Electronic Research Archive

ERA, 30(5): 1775–1798.
DOI: 10.3934/era.2022090
Received: 22 December 2021
Revised: 14 March 2022
Accepted: 15 March 2022
Published: 30 March 2022

http://www.aimspress.com/journal/ERA

Research article

An iterative spectral strategy for fractional-order weakly singular integro-partial differential equations with time and space delays

M. Usman[1], T. Zubair[2], J. Imtiaz[3], C. Wang[4] and W. Wu[5,*]

[1] Department of Mathematics, National University of Modern Languages, Islamabad 44790, Pakistan
[2] School of Mathematical Sciences, Peking University, Beijing 100871, China
[3] Department of Electrical Engineering, Bahria University, Islamabad, Pakistan
[4] Department of Mathematics, Southern University of Science and Technology, China
[5] Department of Mathematics, Hong Kong Baptist University, Hong Kong SAR, China

* **Correspondence:** Email: wei-wu@hkbu.edu.hk; Tel: +85234112531.

Abstract: This study aims at extending and implementing an iterative spectral scheme for fractional-order unsteady nonlinear integro-partial differential equations with weakly singular kernel. In this scheme, the unknown function $u(\mathrm{x}, t)$ is estimated by using shifted Gegenbauer polynomials vector $\Lambda(\mathrm{x}, t)$, and Picard iterative scheme is used to handle underlying nonlinearities. Some novel operational matrices are developed for the first time in order to approximate the singular integral like, $\int_0^x \int_0^y u(pa_1 + b_1, qa_2 + b_2, t) / (x^{\rho_1} - p^{\rho_1})^{\alpha_1} (y^{\rho_2} - q^{\rho_2})^{\alpha_2} \, \mathrm{d}q\mathrm{d}p$ and $\int_0^t u^{\gamma}(\mathbf{x}, \xi) / (t^{\rho_3} - \xi^{\rho_3})^{\alpha_3} \, \mathrm{d}\xi$, where ρ's > 1, 0 < α's < 1 by means of shifted Gegenbauer polynomials vector. The advantage of this extended method is its ability to convert nonlinear problems into systems of linear algebraic equations. A computer program in Maple for the proposed scheme is developed for a sample problem, and we validate it to compare the results with existing results. Six new problems are also solved to illustrate the effectiveness of this extended computational method. A number of simulations are performed for different ranges of the nonlinearity n, α, fractional-order, ρ, and convergence control M, parameters. Our results demonstrate that the extended scheme is stable, accurate, and appropriate to find solutions of complex problems with inherent nonlinearities.

Keywords: spectral methods; shifted gegenbauer polynomials; fractional calculus; weakly singular integral equations

1. Introduction

In the last few decades, it is observed that mathematical modeling of physical processes through fractional-order derivatives provides several advantages and reveals important information about complex systems, which could be limited with modeling using positive integer-order derivatives. The leverage is due to the fractional-order modelling, defining all nonlocal axioms, containing the present state and all preceding dynamical or physical states. Fractional-order modeling can be beneficial in several disciplines of science [1,2]. Recently, it has been observed that fractional calculus is used frequently in the modeling of weakly singular Volterra integral equations, such as the logistic growth of human population with migration, spreading disease, and biofluids flow in fractured biomaterials. In this context, this work is devoted to the numerical study of two-dimensional unsteady time-fractional integro-partial differential equations with singular kernel, having time-space delays as given in Eq (1), which is defined in the domain $\Omega=\{\mathbf{x}=(x,y),0\le x,y\le 1\}$ and $0\le t\le \mathrm{T}=1$ as [3].

$$\begin{aligned}{}_0^C\mathcal{D}_t^{\mu}u_M(\mathbf{x},t)+\lambda_1 u_M(\mathbf{x},a_1t+b_1)&=\phi_M(\mathbf{x},t)+\lambda_2\Delta u_M-\lambda_3F(u_M(\mathbf{x},t))-\lambda_4\int_0^t\frac{u_M^n(\mathbf{x},t)}{\left(t^{\rho_3}-\xi^{\rho_3}\right)^{\alpha_3}}\mathrm{d}\xi\\&-\lambda_5\int_0^x\int_0^y\frac{1}{\left(x^{\rho_1}-p^{\rho_1}\right)^{\alpha_1}\left(y^{\rho_2}-q^{\rho_2}\right)^{\alpha_2}}u_M(a_2p+b_2,a_3p+q_3,t)\mathrm{d}q\mathrm{d}p.\end{aligned}\tag{1}$$

The conditions associated with problem Eq (1) are given as,

$$u(\mathbf{x},0)=\phi(\mathbf{x})\quad\text{where}\quad \mathbf{x}=(x,y)\in\Omega,$$

$$u(0,y,t)=f_1(y,t),u(1,y,t)=f_2(y,t),u(x,0,t)=f_3(x,t),u(x,1,t)=f_4(x,t).$$

Here, Ω is the computation domain of the problem (1), ${}_0^C\mathcal{D}_t^{\mu}$ is the derivative operator of fractional-order under Caputo logic, F(u(x, t)) = [sin(u), cos(u), exp(u)] is the nonlinear term. Also, ϕ(x, t) and u(x, t) are assumed to be sufficiently smooth to promise the existence and uniqueness of the solution $u(\mathbf{x},t)\in C(\Omega)$.

Obtaining accurate solutions of integro-partial differential equations of a fractional-order is a challenging task due to the involvement of a weakly singular kernel, nonlinear terms, time and space delay terms, and fractional-order derivatives. It is often difficult to attain the exact solution of these equations. This point, together with the need to develop an efficient numerical scheme, inspires us to propose more accurate computational algorithms. In recent years, numerous methods, e.g., finite difference, semi-analytical, meshless, finite element, and spectral methods were introduced to inspect perfect solutions of fractional-order unsteady nonlinear integro-partial differential equations with weakly singular kernel. Asgari and Ezzati [4] developed a method based on two-dimensional Bernstein polynomials operational matrix to explore the numerical solutions of two-dimensional fractional integral equations. By solving three different problems, they proved that the suggested scheme was very accurate. Another efficient scheme based on the coupling of block-pulse and Lagrange polynomials functions was designed by Mollahasani et al. [5]. In their study, they used some special properties of the functions in order to find integrals and derivative. An effective matrix method by means of shifted Legendre polynomials was proposed by Singh et al. [6] to find approximate solutions of nonlinear weakly singular Volterra partial integro-differential equations. They also established operational matrices to compute derivatives, integrals, and products, to prove that their scheme was convergent. In that study, error bounded were also computed. Areshed [7] coupled finite difference algorithm and cubic B-spline collocation method to investigate accurate solutions of fractional-order integro-partial differential equations. Temporal and spatial terms were discretized by finite difference approach. She also demonstrated that the proposed numerical method was stable able to converge with respect to time. Recently, a convergent scheme is developed by Bebei et al. [3] to attain solutions of multi-dimensional variable-order fractional integro-partial differential equations. They proposed piecewise linear interpolation and upwind method, respectively, to approximate the integral term and variable-order derivative. It is noted that the spectral methods are highly accurate and widely used to examine the solution of fractional-order problem arising in mathematical physics. Therefore, the development of operational matrices in this context is the challenging task for the researcher. Recently Zaky et al. developed the operational matrices for the positive integer and fractional-order derivative of Legendre [8], Chebyshev [9], Jacobi [10] and shifted Jacobi [11–13] polynomials in order to investigate the novel behavior of diffusion problem [8], telegraph model [9], Schrodinger problem [10,13], cable equation [12] and partial differential differential model associated with Dirichlet conditions [11]. The readers are referred [14–17] and [18–21] to find some more inclusive works on the development and its applications of spectral-collocation approach for a verity of physical problem arising in mathematical physics and engineering.

This comprehensive literature analysis stated shows that all existing schemes are only valid for nonlinear weakly singular integro-partial differential equations. It identifies a big gap in our knowledge that needs to be covered. Thus far, no researcher presented a powerful tool for an appropriate solutions of two-dimensional unsteady time-fractional integro-partial differential equations with singular kernel and time-space delays, when ρ1, ρ2, ρ3 > 1.

The primary aim of our current study is to present an accurate and higher-order tool to investigate numerical solutions of multi-dimensional unsteady time-fractional integro-partial differential equations with singular kernel, having time-space delays Eq (1). An iterative spectral scheme based on shifted Gegenbauer polynomials is presented to achieve this goal. First, an unknown function u(x, t) is approximated, using shifted Gegenbauer polynomials. Then, novel operational matrices to approximate the time-space delay, fractional/integer-order derivative and weakly singular integral terms are presented for the first time. Picard iteration scheme is incorporated for the iterative spectral method until the required accuracy level is obtained. In order to generate a system of linear equations, collocation approach is utilized. A computer program is developed in Maple software and the scheme is validated, using existing results in literature. Some more problems are also solved to show accuracy, stability, and reliability of the extended computational scheme. A number of simulations are carried out for different nonlinear n, α, fractional-order μ, ρ, convergence control M1, M2 and M3, and iterations via Picard scheme r parameters, r is the iteration via Picard scheme. The obtained results exhibit that the extended scheme is very accurate, stable, and appropriate to seek the solutions of various problems. Furthermore, the proposed method can be employed for numerical solutions of other multi-dimensional and highly nonlinear problems of fractional or variable orders of physical nature in complex geometry.

2. Preliminaries and basic definitions

Some basic definitions regarding shifted Gegenbauer polynomials (SGPs) and fractional calculus are explained in this section.

2.1. Fractional calculus

The elementary definitions concerning Riemann-Liouville (RL) and Caputo's fractional derivative are given below.

Definition 1. The derivative of a fractional-order μ in Riemann-Liouville sense [22] is given as follows:

$$ {}_0^{\mathrm{RL}}\mathcal{D}_s^{\mu}u(s)=\frac{1}{\Gamma(k-\mu)}\frac{\mathrm{d}^k}{\mathrm{d}s^k}\int_0^s\frac{u(t)}{(s-t)^{\mu-k+1}}\mathrm{d}t,\ \text{for}\, k-1\le\mu<k\in Z^+. \tag{2}$$

Definition 2. The derivative of a fractional-order μ s.t. $k-1<\mu<k,$ in Caputo's sense [23,24] is specified as:

$$ {}_0^{C}\mathcal{D}_s^{\mu}u(s)=\frac{1}{\Gamma(k-\mu)}\int_{0+}^s\frac{1}{(s-t)^{\mu-k+1}}u^{(k)}(t)\mathrm{d}t, k\in \mathrm{R}. \tag{3}$$

where R is set of real number and k from natural number set. The operator ${}_0^{C}\mathcal{D}_s^{\mu}$ satisfies the following properties:

$$\begin{gathered} {}_0^{C}\mathcal{D}_s^{\mu}\left(\lambda_1 f(s)+\lambda_2 g(s)\right)=\lambda_1\,{}_0^{C}\mathcal{D}_s^{\mu}f(s)+\lambda_2\,{}_0^{C}\mathcal{D}_s^{\mu}g(s),\\ {}_0^{C}\mathcal{D}_s^{\mu}s^m=\begin{cases}\dfrac{\Gamma(m+1)}{\Gamma(m-\mu+1)}s^{m-\mu}, & \text{otherwise},\\ 0, & \mu\in\mathrm{R}_0,\ m<\mu, m\in\mathrm{R}.\end{cases}\\ {}_0^{C}\mathcal{D}_s^{\mu}Const.=0,\end{gathered} \tag{4}$$

where λ_1 and λ_2 are the constants and $\mathrm{R}_0=\mathrm{R}-\{0\}$.

2.2. Shifted gegenbauer polynomials (SGPs)

The *m*th-order shifted Gegenbauer polynomials (SGPs) $G_m^{\lambda}(s)$ defined in $[0,1]$ can be computed, using the relation given in Eq (5) [25,26]:

$$G_m^{\lambda}(x)=\sum_{n=0}^{m}\frac{\Gamma(1/2+\lambda)}{\Gamma(2\lambda)}\frac{(-1)^{m-n}\Gamma(n+m+2\lambda)}{(m-n)!\Gamma(n+\lambda+1/2)n!}x^n=\frac{\Gamma(1/2+\lambda)}{\Gamma(2\lambda)}\sum_{n=0}^{m}X_{n,m}^{\lambda}x^n. \tag{5}$$

The shifted Gegenbauer polynomials $G_m^{\lambda}(x)$ are orthogonal [25,26] with respect to the space- L^2 in the interval [0, 1] , i.e.,

$$\int_0^1 G_l^{\lambda}(x) G_m^{\lambda}(x) \vartheta^{\lambda}(x) \mathrm{d}x = \begin{cases} 0, \text{ for } l \neq m, \\ \aleph_m^{\lambda}, \text{for } l = m, \end{cases}$$

where $\aleph_m^{\lambda}$ and $\vartheta^{\lambda}(x)$ denotes the normalizing factor and the weight function correspondingly and are defined as,

$$\vartheta^{\lambda}(x) = (x - x^2)^{-\frac{1}{2}+\lambda}, \aleph_m^{\lambda} = \frac{\Gamma(m+2\lambda)(\Gamma(\lambda+1/2))^2}{m!(2m+2\lambda)(\Gamma(2\lambda))^2}.$$

It is important to mention that the shifted Gegenbauer polynomials $G_m^{\lambda}(x)$ must fulfill the subsequent properties [27].

i. $G_m^{\lambda}(-x) = (-1)^m G_m^{\lambda}(x)$.

ii. $\frac{\mathrm{d}}{\mathrm{d}x}\left[G_m^{\lambda}(x)\right] = 4\lambda G_{m-1}^{\lambda+1}(x), m \geq 1.$

iii. $\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[G_m^{\lambda}(x)\right] = 4^k \lambda_k G_{m-k}^{\lambda+k}(x), m \geq k.$

iv. $\int (x - x^2)^{\lambda-\frac{1}{2}} G_m^{\lambda}(x) \mathrm{d}x = -\frac{4\lambda (x - x^2)^{\lambda+1/2}}{m(m+2\lambda)} G_{m-1}^{\lambda+1}(x), m > 1.$

SGPs can be defined in the multi-dimensions $G_{i,j,k}^{\lambda}(\mathbf{x},t) = G_{i,j,k}^{\lambda}(x,y,t)$ and are given as [28],

$$G_{i,j,k}^{\lambda}(\mathbf{x},t) = G_{i,j,k}^{\lambda}(x,y,t) = G_i^{\lambda}(x) G_j^{\lambda}(y) G_k^{\lambda}(t).$$

2.3. Function approximation

Consider a function $u(x)$ from $L^2(\mathrm{R})$ -space; then, it can be predicted by means of the truncated SGPs given in Eq (6) as [29,30],

$$u(x) \cong \tilde{u}(x) = \sum_{j=0}^{M-1} u_j G_j^{\lambda}(x) = C^T \Lambda(x) = \Lambda(x) C^T, \tag{6}$$

where $u_j = (\aleph_j^{\lambda})^{-1} \int_0^1 u(x) G_j^{\lambda}(x) \vartheta^{\lambda}(x) \mathrm{d}x.$ Furthermore, the vectors $\Lambda(x)$ and C are of the order $M \times 1$ and assumed as follows:

$$C = [u_0, u_1, u_2, \ldots, u_{M-1}]^T, \Lambda(x) = \left[G_0^{\lambda}, G_1^{\lambda}, G_2^{\lambda}, \ldots, G_{M-1}^{\lambda}\right]^T. \tag{7}$$

Similarly, the function $u(\mathbf{x},t) = u(x,y,t)$ of three variables can be computed by means of three-dimensional truncated SGPs $G_{i,j,k}^{\lambda}(\mathbf{x},t)=G_{i,j,k}^{\lambda}(x,y,t)$ as,

$$u(\mathbf{x},t)=\tilde{u}(\mathbf{x},t)=\sum_{i=0}^{M_1-1}\sum_{j=0}^{M_2-1}\sum_{k=0}^{M_3-1}u_{i,j,k}G_{i,j,k}^{\lambda}(\mathbf{x},t)=U^T\Lambda(\mathbf{x},t), \tag{8}$$

Here, $u_{i,j,k}=\left\langle u(\mathbf{x},t),G_{i,j,k}^{\lambda}(\mathbf{x},t)\right\rangle_{L_{\vartheta}^2}$ and $\langle\cdot,\cdot\rangle$ means the inner product, i.e.,

$$u_{i,j,k}=\frac{1}{\aleph_i^{\lambda}\aleph_j^{\lambda}\aleph_k^{\lambda}}\int_0^1\int_0^1\int_0^1 u(\mathbf{x},t)G_{i,j,k}^{\lambda}(\mathbf{x},t)\vartheta_{i,j,k}^{\lambda}(\mathbf{x},t)\mathrm{d}x\mathrm{d}y\mathrm{d}t, \tag{9}$$

where $\vartheta_{i,j,k}^{\lambda}(\mathbf{x},t)=\vartheta_i^{\lambda}(x)\vartheta_j^{\lambda}(y)\vartheta_k^{\lambda}(t)$. In Eqs (8) and (9), vectors U and $\Lambda(\mathbf{x},t)$ are of the order $M_1M_2M_3\times 1$ and are given as,

$$U=\begin{bmatrix} u_{0,0,0},\ldots,u_{0,0,M_3-1}\left|u_{0,1,0},\ldots,u_{0,1,M_3-1}\right|,\ldots,\left|u_{0,M_2-1,0},\ldots,u_{0,M_2-1,M_3-1}\right. \\ u_{1,0,0},\ldots,u_{1,0,M_3-1}\left|u_{1,1,0},\ldots,u_{1,1,M_3-1}\right|,\ldots,\left|u_{1,M_2-1,0},\ldots,u_{1,M_2-1,M_3-1}\right. \\ \vdots \\ u_{M_1-1,0,0},\ldots,u_{M_1-1,0,M_3-1}\left|u_{M_1-1,1,0},\ldots,u_{M_1-1,1,M_3-1}\right|,\ldots,\left|u_{M_1-1,M_2-1,0},\ldots,u_{M_1-1,M_2-1,M_3-1}\right. \end{bmatrix}^T,$$

$$\Lambda(\mathbf{x},t)=\begin{bmatrix} G_{0,0,0}^{\lambda},\ldots,G_{0,0,M_3-1}^{\lambda}\left|G_{0,1,0}^{\lambda},\ldots,G_{0,1,M_3-1}^{\lambda}\right|,\ldots,\left|G_{0,M_2-1,0}^{\lambda},\ldots,G_{0,M_2-1,M_3-1}^{\lambda}\right. \\ G_{1,0,0}^{\lambda},\ldots,G_{1,0,M_3-1}^{\lambda}\left|G_{1,1,0}^{\lambda},\ldots,G_{1,1,M_3-1}^{\lambda}\right|,\ldots,\left|G_{1,M_2-1,0}^{\lambda},\ldots,G_{1,M_2-1,M_3-1}^{\lambda}\right. \\ \vdots \\ G_{M_1-1,0,0}^{\lambda},\ldots,G_{M_1-1,0,M_3-1}^{\lambda}\left|G_{M_1-1,1,0}^{\lambda},\ldots,G_{M_1-1,1,M_3-1}^{\lambda}\right|,\ldots,\left|G_{M_1-1,M_2-1,0}^{\lambda},\ldots,G_{M_1-1,M_2-1,M_3-1}^{\lambda}\right. \end{bmatrix}^T.$$

3. Development of novel operational matrices

In this section, we develop some novel operational matrices to approximate positive-integer and fractional-order derivatives, space-time delay terms, integral terms, including singular kernel, and nonlinear terms with the help of shifted Gegenbauer polynomials vector $\Lambda(\mathbf{x},t)$.

Theorem 1. Let $\Lambda(x)$ be an one-dimensional shifted Gegenbauer polynomials. Then, there exists a square matrix $\mathbf{A}_1$, such that [25,26],

$$\Lambda(x)=\mathbf{\Delta}^{-1}N(x)=\mathbf{A}_1N(x).$$

here, $\mathbf{A}_1$ and $N(x)$ are matrices of the order $M\times M$ and $M\times 1$, respectively, and are defined as,

$$\mathbf{\Delta} = \begin{bmatrix} \sigma_0^0 & \sigma_1^0 & \cdots & \sigma_{M-1}^0 \\ \sigma_0^1 & \sigma_1^1 & \cdots & \sigma_{M-1}^1 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_0^{M-1} & \sigma_1^{M-1} & \cdots & \sigma_{M-1}^{M-1} \end{bmatrix}, N(x) = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{M-1} \end{bmatrix}. \tag{10}$$

The elements σ_j^i existing in $\mathbf{\Delta}^{-1} = \mathbf{A}_1$ must satisfy the following relation.

$$\sigma_j^i = \sum_{k=0}^{j} \frac{2(-1)^{j-k}\Gamma(2\lambda)(j+\lambda)j!\Gamma(k+j+2\lambda)\Gamma(k+i+\lambda+1/2)}{(j-k)!k!\Gamma(k+\lambda+1/2)\Gamma(j+2\lambda)\Gamma(k+i+2\lambda+1)} = \sum_{k=0}^{j} \Theta_{k,j}^{i,\lambda},$$

In the same way, a three-dimensional SGP vector $\Lambda(\mathbf{x},t) = \Lambda(x,y,t)$ can be stated as,

$$\Lambda(\mathbf{x},t) = \mathbf{A}N(\mathbf{x},t).$$

here, $\mathbf{A}$ and $N(\mathbf{x},t)$ are matrices of order $M_1M_2M_3 \times M_1M_2M_3$ and $M_1M_2M_3 \times 1$, respectively, and are given below:

$$\mathbf{A} = \mathbf{A}_1 \otimes \mathbf{A}_1 \otimes \mathbf{A}_1\,; N(\mathbf{x},t) = N(x) \otimes N(y) \otimes N(t).$$

Theorem 2. Let $\Lambda(x)$ be a one-dimensional shifted Gegenbauer polynomials (SGPs) vector given in above (7) and

$$\Lambda(x)\Lambda^T(x)C = \mathbf{B}_1\Lambda(x).$$

Then, the elements in the matrix $\mathbf{B}_1$ satisfy the following relation.

$$l_{j+1,k+1} = \frac{1}{\aleph_k^\lambda} \times \sum_{i=0}^{M-1} l_i \varpi_{i,j,k}^\lambda, \text{ for } j,k = 0,1,2,...,M-1,$$

with $\varpi_{i,j,k}^\lambda = \int_0^1 G_i^{\lambda}(x) G_j^{\lambda}(x) G_k^{\lambda}(x) \vartheta^\lambda(x)\,\mathrm{d}x$.

In a similar way, let $\Lambda(\mathbf{x},t)$ be a three-dimensional SGPs vector, then, we have,

$$\Lambda(\mathbf{x},t)\Lambda^T(\mathbf{x},t)U = \mathbf{B}\Lambda(\mathbf{x},t),$$

with the square matrix $\mathbf{B}$ with the order $M_1M_2M_3 \times M_1M_2M_3$ and it is given as,

$$\mathbf{B} = \begin{bmatrix} b_1^1 & b_2^1 & \cdots & b_{M_1M_2M_2}^1 \\ b_1^2 & b_2^2 & \cdots & b_{M_1M_2M_3}^2 \\ \vdots & \vdots & \ddots & \vdots \\ b_1^{M_1M_2M_3} & b_2^{M_1M_2M_3} & \cdots & b_{M_1M_2M_3}^{M_1M_2M_3} \end{bmatrix}. \tag{11}$$

The elements b_j^i in **B** are shown below, with $r_i = M_3(M_2 i_1 + i_2) + i_3 + 1,$ for r_j and r_k can be computed by just changing the suffices.

$$b_{r_j}^{r_k} = \prod_{l=1}^{3} \frac{\left(\Gamma(2\lambda)\right)^2 \left(2\lambda + 2k_l\right) k_l!}{\left(\Gamma\left(\lambda + 1/2\right)\right)^2 \Gamma\left(2\lambda + k_l\right)} \sum_{i_1=0}^{M_1-1} \sum_{i_2=0}^{M_2-1} \sum_{i_3=0}^{M_3-1} u_{i_1,i_2,i_3} q_{r_i,r_j,r_k},$$

with $q_{r_i,r_j,r_k} = \int_0^1 \int_0^1 \int_0^1 G_{i_1,j_1,k_1}^{\lambda}(\mathbf{x},t) G_{i_2,j_2,k_2}^{\lambda}(\mathbf{x},t) G_{i_3,j_3,k_3}^{\lambda}(\mathbf{x},t) \left(x - x^2\right)^{\lambda - \frac{1}{2}} \left(y - y^2\right)^{\lambda - \frac{1}{2}} \left(t - t^2\right)^{\lambda - \frac{1}{2}} \mathrm{d}t\mathrm{d}y\mathrm{d}x,$ and

for $0 \le i_l, j_l, k_l \le M_l - 1; l = 1,2,3.$

Theorem 3. Let $\int_0^x t^n / (x^\rho - t^\rho)^\alpha \mathrm{d}t; 0 < \alpha < 1, \rho > 1$ be the kernel of weakly singular integral equations. Then, the following relation must hold.

$$\int_0^x \frac{t^n}{\left(x^\rho - t^\rho\right)^\alpha} \mathrm{d}t = \frac{x^{n-\rho\alpha+1}}{\rho} \mathrm{B}\left(\frac{n+1}{\rho}, 1-\alpha\right),$$

where $\mathrm{B}(x, y)$ is the beta function.

Proof. It is very difficult to compute this integral directly. Thus, we assume the substitution $z = t^\rho / x^\rho$ and $\mathrm{d}t = x^\rho / \rho t^{\rho-1} \mathrm{d}z$. Then, the above integral takes the following form:

$$\int_0^x \frac{t^n}{(x^\rho - t^\rho)^\alpha} \mathrm{d}t = \int_0^1 \frac{x^n z^{n/\rho}}{(x^\rho - x^\rho z)^\alpha} \frac{x^\rho}{\rho (x z^{1/\rho})^{\rho-1}} \mathrm{d}z,$$

$$= \frac{x^{n+\rho}}{\rho x^{\rho\alpha+\rho-1}} \int_0^1 \frac{z^{n/\rho}}{(1-z)^\alpha} \frac{1}{z^{\rho-1/\rho}} \mathrm{d}z = \frac{x^{n+\rho}}{\rho x^{\rho\alpha+\rho-1}} \int_0^1 \frac{z^{(n-\rho+1)/\rho}}{(1-z)^\alpha} \mathrm{d}z = \frac{1}{\rho} x^{n-\rho\alpha+1} \mathrm{B}\left(\frac{n+1}{\rho}, 1-\alpha\right).$$

So, this is the required result.

Remark 1. However, using Theorem 3, the matrix approximation of an integral part is given as follow:

$$\int_0^x \frac{u(t)}{\left(x^\rho - t^\rho\right)^\alpha} \mathrm{d}t = C^T \mathbf{A}_1 \int_0^x \frac{N(t)}{\left(x^\rho - t^\rho\right)^\alpha} \mathrm{d}t = C^T \mathbf{A}_1 \mathbf{I}_\alpha^\rho(x).$$

here, the vector $\mathbf{I}_\alpha^\rho(x)$ is given as,

$$\mathbf{I}_\alpha^\rho\left(x\right) = \left[\frac{1}{\rho} x^{1-\rho\alpha} \mathrm{B}\left(\frac{1}{\rho}, 1-\alpha\right) \quad \frac{1}{\rho} x^{2-\rho\alpha} \mathrm{B}\left(\frac{2}{\rho}, 1-\alpha\right) \quad \cdots \quad \frac{1}{\rho} x^{M-\rho\alpha} \mathrm{B}\left(\frac{M}{\rho}, 1-\alpha\right)\right]. \qquad (12)$$

Similarly, the matrix approximation of $u(\mathbf{x},t)$, the weakly singular integral part is as follows,

$$\int_0^x \int_0^y \frac{u(\mathbf{p},t)}{\left(x^{\rho_1} - p^{\rho_1}\right)^{\alpha_1} \left(y^{\rho_2} - q^{\rho_2}\right)^{\alpha_2}} \mathrm{d}q\mathrm{d}p = U^T \mathbf{A} \mathbf{I}_{\alpha_1,\alpha_2}^{\rho_1,\rho_2}(\mathbf{x},t).$$

where **A** is a matrix of the order $M_1M_2M_3 \times M_1M_2M_3$ and $\mathbf{I}_{\alpha_1,\alpha_2}^{\rho_1,\rho_2}(\mathbf{x},t) = \mathbf{I}_{\alpha_1}^{\rho_1}(x) \otimes \mathbf{I}_{\alpha_2}^{\rho_2}(y) \otimes N(t)$.

Theorem 4. Assume $u(\mathbf{x},t)$ from $C([0,1]^3)$, and $u(\mathbf{x},t) \cong \Lambda^T(\mathbf{x},t)U$. Then, the following equality holds:

$$\int_0^x\int_0^y \frac{1}{\left(x^{\rho_1}-p^{\rho_1}\right)^{\alpha_1}\left(y^{\rho_2}-q^{\rho_2}\right)^{\alpha_2}} u^2(\mathbf{p},t)\mathrm{d}\mathbf{p} = U^T\mathbf{B}\mathbf{A}\mathbf{I}_{\alpha_1,\alpha_2}^{\rho_1,\rho_2}(\mathbf{x},t)$$

where the product operational matrix **B** corresponds to the vector U and is given in Eq (11).
Proof. Using the given information and function approximation, we get,

$$\int_0^x\int_0^y \frac{1}{\left(x^{\rho_1}-p^{\rho_1}\right)^{\alpha_1}\left(y^{\rho_2}-q^{\rho_2}\right)^{\alpha_2}} u^2(\mathbf{p},t)\mathrm{d}\mathbf{p} = \int_0^x\int_0^y \frac{1}{\left(x^{\rho_1}-p^{\rho_1}\right)^{\alpha_1}\left(y^{\rho_2}-q^{\rho_2}\right)^{\alpha_2}} u(\mathbf{p},t)u(\mathbf{p},t)\mathrm{d}\mathbf{p},$$

$$= U^T\int_0^x\int_0^y \frac{\Lambda(\mathbf{x},t)\Lambda^T(\mathbf{x},t)U}{\left(x^{\rho_1}-p^{\rho_1}\right)^{\alpha_1}\left(y^{\rho_2}-q^{\rho_2}\right)^{\alpha_2}}\mathrm{d}\mathbf{p} = U^T\mathbf{B}\int_0^x\int_0^y \frac{\Lambda(\mathbf{x},t)}{\left(x^{\rho_1}-p^{\rho_1}\right)^{\alpha_1}\left(y^{\rho_2}-q^{\rho_2}\right)^{\alpha_2}}\mathrm{d}\mathbf{p},$$

$$= U^T\mathbf{B}\mathbf{A}\int_0^x\int_0^y \frac{N(\mathbf{x},t)}{\left(x^{\rho_1}-p^{\rho_1}\right)^{\alpha_1}\left(y^{\rho_2}-q^{\rho_2}\right)^{\alpha_2}}\mathrm{d}\mathbf{p} = U^T\mathbf{B}\mathbf{A}\mathbf{I}_{\alpha_1,\alpha_2}^{\rho_1,\rho_2}.$$

It is the necessary result. In the same way, the matrix representation for the nonlinearity of the order $n > 1$ is given as;

$$\int_0^x\int_0^y \frac{1}{\left(x^{\rho_1}-p^{\rho_1}\right)^{\alpha_1}\left(y^{\rho_2}-q^{\rho_2}\right)^{\alpha_2}} u^n(\mathbf{p},t)\mathrm{d}\mathbf{p} = U^T\mathbf{B}^{n-1}\mathbf{A}\mathbf{I}_{\alpha_1,\alpha_2}^{\rho_1,\rho_2}(\mathbf{x},t) \tag{13}$$

Theorem 5. Let $\Lambda(x)$ be an one-dimensional SGPs vector. Then, fractional-order differentiation of the order $\beta > 0; \gamma - 1 < \beta < \gamma$ of $\Lambda(x)$ is given as [25,26],

$${}_0^C\mathcal{D}_x^{\beta}\left[\Lambda(x)\right] = \mathbf{D}_x^{\beta}\Lambda(x) = \left(\mathbf{\Delta}^{-1}\mathbf{P}^{\beta}\mathbf{\Delta}\right)\Lambda(x).$$

In this relation, $\mathbf{D}_x^{\beta}$ is an operational matrix of a fractional-order derivative of the order β. Also, the matrix $\mathbf{\Delta}$ is defined above and the diagonal matrix $\mathbf{P}^{\beta}$ is,

$$\mathbf{P}^{\beta}=x^{-\beta}\begin{bmatrix}0 & \cdots & 0 & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots\\ 0 & \cdots & \frac{\gamma!}{\Gamma(\gamma-\beta+1)} & 0 & \cdots & 0\\ 0 & \cdots & 0 & \frac{(\gamma+1)!}{\Gamma(\gamma-\beta+2)} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \cdots & \cdots & \cdots & \cdots & \frac{(M-1)!}{\Gamma(M-\beta)}\end{bmatrix}..$$

Theorem 6. Let $\Lambda(\mathbf{x}, t) = \Lambda(x, y, t)$ be the SGPs vector in three dimensions. Then, the fractional-order derivatives of $\Lambda(\mathbf{x}, t)$ of orders α,β and γ with respect to x, y, and t, respectively, are given as,

$$ {}_{0}^{C}\mathcal{D}_{\xi_l}^{\alpha_l}\left[\Lambda(\mathbf{x},t)\right]=\mathbf{F}_{x_l}^{\alpha_l}\Lambda(\mathbf{x},t)=\left[\left(\mathbf{A}_{x_l}\right)^{-1}\mathbf{P}_{\xi_l}^{\alpha_l}\mathbf{A}_{x_l}\right]\Lambda(\mathbf{x},t), l=1,2,3. \tag{14}$$

here, the operational matrices of fractional-order derivatives with respect to x, y, and t are represented by $\mathbf{F}_{x_1}^{\alpha_1}=\mathbf{F}_{x}^{\alpha},\mathbf{F}_{x_2}^{\alpha_2}=\mathbf{F}_{y}^{\beta}$ and $\mathbf{F}_{x_3}^{\alpha_3}=\mathbf{F}_{t}^{\gamma}$, respectively. Furthermore, $\mathbf{A}_{x_1}=\mathbf{A}_{x},\mathbf{P}_{x_1}^{\alpha_1}=\mathbf{P}_{x}^{\alpha}$, $\mathbf{A}_{x_2}=\mathbf{A}_{y},\mathbf{P}_{x_2}^{\alpha_2}=\mathbf{P}_{y}^{\beta},\mathbf{A}_{x_3}=\mathbf{A}_{t}$, and $\mathbf{P}_{x_3}^{\alpha_3}=\mathbf{P}_{t}^{\gamma}$ are the matrices of the order $M_1M_2M_3\times M_1M_2M_3$, and are given as,

$$\mathbf{A}_{x}=\begin{bmatrix}\boldsymbol{\eta}_0^0 & \boldsymbol{\eta}_0^1 & \cdots & \boldsymbol{\eta}_0^{M_1-1}\\ \boldsymbol{\eta}_1^0 & \boldsymbol{\eta}_1^1 & \cdots & \boldsymbol{\eta}_1^{M_1-1}\\ \vdots & \vdots & \ddots & \vdots\\ \boldsymbol{\eta}_{M_1-1}^0 & \boldsymbol{\eta}_{M_1-1}^1 & \cdots & \boldsymbol{\eta}_{M_1-1}^{M_1-1}\end{bmatrix},\mathbf{p}_x^{\alpha}(m)=\frac{1}{x^{\alpha}}\begin{cases}\frac{(m-1)!}{\Gamma(m-\alpha)}\mathbf{I}_x^{\alpha} & m=\lceil\alpha+1\rceil,\ldots,M_1,\\ 0 & \text{otherwise,}\end{cases}$$

$$\mathbf{A}_{y}=\begin{bmatrix}\tilde{\mathbf{A}}_y & \mathbf{O} & \cdots & \mathbf{O}\\ \mathbf{O} & \tilde{\mathbf{A}}_y & \cdots & \mathbf{O}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{O} & \mathbf{O} & \cdots & \tilde{\mathbf{A}}_y\end{bmatrix},\mathbf{P}_{y}^{\beta}=\begin{bmatrix}\tilde{\mathbf{P}}_y^{\beta} & \mathbf{O} & \cdots & \mathbf{O}\\ \mathbf{O} & \tilde{\mathbf{P}}_y^{\beta} & \cdots & \mathbf{O}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{O} & \mathbf{O} & \cdots & \tilde{\mathbf{P}}_y^{\beta}\end{bmatrix},$$

$$\mathbf{A}_{t}=\begin{bmatrix}\tilde{\mathbf{A}}_t & \mathbf{O} & \cdots & \mathbf{O}\\ \mathbf{O} & \tilde{\mathbf{A}}_t & \cdots & \mathbf{O}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{O} & \mathbf{O} & \cdots & \tilde{\mathbf{A}}_t\end{bmatrix},\mathbf{P}_{t}^{\gamma}=\begin{bmatrix}\tilde{\mathbf{P}}_t^{\gamma} & \mathbf{O} & \cdots & \mathbf{O}\\ \mathbf{O} & \tilde{\mathbf{P}}_t^{\gamma} & \cdots & \mathbf{O}\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{O} & \mathbf{O} & \cdots & \tilde{\mathbf{P}}_t^{\gamma}\end{bmatrix}.$$

In the relations given above, $\mathbf{I}_x^{\alpha}$ is an identity matrix of the order $M_2M_3\times M_2M_3$. Components in the matrix $\mathbf{A}_x$ can be computed by using Eq (15). The matrices $\tilde{\mathbf{A}}_y$ and $\tilde{\mathbf{P}}_y^{\beta}$ are of the order $M_2M_3\times M_2M_3$, the components of which must fulfil the relations given below. Likewise, the matrices $\tilde{\mathbf{A}}_t$ are square and $\tilde{\mathbf{P}}_t^{\gamma}$ is a diagonal matrix with elements $p_t^{\gamma}(m)$, which are given as,

$$\tilde{\mathbf{A}}_y = \begin{bmatrix} \xi_0^0 & \xi_0^1 & \cdots & \xi_0^{M_2-1} \\ \xi_1^0 & \xi_1^1 & \cdots & \xi_1^{M_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{M_2-1}^0 & \xi_{M_2-1}^1 & \cdots & \xi_{M_2-1}^{M_2-1} \end{bmatrix}, p_y^{\beta}(m) = \frac{1}{y^{\beta}} \begin{cases} \dfrac{(m-1)!}{\Gamma(m-\beta)} \mathbf{I}_y^{\beta} & m = \lceil \beta+1 \rceil, \ldots, M_2, \\ 0 & \text{otherwise}, \end{cases}$$

$$\tilde{\mathbf{A}}_t = \begin{bmatrix} \zeta_0^0 & \zeta_0^1 & \cdots & \zeta_0^{M_3-1} \\ \zeta_1^0 & \zeta_1^1 & \cdots & \zeta_1^{M_3-1} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{M_3-1}^0 & \zeta_{M_3-1}^1 & \cdots & \zeta_{M_3-1}^{M_3-1} \end{bmatrix}, p_t^{\gamma}(m) = \frac{1}{t^{\gamma}} \begin{cases} \dfrac{(m-1)!}{\Gamma(m-\gamma)} & m = \lceil \gamma+1 \rceil, \ldots, M_3, \\ 0 & \text{otherwise}, \end{cases}$$

$$\boldsymbol{\eta}_j^k = \mathbf{I}_x^{\alpha} \times \sum_{i=0}^{j} \Theta_{k,j}^{i,\lambda}, \ \boldsymbol{\xi}_j^k = \mathbf{I}_y^{\beta} \times \sum_{i=0}^{j} \Theta_{k,j}^{i,\lambda}, \zeta_j^k = \sum_{i=0}^{j} \Theta_{k,j}^{i,\lambda}, \tag{15}$$

where $\mathbf{I}_y^{\beta}$ is an identity matrix of the order $M_3 \times M_3$.

Theorem 7. Let $\Lambda(\mathbf{x}, t)$ be a SGPs vector in three-dimensions. Then the derivatives of $\Lambda(\mathbf{x}, t)$ with respect to x, y and t fulfill the following relation.

$$\frac{\mathrm{d}}{\mathrm{d}\xi_l}\left[\Lambda(\mathbf{x},t)\right] = \mathbf{D}_{\xi_l} \Lambda(\mathbf{x},t), l = 1,2,3. \tag{16}$$

where $\mathbf{D}_{\xi_n} = \mathbf{D}_x, \mathbf{D}_{\xi_2} = \mathbf{D}_y$, and $\mathbf{D}_{\xi_3} = \mathbf{D}_t$ denote the derivative operational matrices with respect to x, y and t with the same order $M_1 M_2 M_3 \times M_1 M_2 M_3$. Here, the square matrix $\mathbf{D}_x$ holds the elements $\mathbf{r}_{l,m}^x$. Moreover, $\mathbf{D}_y$ and $\mathbf{D}_t$ are diagonal matrices, which are given below.

$$\mathbf{r}_{l,m}^x = \begin{cases} 4\mathbf{I}_1(m+\lambda-1), \text{ if } l = 2,\ldots,M_1, m = 1,\ldots,l-1 \text{ and } (l+m) \text{ odd}, \\ 0, \quad \text{else where}. \end{cases}$$

$$\mathbf{D}_y = \begin{bmatrix} \tilde{\mathbf{D}}_y & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \tilde{\mathbf{D}}_y & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \tilde{\mathbf{D}}_y \end{bmatrix}, \mathbf{D}_t = \begin{bmatrix} \tilde{\mathbf{D}}_t & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \tilde{\mathbf{D}}_t & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \tilde{\mathbf{D}}_t \end{bmatrix},$$

where $\mathbf{I}_1$ is an identity matrix of the order $M_1 M_2 \times M_1 M_2$. Entries of $\tilde{\mathbf{D}}_y$ and $\tilde{\mathbf{D}}_t$ represented by $\mathbf{r}_{l,m}^y$ and $r_{l,m}^t$, respectively, and can be computed by the following relations.

$$\mathbf{r}_{l,m}^y = \begin{cases} 4\mathbf{I}_2(m+\lambda-1), \text{ if } l = 2,\ldots,M_2, m = 1,\ldots,l-1 \text{ and } (l+m) \text{ odd}, \\ 0, \quad \text{else where}, \end{cases}$$

$$r_{l,m}^t = \begin{cases} 4(m+\lambda-1), \text{ if } l = 2,3,\ldots,M_3, m = 1,2,\ldots,l-1 \text{ and } (l+m) \text{ odd}, \\ 0, \quad \text{else where}. \end{cases}$$

here, $\mathbf{I}_2$ is an identity matrix of the order $M_1 \times M_1$.

Theorem 8. Let $\Lambda(\mathbf{x},t) = \Lambda(x,y,t)$ be a SGPs vector. Then, the vector of time and space delay expression $\Lambda(a_1 x + b_1, a_2 y + b_2, a_3 t + b_3)$ must satisfy the following relation.

$$\Lambda(a_1x+b_1,a_2y+b_2,a_3t+b_3)=\mathbf{D}_{b_1}^{a_1}\mathbf{D}_{b_2}^{a_2}\mathbf{D}_{b_3}^{a_3}\Lambda(\mathbf{x},t), \tag{17}$$

where $\mathbf{D}_{b_1}^{a_1},\mathbf{D}_{b_2}^{a_2}$ and $\mathbf{D}_{b_3}^{a_3}$ are the delay operational matrices of the order $M_1M_2M_3\times M_1M_2M_3$, which are defined as,

$$\mathbf{D}_{b_1}^{a_1}=\begin{bmatrix} {}_{b_1}^{a_1}\mathbf{d}_1^1 & {}_{b_1}^{a_1}\mathbf{d}_1^2 & \cdots & {}_{b_1}^{a_1}\mathbf{d}_1^{M_1} \\ {}_{b_1}^{a_1}\mathbf{d}_2^1 & {}_{b_1}^{a_1}\mathbf{d}_2^2 & \cdots & {}_{b_1}^{a_1}\mathbf{d}_2^{M_1} \\ \vdots & \vdots & \ddots & \vdots \\ {}_{b_1}^{a_1}\mathbf{d}_{M_1}^1 & {}_{b_1}^{a_1}\mathbf{d}_{M_1}^2 & \cdots & {}_{b_1}^{a_1}\mathbf{d}_{M_1}^{M_1} \end{bmatrix},\mathbf{D}_{b_2}^{a_2}=\begin{bmatrix} \tilde{\mathbf{D}}_{b_2}^{a_2} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \tilde{\mathbf{D}}_{b_2}^{a_2} & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \tilde{\mathbf{D}}_{b_2}^{a_2} \end{bmatrix},\mathbf{D}_{b_3}^{a_3}=\begin{bmatrix} \tilde{\mathbf{D}}_{b_3}^{a_3} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \tilde{\mathbf{D}}_{b_3}^{a_3} & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \tilde{\mathbf{D}}_{b_3}^{a_3} \end{bmatrix}.$$

In the above expression, the elements ${}_{b_1}^{a_1}\mathbf{d}_i^j$, $i,j=1,2,3,\ldots,M_1$ in the matrix $\mathbf{D}_{b_1}^{a_1}$ must satisfy Eq (18). The matrices $\tilde{\mathbf{D}}_{b_2}^{a_2}$ and $\tilde{\mathbf{D}}_{b_3}^{a_3}$ are of order $M_2M_3\times M_2M_3$ and $M_3\times M_3$, respectively, and are computed, using the following relations.

$${}_{b_1}^{a_1}\mathbf{d}_i^j=\begin{cases} \mathbf{I}_x\times\sum\limits_{k=0}^{i}\sum\limits_{s=0}^{j}\dfrac{\Gamma(\lambda+1/2)\Gamma(k+s+\lambda+1/2)}{\Gamma(k+2\lambda+s+1)}\varpi_{i,j}^{\lambda,a_1}, & \text{if } b_1=0, \\ \mathbf{I}_x\times\sum\limits_{k=0}^{i}\sum\limits_{s=0}^{j}\varpi_{i,j}^{\lambda,a_1}\sum\limits_{l}^{k}\dbinom{k}{l}\left(\dfrac{b_1}{a_1}\right)^{k-l}\dfrac{\Gamma(\lambda+1/2)\Gamma(k+s+\lambda+1/2)}{\Gamma(k+2\lambda+s+1)}, & \text{if } b_1\neq 0. \end{cases} \tag{18}$$

$$\tilde{\mathbf{D}}_{b_2}^{a_2}=\begin{bmatrix} {}_{b_2}^{a_2}\mathbf{d}_1^1 & {}_{b_2}^{a_2}\mathbf{d}_1^2 & \cdots & {}_{b_2}^{a_2}\mathbf{d}_1^{M_2} \\ {}_{b_2}^{a_2}\mathbf{d}_2^1 & {}_{b_2}^{a_2}\mathbf{d}_2^2 & \cdots & {}_{b_2}^{a_2}\mathbf{d}_2^{M_2} \\ \vdots & \vdots & \ddots & \vdots \\ {}_{b_2}^{a_2}\mathbf{d}_{M_2}^1 & {}_{b_2}^{a_2}\mathbf{d}_{M_2}^2 & \cdots & {}_{b_2}^{a_2}\mathbf{d}_{M_2}^{M_2} \end{bmatrix},\tilde{\mathbf{D}}_{b_3}^{a_3}=\begin{bmatrix} {}_{b_3}^{a_3}d_1^1 & {}_{b_3}^{a_3}d_1^2 & \cdots & {}_{b_3}^{a_3}d_1^{M_3} \\ {}_{b_3}^{a_3}d_2^1 & {}_{b_3}^{a_3}d_2^2 & \cdots & {}_{b_3}^{a_3}d_2^{M_3} \\ \vdots & \vdots & \ddots & \vdots \\ {}_{b_3}^{a_3}d_{M_3}^1 & {}_{b_3}^{a_3}d_{M_3}^2 & \cdots & {}_{b_3}^{a_3}d_{M_3}^{M_3} \end{bmatrix},$$

where $\mathbf{I}_x$ is identity matrix of order $M_1M_2\times M_1M_2$. The elements ${}_{b_2}^{a_2}\mathbf{d}_i^j$, $i,j=1,2,3,\ldots,M_2$ and ${}_{b_3}^{a_3}d_i^j$, $i,j=1,2,3,\ldots,M_3$ present in $\tilde{\mathbf{D}}_{b_2}^{a_2}$ and $\tilde{\mathbf{D}}_{b_3}^{a_3}$, respectively, are given below.

$${}_{b_2}^{a_2}\mathbf{d}_i^j=\begin{cases} \mathbf{I}_y\times\sum\limits_{k=0}^{i}\sum\limits_{s=0}^{j}\dfrac{\Gamma(\lambda+1/2)\Gamma(k+s+\lambda+1/2)}{\Gamma(k+2\lambda+s+1)}\varpi_{i,j}^{\lambda,a_2}, & \text{if } b_2=0, \\ \mathbf{I}_y\times\sum\limits_{k=0}^{i}\sum\limits_{s=0}^{j}\varpi_{i,j}^{\lambda,a_2}\sum\limits_{l=0}^{k}\dbinom{k}{l}\left(\dfrac{b_2}{a_2}\right)^{k-l}\dfrac{\Gamma(\lambda+1/2)\Gamma(k+s+\lambda+1/2)}{\Gamma(k+2\lambda+s+1)}, & \text{if } b_2\neq 0. \end{cases}$$

$${}_{b_3}^{a_3}d_i^j=\begin{cases} \sum\limits_{k=0}^{i}\sum\limits_{s=0}^{j}\dfrac{\Gamma(\lambda+1/2)\Gamma(k+s+\lambda+1/2)}{\Gamma(k+2\lambda+s+1)}\varpi_{i,j}^{\lambda,a_3}, & \text{if } b_3=0, \\ \sum\limits_{k=0}^{i}\sum\limits_{s=0}^{j}\varpi_{i,j}^{\lambda,a_3}\sum\limits_{l=0}^{k}\dbinom{k}{l}\left(\dfrac{b_3}{a_3}\right)^{k-l}\dfrac{\Gamma(\lambda+1/2)\Gamma(k+s+\lambda+1/2)}{\Gamma(k+2\lambda+s+1)}, & \text{if } b_3\neq 0. \end{cases}$$

where $\mathbf{I}_y$ is the identity matrix of the order $M_3 \times M_3$ and $\varpi_{i,j}^{\lambda,a_1}$ is given as,

$$\varpi_{i,j}^{\lambda,a_1} = \frac{2(j+\lambda)j!(-1)^{i-k}\Gamma(i+k+2\lambda)(-1)^{j-s}\Gamma(j+s+2\lambda)a_1^{\,k}}{\Gamma(j+2\lambda)\Gamma(k+\lambda+1/2)(i-k)!k!\Gamma(s+\lambda+1/2)(j-s)!s!}.$$

4. Iterative spectral scheme

In this section, an iterative spectral procedure based on shifted Gegenbauer polynomials is presented to obtain accurate solutions of multi-dimensional unsteady time-fractional integro-partial differential equations with singular kernel and time-space delays Eq (1). This iterative strategy consists of following steps:

Step 1. First of all, to handle the nonlinear terms, we linearize the problem at hand by applying the Picard iterative method as:

$$\begin{aligned}{}_0^C\mathcal{D}_t^{\,\mu}u_r(\mathbf{x},t)+\lambda_1 u_r(\mathbf{x},a_1t+b_1)&=\phi(\mathbf{x},t)+\lambda_2\Delta u_r-\lambda_3F\left(u_{r-1}(\mathbf{x},t)\right)-\lambda_4\int_0^t\frac{u_{r-1}^n(\mathbf{x},t)}{\left(t^{\rho_3}-\xi^{\rho_3}\right)^{\alpha_3}}\mathrm{d}\xi\\&-\lambda_5\int_0^x\int_0^y\frac{1}{\left(x^{\rho_1}-p^{\rho_1}\right)^{\alpha_1}\left(y^{\rho_2}-q^{\rho_2}\right)^{\alpha_2}}u_r\left(a_2p+b_2,a_3p+q_3,t\right)\mathrm{d}q\mathrm{d}p,\end{aligned} \tag{19}$$

where $r=1,2,3,...,R.$ For $r=0$, we have to choose $u_0(\mathbf{x},t)$ appropriately that satisfies the initial condition associated with the problem.

Step 2. Now, to examine the approximate/analytic solution of the problem (19) by means of the proposed scheme, we first assume the following trial solution as:

$$u_r(\mathbf{x},t)\cong\tilde{u}_r(\mathbf{x},t)=\Lambda^T(\mathbf{x},t)U_r. \tag{20}$$

After collocating the assumed trial solution at uniformed collocation points $x=x_i=i-1/M_1-1$, $y=y_j=j-1/M_2-1$, $t=t_k=k-1/M_3-1$ when $i=1,2,...,M_1$, $j=1,2,...,M_2$, $k=1,2,...,M_3$, the trial solution in Eq (20) takes the following form,

$$u_r(\mathbf{x},t)=\mathbf{Q}U_r=\left[\mathbf{Q}_\Omega+\mathbf{Q}_{\partial\Omega}+\mathbf{Q}_\Gamma\right]U_r. \tag{21}$$

here, $\mathbf{Q}_\Omega$ and $\mathbf{Q}_{\partial\Omega}$ are the matrices with elements lying in the entire domain and boundaries, $\mathbf{Q}_\Gamma$ is the matrix associated with initial conditions. Also, U_r and $\mathbf{Q}$ are matrices of the order $M_1M_2M_3\times 1$ and $M_1M_2M_3\times M_1M_2M_3$, respectively, and are given below. Here, the vector U_r also needs to be computed.

$$U_r = \begin{bmatrix} u^r_{0,0,0},\ldots,u^r_{0,0,M_3-1}\left|u^r_{0,1,0},\ldots,u^r_{0,1,M_3-1}\right|,\ldots,\left|u^r_{0,M_2-1,0},\ldots,u^r_{0,M_2-1,M_3-1}\right. \\ u^r_{1,0,0},\ldots,u^r_{1,0,M_3-1}\left|u^r_{1,1,0},\ldots,u^r_{1,1,M_3-1}\right|,\ldots,\left|u^r_{1,M_2-1,0},\ldots,u^r_{1,M_2-1,M_3-1}\right. \\ \vdots \\ u^r_{M_1-1,0,0},\ldots,u^r_{M_1-1,0,M_3-1}\left|u^r_{M_1-1,1,0},\ldots,u^r_{M_1-1,1,M_3-1}\right|,\ldots,\left|u^r_{M_1-1,M_2-1,0},\ldots,u^r_{M_1-1,M_2-1,M_3-1}\right. \end{bmatrix}^T,$$

$$\mathbf{Q} = \begin{bmatrix} G_1^{\lambda}\left(\mathbf{x}_{1,1},t_1\right) & G_2^{\lambda}\left(\mathbf{x}_{1,1},t_1\right) & \cdots & G_{\ell-1}^{\lambda}\left(\mathbf{x}_{1,1},t_1\right) & G_{\ell}^{\lambda}\left(\mathbf{x}_{1,1},t_1\right) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ G_1^{\lambda}\left(\mathbf{x}_{1,1},t_{M_3}\right) & G_2^{\lambda}\left(\mathbf{x}_{1,1},t_{M_3}\right) & \cdots & G_{\ell-1}^{\lambda}\left(\mathbf{x}_{1,1},t_{M_3}\right) & G_{\ell}^{\lambda}\left(\mathbf{x}_{1,1},t_{M_3}\right) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ G_1^{\lambda}\left(\mathbf{x}_{M_1,1},t_1\right) & G_2^{\lambda}\left(\mathbf{x}_{M_1,1},t_1\right) & \cdots & G_{\ell-1}^{\lambda}\left(\mathbf{x}_{M_1,1},t_1\right) & G_{\ell}^{\lambda}\left(\mathbf{x}_{M_1,1},t_1\right) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ G_1^{\lambda}\left(\mathbf{x}_{M_1,1},t_{M_3}\right) & G_2^{\lambda}\left(\mathbf{x}_{M_1,1},t_{M_3}\right) & \cdots & G_{\ell-1}^{\lambda}\left(\mathbf{x}_{M_1,1},t_{M_3}\right) & G_{\ell}^{\lambda}\left(\mathbf{x}_{M_1,1},t_{M_3}\right) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ G_1^{\lambda}\left(\mathbf{x}_{M_1,M_2},t_1\right) & G_2^{\lambda}\left(\mathbf{x}_{M_1,M_2},t_1\right) & \cdots & G_{\ell-1}^{\lambda}\left(\mathbf{x}_{M_1,M_2},t_1\right) & G_{\ell}^{\lambda}\left(\mathbf{x}_{M_1,M_2},t_1\right) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ G_1^{\lambda}\left(\mathbf{x}_{M_1,M_2},t_{M_3}\right) & G_2^{\lambda}\left(\mathbf{x}_{M_1,M_2},t_{M_3}\right) & \cdots & G_{\ell-1}^{\lambda}\left(\mathbf{x}_{M_1,M_2},t_{M_3}\right) & G_{\ell}^{\lambda}\left(\mathbf{x}_{M_1,M_2},t_{M_3}\right) \end{bmatrix},$$

Step 3. Next, each term presented in previous equation (19) is approximated by employing the operational matrices developed in Section 3 in the following manner.

$$\begin{gathered} {}_0^C\mathcal{D}_t^{\mu}u_r(\mathbf{x},t) = \Lambda^T\left[\mathbf{F}_t^{\mu}\right]^T U_r = \mathbf{Q}_{\Omega}\left[\mathbf{F}_t^{\mu}\right]^T U_r, u_r\left(\mathbf{x},a_1t+b_1\right) = \Lambda^T\left[\mathbf{D}_{b_1}^{a_1}\right]^T U_r = \mathbf{Q}_{\Omega}\left[\mathbf{D}_{b_1}^{a_1}\right]^T U_r, \\ \int_0^x\int_0^y \frac{u_r\left(a_2p+b_2,a_3p+q_3,t\right)}{\left(x^{\rho_1}-p^{\rho_1}\right)^{\alpha_1}\left(y^{\rho_2}-q^{\rho_2}\right)^{\alpha_2}}\mathrm{d}q\mathrm{d}p = \left[\mathbf{I}_{\rho_1,\rho_2}^{\alpha_1,\alpha_2}\right]^T\mathbf{A}^T\left[\mathbf{D}_{b_2}^{a_2}\right]^T\left[\mathbf{D}_{b_3}^{a_3}\right]^T U_r = \mathbf{G}_2\mathbf{A}^T\left[\mathbf{D}_{b_2}^{a_2}\right]^T\left[\mathbf{D}_{b_3}^{a_3}\right]^T U_r, \\ \Delta u_r(\mathbf{x},t) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u_r(\mathbf{x},t) = \Lambda^T(\mathbf{x},t)\left(\left[\mathbf{D}_x^2+\mathbf{D}_y^2\right]^T\right)U_r = \mathbf{Q}_{\Omega}\left(\left[\mathbf{D}_x^2+\mathbf{D}_y^2\right]^T\right)U_r, \\ \int_0^t \frac{1}{\left(t^{\rho_3}-\xi^{\rho_3}\right)^{\alpha_3}}u_{r-1}^n(\mathbf{x},t)\mathrm{d}\xi = U_{r-1}^T\mathbf{B}_{r-1}^{n-1}\mathbf{A}\mathbf{I}_{\rho_3}^{\alpha_3}(\mathbf{x},t) = U_{r-1}^T\mathbf{B}_{r-1}^{n-1}\mathbf{A}\mathbf{G}_1^T, \end{gathered}$$

where $\mathbf{G}_1$ and $\mathbf{G}_2$ are obtained after collocating $\mathbf{I}_{\rho_3}^{\alpha_3}$ and $\mathbf{I}_{\rho_1,\rho_2}^{\alpha_1,\alpha_2}$, respectively, at equally spaced collocation points.

Step 4. After incorporating the trial solution given in Eq (20) and approximated terms given above into the problem, the matrix form is given as,

$$\begin{gathered} \left(\mathbf{Q}_{\Omega}\left(\left[\mathbf{F}_t^{\mu}\right]^T+\lambda_1\left[\mathbf{D}_{b_1}^{a_1}\right]^T-\lambda_2\left(\left[\mathbf{D}_x^2+\mathbf{D}_y^2\right]^T\right)\right)+\lambda_5\mathbf{G}_2\mathbf{A}^T\left[\mathbf{D}_{b_2}^{a_2}\right]^T\left[\mathbf{D}_{b_3}^{a_3}\right]^T+\mathbf{Q}_{\partial\Omega}+\mathbf{Q}_{\Gamma}\right)U_r \\ = -\lambda_3F\left(\mathbf{Q}_{\Omega}U_{r-1}\right)-\lambda_4U_{r-1}^T\mathbf{B}_{r-1}^{n-1}\mathbf{A}\mathbf{G}_1^T+\mathbf{b}_{\Gamma}+\mathbf{b}_{\partial\Omega}+\mathbf{b}_{\Omega}, \end{gathered} \tag{22}$$

where $\mathbf{b}_{\partial\Omega}$ and $\mathbf{b}_{\Omega}$ are the vectors obtained after collocating boundary conditions and source term $\phi(\mathbf{x},t)$ at boundaries and entire domain, respectively. It can be rewritten as,

$$U_r = \mathbf{K}^{-1}\tilde{U}_{r-1}, \tag{23}$$

where $\mathbf{K}$ and $\tilde{U}_{r-1}$ are the matrix and vector of orders $M_1M_2M_3 \times M_1M_2M_3$ and $M_1M_2M_3 \times 1$, respectively, and are given as,

$$\mathbf{K} = \mathbf{Q}_{\Omega}\left(\left[\mathbf{F}_t^{\mu}\right]^T + \lambda_1\left[\mathbf{D}_{b_1}^{a_1}\right]^T - \lambda_2\left(\left[\mathbf{D}_x^2 + \mathbf{D}_y^2\right]^T\right)\right) + \lambda_5\mathbf{G}_2\mathbf{A}^T\left[\mathbf{D}_{b_2}^{a_2}\right]^T\left[\mathbf{D}_{b_3}^{a_3}\right]^T + \mathbf{Q}_{\partial\Omega} + \mathbf{Q}_{\Gamma},$$
$$\tilde{U}_{r-1} = -\lambda_3F\left(\mathbf{Q}_{\Omega}U_{r-1}\right) - \lambda_4U_{r-1}^T\mathbf{B}_{r-1}^{n-1}\mathbf{A}\mathbf{G}_1^T + \mathbf{b}_{\Gamma} + \mathbf{b}_{\partial\Omega} + \mathbf{b}_{\Omega},$$

Step 5. In order to obtain the accurate solution, iterate the procedure given in Eqs (22) and (23) until the required accuracy is attained using Maple 15. Estimated solution can be explored by setting U_r into the assumed trial solution give in Eq (20).

5. Error bound analysis

This section is devoted to studying the error-bound and convergence analysis of the proposed iterative spectral scheme. For this section, we consider that M = M1 = M2 = M3 for simplicity.

Theorem 9. Consider $\partial^{l+m+n}\phi(\mathbf{x},t)/\partial x^l\partial y^m\partial t^n \in C([0,1]^3)$ and $l = m = n = 0, 1, 2, \ldots, M$. If the estimated solution of $\phi(\mathbf{x},t)$ is $\phi_M(\mathbf{x},t)$ and $\phi_M(\mathbf{x},t) \in S_M = \operatorname{span}\{G_{i,j,k}^{\lambda}(\mathbf{x},t), 0 \le i,j,k \le M\}$, assume that the Taylor series of *M*th-order of $u(\mathbf{x},t)$ is $\phi_M(\mathbf{x},t)$ with respect to *x*, *y* and *t*. Then, the error bound is given by Eq (24) [25].

$$\left\|\phi(\mathbf{x},t) - \phi_M(\mathbf{x},t)\right\|_2 \le \frac{3^{M+1}}{(M+1)!}\beta \tag{24}$$

where β is given as,

$$\beta = \max_{0\le i,j\le M+1}\left\{\beta_{i,j}\right\}, \beta_{i,j} = \max_{(x,y,z)\in[0,1]^3}\left|\frac{\partial^{M+1}\phi(\mathbf{x},t)}{\partial^{M+1-i}x\partial^{i-j}y\partial^j t}\right|.$$

6. Test problems and code validations

This segment comprises some numerical problems for code validation and comparative analysis. Some new numerical problems are taken to authenticate the efficiency of the proposed method based on novel operational matrices. We use Maple 2015 for numerical computation. Here, the absolute-error, relative error, L2-norms, L∞-norms, and RMS-norms can be calculated by using the following relations at t = T = 1.

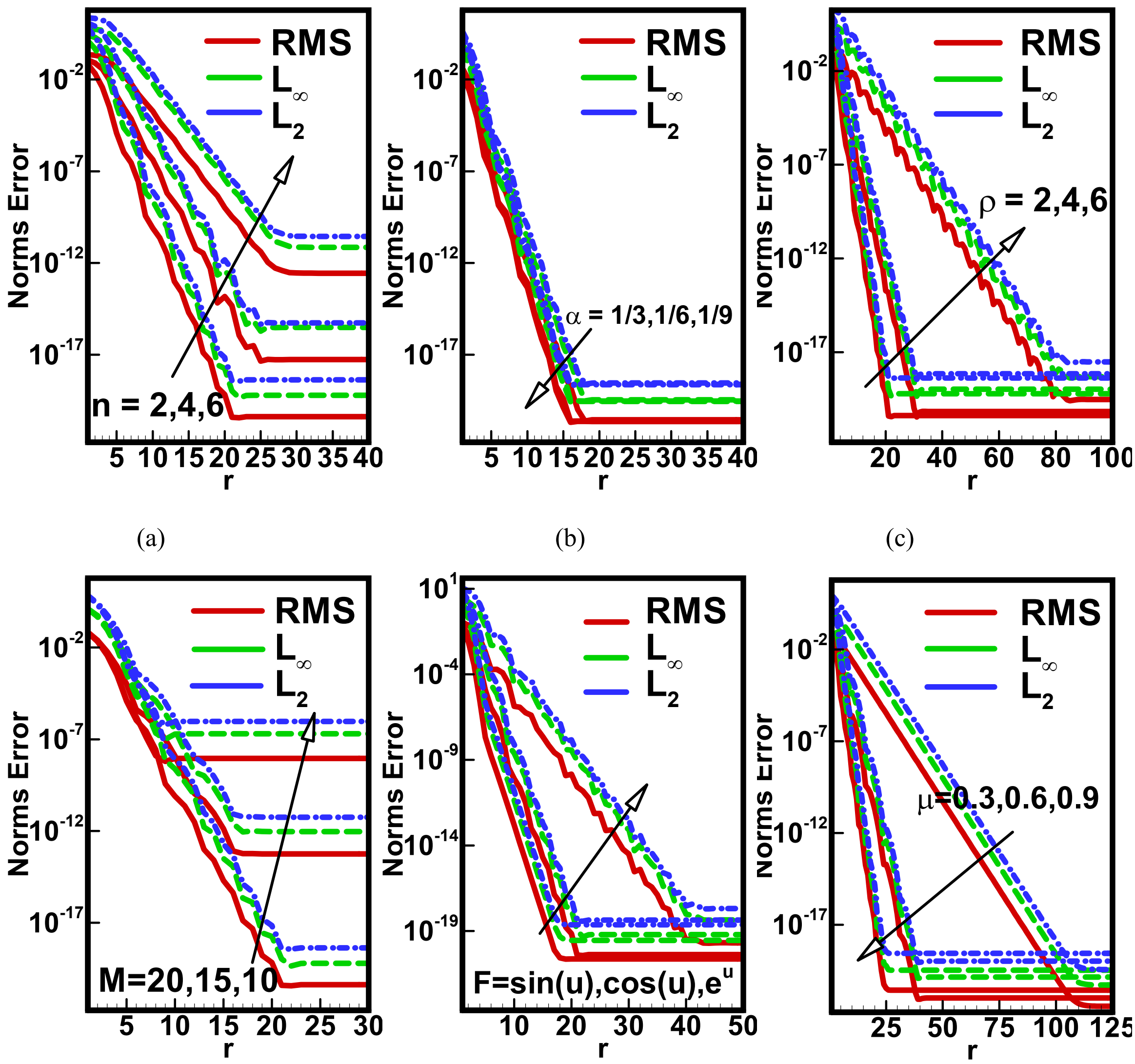
$$|\mathbf{AE}| = \left|u\left(\mathbf{x}_{i,j},t\right) - \tilde{u}\left(\mathbf{x}_{i,j},t\right)\right|,\ L_\infty = \max_{i,j,k}\left(|\mathbf{AE}|\right),$$

$$L_2 = \sqrt{\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\sum_{k=1}^{M_3}\left|u\left(\mathbf{x}_{i,j},t\right) - \tilde{u}\left(\mathbf{x}_{i,j},t\right)\right|^2},\mathrm{RMS} = \frac{1}{M_1M_2M_3+1}\sqrt{\sum_{i=1}^{M_1}\sum_{j=1}^{M_2}\sum_{k=1}^{M_3}\left|u\left(\mathbf{x}_{i,j},t\right) - \tilde{u}\left(\mathbf{x}_{i,j},t\right)\right|^2}.$$

6.1. Code validation study

In this section, one-dimensional fractional-order nonlinear weakly singular problem is considered as given below.

$${}_0^C\mathcal{D}_t^{\,\mu}u(t) + u\left(3t - \frac{2}{7}\right) = \sin\left(u(t)\right) + \int_0^t \frac{1}{\left(t^\rho - \tau^\rho\right)^\alpha}u^n(\tau)\,\mathrm{d}\tau + \phi(t), \tag{25}$$



(a) (b) (c)

(d) (e) (f)

Figure 1. Convergence analysis and code validation of the proposed method by solving problem (25) against the various choices of (a) n, (b) α, (c) ρ, (d) M, (e) $F(u)$ and (f) μ.

First, a computer program is developed in Maple, using the iterative spectral scheme. Then, various simulation are performed to analyze the stability and convergence by plotting the RMS and L2 and L∞-norms versus various parameters as shown in Figure 1. It is important to point out that the numerical solution of the problem in Eq (25) are stable and converges upon varying different parameters, including n, α, ρ, M, F(u) and μ against the number of iterations r. For some cases, the proposed method needs more iterations to make the solution stable and convergent. The level of accuracy is not effected by changing the values of α, ρ, F(u) and μ, when r = 120, where the accuracy increases with an increasing M and decreasing n gradually.

6.2. Comparative analysis

A comparative study and its detail explanation are presented in this subsection. For this purpose, consider the following nonlinear partial integro-differential equation of singular kernel when [0, 1] × Ω = [-1, 1] × [-1, 1],

$$ {}_0^C\mathcal{D}_t^{\mu} u(\mathbf{x},t) = \Delta u(\mathbf{x},t) + t u(\mathbf{x},t)\sin\left(u(\mathbf{x},t)\right) + f(\mathbf{x},t) - \int_0^t \frac{1}{\sqrt{t-\xi}} u^2(\mathbf{x},\xi)\,\mathrm{d}\xi, \quad (26) $$

Its initial and boundary conditions are $u(x,-1,t) = u(x,1,t) = u(-1,y,t) = u(1,y,t) = 0$, and $u(x,y,0) = 0$. The function $f(\mathbf{x},t)$ is given below.

$$ f(\mathbf{x},t) = (1-x^2)(1-y^2)\frac{1}{\Gamma(2-\mu)}t^{1-\mu} + \frac{16}{15}t^{5/2}(1-x^2)(1-y^2) - 2t(x^2+y^2-2) $$
$$ t^2(1-x^2)(1-y^2)\sin\left(t(1-x^2)(1-y^2)\right). $$

The exact solution of Eq (30) is u(x, y) = t(1–x^2)(1–y^2). We utilize the proposed iterative spectral scheme for various values of M1 = M2, when M2 = 3, R = 30 and taking Digits = 30 for solving the problem in Eq (30). A comparison is made in the form of L2-norm with the results attained by Sinc collocation scheme [3]. This comparison shown in Table 1 confirms that the suggested scheme provides a more appropriate tool to seek the solution of this problem than the other schemes published previously [3]. These earlier schemes fail to attain the obtained level of accuracy with the same level of computations. We obtain the analytical solution, when M1 = M2 > 4. In other words, the proposed method gives more accurate solutions with a significantly reduced computational cost.

Table 1. Comparison of L_2-norm obtained from proposed scheme with the published work [3] for problem (26).

L_2-norm using Sinc scheme		L_2-norm using proposed scheme			
N	[3]	$M_1 = M_2$	$\alpha = 1/2$	$\alpha = 1/3$	$\alpha = 1/4$
9	2.7486×10^{-3}	3	4.4290×10^{-5}	4.6080×10^{-5}	4.6883×10^{-5}
12	1.7565×10^{-4}	4	6.7400×10^{-6}	3.4547×10^{-6}	7.8004×10^{-6}
15	1.1880×10^{-5}	≥5	0	0	0

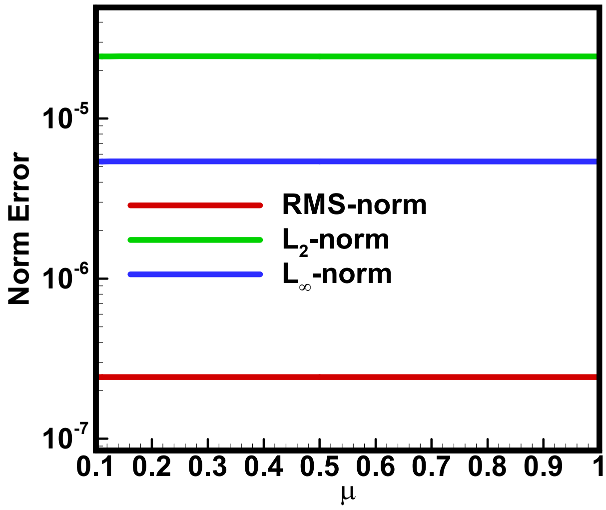
6.3. Some novel results and discussion

After validating our computer program and presenting the comparative study, this subsection reports some novel results for problems, such as presented in Eq (31) when ρ1, ρ2, ρ3 ≥ 1, by means of the suggested iterative spectral scheme.

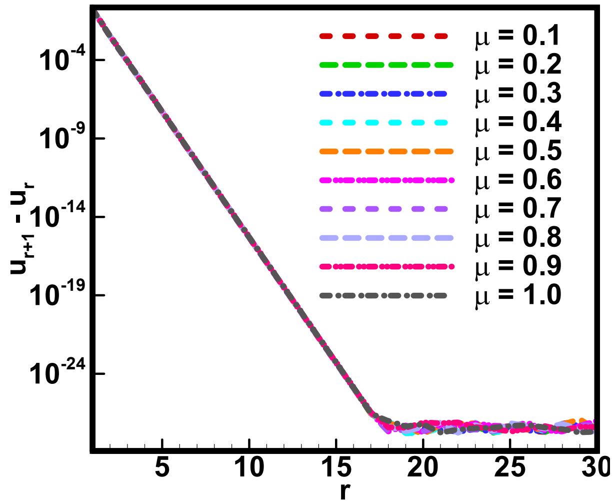
Problem 1. First, consider the time-fractional unsteady nonlinear partial integro-differential equation of weakly singular kernel in two dimensions defined on $\Omega = [0, 1] \times [0, 1]$, $t > 0$ when $0 < \mu < 1$ as given below.

$$
\begin{aligned}
{}_0^C\mathcal{D}_t^{\mu} u(\mathbf{x},t) + \frac{1}{2} u\left(\mathbf{x}, t+\frac{1}{2}\right) &= 2\nabla^2 u(\mathbf{x},t) - \sin\left(u(\mathbf{x},t)\right) + \frac{1}{2}\int_0^t \frac{1}{\left(t^2-\xi^2\right)^{1/3}} u\left(\mathbf{p},\xi\right) \mathrm{d}\xi \\
&+ \phi\left(\mathbf{x},t\right) + \int_0^x\int_0^y \frac{1}{\left(x^4-p^4\right)^{1/4}\left(y^2-q^2\right)^{1/3}} u\left(2p,-3q,t\right)\mathrm{d}q\mathrm{d}p
\end{aligned}
\qquad (27)
$$

The initial guess related to the problem in Eq (31) is u(x) = 0. The analytical solution of Eq (31) is u(x, t) = tsin(xy), and the function ϕ(x, t) can be chosen such that Eq (27) is satisfied. Figure 2 illustrates the behavior of error norms L2, L∞ and RMS for the fractional-order parameter by using our new extended scheme, when M1 = M2 = 9 and M3 = 2.



(a)



(b)

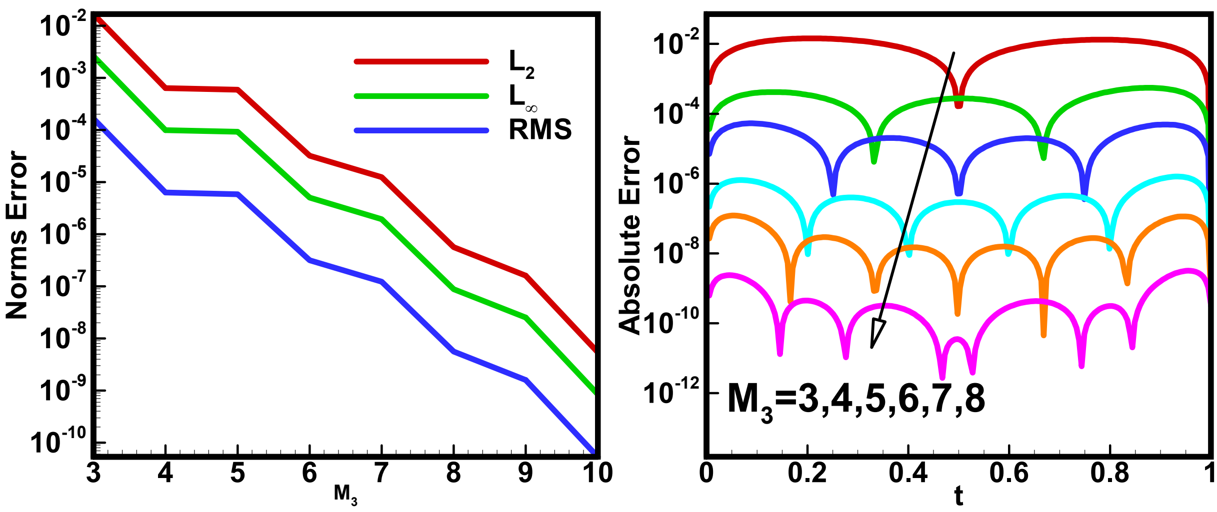
Figure 2. Behavior of (a) norms error L_2, L_∞ and RMS and (b) L_2-norm of two consecutive solutions i.e., $|u_{r+1} - u_r|_2$ for solving Eq (31) as varying fractional-order parameter.

It can be deduced that the results are stable for the varying fractional-order parameter, and the accuracy is not affected by its variations. On the other hand, our results also demonstrate that the L2-norm of two consecutive solutions i.e., $|u_{r+1} - u_r|_2$ is not affected significantly by the varying μ. It can also be noted that the reasonable accuracy that can be increased by enhancing either M1 or M2.

Problem 2. Now, assume a fractional-order nonlinear weakly singular partial integro-differential equation defined on $[0, 1] \times \Omega = [0, 1] \times [0, 1]$ as given below.

$$
\begin{aligned}
&{}_{0}^{C}D_{t}^{2/3}u(\mathbf{x},t)-u\left(\mathbf{x},-t+\frac{1}{2}\right)=\frac{1}{2}\nabla^{2}u(\mathbf{x},t)+\frac{1}{3}\sin\left(u(\mathbf{x},t)\right)+\frac{1}{10}\int_{0}^{t}\frac{1}{\left(t^{2}-\xi^{2}\right)^{1/10}}u\left(\mathbf{p},\xi\right)\mathrm{d}\xi \\
&-\int_{0}^{x}\int_{0}^{y}\frac{1}{\left(x^{2}-p^{2}\right)^{1/2}\left(y^{3}-q^{3}\right)^{1/3}}u\left(-p,-q,t\right)\mathrm{d}q\mathrm{d}p++\frac{3}{2}\frac{t^{1/3}x^{3}y^{2}\sqrt{3}}{\pi}\Gamma\left(\frac{2}{3}\right)-x^{3}y^{2}\left(-t+\frac{1}{2}\right) \qquad (28) \\
&-tx^{3}-3txy^{2}-\frac{1}{3}\sin\left(x^{3}y^{2}t\right)-\frac{1}{18}t^{9/5}x^{3}y^{2}-\frac{1}{3}x^{3}y^{2}t\,\mathrm{csgn}\left(x\right).
\end{aligned}
$$

The exact solution of this problem is $u(x, t) = x^3y^2t$. Initial and boundary conditions may be computed using the exact solution. For this problem, we apply the scheme with M1 = 5, M2 = 4, M3 = 3 and R = 30. After solving only 60 linear algebraic equations at each iteration, we attain the exact solution via the proposed scheme that clearly shows its effectiveness and accuracy for solving this kind of complex problems.



(a) (b)

Figure 3. Impact of order of approximation parameters M_3 on (a) L_2, L_∞, RMS and (b) absolute error.

Problem 3. Now, consider the two-dimensional system of nonlinear unsteady weakly singular integral equations when $\Omega = [0, 1] \times [0, 1]$, $t > 0$ as given below.

$$
\begin{aligned}
&{}_{0}^{C}D_{t}^{1/2}u(\mathbf{x},t)-\frac{1}{3}u\left(\mathbf{x},-t+\frac{1}{2}\right)=\frac{1}{10}\nabla^{2}u(\mathbf{x},t)-\exp\left(u(\mathbf{x},t)\right)+\frac{1}{10}\int_{0}^{t}\frac{1}{\left(t^{3}-\xi^{3}\right)^{1/3}}u^{2}\left(\mathbf{p},\xi\right)\mathrm{d}\xi \\
&+2\int_{0}^{x}\int_{0}^{y}\frac{1}{\sqrt{(x-p)(y-q)}}u\left(p-\frac{1}{10},-\frac{1}{3}q+2,t\right)\mathrm{d}q\mathrm{d}p
\end{aligned}
\qquad (29)
$$

The analytical solution of this problem is $u(x, t) = (x + y)\sin(t)$. Its initial and boundary conditions are given as,

$$u(\mathbf{x}) = 0;$$
$$u(0,y,t) = y\sin(t), u(1,y,t) = (1+y)\sin(t), u(x,0,t) = x\sin(t), u(x,1,t) = (x+1)\sin(t).$$

The Maple code of the suggested iterative scheme is used for multiple values of the order of convergence M1 = M2 = 3 and M3. Figure 3 shows that the attained solution via proposed schemes are in excellent agreement with the exact solution and the accuracy is much effected against the choice of the convergence control parameter. The accuracy enhances sharply with an increase in the values of M3, which shows that the proposed method is convergent and converges to zero as M3 → ∞. Furthermore, the absolute error between the exact and approximate solutions decreases as M3 is increased.

Table 2. Absolute error analysis for problem 3 when $M_1 = M_2 = 3$ as varying M_3.

$x = y = t$	$M_3 = 3$	$M_3 = 4$	$M_3 = 5$	$M_3 = 6$	$M_3 = 7$
0.20	3.39E-08	2.36E-08	5.51E-08	4.12E-10	0
0.40	1.66E-07	4.94E-07	6.81E-07	2.22E-09	0
0.60	2.83E-07	1.45E-06	2.40E-06	3.18E-09	0
0.80	1.95E-07	1.40E-06	2.92E-06	2.48E-09	0
1.00	4.27E-10	2.55E-10	1.65E-09	1.71E-09	0

Problem 4. Next, a nonlinear partial integro-differential equation is considered on Ω = [0, 1] × [0, 1] × [0, 1] and $t > 0$ and is given as,

$$\begin{aligned} {}_0^C\mathcal{D}_t^{1/3}u(\mathbf{x},t) - 2u(\mathbf{x},2t-1) &= 5\nabla^2 u(\mathbf{x},t) + \frac{1}{10}\exp(-u(\mathbf{x},t)) + \frac{7}{100}\int_0^t \frac{1}{\sqrt{t-\xi}}u^3(\mathbf{p},\xi)\,\mathrm{d}\xi \\ &-5\int_0^x\int_0^y \frac{1}{(x^3-p^3)^{1/4}(y^2-q^2)^{1/3}}u(a_1p+b_1,a_2q+b_2,\xi)\,\mathrm{d}q\mathrm{d}p \end{aligned} \tag{30}$$

Its analytical solution is $u(x, t) = x^2y^2(1 + t)$. The condition associated with this problem and the inhomogeneous term $\phi(x, t)$ can be obtained through the exact solution. Now, for this problem, simulations are performed for various values of M3 and M1 = M2 = 3. Table 2 presents the absolute error growth with an increasing M3 and increase in x = y = z. It can be found the realistic accuracy, which is not reported before obtained when M3 > 6.

Problem 5. Now, the following nonlinear partial integro-differential equation is considered on Ω = [0, 1] × [0, 1] × [0, 1] and $t > 0$ as below,

$${}_0^C\mathcal{D}_t^{\mu}u(\mathbf{x},t) = \frac{1}{2}\nabla^2 u(\mathbf{x},t) - \sin(u(\mathbf{x},t)) + \frac{1}{2}\int_0^t \frac{1}{\sqrt{t-\xi}}u(\mathbf{p},\xi)\,\mathrm{d}\xi + \phi(\mathbf{x},t). \tag{35}$$

It has the non-smooth solution as $u(x, t) = x^{1/2}y^{1/2}t^{1/2}$. The term $\phi(x, t)$ is computed accordingly. In order to see the behavior, effectiveness of the proposed method to examine the solution of discussed problem. For this purpose, simulations have been performed using the suggested method against fractional order parameter and M. It is found that the suggested method gives the accurate results if the solution is non-smooth, and technique converges as M increases gradually. However, the rate of convergence is less as compared to solve those problems which have smooth solution. On the other hand, it is also noted that the fractional-order parameter have insignificant impact on the convergence.

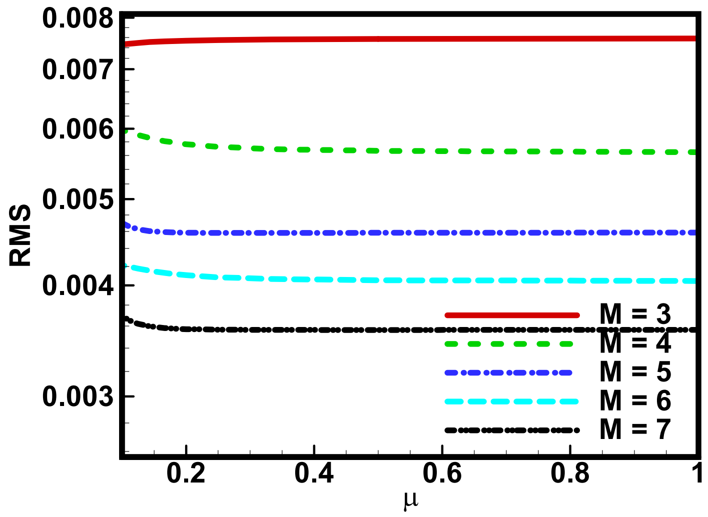


Figure 4. Relationship between RMS and μ with different M.

7. Conclusions

In this study, an efficient iterative spectral method is extended and applied to a class of nonlinear multi-dimensional fractional-order integro-differential equations with weakly singular kernel, having time-space delays. The key outcomes of this manuscript are as follows:

- The spectral scheme is coupled with Picard iterative schemes for a highly nonlinear multi-dimensional fractional-order integro-differential equations with weakly singular kernel, having time-space delays when $\rho_1, \rho_2, \rho_3 > 1$, which was not reported before.
- Some novel operational matrices are developed to approximate the weakly singular integral like,

$$\int_0^x \int_0^y u(pa_1 + b_1, qa_2 + b_2, t) / (x^{\rho_1} - p^{\rho_1})^{\alpha_1} (y^{\rho_2} - q^{\rho_2})^{\alpha_2} \, \mathrm{d}q \mathrm{d}p$$

and $\int_0^t u^{\gamma}(\mathbf{x}, \xi) / (t^{\rho_3} - \xi^{\rho_3})^{\alpha_3} \, \mathrm{d}\xi$, for ρ's > 1, $0 < \alpha$'s < 1.

- The proposed iterative strategy transforms a highly nonlinear problem into a system of linear algebraic equations, which is its main advantage.
- The proposed method can be applied those physical problems whose exact solution is unknown.
- The error bound and convergence of the established scheme are verified theoretically and numerically by performing various simulations for different values of n, α, ρ, M, $F(u)$ and μ.
- A comparative study with the published results illustrates that the newly developed schemes are the best tools to find solutions of complex nonlinear problems, and no previous method attains the presented accuracy level.
- To summarize, the current approaches can be used conveniently for more types of multidimensional nonlinear fractional or variable-order complex problems, arising in mechanics.
- The suggested method is slow convergent for the problem have non-smooth solution.

- Our study paves the way for the numerical investigation of inverse problems associated with fractional PDEs [31,32], and we also refer to [33–36] for more related physical background on inverse problems.

Acknowledgment

The work of W. Wu was supported by the Hong Kong RGC General Research Funds (projects 12302219, 12300520) and Tier 2 Start-up Grant of HKBU. The work of Wang was partially supported by the NSF of China No. 11971221 and the Shenzhen Sci-Tech Fund No. RCJC20200714114556020, JCYJ20200109115422828, JCYJ20190809150413261 and JCYJ20170818153840322, and Guangdong Provincial Key Laboratory of Computational Science.

Conflict of interest

All the authors declare no actual or potential conflict of interest, including any financial, personal, or other relationships with other people or organizations.

References

1. K. Oldham, J. Spanier, *The fractional calculus theory and applications of differentiation and integration to arbitrary order*, 1*st* edition, Elsevier Science, Netherland, 1971.
2. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations,* 1*st* edition, Wiley-Interscience, New York, 1993.
3. A Babaei, B. P. Moghaddam, S. Banihashemi, J. A. T. Machado, Numerical solution of variable-order fractional integro-partial differential equations via Sinc collocation method based on single and double exponential transformations, *Commun. Nonlinear Sci. Numer. Simul.*, **82** (2020), 104985. https://doi.org/10.1016/j.cnsns.2019.104985
4. M. Asgari, R. Ezzati, Using operational matrix of two-dimensional Bernstein polynomials for solving two-dimensional integral equations of fractional order, *Appl. Math. Comput.*, **307** (2017), 290–298. https://doi.org/10.1016/j.amc.2017.03.012
5. N. Mollahasani, M. M. Moghadam, G. Chuev, Hybrid Functions of Lagrange Polynomials and Block-Pulse Functions for Solving Integro-partial Differential Equations, *Iran. J. Sci. Technol., Trans. A: Sci.*, (2018), 1–9. https://dx.doi.org/10.22099/ijsts.2015.3377
6. S. Singh, V. K. Patel, V. K. Singh, E. Tohidi, Numerical solution of nonlinear weakly singular partial integro-differential equation via operational matrices, *Appl. Math. Comput.*, **298** (2017), 310–321. https://doi.org/10.1016/j.amc.2016.11.012
7. S. Arshed, B-spline solution of fractional integro partial differential equation with a weakly singular kernel, *Numer. Methods Partial Differ. Equations*, **33** (2017), 1565–1581. https://doi.org/10.1002/num.22153
8. M. A. Zaky, A Legendre spectral quadrature tau method for the multi-term time-fractional diffusion equations, *Comput. Appl. Math.*, **37** (2018), 3525–3538. https://doi.org/10.1007/s40314-017-0530-1

9. A. H. Bhrawy, M. A. Zaky, J. A. T. Machado, Numerical solution of the two-sided space-time fractional telegraph equation via Chebyshev tau approximation, *J. Optim. Theory Appl.*, **174** (2017), 321–341. https://doi.org/10.1007/s10957-016-0863-8
10. A. H. Bhrawy, M. A. Zaky, Highly accurate numerical schemes for multi-dimensional space variable-order fractional Schrödinger equations, *Comput. Math. Appl.*, **73** (2017), 1100–1117. https://doi.org/10.1016/j.camwa.2016.11.019
11. A. H. Bhrawy, M. A. Zaky, A method based on the Jacobi tau approximation for solving multi-term time–space fractional partial differential equations, *J. Comput. Phys.*, **281** (2015), 876–895. https://doi.org/10.1016/j.jcp.2014.10.060
12. A. H. Bhrawy, M. A. Zaky, Numerical simulation for two-dimensional variable-order fractional nonlinear cable equation, *Nonlinear Dyn.*, **80** (2015), 101–116. http://doi.org/10.1007/S11071-014-1854-7
13. A. H. Bhrawy, M. A. Zaky, An improved collocation method for multi-dimensional space–time variable-order fractional Schrödinger equations, *Appl. Numer. Math.*, **111** (2017), 197–218. https://doi.org/10.1016/j.apnum.2016.09.009
14. M. A. Zaky, Existence, uniqueness and numerical analysis of solutions of tempered fractional boundary value problems, *Appl. Mumer. Math.*, **145** (2019), 429–457. https://doi.org/10.1016/j.apnum.2019.05.008
15. M. A. Zaky, An accurate spectral collocation method for nonlinear systems of fractional differential equations and related integral equations with nonsmooth solutions, *Appl. Numer. Math.*, **154** (2020), 205–222. https://doi.org/10.1016/j.apnum.2020.04.002
16. N. A. Elkot, M. A. Zaky, E. H. Doha, I. G. Ameen, On the rate of convergence of the Legendre spectral collocation method for multi-dimensional nonlinear Volterra-Fredholm integral equations, *Commun. Theor. Phys.*, **73** (2021), 025002. https://doi.org/10.1088/1572-9494/abcfb3
17. I. G. Ameen, M. A. Zaky, E. H. Doha, Singularity preserving spectral collocation method for nonlinear systems of fractional differential equations with the right-sided Caputo fractional derivative, *J. Comput. Appl. Math.*, **392** (2021), 113468. https://doi.org/10.1016/j.cam.2021.113468
18. F. Salehi, H. Saeedi, M. M. Moghadam, Discrete Hahn polynomials for numerical solution of two-dimensional variable-order fractional Rayleigh–Stokes problem, *Comput. Appl. Math.*, **37** (2018), 5274–5292. https://doi.org/10.1007/S40314-018-0631-5
19. Z. Abdollahi, A computational approach for solving fractional Volterra integral equations based on two-dimensional Haar wavelet method, *Int. J. Comput. Math.*, (2021), 1–17. https://doi.org/10.1080/00207160.2021.1983549
20. S. Zaeri, H. Saeedi, M. Izadi, Fractional integration operator for numerical solution of the integro-partial time fractional diffusion heat equation with weakly singular kernel, *Asian-Europ. J. Math.*, **10** (2017), 1750071. https://doi.org/10.1142/S1793557117500711
21. H. Saeedi, N. Mollahasani, M. M. Moghadam, G. N. Chuev, An operational Haar wavelet method for solving fractional Volterra integral equations, *Int. J. Appl. Math. Comput. Sci.*, **21** (2011), 535–547. https://doi.org/10.2478/v10006-011-0042-x
22. R. Agarwal, M. Belmekki, M. Benchohra, A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative, *Adv. Differ. Equations*, **1** (2009), 981728. https://doi.org/10.1155/2009/981728

23. M. Usman, S. T. Mohyud-Din, Traveling wave solutions of 7 th order Kaup Kuperschmidt and Lax equations of fractional-order, *Int. J. Adv. Appl. Math. Mech.*, **1** (2013), 17–34.
24. M. Usman, H. Muhammad, U. H. Rizwan, W. Wang, An efficient algorithm based on Gegenbauer wavelets for the solutions of variable-order fractional differential equations, *Europ. Phys. J. Plus*, **133** (2018), 327. https://doi.org/10.1140/epjp/i2018-12172-1
25. M. Usman, Novel operational matrices-based method for solving fractional-order delay differential equations via shifted Gegenbauer polynomials, *Appl. Math. Comput.*, **372** (2020), 124985. https://doi.org/10.1016/j.amc.2019.124985
26. M. Usman, A robust scheme based on novel-operational matrices for some classes of time-fractional nonlinear problems arising in mechanics and mathematical physics, *Numer. Methods Partial Differ. Equations*, **36** (2020), 1566–1600. https://doi.org/10.1002/num.22492
27. D. S. Kim, T. Kim, S. H. Rim, Some identities involving Gegenbauer polynomials, *Adv. Differ. Equations*, **1** (2012), 219. https://doi.org/10.1186/1687-1847-2012-219
28. M. Usman, M. Hamid, R. U. Haq, M. Liu, Linearized novel operational matrices-based scheme for classes of nonlinear time-space fractional unsteady problems in 2D, *Appl. Numer. Math.*, **162** (2021), 351–373. https://doi.org/10.1016/j.apnum.2020.12.021
29. M. Hosseininia, M. H. Heydari, F. M. M. Ghaini, Z. Avazzadeh, A wavelet method to solve nonlinear variable-order time fractional 2D Klein-Gordon equation, *Comput. Math. Appl.*, **78** (2019), 3713–3730. https://doi.org/10.1016/j.camwa.2019.06.008
30. M. Heydari, M. Hooshmandasl, C. Cattani, A new operational matrix of fractional order integration for the Chebyshev wavelets and its application for nonlinear fractional Van der Pol oscillator equation, *Proceed. Math. Sci.*, **128** (2018), 1–26. https://doi.org/10.1007/s12044-018-0393-4
31. X. Cao, Y. Lin, H. Liu, Simultaneously recovering potentials and embedded obstacles for anisotropic fractional Schrödinger operators, *Inverse Probl. Imaging*, **13** (2019), 197–210.
32. X. Cao, H. Liu, Determining a fractional Helmholtz equation with unknown source and scattering potential, *Commun. Math. Sci.*, **17** (2019), 1861–1876. https://doi.org/10.4310/CMS.2019.v17.n7.a5
33. H. Liu, J. Zou, Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers, *Inverse Probl.*, **22** (2006), 515–524.
34. J. Li, H. Liu, J. Zou, Strengthened linear sampling method with a reference ball, *SIAM J. Sci. Comput.*, **31** (2009/10), 4013–4040. https://doi.org/10.1137/080734170
35. H. Diao, X. Cao, H. Liu, On the geometric structures of transmission eigenfunctions with a conductive boundary condition and applications, *Comm. Partial Differ. Equations*, **46** (2021), 630–679.
36. Y. Chow, Y. Deng, Y. He, H. Liu, X. Wang, Surface-localized transmission eigenstates, super-resolution imaging, and pseudo surface plasmon modes, *SIAM J. Imaging Sci.*, **14** (2021), 946–975. https://doi.org/10.1137/20M1388498

AIMS Press

©2022 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)