit{ for all } t \in \mathbb{R}, \tag{1.14}$$

$$t\varphi''(t) \le 0 \textit{ for all } t \in \mathbb{R}, \tag{1.15}$$

and moreover that its range over $\mathbb{R}$ is bounded, while the function $f(u) \in C^2(\bar{\mathbb{R}}_+)$ is convex, it satisfies $f(u) > 0$ for $u > 0$ and it is bounded below by a positive constant on $[0, \infty)$. Then (1.1) has at most two positive solutions for any $\lambda > 0$. Moreover, all positive solutions lie on a unique bifurcation curve C_L on the $(\lambda, \|u\|_\infty)$-plane. This curve C_L emanates from the origin $(0, 0)$ and either it tends to infinity at some $\lambda_0 > 0$, or at λ_0 it develops infinite slope at $x = \pm L$ and stops, or else it bends back at some $\lambda_0 > 0$. After the turn, the curve continues without any more turns, and it either tends to infinity for decreasing λ, or else it develops infinite slope at $x = \pm L$ and stops at some nonnegative $\bar{\lambda} < \lambda_0$.

Pan and Xing [33], in the next theorem, extended the first conclusion of Theorem 1.1, which requires more assumptions in [4] — the boundedness of φ, $\eta = \infty$, and $f(0) > 0$ [33, p. 3632, lines 5 and 6].

Theorem 1.2. *Consider (1.1) where φ satisfies (1.14) and (1.15) and f satisfies the following conditions:*

$$\begin{gathered} f \in C[0,\eta),\ f(u) > 0 \text{ on } (0,\eta) \text{ for some positive } \eta \leq \infty, \\ \text{where either } \eta = \infty,\ \text{or } \eta < \infty \text{ with } \lim_{u\to\eta^-} f(u) = \infty, \\ f \in C^2[0,\eta) \text{ satisfying } f''(u) > 0 \text{ on } (0,\eta), \\ f \in C^1[0,\eta) \text{ satisfying } f'(u) > 0 \text{ on } (0,\eta). \end{gathered} \tag{1.16}$$

In addition, one of the inequalities in (1.15) and (1.16) is strict, except for at most finite number of t and u. Then (1.1) has at most two non-trivial positive solutions for any $\lambda > 0$.

For one-dimensional φ-Laplacian Dirichlet problem (1.1), in the next theorem, Pan and Xing [31] proved the existence and uniqueness of positive solution. They also established various results on the exact number of positive solutions as well as global bifurcation diagrams, see [31] for details.

Theorem 1.3. *([31, Theorem 2.1]) Consider (1.1) where φ satisfies (1.14) and (1.15) and f satisfies $f \in C^1[0,\eta)$, $f(0) \geq 0$, $f(u) > 0$ on $(0,\eta)$ for some positive $\eta \leq \infty$, where either $\eta = \infty$, or $\eta < \infty$ with $\lim_{u\to\eta^-} f(u) = \infty$. Moreover,*

$$f(u) - uf'(u) \leq 0 \text{ for } u \in [0,\eta). \tag{1.17}$$

In addition, one of the inequalities in (1.15) and (1.17) is strict, except for at most finite number of t and u. Then (1.1) has at most one positive solution for any $\lambda > 0$.

We remark that Boscaggin et al. [34] has recently proved the uniqueness of positive solution for one-dimensional φ-Laplacian equation associated with the Neumann or periodic boundary conditions; see [34, Theorems 1.1, 1.2 and Section 2] for details.

We end this section by giving next Theorems 1.4 and 1.5 which are main motivation of this paper. Theorem 1.4 on ordering properties of positive solutions for φ-Laplacian Dirichlet problem (1.1) is due to Korman and Li [4, Corollary 2.5 and Lemma 2.7] after some slight generalization for φ and f satisfying (H1) and (H2) respectively. Theorem 1.4 ([4, Corollary 2.5 and Lemma 2.7]) which was applied in [4] to prove Theorem 1.1 says that any two positive solutions of (1.1) are strictly ordered on $(-L, L)$. Theorem 1.5 on ordering properties of positive solutions for *Laplacian* Dirichlet problem (1.5) is due to Liu and Zhang [21, Theorem 1(iv),(v)] and Wang and Yeh [25, Theorem1.2] after some slight generalization.

Theorem 1.4. *([4, Corollary 2.5 and Lemma 2.7]) Consider (1.1) where φ satisfies (H1) and f satisfies (H2). Suppose that, for two fixed positive numbers $\lambda_1 < \lambda_2$, $u_{\lambda_1}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_1$ and $u_{\lambda_2}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_2$. Then the following assertions (I) and (II) hold:*

(I) If $\left\|u_{\lambda_1}\right\|_\infty < \left\|u_{\lambda_2}\right\|_\infty$, then $u_{\lambda_1}(x) < u_{\lambda_2}(x)$ for $x \in (-L, L)$.
(II) If $\left\|u_{\lambda_1}\right\|_\infty > \left\|u_{\lambda_2}\right\|_\infty$, then $u_{\lambda_1}(x) > u_{\lambda_2}(x)$ for $x \in (-L, L)$.

The steps of the sketch of the proof of Theorem 1.4 are as follows (Cf. [4, Lemmas 2.1, 2.3, 2.7 and Corollaries 2.4, 2.5]):

Step 1. Assume that φ satisfies (H1). Show that any positive solution of (1.7) is an even function, with $u'(x) < 0$ for $x > 0$.

Step 2. Assume that φ satisfies (H1), that $v(x)$ is a supersolution and $u(x)$ is a subsolution of (1.1) and that both functions are positive on $(-L, L)$ and even. Assume that $|u'(x)| > |v'(x)|$. Show that $u(x) > v(x)$ for all $x \in (-L, L)$. Moreover, if $u(\eta) = v(\eta)$ for some $\eta \in (0, L)$, then show that $u(x) > v(x)$ for all $x \in (-\eta, \eta)$.

Step 3. Show that any two positive solutions of (1.1) cannot intersect, and hence they are strictly ordered on $(-L, L)$.

Step 4. Assume that φ satisfies (H1) and f satisfies (H2). Show that the value of $u(0) = \alpha$ uniquely identifies the solution pair $(\lambda, u(x))$ of (1.1) (i.e., there is at most one λ, with at most one positive solution $u(x)$, so that $u(0) = \alpha$).

Theorem 1.5. *Consider Laplacian problem (1.5) where f satisfies (H2). Suppose that, for fixed two positive numbers $\lambda_1 < \lambda_2$, $u_{\lambda_1}(x)$ is a positive solution of (1.5) for $\lambda = \lambda_1$, $u_{\lambda_2}(x)$ is a positive solution of (1.5) for $\lambda = \lambda_2$. Then the following assertions (I) and (II) hold:*

(I) If $\left\|u_{\lambda_1}\right\|_\infty < \left\|u_{\lambda_2}\right\|_\infty$, then

$$u_{\lambda_1}(x) < u_{\lambda_2}(x) \ \ \text{for } x \in (-L, L).$$

Moreover, if f is a strictly increasing function of u on $[0, \eta)$, then

$$u_{\lambda_1}(x) < \left(\frac{\lambda_1}{\lambda_2}\right) u_{\lambda_2}(x) \ \ \text{for } x \in (-L, L).$$

In particular, if $f(u) = \sum_{i=1}^{m} a_i u^{p_i} + \sum_{j=1}^{n} b_j u^{q_j}$ satisfies

$$\begin{cases} 0 < p_1 < p_2 < \cdots < p_m < 1 \le q_1 < q_2 < \cdots < q_n, \ m, n \ge 1, \ q_n > 1, \\ a_i > 0 \ \text{for } i = 1, 2, \ldots, m \ \text{and } b_j > 0 \ \text{for } j = 1, 2, \ldots, n, \end{cases} \tag{1.18}$$

then

$$u_{\lambda_1}(x) < \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-p_1}} u_{\lambda_2}(x) \ \ \text{for } x \in (-L, L).$$

(II) If $\left\|u_{\lambda_1}\right\|_\infty > \left\|u_{\lambda_2}\right\|_\infty$, then

$$u_{\lambda_1}(x) > \sqrt{\frac{\lambda_1}{\lambda_2}} u_{\lambda_2}(x) \ \ \text{for } x \in (-L, L).$$

We finally remark in this section that it is also interesting to study ordering properties of positive solutions for one-dimensional φ-Laplacian problems with *nonlinear* boundary conditions. Cf. e.g., [35, 36] in which multiplicity results of positive solutions were obtained. Further research is needed.

The rest of this paper is organized as follows. Section 2 contains the main theorem (Theorem 2.1), its several applications (Corollaries 2.2–2.6), and a simple example of numerical computation for Laplacian problem (1.5). Section 3 contains the proofs of the main results.

2. Main results

The main result in this paper is the next Theorem 2.1 for one-dimensional φ-Laplacian Dirichlet problem (1.1), in which we study ordering properties of positive solutions $u \in C^2(-L, L) \cap C[-L, L]$. Theorem 2.1 improves Theorem 1.4(I) by further analysis on the positivity of the term $t\varphi''(t)$ on $(-\kappa, 0) \cup (0, \kappa)$. Theorem 2.1 also generalizes and improves Theorem 1.5(I) for Laplacian Dirichlet problem (1.5). Thus we are able to provide practical applications for (1.1) with $\varphi(t) = \frac{t}{\sqrt{1+t^2}}$ of the prescribed mean curvature problem in Euclidean space, $\varphi(t) = \frac{t}{\sqrt{1-t^2}}$ of the prescribed mean curvature problem in Minkowski space, $\varphi(t) = |t|^{m-2}t$ of the m-Laplacian problem, and $\varphi(t) = |t|^{m-2}t + |t|^{n-2}t$ of the (m, n)-Laplacian problem, see Remark 2.7. We then give some applications for some nonlinearities f, including $f(u) = u^p$ $(p > 0)$ (Corollary 2.2), $\sum_{i=1}^{m} a_i u^{p_i} + \sum_{j=1}^{n} b_j u^{q_j}$ satisfying (2.11) stated behind (Corollary 2.3), $\frac{1}{(1-u)^p}$ $(p > 0)$, $\exp(u)$, $\exp\left(\frac{au}{a+u}\right)$ $(a > 0)$ (Corollary 2.4), and $\frac{1}{(1-u)^2} - \frac{\varepsilon^2}{(1-u)^4}$ $(\varepsilon \in (0, 1))$ (Corollaries 2.5 and 2.6).

Theorem 2.1. *Consider (1.1) where φ satisfies (H1) and f satisfies (H2). Suppose that, for two fixed positive numbers $\lambda_1 < \lambda_2$, $u_{\lambda_1}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_1$ and $u_{\lambda_2}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_2$. If $\left\|u_{\lambda_1}\right\|_\infty < \left\|u_{\lambda_2}\right\|_\infty$, then*

$$u_{\lambda_1}(x) < u_{\lambda_2}(x) \ \text{ for } x \in (-L, L). \tag{2.1}$$

Moreover, the following assertions (i) and (ii) hold:

(i) Suppose that

$$t\varphi''(t) \le 0 \ \text{ for all } t \in (-\kappa, 0) \cup (0, \kappa). \tag{2.2}$$

If f is a strictly increasing function of u on $[0, \eta)$, then

$$u_{\lambda_1}(x) < \left(\frac{\lambda_1}{\lambda_2}\right) u_{\lambda_2}(x) \ \text{ for } x \in (-L, L). \tag{2.3}$$

Moreover, if there exists a constant $\hat{p} \in (0, 1)$ such that

$$\frac{f(u)}{u^{\hat{p}}} \text{ is an increasing function of } u \ \text{ on } [0, \eta), \tag{2.4}$$

then

$$u_{\lambda_1}(x) \le \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\hat{p}}} u_{\lambda_2}(x) \ \text{ for } x \in (-L, L). \tag{2.5}$$

In particular, if $t\varphi''(t) < 0$ for all $t \in (-\kappa, 0) \cup (0, \kappa)$, then

$$u_{\lambda_1}(x) < \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\hat{p}}} u_{\lambda_2}(x) \ \text{ for } x \in (-L, 0) \cup (0, L). \tag{2.6}$$

Furthermore, if $\varphi'(t) > 0$ for all $t \in (-\kappa, \kappa)$, then (2.6) holds for all $x \in (-L, L)$.

(ii) Suppose that

$$t\varphi''(t) \ge 0 \ \text{ for all } t \in (-\kappa, 0) \cup (0, \kappa). \tag{2.7}$$

If f is a strictly decreasing function of u on $[0, \eta)$, then

$$\left(\frac{\lambda_1}{\lambda_2}\right) u_{\lambda_2}(x) < u_{\lambda_1}(x) < u_{\lambda_2}(x) \ \text{ for } x \in (-L, L). \tag{2.8}$$

Corollary 2.2. *(Cf. [27, Figure 1] with $f(u) = u^p$ ($p > 0$) for (1.6).) Consider (1.1) where φ satisfies (H1) and (2.2) and $f(u) = u^p$, $p > 0$. Suppose that, for two fixed positive numbers $\lambda_1 < \lambda_2$, $u_{\lambda_1}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_1$ and $u_{\lambda_2}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_2$, and $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$. Then (2.3) holds. Moreover, if $0 < p < 1$, then*

$$u_{\lambda_1}(x) \leq \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-p}} u_{\lambda_2}(x) \text{ for } x \in (-L, L). \tag{2.9}$$

In particular, if $t\varphi''(t) < 0$ for all $t \in (-\kappa, 0) \cup (0, \kappa)$, then

$$u_{\lambda_1}(x) < \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-p}} u_{\lambda_2}(x) \text{ for } x \in (-L, 0) \cup (0, L). \tag{2.10}$$

Corollary 2.3. *(Cf. [3, Figures 2–5] with $f(u) = u^p + u^q$, $0 < p < q < \infty$ for (1.2).) Consider (1.1) where φ satisfies (H1) and (2.2) and $f(u) = \sum_{i=1}^m a_i u^{p_i} + \sum_{j=1}^n b_j u^{q_j}$ satisfies*

$$\begin{cases} 0 < p_1 < p_2 < \cdots < p_m < 1 \leq q_1 < q_2 < \cdots < q_n, \ m, n \in \mathbb{N} \cup \{0\}, \ m^2 + n^2 \geq 1, \ q_n > 1, \\ a_i > 0 \text{ for } i = 1, 2, \ldots, m \text{ and } b_j > 0 \text{ for } j = 1, 2, \ldots, n; \end{cases} \tag{2.11}$$

cf. (1.18). Suppose that, for two fixed positive numbers $\lambda_1 < \lambda_2$, $u_{\lambda_1}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_1$ and $u_{\lambda_2}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_2$, and $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$. Then (2.3) holds. Moreover, if $m \geq 1$, then

$$u_{\lambda_1}(x) \leq \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-p_1}} u_{\lambda_2}(x) \text{ for } x \in (-L, L).$$

In particular, if $t\varphi''(t) < 0$ for all $t \in (-\kappa, 0) \cup (0, \kappa)$, then

$$u_{\lambda_1}(x) < \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-p_1}} u_{\lambda_2}(x) \text{ for } x \in (-L, 0) \cup (0, L).$$

Corollary 2.4. *(Cf. [14, Figure 4] with $f(u) = \exp(u)$ for (1.3).) Consider (1.1) where φ satisfies (H1) and (2.2) and*

$$f(u) = \frac{1}{(1-u)^p} (p > 0), \ \exp(u), \ \text{and } \exp\left(\frac{au}{a+u}\right) (a > 0).$$

Suppose that, for two fixed positive numbers $\lambda_1 < \lambda_2$, $u_{\lambda_1}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_1$ and $u_{\lambda_2}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_2$, and $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$. Then (2.3) holds.

Corollary 2.5. *(Cf. [20, Figure 1] for (1.5) with $L = 1$.) Consider (1.1) where φ satisfies (H1) and (2.2) and $f_\varepsilon(u) = \frac{1}{(1-u)^2} - \frac{\varepsilon^2}{(1-u)^4}$, $\varepsilon \in (0, 1)$. Suppose that, for two fixed positive numbers $\lambda_1 < \lambda_2$, $u_{\lambda_1}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_1$ and $u_{\lambda_2}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_2$, and $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$. Then for $0 < \varepsilon < \frac{1}{\sqrt{2}} \approx 0.707$, if $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty \leq 1 - \sqrt{2}\varepsilon$, then (2.3) holds.*

Corollary 2.6. *(Cf. [20, Figure 1(c)] for (1.5) with $L = 1$.) Consider (1.1) where φ satisfies (H1) and (2.7) and $f_\varepsilon(u) = \frac{1}{(1-u)^2} - \frac{\varepsilon^2}{(1-u)^4}$, $\varepsilon \in (0, 1)$. Suppose that, for two fixed positive numbers $\lambda_1 < \lambda_2$, $u_{\lambda_1}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_1$ and $u_{\lambda_2}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_2$, and $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$. Then for $1 > \varepsilon \geq \frac{1}{\sqrt{2}} \approx 0.707$, (2.8) holds.*

Finally, in this section, we give several remarks to Theorem 2.1 and Corollary 2.3.

Remark 2.7. *Theorem 2.1(i) holds for φ-Laplacian operators*

$$\varphi(t) = \frac{t}{\sqrt{1+t^2}}, \ \ |t|^{m-2}t \ (1 < m \le 2) \text{ and } |t|^{m-2}t + |t|^{n-2}t \ (1 < m < n \le 2)$$

since (2.2) holds by (1.11)–(1.13). In addition, Theorem 2.1(ii) holds for φ-Laplacian operators

$$\varphi(t) = \frac{t}{\sqrt{1-t^2}}, \ \ |t|^{m-2}t \ (2 \le m) \ \text{ and } |t|^{m-2}t + |t|^{n-2}t \ (2 \le m < n < \infty)$$

since (2.7) holds by (1.11)–(1.13).

Remark 2.8. *Corollary 2.3 applies to $f(u) = au^p + bu^q$ with $a, b > 0$ and $0 < p < q < \infty$. So (2.3) holds. Moreover, (2.9) holds if $0 < p < 1$.*

Remark 2.9. *(Cf. Corollary 2.3.) It is interesting to note that Theorem 2.1(i) can apply to polynomial nonlinearities $f(u) = \sum_{i=1}^{m} a_i u^{p_i} + \sum_{j=1}^{n} b_j u^{q_j}$ with some negative coefficients a_i or b_j. For example, let*

$$f = \hat{f}(u) = u^{\frac{1}{4}} - \hat{a}u^{\frac{1}{3}} + u^{\frac{1}{2}} + \sum_{i=4}^{m} a_i u^{p_i} + \sum_{j=1}^{n} b_j u^{q_j}$$

satisfying

$$\begin{cases} \frac{1}{2} < p_1 < p_2 < \cdots < p_m < 1 \le q_1 < q_2 < \cdots < q_n, \ m \ge 4, \ n \ge 0, \ q_n > 1, \\ \hat{a} \ge 0, \ a_i > 0 \ \text{for } i = 4, 5, \ldots, m \ \text{ and } b_j > 0 \ \text{ for } j = 1, 2, \ldots, n. \end{cases}$$

We choose constant $\hat{p} = \frac{1}{5}$ in (2.4). Then it can be easily shown that, for $0 < \hat{a} < \frac{9}{8}\left(\frac{3}{2}\right)^{\frac{1}{3}} \approx 1.288$, $\hat{f}(u)$ is a positive, strictly increasing function of u on $[0, \infty)$ and it satisfies

$$\frac{\hat{f}(u)}{u^{\frac{1}{5}}} = u^{\frac{1}{20}} - \hat{a}u^{\frac{2}{15}} + u^{\frac{3}{10}} + \sum_{i=4}^{m} a_i u^{p_i - \frac{1}{5}} + \sum_{j=1}^{n} b_j u^{q_j - \frac{1}{5}}$$

is a strictly increasing function of u on $[0, \infty)$. Thus, by Theorem 2.1(i), for two fixed positive numbers $\lambda_1 < \lambda_2$, suppose that, $u_{\lambda_1}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_1$ and $u_{\lambda_2}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_2$ satisfying $\left\|u_{\lambda_1}\right\|_\infty < \left\|u_{\lambda_2}\right\|_\infty$, we have that

$$u_{\lambda_1}(x) < \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{5}{4}} u_{\lambda_2}(x) \ \text{ for } x \in (-L, L).$$

Remark 2.10. *Consider (1.1) where φ satisfies (H1) and (2.2) and $f(u)$ satisfies (H2) and (2.4) for some $\hat{p} \ge 1$. Suppose that, for two fixed positive numbers $\lambda_1 < \lambda_2$, $u_{\lambda_1}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_1$ and $u_{\lambda_2}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_2$. Then it can be shown that $\left\|u_{\lambda_1}\right\|_\infty > \left\|u_{\lambda_2}\right\|_\infty$, cf. [31, Theorem 2.1].*

Remark 2.11. *If $\varphi(t) = t$ and hence $\varphi'(t) = 1 > 0$ and $\varphi''(t) = 0$ for all $t \in (-\infty, \infty)$. Then φ-Laplacian Dirichlet problem (1.1) reduces to the Laplacian Dirichlet problem (1.5) and both results in Theorem 2.1(i),(ii) hold.*

2.1. A simple example of numerical computation for Laplacian problem (1.5)

We study ordering of positive solutions for the one-dimensional Laplacian problem (1.5) with $L = 1$ and $f(u) = \sqrt{u}$, $\eta = \infty$. Function $f(u) = \sqrt{u}$ satisfies $f(0) = 0$ and is strictly increasing, concave on $[0, \infty)$. So it is easy to show that, on the $(\lambda, \|u\|_\infty)$-plane, the bifurcation curve is a strictly increasing bifurcation curve which emanates at the origin and tends to infinity as $\lambda \to \infty$. Theorem 2.1(i) applies for $f(u) = \sqrt{u}$ and (2.4) holds with $\hat{p} = 1/2 \in (0, 1)$. In Figure 1, we give numerical bifurcation curve for (1.5) produced by MATHEMATICA. In Figure 2 we choose two solutions $u_{\lambda_1}(x)$ and $u_{\lambda_2}(x)$ for (1.5) satisfying $u_{\lambda_2}(0) = 2 > 1 = u_{\lambda_1}(0)$, $\lambda_1 \approx 7.05518$,$\lambda_2 \approx 9.97754$. So by (2.5) we obtain that

$$\frac{u_{\lambda_2}(x)}{u_{\lambda_1}(x)} \geq \left(\frac{\lambda_2}{\lambda_1}\right)^2 \approx 2 \ \text{ for } x \in (-1, 1).$$

The *numerical simulation* graph of $u_{\lambda_2}(x)/u_{\lambda_1}(x)$ in Figure 3 suggests that the value $\left(\frac{\lambda_2}{\lambda_1}\right)^2$ gives a pretty close lower bound for $\frac{u_{\lambda_2}(x)}{u_{\lambda_1}(x)}$ on $(-1, 1)$ for any two positive solutions $u_{\lambda_1}(x)$ and $u_{\lambda_2}(x)$ for (1.5) with any positive $\lambda_1 < \lambda_2$. Both Figures 2 and 3 are also produced by MATHEMATICA.

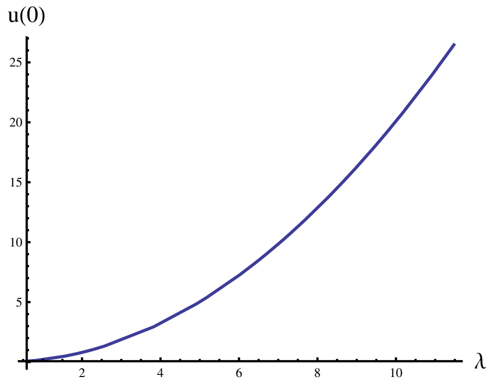

Figure 1. Bifurcation curve for (1.5) with $f(u) = \sqrt{u}$, $L = 1$.

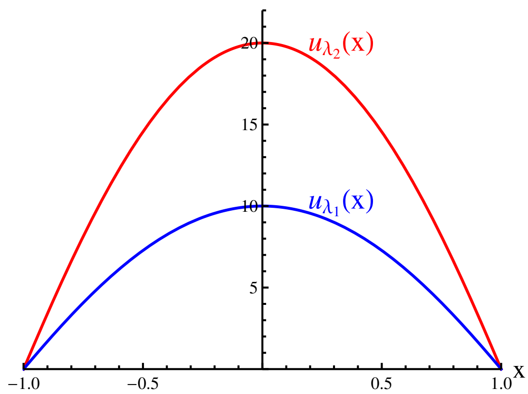

Figure 2. Solutions $u_{\lambda_1}(x)$ and $u_{\lambda_2}(x)$ for (1.5) with $f(u) = \sqrt{u}$, $L = 1$, $u_{\lambda_2}(0) = 2 > 1 = u_{\lambda_1}(0)$, $\lambda_1 \approx 7.05518$, $\lambda_2 \approx 9.97754$.

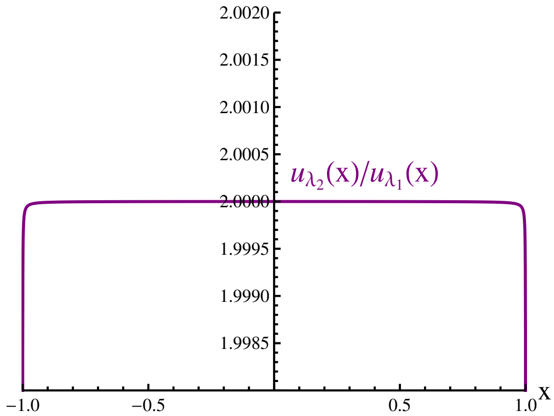

Figure 3. The *numerical simulation* graph of $u_{\lambda_2}(x)/u_{\lambda_1}(x)$ with $u_{\lambda_1}(x), u_{\lambda_2}(x)$ in Figure 2.

3. Proof of main results

To prove Theorem 2.1, we need the next lemma which is well-known.

Lemma 3.1. *Let $\tilde{\varphi}$ be a continuous, strictly increasing function on the open interval $I \subset \mathbb{R}$ and $J \equiv \tilde{\varphi}(I) \subset \mathbb{R}$. Then $\tilde{\varphi}^{-1}$ is a continuous, strictly increasing function on J. Moreover, the following assertions (i) and (ii) hold:*

(i) If $\tilde{\varphi}$ is convex on I, then $\tilde{\varphi}^{-1}$ is concave on J.
(ii) If $\tilde{\varphi}$ is concave on I, then $\tilde{\varphi}^{-1}$ is convex on J.

Proof of Theorem 2.1. Consider (1.1) where φ satisfies (H1) and f satisfies (H2). Assume that $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$ for positive numbers $\lambda_1 < \lambda_2$. First, inequality (2.1) follows by Theorem 1.4(I).

(I) We prove Theorem 2.1(i). Suppose that φ satisfies (2.2) and f is a strictly increasing function of u on $[0, \eta)$. Then, for $-L < x \le 0$, by (2.1),

$$0 > \left(\varphi(u'_{\lambda_1}(x))\right)' = -\lambda_1 f(u_{\lambda_1}(x)) > -\left(\frac{\lambda_1}{\lambda_2}\right)\lambda_2 f(u_{\lambda_2}(x)) = \frac{\lambda_1}{\lambda_2}\left(\varphi(u'_{\lambda_2}(x))\right)'.$$

Since $u'_{\lambda_1}(0) = u'_{\lambda_2}(0) = 0$, we have that

$$0 < \varphi(u'_{\lambda_1}(x)) < \frac{\lambda_1}{\lambda_2}\varphi(u'_{\lambda_2}(x)) \text{ for } -L < x < 0. \tag{3.1}$$

Since $\varphi(t)$ is continuous, strictly increasing on $I \equiv (0, \kappa)$, and $\varphi''(t) \le 0$ for all $t \in I = (0, \kappa)$ by (2.2), we obtain that φ^{-1} is continuous, strictly increasing and convex on $J \equiv \varphi(I)$ by Lemma 3.1(ii). Thus, for $-L < x < 0$,

$$\begin{aligned} u'_{\lambda_1}(x) &= \varphi^{-1}(\varphi(u'_{\lambda_1}(x))) \\ &< \varphi^{-1}\left(\frac{\lambda_1}{\lambda_2}\varphi(u'_{\lambda_2}(x))\right) \text{ (by (3.1))} \\ &\le \frac{\lambda_1}{\lambda_2}\varphi^{-1}(\varphi(u'_{\lambda_2}(x))) \text{ (since } \varphi^{-1}(0) = 0 \text{ and } \varphi^{-1} \text{ is convex on } J = \varphi(I)) \\ &= \frac{\lambda_1}{\lambda_2}u'_{\lambda_2}(x). \end{aligned}$$

This implies that

$$u_{\lambda_1}(x) < \left(\frac{\lambda_1}{\lambda_2}\right) u_{\lambda_2}(x) \text{ for } x \in (-L, L) \tag{3.2}$$

since $u_{\lambda_1}(-L) = u_{\lambda_2}(-L) = 0$ and $u_{\lambda_1}(x), u_{\lambda_2}(x)$ are both symmetric on $(-L, L)$ with respect to $x = 0$. So (2.3) holds.

Moreover, suppose that there exists a constant $\hat{p} \in (0, 1)$ such that $\frac{f(u)}{u^{\hat{p}}}$ is an increasing function of u on $[0, \eta)$. Then, for any two positive numbers $u_1 < u_2 < \eta$, we have that

$$\frac{f(u_1)}{u_1^{\hat{p}}} \leq \frac{f(u_2)}{u_2^{\hat{p}}}.$$

Thus, for $-L < x < 0$,

$$\begin{aligned}
(\ 0 > \ )\left(\varphi(u'_{\lambda_1}(x))\right)' &= -\lambda_1 f(u_{\lambda_1}(x)) \\
&\geq -\lambda_2 \left(\frac{\lambda_1}{\lambda_2}\right) f(u_{\lambda_2}(x)) \left(\frac{u_{\lambda_1}(x)}{u_{\lambda_2}(x)}\right)^{\hat{p}} \\
&> -\left(\frac{\lambda_1}{\lambda_2}\right)^{1+\hat{p}} \lambda_2 f(u_{\lambda_2}(x)) \text{ (by (3.2))} \\
&= \left(\frac{\lambda_1}{\lambda_2}\right)^{1+\hat{p}} \left(\varphi(u'_{\lambda_2}(x))\right)'.
\end{aligned}$$

By similar argument as above, we have that

$$u_{\lambda_1}(x) < \left(\frac{\lambda_1}{\lambda_2}\right)^{1+\hat{p}} u_{\lambda_2}(x) \text{ for } x \in (-L, L).$$

Then an inductive argument leads to, for any $k \in \mathbb{N}$,

$$u_{\lambda_1}(x) < \left(\frac{\lambda_1}{\lambda_2}\right)^{1+\hat{p}+\hat{p}^2+\cdots\hat{p}^k} u_{\lambda_2}(x) \text{ for } x \in (-L, L).$$

Letting $k \to \infty$, we obtain that

$$u_{\lambda_1}(x) \leq \left(\frac{\lambda_1}{\lambda_2}\right)^{\sum_{k=0}^{\infty} \hat{p}^k} u_{\lambda_2}(x) = \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\hat{p}}} u_{\lambda_2}(x) \text{ for } x \in (-L, L). \tag{3.3}$$

Now suppose that $t\varphi''(t) < 0$ for $t \in (-\kappa, 0) \cup (0, \kappa)$. We prove that the inequality (3.3) is strict for $x \in (-L, 0) \cup (0, L)$ by the method of contradiction. Suppose that there exists $\xi \in (-L, 0) \cup (0, L)$ such that

$$u_{\lambda_1}(\xi) = \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\hat{p}}} u_{\lambda_2}(\xi). \tag{3.4}$$

Since the solutions $u_{\lambda_1}(x)$ and $u_{\lambda_2}(x)$ are symmetric with respect to $x = 0$, we only need to consider the case $\xi \in (-L, 0)$. The proof for the case $\xi \in (0, L)$ is similar. Then $u'_{\lambda_1}(\xi), u'_{\lambda_2}(\xi) > 0$ by (H1) and (1.7), and hence

$$
\begin{aligned}
\lambda_1 f(u_{\lambda_1}(\xi)) &= -\left(\varphi(u'_{\lambda_1}(\xi))\right)' \text{ (by (1.1))} \\
&= -u''_{\lambda_1}(\xi)\varphi'(u'_{\lambda_1}(\xi)) \\
&= -\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\hat{p}}} u''_{\lambda_2}(\xi)\varphi'\left[\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\hat{p}}} u'_{\lambda_2}(\xi)\right] \\
&> -\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\hat{p}}} u''_{\lambda_2}(\xi)\varphi'(u'_{\lambda_2}(\xi)) \qquad (3.5)\\
&\text{(since } -u''_{\lambda_2}(\xi) > 0 \text{ and } \varphi''(t) < 0 \text{ on } (0,\kappa) \\
&= -\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\hat{p}}} \left(\varphi(u'_{\lambda_2}(\xi))\right)' \\
&= \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\hat{p}}} \lambda_2 f(u_{\lambda_2}(\xi))
\end{aligned}
$$

by (1.1). This implies that

$$
\frac{\lambda_1}{\lambda_2}\left(\frac{u_{\lambda_1}(\xi)}{u_{\lambda_2}(\xi)}\right)^{\hat{p}} > \frac{\lambda_1}{\lambda_2}\frac{f(u_{\lambda_1}(\xi))}{f(u_{\lambda_2}(\xi))} > \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\hat{p}}}.
$$

So, by (3.4), we have that

$$
\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\hat{p}}} > \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\hat{p}}}
$$

which is a contradiction. So we have that $u_{\lambda_1}(x) < \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\hat{p}}} u_{\lambda_2}(x)$ for $x \in (-L, 0) \cup (0, L)$.

Furthermore, if $\varphi'(t) > 0$ for all $t \in (-\kappa, \kappa)$, then the proof for the case $\xi = 0$ in (3.5) still works. Hence, (2.6) holds for all $x \in (-L, L)$.

(II) We prove Theorem 2.1(ii). Suppose that φ satisfies (2.7) and f is a strictly decreasing function of u on $[0, \eta)$. Then, for $-L < x \leq 0$, by (2.1),

$$
\left(\varphi(u'_{\lambda_1}(x))\right)' = -\lambda_1 f(u_{\lambda_1}(x)) < -\left(\frac{\lambda_1}{\lambda_2}\right)\lambda_2 f(u_{\lambda_2}(x)) = \frac{\lambda_1}{\lambda_2}\left(\varphi(u'_{\lambda_2}(x))\right)' < 0.
$$

Since $u'_{\lambda_1}(0) = u'_{\lambda_2}(0) = 0$, we have that

$$
\varphi(u'_{\lambda_1}(x)) > \frac{\lambda_1}{\lambda_2}\varphi(u'_{\lambda_2}(x)) > 0 \text{ for } -L < x < 0. \qquad (3.6)
$$

Since $\varphi(t)$ is strictly increasing on $I \equiv (0, \kappa)$ and $\varphi''(t) \geq 0$ for all $t \in I = (0, \kappa)$ by (2.2), we obtain that φ^{-1} is continuous, strictly increasing and concave on $J = \varphi(I)$ by Lemma 3.1(i). Thus, for $-L < x < 0$,

$$\begin{aligned} u'_{\lambda_1}(x) &= \varphi^{-1}(\varphi(u'_{\lambda_1}(x))) \\ &> \varphi^{-1}(\frac{\lambda_1}{\lambda_2}\varphi(u'_{\lambda_2}(x))) \text{ (by (3.6))} \\ &\geq \frac{\lambda_1}{\lambda_2}\varphi^{-1}(\varphi(u'_{\lambda_2}(x))) \text{ (since } \varphi^{-1}(0) = 0 \text{ and } \varphi^{-1} \text{ is concave on } \varphi(I)) \\ &= \frac{\lambda_1}{\lambda_2}u'_{\lambda_2}(x). \end{aligned}$$

This implies that

$$u_{\lambda_1}(x) > \left(\frac{\lambda_1}{\lambda_2}\right)u_{\lambda_2}(x) \text{ for } x \in (-L, L)$$

since $u_{\lambda_1}(-L) = u_{\lambda_2}(-L) = 0$ and $u_{\lambda_1}(x), u_{\lambda_2}(x)$ are both symmetric on $(-L, L)$ with respect to $x = 0$. So (2.7) holds.

The proof of Theorem 2.1 is now complete.

Proof of Corollary 2.2. Consider (1.1) where φ satisfies (H1) and (2.2) and $f(u) = u^p$, $p > 0$. Suppose that, for two fixed positive numbers $\lambda_1 < \lambda_2$, $u_{\lambda_1}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_1$ and $u_{\lambda_2}(x)$ is a positive solution of (1.1) for $\lambda = \lambda_2$, and $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$. Since $f(u) = u^p$, $p > 0$ satisfies (H2) with $\eta = \infty$ and is a strictly increasing function of u on $[0, \infty)$, inequality (2.3) holds by Theorem 2.1(i). Moreover, if $0 < p < 1$, we have that

$$\frac{f(u)}{u^{p-\varepsilon}} = \frac{u^p}{u^{p-\varepsilon}} = u^\varepsilon$$

is a positive, strictly increasing function of u on $[0, \infty)$, where ε is any small enough positive constant. Thus

$$u_{\lambda_1}(x) \leq \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-(p-\varepsilon)}} u_{\lambda_2}(x) \text{ for } x \in (-L, L) \tag{3.7}$$

by (2.5). Inequality (3.7) holds for any positive ε small enough. This implies that

$$u_{\lambda_1}(x) \leq \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-p}} u_{\lambda_2}(x) \text{ for } x \in (-L, L).$$

In particular, if $t\varphi''(t) < 0$ for all $t \in (-\kappa, 0) \cup (0, \kappa)$, then the same arguments used to prove (2.6) can be applied to prove (2.10).

The proof of Corollary 2.2 is complete.

Corollary 2.3 follows by Theorem 2.1(i), or by slight modification of the arguments in the proof of Corollary 2.2; we omit its proof.

Corollary 2.4 follows immediately by Theorem 2.1(i) since nonlinearities $f(u) = \frac{1}{(1-u)^p}$ $(p > 0)$, $\exp(u)$, and $\exp\left(\frac{au}{a+u}\right)$ $(a > 0)$ all satisfies (H2) with $\eta = 1, \infty$ and ∞ respectively, and all are strictly increasing functions of u on $[0, \eta)$.

Proof of Corollary 2.5. For $0 < \varepsilon < \frac{1}{\sqrt{2}} \approx 0.707$, it is easy to see that $f_\varepsilon(u) = \frac{1}{(1-u)^2} - \frac{\varepsilon^2}{(1-u)^4}$ satisfies (H2) with $\eta_\varepsilon = 1 - \varepsilon \in (0, 1)$ and is a strictly increasing function of u on $[0, 1 - \sqrt{2}\varepsilon]$ with $f_\varepsilon(0) = 1 - \varepsilon^2 > 0$, $f'_\varepsilon(0) = 1 - 2\varepsilon^2 > 0$, $f_\varepsilon(\eta_\varepsilon) = 0$, $f_\varepsilon(1 - \sqrt{2}\varepsilon) = \frac{1}{4\varepsilon} > 0$, and

$$f'_\varepsilon(u) = \frac{2(1-2u+u^2-2\varepsilon^2)}{(1-u)^5}\begin{cases} >0 & \text{if } 0 \le u < 1-\sqrt{2}\varepsilon, \\ =0 & \text{if } u = 1-\sqrt{2}\varepsilon. \end{cases}$$

Thus, if $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty \le 1-\sqrt{2}\varepsilon$, then (2.3) holds by applying modified arguments in the proof of Theorem 2.1(i).

The proof of Corollary 2.5 is complete.

Proof of Corollary 2.6. For $1 > \varepsilon \ge \frac{1}{\sqrt{2}} \approx 0.707$, it is easy to check that $f_\varepsilon(u) = \frac{1}{(1-u)^2} - \frac{\varepsilon^2}{(1-u)^4}$ satisfies (H2) with $\eta_\varepsilon = 1-\varepsilon \in (0,1)$ and is a strictly decreasing function of u on $[0,\eta_\varepsilon]$ with $f_\varepsilon(0) = 1-\varepsilon^2 > 0$, $f'_\varepsilon(0) = 1-2\varepsilon^2 \le 0$, $f''_\varepsilon(u) < 0$ for $0 < u < \eta_\varepsilon$, and $f(\eta_\varepsilon) = 0$. Thus, (2.7) holds by Theorem 2.1(ii).

The proof of Corollary 2.6 is complete.

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Conflict of interest

The authors declare no conflict of interests.

References

1. Y. H. Cheng, K. C. Hung, S. H. Wang, Global bifurcation diagrams and exact multiplicity of positive solutions for a one-dimensional prescribed mean curvature problem arising in MEMS, *Nonlinear Anal. Theory Methods Appl.*, **89** (2013), 284–298. http://dx.doi.org/https://doi.org/10.1016/j.na.2013.04.018
2. P. Habets, P. Omari, Multiple positive solutions of a one-dimensional prescribed mean curvature problem, *Commun. Contemp. Math.*, **9** (2007), 701–730. https://doi.org/10.1142/S0219199707002617
3. K. C. Hung, Y. H. Cheng, S. H. Wang, C. H. Chuang, Exact multiplicity and bifurcation diagrams of positive solutions of a one-dimensional prescribed mean curvature problem, *J. Differ. Equations*, **257** (2014), 3272–3299. https://doi.org/10.1016/j.jde.2014.06.013
4. P. Korman, Y. Li, Global solution curves for a class of quasilinear boundary value problem, *Proc. R. Soc. Edinburgh Sect. A: Math.*, **140** (2010), 1197–1215. https://doi.org/10.1017/S0308210509001449
5. F. Obersnel, Classical and non-classical sign-changing solutions of a one-dimensional autonomous prescribed curvature equation, *Adv. Nonlinear Stud.*, **7** (2007), 671–682. https://doi.org/10.1515/ans-2007-0409
6. H. Pan, One-dimensional prescribed mean curvature equation with exponential nonlinearity, *Nonlinear Anal. Theory Methods Appl.*, **70** (2009), 999–1010. https://doi.org/10.1016/j.na.2008.01.027

7. H. Pan, R. Xing, Time maps and exact multiplicity results for one-dimensional prescribed mean curvature equations, *Nonlinear Anal. Theory Methods Appl.*, **74** (2011), 1234–1260. https://doi.org/10.1016/j.na.2010.09.063

8. H. Pan, R. Xing, Time maps and exact multiplicity results for one-dimensional prescribed mean curvature equations, II, *Nonlinear Anal. Theory Methods Appl.*, **74** (2011), 3751–3768. https://doi.org/10.1016/j.na.2011.03.020

9. C. Bereanu, P. Jebelean, P. J. Torres, Multiple positive radial solutions for a Dirichlet problem involving the mean curvature operator in Minkowski space, *J. Funct. Anal.*, **265** (2013), 644–659. https://doi.org/10.1016/j.jfa.2013.04.006

10. A. Boscaggin, G. Feltrin, Pairs of positive radial solutions for a Minkowski-curvature Neumann problem with indefinite weight, *Nonlinear Anal.*, **196** (2020), 111807. https://doi.org/10.1016/j.na.2020.111807

11. I. Coelho, C. Corsato, F. Obersnel, P. Omari, Positive solutions of the Dirichlet problem for the one-dimensional Minkowski-curvature equation, *Adv. Nonlinear Stud.*, **12** (2012), 621–638. https://doi.org/10.1016/j.jmaa.2013.04.003

12. S. Y. Huang, Classification and evolution of bifurcation curves for the one-dimensional Minkowski-curvature problem and its applications, *J. Differ. Equations*, **264** (2018), 5977–6011. https://doi.org/10.1016/j.jde.2018.01.021

13. S. Y. Huang, Bifurcation diagrams of positive solutions for one-dimensional Minkowski-curvature problem and its applications, *Discrete Contin. Dyn. Syst.*, **39** (2019), 3443–3462. https://doi.org/10.3934/dcds.2019142

14. S.Y. Huang, Global bifurcation diagrams for Liouville-Bratu-Gelfand problem with Minkowski curvature operator, *J. Dyn. Differ. Equations*, (2021), 1–16. https://doi.org/10.1007/s10884-021-09982-4

15. R. Ma, L. Wei, Z. Chen, Evolution of bifurcation curves for one-dimensional Minkowski-curvature problem, *Appl. Math. Lett.*, **103** (2020), 106176. https://doi.org/10.1016/j.aml.2019.106176

16. X. Zhang, M. Feng, Bifurcation diagrams and exact multiplicity of positive solutions of one-dimensional prescribed mean curvature equation in Minkowski space, *Commun. Contemp. Math.*, **21** (2019), 1850003. https://doi.org/10.1142/S0219199718500037

17. J. I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries, Volume I, Elliptic Equations*, *SERBIULA (sistema Librum 2.0)*, Pitman, Boston, MA, 1985.

18. J. I. Díaz, Qualitative study of nonlinear parabolic equations: an introduction, *Extracta Math.*, **16** (2001), 303–341. Available from: https://www.eweb.unex.es/eweb/extracta/Vol-16-3/16a3diaz.pdf.

19. S. Y. Huang, S. H. Wang, Proof of a conjecture for the one-dimensional perturbed Gelfand problem from combustion theory, *Arch. Ration. Mech. Anal.*, **222** (2016), 769–825. https://doi.org/10.1007/s00205-016-1011-1

20. X. Lao, H. Pan, R. Xing, Global bifurcation curves of a regularized MEMS model, *Appl. Math. Lett.*, **111** (2021), 106688. https://doi.org/10.1016/j.aml.2020.106688

21. Z. Liu, X. Zhang, A class of two-point boundary value problems, *J. Math. Anal. Appl.*, **254** (2001), 599–617. https://doi.org/10.1006/jmaa.2000.7258

22. R. Manásevich, F. I. Njoku, F. Zanolin, Positive solutions for the one-dimensional p-Laplacian, *Differ. Integr. Equations*, **8** (1995), 213–222. Available from: https://projecteuclid.org/journals/differential-and-integral-equations/volume-8/issue-1/Positive-solutions-for-the-one-dimensional-p-Laplacian/die/1411134162.full.

23. B. Rynne, Exact multiplicity and stability of solutions of a 1-dimensional, p-Laplacian problem with positive convex nonlinearity, *Nonlinear Anal.*, **183** (2019), 271–283. https://doi.org/10.1016/j.na.2019.01.023

24. S. H. Wang, D. M. Long, An exact multiplicity theorem involving concave-convex nonlinearities and its application to stationary solutions of a singular diffusion problem, *Nonlinear Anal. Theory Methods Appl.*, **44** (2001), 469–486. https://doi.org/10.1016/S0362-546X(99)00272-2

25. S. H. Wang, T. S. Yeh, Exact multiplicity and ordering properties of positive solutions of a p-Laplacian Dirichlet problem and their applications, *J. Math. Anal. Appl.*, **287** (2003), 380–398. https://doi.org/10.1016/S0022-247X(02)00520-6

26. U. Das, A. Muthunayake, R. Shivaji, Existence results for a class of p-q Laplacian semipositone boundary value problems, *Electron. J. Qual. Theory Differ. Equations*, (2020), 1–7. https://doi.org/10.14232/ejqtde.2020.1.88

27. R. Kajikiya, I. Sim, S. Tanaka, A complete classification of bifurcation diagrams for a class of (p, q)-Laplace equations, *J. Math. Anal. Appl.*, **462** (2018), 1178–1194. https://doi.org/10.1016/j.jmaa.2018.02.049

28. R. Kajikiya, M. Tanaka, S. Tanaka, Bifurcation of positive solutions for the one-dimensional (p, q)-Laplace equation, *Electron. J. Differ. Equations*, **2017** (2017), 1–37. Available from: https://ejde.math.txstate.edu/Volumes/2017/107/kajikija.pdf.

29. Y. Komiya, R. Kajikiya, Existence of infinitely many solutions for the (p, q)-Laplace equation, *Nonlinear Differ. Equations Appl.*, **23** (2016), 49. https://doi.org/10.1007/s00030-016-0402-1

30. I. Sim, B. Son, Positive solutions to classes of infinite semipositone (p, q)-Laplace problems with nonlinear boundary conditions, *J. Math. Anal. Appl.*, **494** (2021), 124577. https://doi.org/10.1016/j.jmaa.2020.124577

31. H. Pan, R. Xing, On the existence of positive solutions for some nonlinear boundary value problems II, preprint, arXiv:1501.02882.

32. M. G. Crandall, P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, *Arch. Ration. Mech. Anal.*, **52** (1973), 161–180. https://doi.org/10.1007/BF00282325

33. H. Pan, R. Xing, On the existence of positive solutions for some nonlinear boundary value problems and applications to MEMS models, *Discrete Contin. Dyn. Syst.*, **35** (2015), 3627–3682. https://doi.org/10.3934/dcds.2015.35.3627

34. A. Boscaggin, G. Feltrin, F. Zanolin, Uniqueness of positive solutions for boundary value problems associated with indefinite Φ-Laplacian-type equations, *Open Math.*, **19** (2021), 163–183. https://doi.org/10.1515/math-2021-0003

35. D. D. Hai, X. Wang, Existence and multiplicity of positive solutions for singular ϕ-Laplacian superlinear problems with nonlinear boundary conditions, *Electron. J. Qual. Theory Differ. Equations*, (2021).

36. D. D. Hai, X. Wang, Positive solutions for singular superlinear ϕ-Laplacian problems with nonlinear boundary conditions, *Mediterr. J. Math.*, **19** (2022), 42. https://doi.org/10.1007/s00009-021-01963-7

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