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Research article

Extended incomplete Riemann-Liouville fractional integral operators and related special functions

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Abstract: In this study, we introduce the extended incomplete versions of the Riemann-Liouville (R-L) fractional integral operators and investigate their analytical properties rigorously. More precisely, we investigate their transformation properties in L_1 and L_∞ spaces, and we observe that the extended incomplete fractional calculus operators can be used in the analysis of a wider class of functions than the extended fractional calculus operator. Moreover, by considering the concept of analytical continuation, definitions for extended incomplete R-L fractional derivatives are given and therefore the full fractional calculus model has been completed for each complex order. Then the extended incomplete τ-Gauss, confluent and Appell's hypergeometric functions are introduced by means of the extended incomplete beta functions and some of their properties such as integral representations and their relations with the extended R-L fractional calculus has been given. As a particular advantage of the new fractional integral operators, some generating relations of linear and bilinear type for extended incomplete τ-hypergeometric functions have been derived.

Keywords: extended incomplete hypergeometric functions; extended incomplete Appell's functions; incomplete fractional calculus; generating functions

1. Introduction

There is a close relationship between the special functions and fractional calculus, which is a branch gaining popularity, especially in the last decades, because of its potential usefulness in real world applications (see [1–10]). Different definitions of fractional calculus have been introduced in the literature, each of which has their own advantages or disadvantages. One direction of research has been to add more parameters, for instance the Erdelyi-Kober fractional model, and another direction is to consider some analytic functions in the kernel of fractional calculus operators such as the Prabhakar model.

Recently, the extended R-L fractional integral of order μ [11] was defined by

$$D_z^{\mu,p}(f(z)) = \frac{1}{\Gamma(-\mu)} \int_0^z f(t)(z-t)^{-\mu-1} \exp\left(\frac{-pz^2}{t(z-t)}\right) dt, \quad \mathrm{Re}(\mu) < 0, \ \ \mathrm{Re}(p) > 0,$$

and it has been shown that these operators are useful in the analysis of certain extensions of special functions defined in [12–14].

Another interesting approach in generalizing fractional calculus is given in the papers [15, 16], where the authors, instead of integrating over a full interval $[0, z]$, introduced two integral operators by separeting the interval by a variable yz $(0 < y < 1)$, therefore this approach provides a general definition of fractional integrals, in which the singular and nonsingular parts of the integral can be separated. More precisely they introduced the operators

$$D_z^{\mu}[f(z); y] = \frac{z^{-\mu}}{\Gamma(-\mu)} \int_0^y f(uz)(1-u)^{-\mu-1} du, \quad \mathrm{Re}(\mu) < 0 \tag{1.1}$$

and

$$D_z^{\mu}\{f(z); y\} = \frac{z^{-\mu}}{\Gamma(-\mu)} \int_y^1 f(uz)(1-u)^{-\mu-1} du, \quad \mathrm{Re}(\mu) < 0. \tag{1.2}$$

The incomplete fractional integrals and derivatives have been subjected to an in-depth analysis in the papers [15–18].

On the other hand, in recent papers such as [11–14, 19–28], several extensions of the well-known special functions have been considered, many of which have close relationships with fractional calculus. Also we should refer the recent surveys on the transcendental functions with their connections between the fractional calculus [29, 30].

Very recently, in order to introduce a different variant of incomplete Gauss hypergeometric functions which is more suitable for the fractional calculus results as well, the authors introduced the incomplete Pochhammer ratios as follows [15]:

$$[b, c; y]_n := \frac{B_y(b+n, c-b)}{B(c-b, b)}$$

and

$$\{b, c; y\}_n := \frac{B_{1-y}(c-b, b+n)}{B(c-b, b)}$$

where $0 \leq y < 1$ and

$$B_y(x, z) = \int_0^y t^{x-1}(1-t)^{z-1} dt, \ \mathrm{Re}(x) > 0, \mathrm{Re}(z) > 0, \ 0 < y < 1. \tag{1.3}$$

is the incomplete beta function.

They defined the incomplete Gauss hypergeometric functions as follows:

$${}_2F_1(a, [b, c; y]; x) := \sum_{n=0}^{\infty} (a)_n [b, c; y]_n \frac{x^n}{n!}, \tag{1.4}$$

and

$$ {}_2F_1(a,\{b,c;y\};x) := \sum_{n=0}^{\infty}(a)_n \{b,c;y\}_n \frac{x^n}{n!}. \tag{1.5} $$

Several properties of these functions were obtained, such as integral representations, derivative formulae, transformation formulae, and recurrence relations. Also, the incomplete Appell's functions were defined and expressed using integral representations. It should be mentioned that, in a recent paper, the incomplete Gauss hypergeometric function was used in the derivation of some new estimates for the generalized Simpson's quadrature rule [31].

One of the generalisation of Gauss hypergeometric function was defined by Chaudhry [13, 14]

$$ F_p(a,b;c;z) = \sum_{n=o}^{\infty}(a)_n \frac{B_p(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!},\ p \geq 0\ ,\ Re(c) > Re(b) > 0 \tag{1.6} $$

where

$$ B_p(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left[\frac{-p}{t(1-t)}\right]dt,\ Re(p) > 0,\ Re(x) > 0,\ Re(y) > 0. \tag{1.7} $$

is the extended beta function. On the other hand, the incomplete version of the extended beta function has been defined as

$$ B_y(x,z;p) = \int_0^y t^{x-1}(1-t)^{z-1}\exp\left(\frac{-p}{t(1-t)}\right)dt,\ \mathrm{Re}(p) > 0,\ 0 \leq y < 1 \tag{1.8} $$

and investigated in [14]. Some applications, where these functions are used, can be found in [14].

We saw from the above discussion that both structures of generalisations by adding new parameters or incomplitifications are interesting topics of study with potential real world applications. By combining both forms of generalisations, we are able to construct new functions and operators which have the advantages of both the incomplete versions and the parametric versions. Therefore in this paper, in Sections 2 and 3 we combine these generalisations and investigate them thoroughly. More precisely in Section 2, we introduce the extended incomplete versions of the Riemann-Liouville (R-L) fractional integral operator and investigate their transformation properties in L_1 and L_∞ spaces. We observe that the extended incomplete fractional calculus operators can be used in the analysis of a wider class of functions than the extended fractional calculus operator. Moreover, by considering the concept of analytical continuation, definitions for extended incomplete R-L fractional derivatives are given and therefore the full fractional calculus model has been completed for each complex order. In Section 3, similar treatment has been considered to introduce extended incomplete τ-Gauss, confluent and Appell's hypergeometric functions. Some of their properties such as integral representations and their relations with the extended R-L fractional calculus has been given. In the last section, one particular advantage of the new fractional integral operators has been exhibited by deriving some generating relations of linear and bilinear type for extended incomplete τ-hypergeometric functions.

2. Extended incomplete R-L fractional integral operator

The extended incomplete R-L fractional integral operators $D_z^{\mu,p}\left[f(z);y\right]$ and $D_z^{\mu,p}\left\{f(z);y\right\}$ are introduced by

$$D_z^{\mu,p}\left[f(z);y\right] := \frac{z^{-\mu}}{\Gamma(-\mu)}\int_0^y f(uz)(1-u)^{-\mu-1}\exp\left(\frac{-p}{u(1-u)}\right)du,\ \mathrm{Re}(\mu)<0,\ \ \mathrm{Re}(p)>0, 0\le y\le 1, \tag{2.1}$$

and

$$D_z^{\mu,p}\{f(z);y\} := \frac{z^{-\mu}}{\Gamma(-\mu)}\int_y^1 f(uz)(1-u)^{-\mu-1}\exp\left(\frac{-p}{u(1-u)}\right)du,\ \mathrm{Re}(\mu)<0,\ \ \mathrm{Re}(p)>0, 0\le y\le 1. \tag{2.2}$$

Setting $p\to 0$ in (2.1) and (2.2), we obtain the incomplete R-L fractional integral operators which are defined in (1.1) and (1.2), respectively. These extended incomplete R-L fractional integral operators satisfy the following decomposition formula:

$$D_z^{\mu,p}\left[f(z);y\right] + D_z^{\mu,p}\{f(z);y\} = D_z^{\mu,p}\left(f(z)\right).$$

We start the analytical investigation of these operators by considering their transformation properties:

Theorem 2.1. *Let $A>0,\ 0<y<1$ and Re$(\mu)>0$. Then $D_z^{-\mu,p}\left[\cdot;y\right] : L_1\left[0,yA\right]\to L_1\left[0,A\right]$.*

Proof. Fix $0<y<1$ choose any $f\in L_1\left[0,yA\right]$ .For $z\in[0,A]$, since Re$(p)>0$, we can write

$$\begin{aligned}
\left|D_z^{-\mu,p}\left[f(z);y\right]\right| &\le \frac{1}{|\Gamma(\mu)|}\int_0^{zy}\left|f(t)(z-t)^{\mu-1}\exp\left(\frac{-pz^2}{t(z-t)}\right)\right|dt\\
&\le \left[\sup_{[0,yz]}(z-t)^{\mathrm{Re}(\mu)-1}\exp\left(\frac{-\mathrm{Re}\,(p)\,z^2}{t(z-t)}\right)\right]\frac{1}{|\Gamma(\mu)|}\int_0^{zy}|f(t)|\,dt\\
&\le \left[\sup_{[0,yz]}(z-t)^{\mathrm{Re}(\mu)-1}\right]\frac{1}{|\Gamma(\mu)|}\int_0^{zy}|f(t)|\,dt\\
&= \begin{cases}\frac{(z-yz)^{\mathrm{Re}(\mu)-1}}{|\Gamma(\mu)|}\,\|f\|_{L_1\left[0,yA\right]}, & 0<\mathrm{Re}(\mu)<1\\ \frac{z^{\mathrm{Re}(\mu)-1}}{|\Gamma(\mu)|}\,\|f\|_{L_1\left[0,yA\right]}, & \mathrm{Re}(\mu)>1.\end{cases}
\end{aligned}$$

Integrating both sides of this inequality over $z\in[0,A]$, we get

$$\left\|D_z^{-\mu,p}\left[f;y\right]\right\|_{L_1[0,A]} \le \begin{cases}\frac{(1-y)^{\mathrm{Re}(\mu)-1}A^{\mathrm{Re}(\mu)}}{|\Gamma(\mu)|\,\mathrm{Re}(\mu)}\,\|f\|_{L_1\left[0,yA\right]}, & 0<\mathrm{Re}(\mu)<1\\ \frac{A^{\mathrm{Re}(\mu)}}{|\Gamma(\mu)|\,\mathrm{Re}(\mu)}\,\|f\|_{L_1\left[0,yA\right]}, & \mathrm{Re}(\mu)>1.\end{cases}$$

Thus the proof is completed. □

Theorem 2.2. *Let $A>0,\ 0<y<1$ and Re$(\mu)>1$. Then $D_z^{-\mu,p}\{\cdot;y\} : L_1\left[0,A\right]\to L_1\left[0,A\right]$.*

Proof. Fix $0 < y < 1$ choose any $f \in L_1[0, A]$. For $z \in [0, A]$, since $\mathrm{Re}(p) > 0$ we have

$$\begin{aligned}\left|D_z^{-\mu,p}\{f(z);y\}\right| &\leq \frac{1}{|\Gamma(\mu)|}\int_{zy}^{z}\left|f(t)(z-t)^{\mu-1}\exp\left(\frac{-pz^2}{t(z-t)}\right)\right|dt\\ &\leq \left[\sup_{[zy,z]}(z-t)^{\mathrm{Re}(\mu)-1}\exp\left(\frac{-\mathrm{Re}(p)z^2}{t(z-t)}\right)\right]\frac{1}{|\Gamma(\mu)|}\int_{zy}^{z}|f(t)|\,dt\\ &\leq \frac{(z-zy)^{\mathrm{Re}(\mu)-1}}{|\Gamma(\mu)|}\|f\|_{L_1[0,A]}.\end{aligned}$$

Integrating both sides of this inequality over $z \in [0, A]$, we get

$$\left\|D_z^{-\mu,p}\{f;y\}\right\|_{L_1[0,A]} \leq \frac{(1-y)^{\mathrm{Re}(\mu)-1}A^{\mathrm{Re}(\mu)}}{|\Gamma(\mu)|\,\mathrm{Re}(\mu)}\|f\|_{L_1[0,A]}.$$

Whence the result. □

Therefore, using the above two Theorems, we can give the following definitions.

Definition 2.3. Let $A > 0$, $0 < y < 1$ and $Re(\mu) > 0$. Then for all $f \in L_1[0, yA]$ the μ th order extended incomplete lower fractional integral is defined by

$${}_0I_z^{\mu,p}[f(z);y] := \frac{z^\mu}{\Gamma(\mu)}\int_0^y f(uz)(1-u)^{-\mu-1}\exp\left(\frac{-p}{u(1-u)}\right)du.$$

Definition 2.4. Let $A > 0$, $0 < y < 1$ and $Re(\mu) > 1$. Then for all $f \in L_1[0, A]$ the μ th order extended incomplete upper fractional integral is defined by

$${}_0I_z^{\mu,p}\{f(z);y\} := \frac{z^\mu}{\Gamma(\mu)}\int_y^1 f(uz)(1-u)^{-\mu-1}\exp\left(\frac{-p}{u(1-u)}\right)du.$$

There is a gap in the definition of the extended incomplete upper fractional integral for the case $0 < Re(\mu) \leq 1$. In order to fill this gap, in the following theorem, we consider the operators in the space L_∞.

Theorem 2.5. *Let $A > 0$, $0 < y < 1$ and $Re(\mu) > 0$. Then we have*

$$D_z^{-\mu,p}[\cdot;y] : L_\infty[0, yA] \to L_\infty[0, A]$$

and

$$D_z^{-\mu,p}\{\cdot;y\} : L_\infty[0, A] \to L_\infty[0, A].$$

Proof. For any $z \in [0, A]$, since $\mathrm{Re}(p) > 0$, we have

$$\begin{aligned}D_z^{-\mu,p}[f(z);y] &\leq \frac{1}{|\Gamma(\mu)|}\int_0^{zy}|f(t)|\left|(z-t)^{\mu-1}\exp\left(\frac{-pz^2}{t(x-t)}\right)\right|dt\\ &\leq \frac{1}{|\Gamma(\mu)|}es\sup_{[0,yA]}|f|\int_0^{zy}(z-t)^{\mathrm{Re}\,\mu-1}dt\end{aligned}$$

$$\leq \frac{A^{\mathrm{Re}\,\mu}\left(1-(1-y)^{\mathrm{Re}\,\mu}\right)}{\mathrm{Re}\,\mu\,|\Gamma(\mu)|}\,\|f\|_{L_\infty[0,A]}\,.$$

Taking essup over all $z \in [0, A]$ on both sides of the inequality, we complete the proof of the first statement.

Since $\mathrm{Re}(p) > 0$, we have for any $x \in [0, A]$ that

$$\begin{aligned}\left|D_z^{-\mu,p}\{f(z);y\}\right| &\leq \frac{1}{|\Gamma(\mu)|}\int_{zy}^{z}|f(t)|\left|(z-t)^{\mu-1}\exp\left(\frac{-pz^2}{t(x-t)}\right)\right|dt\\ &\leq \frac{1}{|\Gamma(\mu)|}es\sup_{[0,yA]}|f|\int_{zy}^{z}(z-t)^{\mathrm{Re}\,\mu-1}dt\\ &\leq \frac{A^{\mathrm{Re}\,\mu}(1-y)^{\mathrm{Re}\,\mu}}{\mathrm{Re}\,\mu\,|\Gamma(\mu)|}\,\|f\|_{L_\infty[0,A]}\,.\end{aligned}$$

Taking essup over all $z \in [0, A]$ on both sides of the inequality, we complete the proof of the second statement. □

Using the above Theorem, in the following we give definition of the μ th order extended incomplete upper fractional integrals for the case $0 < \mathrm{Re}(\mu) \leq 1$.

Definition 2.6. Let $A > 0,\ 0 < y < 1$ and $0 < \mathrm{Re}(\mu) \leq 1$. Then for all $f \in L_\infty[0, A]$, the μ th order extended incomplete upper fractional integral is defined by

$${}_0I_z^{\mu,p}\{f(z);y\} := \frac{z^\mu}{\Gamma(\mu)}\int_y^1 f(uz)(1-u)^{-\mu-1}\exp\left(\frac{-p}{u(1-u)}\right)du.$$

Remark 2.7. *It should be remarked that the transformation properties of*

$$D_z^{-\mu,p}(f(z)) = \frac{1}{\Gamma(\mu)}\int_0^z f(t)(z-t)^{\mu-1}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt,$$

has not been investigated and it can be easily proved by majorizing the exponential term that $D_z^{-\mu,p}(\cdot) : L_1[0, A] \to L_1[0, A]$ and $D_z^{-\mu,p}(\cdot) : L_\infty[0, A] \to L_\infty[0, A]$ for $Re(\mu) > 0$. Therefore it is clear that the extended incomplete fractional calculus operators can be used in the analysis of a wider class of functions than the extended fractional calculus operator $D_z^{-\mu,p}$.

The extended incomplete R-L fractional integral operators $D_z^{-\mu,p}[f(z);y]$ and $D_z^{-\mu,p}\{f(z);y\}$ are defined in the case $\mathrm{Re}(\mu) > 0$. In order to extend the domain of μ to $\mathrm{Re}(\mu) \leq 0$, and by this way defining the corresponding derivative operators, we consider the concept of analytic continuation in μ. The following theorems will be crucial in this respect.

Theorem 2.8. *Let $A > 0, 0 < y < 1, \mathrm{Re}(\mu) > 2$. For all $f, z^{-1}f \in L_1[0, yA]$, we have*

$$\begin{aligned}\frac{d}{dz}\left(D_z^{-\mu,p}[f(z);y]\right) = &\ \frac{yf(zy)(z-zy)^{\mu-1}\exp\left(\frac{-p}{y(1-y)}\right)}{\Gamma(\mu)} + D_z^{1-\mu,p}[f(z);y]\\ &+ \frac{pz}{(\mu-1)(\mu-2)}D_z^{2-\mu,p}[f(z);y]\end{aligned}$$

$$-\frac{pz}{\mu-1}D_z^{1-\mu,p}[z^{-1}f(z);y].$$

For all $f, z^{-1}f \in L_1[0,A]$, *we have*

$$\begin{aligned}\frac{d}{dz}\left(D_z^{-\mu,p}\{f(z);y\}\right) = {} & \frac{-yf(zy)(z-zy)^{\mu-1}\exp\left(\frac{-p}{y(1-y)}\right)}{\Gamma(\mu)} + D_z^{1-\mu,p}\{f(z);y\} \\ & + \frac{pz}{(\mu-1)(\mu-2)}D_z^{2-\mu,p}\{f(z);y\} \\ & - \frac{pz}{\mu-1}D_z^{1-\mu,p}\{z^{-1}f(z);y\}.\end{aligned}$$

Proof. Firstly, using the usual technique of differentiation under the integral sign, we have

$$\begin{aligned}\frac{d}{dz}\left(D_z^{-\mu,p}[f(z);y]\right) &= \frac{d}{dz}\frac{1}{\Gamma(\mu)}\int_0^{zy} f(t)(z-t)^{\mu-1}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt \\ &= \frac{1}{\Gamma(\mu)}\Bigg[y(z-yz)^{\mu-1}\exp\left(\frac{-p}{y(1-y)}\right)f(yz) \\ &\qquad + (\mu-1)\int_0^{zy} f(t)(z-t)^{\mu-2}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt \\ &\qquad + pz\int_0^{zy} f(t)(z-t)^{\mu-3}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt \\ &\qquad - pz\int_0^{zy} t^{-1}f(t)(z-t)^{\mu-2}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt\Bigg] \\ &= \frac{y(z-yz)^{\mu-1}\exp\left(\frac{-p}{y(1-y)}\right)f(yz)}{\Gamma(\mu)} + D_z^{1-\mu,p}[f(z);y] \\ &\qquad + \frac{pz}{(\mu-1)(\mu-2)}D_z^{2-\mu,p}[f(z);y] - \frac{pz}{\mu-1}D_z^{1-\mu,p}\left[z^{-1}f(z);y\right].\end{aligned}$$

Secondly, taking derivative from the definition of the upper incomplete integral operator, we get

$$\begin{aligned}\frac{d}{dz}\left(D_z^{-\mu,p}\{f(z);y\}\right) &= \frac{1}{\Gamma(\mu)}\frac{d}{dz}\left(\int_{zy}^{z} f(t)(z-t)^{\mu-1}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt\right) \\ &= \frac{1}{\Gamma(\mu)}\Bigg[-yf(zy)(z-zy)^{\mu-1}\exp\left(\frac{-p}{y(1-y)}\right) \\ &\qquad + (\mu-1)\int_{zy}^{z} f(t)(z-t)^{\mu-2}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt \\ &\qquad + pz\int_{zy}^{z} f(t)(z-t)^{\mu-3}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt \\ &\qquad - pz\int_{zy}^{z} t^{-1}f(t)(z-t)^{\mu-2}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt\Bigg] \\ &= \frac{-yf(zy)(z-zy)^{\mu-1}\exp\left(\frac{-p}{y(1-y)}\right)}{\Gamma(\mu)} + D_z^{1-\mu,p}\{f(z);y\}\end{aligned}$$

$$+\frac{pz}{(\mu-1)(\mu-2)}D_z^{2-\mu,p}\{f(z);y\}-\frac{pz}{\mu-1}D_z^{1-\mu,p}\{z^{-1}f(z);y\}.$$

□

Using the above theorem, in the following definitions, we extend the domain of analyticity of both $D_z^{\mu,p}\left[f(z);y\right]$ and $D_z^{\mu,p}\left\{f(z);y\right\}$ to the right half-plane and hence we call them as the μth order extended upper and lower R-L derivative operators.

Definition 2.9. The μth order extended upper R-L derivative operator is defined by

$$\frac{pz}{\mu(\mu-1)}D_z^{\mu,p}\left[f(z);y\right]=\frac{d}{dz}\left(D_z^{\mu-2,p}\left[f(z);y\right]\right)-\frac{y\,(z-yz)^{1-\mu}\exp\left(\frac{-p}{y(1-y)}\right)}{\Gamma(2-\mu)}f(yz)$$
$$-D_z^{\mu-1,p}\left[f(z);y\right]+\frac{pz}{1-\mu}D_z^{\mu-1,p}\left[z^{-1}f(z);y\right]$$

for each successive region $0\le\mathrm{Re}(\mu)<1, 1\le\mathrm{Re}(\mu)<2,\cdots(\mu\neq0)$, provided that $f, z^{-1}f\in L_1\left[0,yA\right]$.

Definition 2.10. The μth order extended lower R-L derivative operator is defined by

$$\frac{pz}{\mu(\mu-1)}D_z^{\mu,p}\left\{f(z);y\right\}=\frac{d}{dz}\left(D_z^{\mu-2,p}\left\{f(z);y\right\}\right)+\frac{y\,(z-yz)^{1-\mu}\exp\left(\frac{-p}{y(1-y)}\right)}{\Gamma(2-\mu)}f(yz)$$
$$-D_z^{\mu-1,p}\left\{f(z);y\right\}+\frac{pz}{1-\mu}D_z^{\mu-1,p}\left\{z^{-1}f(z);y\right\}$$

for each successive region $0\le\mathrm{Re}(\mu)<1, 1\le\mathrm{Re}(\mu)<2,\cdots(\mu\neq0)$, provided that $f, z^{-1}f\in L_1\left[0,A\right]$.

Remark 2.11. *It is important to mention that the definition of $D_z^{\mu,p}\left[f(z);y\right]$ given in (10) does not require the condition Re$(\mu)<0$ since the interval of integration in this definition is $[0,y]$ with $0<y<1$. Therefore the formula (10) is valid for all $\mu\in\mathbb{C}$.*

Example 2.12. Let $\mathrm{Re}(\lambda)>-1$, $\mathrm{Re}(\mu)<0$ and $\mathrm{Re}(p)>0$. Then

$$D_z^{\mu,p}[z^\lambda;y]=\frac{B_y(\lambda+1,-\mu;p)}{\Gamma(-\mu)}z^{\lambda-\mu}.$$

In the next theorem, we present useful representations of the extended upper and lower R-L derivatives of an analytic function.

Theorem 2.13. *If $f(z)$ is an analytic function on the disk $|z|<R$ and has a power series expansion $f(z)=\sum_{n=0}^{\infty}c_nz^n$, then for* $\mathrm{Re}(\lambda)>0$ *and* $\mathrm{Re}(p)>0$ *we have*

$$D_z^{\mu,p}[z^{\lambda-1}f(z);y]=\frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}a_nB_y\left(\lambda+n,-\mu;p\right)z^n, \tag{2.3}$$

and

$$D_z^{\mu,p}\{z^{\lambda-1}f(z);y\}=\frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}a_nB_{1-y}\left(-\mu,\lambda+n;p\right)z^n.$$

Proof. Since the function is analytic in the given disc, its series expansion is uniformly convergent. Using the relation in the above Example,

$$D_z^{\mu,p}[z^{\lambda};y]=\frac{B_y(\lambda+1,-\mu;p)}{\Gamma(-\mu)}z^{\lambda-\mu},$$
$$(\mathrm{Re}(\lambda)>-1,\mathrm{Re}(p)>0),$$

we get

$$\begin{aligned}D_z^{\mu,p}[z^{\lambda-1}f(z);y]&=\sum_{n=0}^{\infty}a_nD_z^{\mu,p}[z^{\lambda+n-1};y]\\&=\sum_{n=0}^{\infty}a_n\left(\frac{z^{-\mu}}{\Gamma(-\mu)}\int_0^y(uz)^{\lambda+n-1}(1-u)^{-\mu-1}\exp\left(\frac{-p}{u(1-u)}\right)du\right)\\&=\frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}a_nB_y(\lambda+n,-\mu;p)\,z^n.\end{aligned}$$

Similarly,

$$\begin{aligned}D_z^{\mu,p}\{z^{\lambda-1}f(z);y\}&=\sum_{n=0}^{\infty}a_nD_z^{\mu,p}\left\{z^{\lambda+n-1};y\right\}\\&=\sum_{n=0}^{\infty}a_n\left(\frac{z^{-\mu}}{\Gamma(-\mu)}\int_y^1(uz)^{\lambda+n-1}(1-u)^{-\mu-1}\exp\left(\frac{-p}{u(1-u)}\right)du\right)\\&=\frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}a_nB_{1-y}(-\mu,\lambda+n;p)\,z^n,\end{aligned}$$

using uniform convergence of the series and absolute convergence of the integral under the given conditions. □

Now we consider the Mellin transform of the extended incomplete beta function. For $Re(s)>0$, we have

$$\begin{aligned}M\left(B_y(x,z;p):p\to s\right)&=\int_0^{\infty}p^{s-1}\left(\int_0^y t^{x-1}(1-t)^{y-1}\exp\left(\frac{-p}{t(1-t)}\right)dt\right)dp\\&=\Gamma(s)B_y(x+s,z+s).\end{aligned}$$

Therefore, from the inverse Mellin transform, we have

$$B_y(x,z;p)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)B_y(x+s,z+s)p^{-s}ds.$$

Using this result, we have

$$D_z^{\mu,p}[z^{\lambda-1}f(z);y]=\frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}a_nB_y(\lambda+n,-\mu;p)\,z^n$$

$$= \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) B_y(\lambda+n+s, -\mu+s) p^{-s} ds z^n$$

$$= \frac{z^{\lambda-\mu-1}}{2\pi i \Gamma(-\mu)} \int_{c-i\infty}^{c+i\infty} \Gamma(s) p^{-s} \left(\sum_{n=0}^{\infty} a_n B_y(\lambda+n+s, -\mu+s) z^n \right) ds,$$

where we have interchanged the contour integral and the series by considering that the function is analytic in the given disc and $\mathrm{Re}(s) > 0$, $\mathrm{Re}(p) > 0$, $\mathrm{Re}(\lambda) > 0$.

Now we consider the case $p \to 0$, which gives

$$D_z^{\mu,p}[f(z); y] := \frac{z^{-\mu}}{\Gamma(-\mu)} \int_0^y f(uz)(1-u)^{-\mu-1} du,$$

the incomplete lower R-L fractional integral operator. Recalling the incomplete beta ratio

$$I_y(p,q) = \frac{B_y(p,q)}{B(p,q)} = \frac{1}{B(p,q)} \int_0^y \tau^{p-1}(1-\tau)^{q-1} d\tau,$$

the following complex contour representation of this function was given in [32] for $p < 1, p+q > 0, 0 < d < 1$;

$$I_y(p,q) = \frac{y^p (1-y)^q}{2\pi i} \int_{d-i\infty}^{d+i\infty} \tau^{-p}(1-\tau)^{-q} \frac{d\tau}{\tau - x}.$$

The condition $p < 1$, which is important for the evaluation of the contour integral, can be cancelled by using the analytic continuation principle. Using the above integral representation, we have for $\lambda - \mu > 0$ that

$$D_z^{\mu}\left[z^{\lambda-1} f(z); y\right] = \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n B_y(\lambda+n, -\mu) z^n$$

$$= \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n \frac{B_y(\lambda+n, -\mu)}{B(\lambda+n, -\mu)} B(\lambda+n, -\mu) z^n$$

$$= \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n I_y(\lambda+n, -\mu) B(\lambda+n, -\mu) z^n$$

$$= \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n \frac{y^{\lambda+n}(1-y)^{-\mu}}{2\pi i} \int_{d-i\infty}^{d+i\infty} \tau^{-\lambda-n}(1-\tau)^{\mu} \frac{d\tau}{\tau - x} B(\lambda+n, -\mu) z^n$$

$$= \frac{z^{\lambda-\mu-1} y^{\lambda} (1-y)^{-\mu}}{2\pi i \Gamma(-\mu)} \int_{d-i\infty}^{d+i\infty} \frac{\tau^{-\lambda}(1-\tau)^{\mu}}{\tau - x} \sum_{n=0}^{\infty} a_n B(\lambda+n, -\mu) \left(\frac{z}{y\tau}\right)^n d\tau$$

$$= \frac{z^{\lambda-\mu-1} y^{\lambda} (1-y)^{-\mu}}{2\pi i B(\lambda, -\mu)} \int_{d-i\infty}^{d+i\infty} \frac{\tau^{-\lambda}(1-\tau)^{\mu}}{\tau - x} \sum_{n=0}^{\infty} a_n \frac{(\lambda)_n}{(\lambda-\mu)_n} \left(\frac{z}{y\tau}\right)^n d\tau.$$

For instance, let's choose $f(z) = {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z)$ with $p \leq q$, which is an entire function on the whole domain. From the above result, we can immediately write an elegant contour integral representation:

$$
\begin{aligned}
D_z^{\mu}&\left[z^{\lambda-1}{}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;z);y\right]\\
&=\frac{z^{\lambda-\mu-1}y^{\lambda}(1-y)^{-\mu}}{2\pi i\Gamma(-\mu)}\int_{d-i\infty}^{d+i\infty}\frac{\tau^{-\lambda}(1-\tau)^{\mu}}{\tau-x}\\
&\qquad\times{}_pF_q\left(\lambda,a_1,\cdots,a_p;b_1,\cdots,b_q,\lambda-\mu;\frac{z}{y\tau}\right)d\tau.
\end{aligned}
$$

The heading levels should not be more than 4 levels. The font of heading and subheadings should be 12 point normal Times New Roman. The first letter of headings and subheadings should be capitalized.

3. Extended incomplete τ-Gauss and confluent hypergeometric functions

The main aim of this section is to initiate the study of the extended incomplete τ-hypergeometric type function and the extended incomplete τ-Appell functions, where, as mentioned in the introduction, the investigation of the usual cases was a concern of the recent years. The second aim of the section is to make the preparation for the next section, where we obtain their generating relations. We should note here that the results obtained in Sections 3 and 4 are reduced to the incomplete versions in the case $p\to 0$, where the reduced results will be new for $\tau-$incomplete special functions discussed in these sections.

We shall introduce the extended incomplete τ-Gauss and confluent hypergeometric functions in terms of the extended incomplete beta function $B_y(x,z;p)$, as follows :

$$
{}_2R_1^{(\tau,p)}[z;y]={}_2R_1^{(\tau,p)}(a,[b,c;y];z)=\sum_{n=0}^{\infty}(a)_n\frac{B_y(b+\tau n,c-b;p)}{B(c-b,b)}\frac{z^n}{n!},\tag{3.1}
$$

$$
(\mathrm{Re}(p)>0\,,\tau>0,\ |z|<1,\ \mathrm{Re}(c)>\mathrm{Re}(b)>0)
$$

and

$$
{}_1\phi_1^{(\tau,p)}[z;y]={}_1\phi_1^{(\tau,p)}([b,c;y];z)=\sum_{n=0}^{\infty}\frac{B_y(b+\tau n,c-b;p)}{B(c-b,b)}\frac{z^n}{n!},\tag{3.2}
$$

$$
(\mathrm{Re}(p)>0\,,\tau>0,\ |z|<1,\ \mathrm{Re}(c)>\mathrm{Re}(b)>0)\,.
$$

Remark 3.1. *The special case of the definitions (3.1) and (3.2) when $\tau=1$ and $p=0$ are easily seen to reduce to the incomplete Gauss and confluent hypergeometric functions [15]:*

$$
{}_2F_1(a,[b,c;y];z)=\sum_{n=0}^{\infty}(a)_n[b,c;y]_n\frac{z^n}{n!}
$$

and

$$
{}_1F_1([b,c;y];z)=\sum_{n=0}^{\infty}[b,c;y]_n\frac{z^n}{n!}.
$$

Also, it should be mentioned that in the special case of (3.1) and (3.2) when $p\to 0$, we arrive at the new definitions which can be called as the incomplete τ-Gauss and confluent hypergeometric functions as follows:

$$
{}_2R_1^{\tau}[z;y]={}_2R_1^{\tau}(a,[b,c;y];z)=\sum_{n=0}^{\infty}(a)_n\frac{B_y(b+\tau n,c-b)}{B(c-b,b)}\frac{z^n}{n!},\tag{3.3}
$$

$$(0 \leq y < 1\ , \tau > 0,\ \ |z| < 1,\ \operatorname{Re}(c) > \operatorname{Re}(b) > 0)$$

and

$$ {}_1\phi_1^{\tau}\left[z; y\right] = {}_1\phi_1^{\tau}(\left[b, c; y\right]; z) = \sum_{n=0}^{\infty} \frac{B_y\left(b + \tau n, c - b\right)}{B\left(c - b, b\right)} \frac{z^n}{n!}. \tag{3.4}$$

$$(0 \leq y < 1\ , \tau > 0,\ \ |z| < 1,\ \operatorname{Re}(c) > \operatorname{Re}(b) > 0)$$

In the following propositions, we obtain integral representations and derivative formulas for incomplete τ-Gauss and confluent hypergeometric functions.

Proposition 3.2. *The extended incomplete τ-Gauss hypergeometric function can be represented by an integral as follows:*

$$ {}_2R_1^{(\tau,p)}\left(a, \left[b, c; y\right]; z\right) = \frac{y^b}{B(c-b,b)} \int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-(uy)^{\tau}z)^{-a} \exp\left(\frac{-p}{uy(1-uy)}\right) du, \tag{3.5}$$

$$p > 0; p = 0 \ \ and \ \left|\arg(1-z)\right| < \pi, Re(c) > Re(b) > 0.$$

Proof. Replacing the extended incomplete beta function in the definition (3.5) by its integral representation given by (1.8), we have

$$ {}_2R_1^{(\tau,p)}\left(a, \left[b, c; y\right]; z\right) = \frac{1}{B(c-b,b)} \sum_{n=0}^{\infty} (a)_n \frac{z^n}{n!} \int_0^y t^{b+\tau n-1}(1-t)^{c-b-1} \exp\left(\frac{-p}{t(1-t)}\right) dt.$$

From the uniform convergence, summation and integration can be interchanged. Then, we have

$$\begin{aligned} {}_2R_1^{(\tau,p)}\left(a, \left[b, c; y\right]; z\right) &= \frac{1}{B(c-b,b)} \int_0^y t^{b+\tau n-1}(1-t)^{c-b-1}(1-zt^{\tau})^{-a} \exp\left(\frac{-p}{t(1-t)}\right) dt \\ &= \frac{y^b}{B(c-b,b)} \int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-(uy)^{\tau}z)^{-a} \exp\left(\frac{-p}{uy(1-uy)}\right) du. \end{aligned}$$

□

Corollary 3.3. *If p is set to 0 in the above proposition, we reach the result corresponding to the incomplete τ-Gauss hypergeometric function which is given as follows:*

$$ {}_2R_1^{\tau}\left(a, \left[b, c; y\right]; z\right) = \frac{y^b}{B(c-b,b)} \int_0^1 u^{b-1}\left(1-uy\right)^{c-b-1}\left(1-(uy)^{\tau} z\right)^{-a} du,$$

$$Re(c) > Re(b) > 0,\ \ |z| < 1.$$

Proposition 3.4. *For the extended incomplete τ-confluent hypergeometric function, we have the following integral representation:*

$$ {}_1\phi_1^{(\tau,p)}(\left[b, c; y\right]; z) = \frac{y^b}{B(c-b,b)} \int_0^1 u^{b-1}\left(1-uy\right)^{c-b-1} \exp\left(\frac{-p}{uy\left(1-uy\right)} + (uy)^{\tau} z\right) du.$$

Corollary 3.5. *If we set $p = 0$ in the above theorem, we can give the corresponding result for the incomplete τ-confluent hypergeometric function as follows:*

$$ {}_1\phi_1^{\tau}([b,c;y];z) = \frac{y^b}{B(c-b,b)} \int_0^1 u^{b-1}(1-uy)^{c-b-1} e^{(uy)^{\tau}z} du. $$

Proposition 3.6. *The equation shown below holds true for the incomplete τ-Gauss hypergeometric function:*

$$ \frac{d^n}{dz^n}\left[{}_2R_1^{\tau}(a,[b,c;y];z)\right] = \frac{(a)_n (b)_{\tau n}}{(c)_{\tau n}} {}_2R_1^{\tau}(a+n,[b+\tau n, c+\tau n; y];z). \tag{3.6} $$

Proof. Using (3.3), differentiating on both sides with respect to z, we get

$$ \begin{aligned} \frac{d}{dz}\left[{}_2R_1^{\tau}(a,[b,c;y];z)\right] &= \frac{a}{B(c-b,b)} \int_0^y t^{b+\tau-1}(1-t)^{c-b-1}(1-zt^{\tau})^{-a-1} dt \\ &= \frac{a}{B(c-b,b)} \int_0^y t^{(b+\tau)-1}(1-t)^{(c+\tau)-(b+\tau)-1}(1-zt^{\tau})^{-(a+1)} dt \\ &= \frac{a(b)_{\tau}}{(c)_{\tau}} \frac{1}{B(c-b,,b+\tau)} \int_0^y t^{(b+\tau)-1}(1-t)^{(c+\tau)-(b+\tau)-1}(1-zt^{\tau})^{-(a+1)} dt. \end{aligned} $$

which is (3.6) for $n = 1$. Recursive application of this procedure yields the general result. □

In a similar manner, we have the following.

Proposition 3.7. *The equation shown below holds true for the incomplete τ-Gauss hypergeometric function:*

$$ \frac{d^n}{dz^n}\left[{}_1\phi_1^{\tau}([b,c;y];z)\right] = \frac{(b)_{\tau n}}{(c)_{\tau n}} {}_1\phi_1^{\tau}([b+\tau n, c+\tau n; y];z). $$

In the following theorem, we give expressions for the Mellin transforms of the extended incomplete τ -Gauss hypergeometric function an expression which involves incomplete τ-Gauss hypergeometric function.

Theorem 3.8. *The extended incomplete τ-hypergeometric function has a Mellin transform which can be written as follows :*

$$ \mathfrak{M}\left\{{}_2R_1^{(\tau,p)}(a,[b,c;y];z) : p \to s\right\} = \frac{\Gamma(s)\, B(c-b+s, b+s)}{B(c-b,b)} {}_2R_1^{\tau}(a,[b+s, c+2s; y];z). \tag{3.7} $$

Proof. To get the Mellin transform, multiplying (3.5) by $p^{s-1}$and integrate over the interval $[0,\infty)$ with respect to p to get

$$ \begin{aligned} \mathfrak{M}\left\{{}_2R_1^{(\tau,p)}(a,[b,c;y];z) : p \to s\right\} &= \int_0^{\infty} p^{s-1}\, {}_2R_1^{(\tau,p)}(a,[b,c;y];z)\, dp \\ &= \frac{y^b}{B(c-b,b)} \int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-z(uy)^{\tau})^{-a} \\ &\times \left[\int_0^{\infty} p^{s-1} \exp\left(\frac{-p}{uy(1-uy)}\right) dp\right] du. \end{aligned} \tag{3.8} $$

Setting $p = tuy(1-uy)$ in (3.8),

$$\int_0^\infty p^{s-1}\exp\left(\frac{-p}{uy(1-uy)}\right)dp = \int_0^\infty t^{s-1}(uy)^s(1-uy)^s\exp(-t)dt$$
$$= (uy)^s(1-uy)^s\int_0^\infty t^{s-1}\exp(-t)dt$$
$$= (uy)^s(1-uy)^s\Gamma(s).$$

Thus we get

$$\mathfrak{M}\left\{{}_2R_1^{(\tau,p)}(a,[b,c;y];z) : p\to s\right\}$$
$$= \frac{y^{b+s}\Gamma(s)}{B(c-b,b)}\int_0^1 u^{b+s-1}(1-uy)^{c-b+s-1}(1-z(uy)^\tau)^{-a}du$$
$$= \frac{\Gamma(s)\,B(c-b+s,b+s)}{B(c-b,b)}\,{}_2R_1^\tau(a,[b+s,c+2s;y];z).$$

□

Remark 3.9. *Setting $s=1$ in (3.7), we get*

$$\int_0^\infty {}_2R_1^{(\tau,p)}(a,[b,c;y];z)\,dp = \frac{b\,(c-b)}{c\,(c+1)}\,{}_2R_1^\tau(a,[b+1,c+2;y];z).$$

By means of the extended incomplete beta function $B_y(x,z;p)$ stated by (1.8) , we introduce the extended incomplete $\tau-$Appell's functions as follows:

$$F_1^{\tau,p}[\lambda,\alpha,\beta;\mu;x,z;y] = \sum_{m,n=0}^\infty \frac{B_y(\lambda+\tau m+\tau n,\mu-\lambda;p)}{B(\mu-\lambda,\lambda)}(\alpha)_m(\beta)_n\frac{x^m}{m!}\frac{z^n}{n!}, \tag{3.9}$$

$$F_1^{\tau,p}\{\lambda,\alpha,\beta;\mu;x,z;y\} = \sum_{m,n=0}^\infty \frac{B_{1-y}(\mu-\lambda,\lambda+\tau m+\tau n;p)}{B(\mu-\lambda,\lambda)}(\alpha)_m(\beta)_n\frac{x^m}{m!}\frac{z^n}{n!}, \tag{3.10}$$

where $\max\{|x|,|z|\}<1$, and

$$F_2^{\tau,p}[\alpha,\beta,\lambda;\gamma,\mu;x,z;y] = \sum_{m,n=0}^\infty (\alpha)_{n+m}\frac{B_y(\beta+\tau m,\gamma-\beta;p)}{B(\gamma-\beta,\beta)}\frac{B_y(\lambda+\tau n,\mu-\lambda;p)}{B(\mu-\lambda,\lambda)}\frac{x^m}{m!}\frac{z^n}{n!}, \tag{3.11}$$

$$F_2^{\tau,p}\{\alpha,\beta,\lambda;\gamma,\mu;x,z;y\} = \sum_{m,n=0}^\infty (\alpha)_{n+m}\frac{B_{1-y}(\gamma-\beta,\beta+\tau m;p)}{B(\gamma-\beta,\beta)}\frac{B_{1-y}(\mu-\lambda,\lambda+\tau n;p)}{B(\mu-\lambda,\lambda)}\frac{x^m}{m!}\frac{z^n}{n!}. \tag{3.12}$$

where $|x|+|z|<1$.

Remark 3.10. *The special case of the definitions (3.9–3.12) when $\tau=1$ and $p\to 0$ are easily seen to reduce to the incomplete Appell's functions [15]:*

$$F_1[\lambda,\alpha,\beta;\mu;x,z;y] := \sum_{m,n=0}^\infty [\lambda,\mu;y]_{m+n}(\alpha)_m(\beta)_n\frac{x^m}{m!}\frac{z^n}{n!},\quad \max\{|x|,|z|\}<1$$

and

$$F_1\{\lambda,\alpha,\beta;\mu;x,z;y\} := \sum_{m,n=0}^{\infty}\{\lambda,\mu;y\}_{m+n}(\alpha)_m(\beta)_n\frac{x^m}{m!}\frac{z^n}{n!},\quad \max\{|x|,|z|\}<1$$

and

$$F_2[\alpha,\beta,\lambda;\gamma,\mu;x,z;y] := \sum_{m,n=0}^{\infty}(\alpha)_{m+n}[\beta,\gamma;y]_m[\lambda,\mu;y]_n\frac{x^m}{m!}\frac{z^n}{n!},\quad |x|+|z|<1$$

and

$$F_2\{\alpha,\beta,\lambda;\gamma,\mu;x,z;y\} := \sum_{m,n=0}^{\infty}(\alpha)_{m+n}\{\beta,\gamma;y\}_m\{\lambda,\mu;y\}_n\frac{x^m}{m!}\frac{z^n}{n!},\quad |x|+|z|<1.$$

Also, in the case of $p\to 0$, we can give the incomplete versions of the $\tau-$Appell's functions as follows:

$$F_1^{\tau}[\lambda,\alpha,\beta;\mu;x,z;y] = \sum_{m,n=0}^{\infty}\frac{B_y(\lambda+\tau m+\tau n,\mu-\lambda)}{B(\mu-\lambda,\lambda)}(\alpha)_m(\beta)_n\frac{x^m}{m!}\frac{z^n}{n!}, \tag{3.13}$$

$$F_1^{\tau}\{\lambda,\alpha,\beta;\mu;x,z;y\} = \sum_{m,n=0}^{\infty}\frac{B_{1-y}(\mu-\lambda,\lambda+\tau m+\tau n)}{B(\mu-\lambda,\lambda)}(\alpha)_m(\beta)_n\frac{x^m}{m!}\frac{z^n}{n!}, \tag{3.14}$$

$$F_2^{\tau}[\alpha,\beta,\lambda;\gamma,\mu;x,z;y] = \sum_{m,n=0}^{\infty}(\alpha)_{n+m}\frac{B_y(\beta+\tau m,\gamma-\beta)}{B(\gamma-\beta,\beta)}\frac{B_y(\lambda+\tau n,\mu-\lambda)}{B(\mu-\lambda,\lambda)}\frac{x^m}{m!}\frac{z^n}{n!}, \tag{3.15}$$

and

$$F_2^{\tau}\{\alpha,\beta,\lambda;\gamma,\mu;x,z;y\} = \sum_{m,n=0}^{\infty}(\alpha)_{n+m}\frac{B_{1-y}(\gamma-\beta,\beta+\tau m)}{B(\gamma-\beta,\beta)}\frac{B_{1-y}(\mu-\lambda,\lambda+\tau n)}{B(\lambda,\mu-\lambda)}\frac{x^m}{m!}\frac{z^n}{n!}. \tag{3.16}$$

and we call these functions as the incomplete τ-Appell's functions.

We can rewrite the series for incomplete τ-Appell's functions in terms of the incomplete τ-Gauss hypergeometric functions, so that

$$F_2^{\tau}[\alpha,\beta,\lambda;\gamma,\mu;x,z;y] = \sum_{n=0}^{\infty}(\alpha)_n\frac{B_y(\lambda+\tau n,\mu-\lambda)}{B(\mu-\lambda,\lambda)}\,{}_2R_1^{\tau}(\alpha+n,[\beta,\gamma;y];x)\frac{z^n}{n!}.$$

In the following proposition, integral representations of the extended incomplete τ-Appell's functions are given.

Proposition 3.11. *The extended incomplete τ-Appell's functions can be represented by an integral as follows:*

$$\begin{aligned}F_1^{\tau,p}[\lambda,\alpha,\beta;\mu;x,z;y] = \frac{y^{\lambda}}{B(\mu-\lambda,\lambda)}\int_0^1 & u^{\lambda-1}(1-uy)^{\mu-\lambda-1}(1-x(uy)^{\tau})^{-\alpha}(1-z(uy)^{\tau})^{-\beta}\\ &\times\exp\left(\frac{-p}{uy(1-uy)}\right)du,\quad p>0;\ p=0\ \text{and}\ \left|\arg(1-x)\right|<\pi,\\ &\left|\arg(1-z)\right|<\pi, Re(\mu)>Re(\lambda)>0, Re(\alpha)>0, Re(\beta)>0,\end{aligned} \tag{3.17}$$

and

$$F_2^{\tau,p}[\alpha,\beta,\lambda;\gamma,\mu;x,z;y] = \frac{y^{\lambda+\beta}}{B(\gamma-\beta,\beta)B(\mu-\lambda,\lambda)} \int_0^1 \int_0^1 u^{\beta-1}(1-uy)^{\gamma-\beta-1} v^{\lambda-1}(1-vy)^{\mu-\lambda-1}$$
$$\times (1-(uy)^\tau x-(vy)^\tau z)^{-\alpha} \exp\left(\frac{-p}{uy(1-uy)}\right) \exp\left(\frac{-p}{vy(1-vy)}\right) dudv,$$
$$p>0; p=0 \ \ and\ \left|\arg(1-x-z)\right| < \pi, Re(\mu) > Re(\lambda) > 0,$$
$$Re(\gamma) > Re(\beta) > 0, Re(\alpha) > 0. \tag{3.18}$$

Proof. Replacing the extended incomplete beta function in the definition (3.9) by its integral representation given by (1.8), then we have

$$F_1^{\tau,p}[\lambda,\alpha,\beta;\mu;x,z;y]$$
$$= \frac{1}{B(\mu-\lambda,\lambda)} \sum_{m,n=0}^{\infty} (\alpha)_m (\beta)_n \frac{x^m}{m!}\frac{z^n}{n!} \int_0^y t^{\lambda+\tau m+\tau n-1}(1-t)^{\mu-\lambda-1} \exp\left(\frac{-p}{t(1-t)}\right) dt.$$

From the uniform convergence condition, summation and integration can be swapped. Then, we get

$$F_1^{\tau,p}[\lambda,\alpha,\beta;\mu;x,z;y]$$
$$= \frac{1}{B(\mu-\lambda,\lambda)} \int_0^y t^{\lambda-1}(1-t)^{\mu-\lambda-1}(1-xt^\tau)^{-\alpha}(1-zt^\tau)^{-\beta} \exp\left(\frac{-p}{t(1-t)}\right) dt$$
$$= \frac{y^\lambda}{B(\mu-\lambda,\lambda)} \int_0^1 u^{\lambda-1}(1-uy)^{\mu-\lambda-1}(1-x(uy)^\tau)^{-\alpha}(1-z(uy)^\tau)^{-\beta} \exp\left(\frac{-p}{uy(1-uy)}\right) du.$$

Whence the result. In a similar manner, formula (3.18) can be proved. □

If p is set to 0 in the above proposition, we have the following corollary:

Corollary 3.12. *The incomplete τ-Appell's functions can be represented by an integral as follows:*

$$F_1^{\tau}[\lambda,\alpha,\beta;\mu;x,z;y] = \frac{y^\lambda}{B(\mu-\lambda,\lambda)} \int_0^1 u^{\lambda-1}(1-uy)^{\mu-\lambda-1}(1-x(uy)^\tau)^{-\alpha}(1-z(uy)^\tau)^{-\beta} du,$$
$$Re(\mu) > Re(\lambda) > 0, Re(\alpha) > 0, Re(\beta) > 0,\ x \notin [1,\infty),\ z \notin [1,\infty), \tag{3.19}$$

and

$$F_2^{\tau}[\alpha,\beta,\lambda;\gamma,\mu;x,z;y] = \frac{y^{\lambda+\beta}}{B(\gamma-\beta,\beta)B(\mu-\lambda,\lambda)} \int_0^1 \int_0^1 u^{\beta-1}(1-uy)^{\gamma-\beta-1} v^{\lambda-1}(1-vy)^{\mu-\lambda-1}$$
$$\times (1-(uy)^\tau x-(vy)^\tau z)^{-\alpha} dudv,$$
$$Re(\mu) > Re(\lambda) > 0, Re(\gamma) > Re(\beta) > 0, Re(\alpha) > 0,\ \left|\arg(1-x-z)\right| < \pi. \tag{3.20}$$

Now we evaluate the following fractional derivative formulas, which we shall need them in the derivation of the generating functions in Section 4.

Proposition 3.13. *Let* $\mathrm{Re}(\mu) > \mathrm{Re}(\lambda) > 0,\ \mathrm{Re}\,(p) > 0,\ \mathrm{Re}\,(\alpha) > 0,\ \tau \in \mathbb{N}$ *and* $|z| < 1$. *Then*

$$D_z^{\lambda-\mu,p}[z^{\lambda-1}\,(1-z^\tau)^{-\alpha}\,;y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1}\ {}_2R_1^{(\tau,p)}\,(\alpha,[\lambda,\mu;y]\,;z^\tau)$$

and

$$D_z^{\lambda-\mu,p}\{z^{\lambda-1}\,(1-z^\tau)^{-\alpha}\,;y\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1}\ {}_2R_1^{(\tau,p)}\,(\alpha,\{\lambda,\mu;y\};z^\tau)\,. \tag{3.21}$$

Proof. We have

$$\begin{aligned}
&D_z^{\lambda-\mu,p}[z^{\lambda-1}\,(1-z^\tau)^{-\alpha}\,;y]\\
&= \frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)}\int_0^y (uz)^{\lambda-1}\,(1-(uz)^\tau)^{-\alpha}\,(1-u)^{\mu-\lambda-1}\exp\left(\frac{-p}{u(1-u)}\right)du\\
&= \frac{z^{\mu-1}y^\lambda}{\Gamma(\mu-\lambda)}\int_0^1 t^{\lambda-1}\,(1-ty)^{\mu-\lambda-1}\,(1-(ty)^\tau\,z^\tau)^{-\alpha}\exp\left(\frac{-p}{ty(1-ty)}\right)dt.
\end{aligned}$$

By (3.5), we can write

$$\begin{aligned}
D_z^{\lambda-\mu,p}[z^{\lambda-1}\,(1-z^\tau)^{-\alpha}\,;y] &= \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}B(\lambda,\mu-\lambda)\ {}_2R_1^{(\tau,p)}\,(\alpha,[\lambda,\mu;y]\,;z^\tau)\\
&= \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}\ {}_2R_1^{(\tau,p)}\,(\alpha,[\lambda,\mu;y]\,;z^\tau)\,.
\end{aligned}$$

Whence the result. In a similar manner, formula (3.21) can be proved. □

Proposition 3.14. *Let* $\mathrm{Re}(\mu) > \mathrm{Re}(\lambda) > 0,\ \mathrm{Re}(\beta) > 0,\ \mathrm{Re}\,(\alpha) > 0,\ \mathrm{Re}\,(p) > 0,\ \tau \in \mathbb{N};\ |z| < \min(\frac{1}{a},\frac{1}{b})$. *Then*

$$D_z^{\lambda-\mu,p}[z^{\lambda-1}\,(1-az^\tau)^{-\alpha}\,(1-bz^\tau)^{-\beta};y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_1^{(\tau,p)}\,[\lambda,\alpha,\beta;\mu;az^\tau,bz^\tau;y]$$

and

$$D_z^{\lambda-\mu,p}\{z^{\lambda-1}\,(1-az^\tau)^{-\alpha}\,(1-bz^\tau)^{-\beta};y\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_1^{(\tau,p)}\{\lambda,\alpha,\beta;\mu;az^\tau,bz^\tau;y\}. \tag{3.22}$$

Proof. Direct calculations yield

$$\begin{aligned}
&D_z^{\lambda-\mu,p}[z^{\lambda-1}\,(1-az^\tau)^{-\alpha}\,(1-bz^\tau)^{-\beta};y]\\
&= \frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)}\int_0^y (uz)^{\lambda-1}\,(1-a\,(uz)^\tau)^{-\alpha}\,(1-b(uz)^\tau)^{-\beta}\,(1-u)^{\mu-\lambda-1}\exp\left(\frac{-p}{u(1-u)}\right)du\\
&= \frac{z^{\mu-1}y^\lambda}{\Gamma(\mu-\lambda)}\int_0^1 t^{\lambda-1}\,(1-a\,(ty)^\tau\,z^\tau)^{-\alpha}\,(1-b\,(ty)^\tau\,z^\tau)^{-\beta}\,(1-ty)^{\mu-\lambda-1}\\
&\times\exp\left(\frac{-p}{ty\,(1-ty)}\right)dt.
\end{aligned}$$

By (3.17), we can write

$$D_z^{\lambda-\mu,p}[z^{\lambda-1}\,(1-az^\tau)^{-\alpha}\,(1-bz^\tau)^{-\beta};y] = \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}B(\lambda,\mu-\lambda)F_1^{(\tau,p)}\,[\lambda,\alpha,\beta;\mu;az^\tau,bz^\tau;y]$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_1^{(\tau,p)}\left[\lambda, \alpha, \beta; \mu; az^{\tau}, bz^{\tau}; y\right].$$

Hence the proof is completed. In a similar manner, formula (3.22) can be proved. □

Proposition 3.15. *For* $\mathrm{Re}(\mu) > \mathrm{Re}(\lambda) > 0,\ \mathrm{Re}(\beta) > 0,\ \mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(p) > 0,\ \mathrm{Re}(\gamma) > 0,\ \tau \in \mathbb{N};\ \left|\frac{t}{1-z}\right| < 1$ *and* $|t| + |z| < 1,$ *we have*

$$D_z^{\lambda-\mu,p}\left[z^{\lambda-1}(1-z^{\tau})^{-\alpha}\ {}_2R_1^{(\tau,p)}\left(\alpha, [\beta, \gamma; y]; \frac{t}{1-z^{\tau}}\right)\right] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_2^{(\tau,p)}\left[\alpha, \beta, \lambda; \gamma, \mu; t, z^{\tau}; y\right].$$

Proof. Using Example 12 and (3.15), we have

$$\begin{aligned}
&D_z^{\lambda-\mu,p}\left[z^{\lambda-1}(1-z^{\tau})^{-\alpha}\ {}_2R_1^{(\tau,p)}\left(\alpha, [\beta, \gamma; y]; \frac{t}{1-z^{\tau}}\right)\right] \\
&= D_z^{\lambda-\mu,p}\left[z^{\lambda-1}(1-z^{\tau})^{-\alpha}\ \frac{1}{B(\gamma-\beta,\beta)} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} B_y(\beta+\tau n, \gamma-\beta; p)\left(\frac{t}{1-z^{\tau}}\right)^n; y\right] \\
&= \frac{1}{B(\gamma-\beta,\beta)} D_z^{\lambda-\mu,p}\left[z^{\lambda-1} \sum_{n=0}^{\infty} (\alpha)_n B_y(\beta+\tau n, \gamma-\beta; p) \frac{t^n}{n!}(1-z^{\tau})^{-\alpha-n}; y\right] \\
&= \frac{1}{B(\gamma-\beta,\beta)} \sum_{m,n=0}^{\infty} B_y(\beta+\tau n, \gamma-\beta; p) \frac{t^n}{n!} \frac{(\alpha)_n(\alpha+n)_m}{m!} D_z^{\lambda-\mu,p}\left[z^{\lambda+\tau m-1}; y\right] \\
&= \frac{1}{B(\gamma-\beta,\beta)} \sum_{m,n=0}^{\infty} B_y(\beta+\tau n, \gamma-\beta; p) \frac{t^n}{n!} \frac{(\alpha)_{n+m}}{m!} \frac{B_y(\lambda+\tau m, \mu-\lambda; p)}{\Gamma(\mu-\lambda)} z^{\mu+\tau m-1} \\
&= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_2^{(\tau,p)}\left[\alpha, \beta, \lambda; \gamma, \mu; t, z^{\tau}; y\right].
\end{aligned}$$

□

In the case $p \to 0$, we list the consequences of the above propositions for the incomplete τ-hypergeometric functions below:

Corollary 3.16. *Let* $\mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\mu) > \mathrm{Re}(\lambda) > 0,\ \tau \in \mathbb{N}$ *and* $|z| < 1$. *Then*

$$D_z^{\lambda-\mu}[z^{\lambda-1}(1-z^{\tau})^{-\alpha}; y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1}\ {}_2R_1^{\tau}(\alpha, [\lambda, \mu; y]; z^{\tau}).$$

Corollary 3.17. *Let* $\mathrm{Re}(\mu) > \mathrm{Re}(\lambda) > 0,\ \mathrm{Re}(\beta) > 0,\ \mathrm{Re}(\alpha) > 0,\ \tau \in \mathbb{N};\ |z| < \min(\frac{1}{a}, \frac{1}{b})$. *Then*

$$D_z^{\lambda-\mu}[z^{\lambda-1}(1-az^{\tau})^{-\alpha}(1-bz^{\tau})^{-\beta}; y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_1^{\tau}[\lambda, \alpha, \beta; \mu; az^{\tau}, bz^{\tau}; y].$$

Corollary 3.18. *For* $\mathrm{Re}(\mu) > \mathrm{Re}(\lambda) > 0,\ \mathrm{Re}(\beta) > 0,\ \mathrm{Re}(\gamma) > 0,\ \mathrm{Re}(\alpha) > 0,\ \tau \in \mathbb{N};\ \left|\frac{t}{1-z}\right| < 1$ *and* $|t| + |z| < 1,$ *we have*

$$D_z^{\lambda-\mu}\left[z^{\lambda-1}(1-z^{\tau})^{-\alpha}\ {}_2R_1^{\tau}\left(\alpha, [\beta, \gamma; y]; \frac{t}{1-z^{\tau}}\right)\right] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_2^{\tau}[\alpha, \beta, \lambda; \gamma, \mu; t, z^{\tau}; y].$$

4. Generating functions

Here we shall obtain linear and bilinear type generating relations for the extended incomplete τ-hypergeometric functions.

Theorem 4.1. *The extended incomplete τ-hypergeometric function can be represented by a linear generating relation as follows:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \, {}_2R_1^{(\tau,p)} \left(\lambda+n, [\alpha,\beta;y]\,; z^\tau\right) t^n = (1-t)^{-\lambda} \; {}_2R_1^{(\tau,p)} \left(\lambda, [\alpha,\beta;y]\,; \frac{z^\tau}{1-t}\right) \tag{4.1}$$

where $\tau \in \mathbb{N}, |z| < \min\{1, |1-t|\}$.

Proof. By expanding as a binomial series, we have for $|t| < |1-z^\tau|$ that

$$(1-z^\tau)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(\frac{t}{1-z^\tau}\right)^n = [(1-z^\tau)-t]^{-\lambda} = (1-t)^{-\lambda} \left[1-\frac{z^\tau}{1-t}\right]^{-\lambda}.$$

Multiplying by $z^{\alpha-1}$ on both sides and applying the extended incomplete fractional derivative operator $D_z^{\alpha-\beta,p}[f(z);y]$ on both sides, we have

$$D_z^{\alpha-\beta,p}\left[\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}(1-z^\tau)^{-\lambda}\left(\frac{t}{1-z^\tau}\right)^n z^{\alpha-1};y\right] = (1-t)^{-\lambda} D_z^{\alpha-\beta,p}\left[z^{\alpha-1}\left[1-\frac{z^\tau}{1-t}\right]^{-\lambda};y\right].$$

Swapping the summation and integration, we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_z^{\alpha-\beta,p}\left[z^{\alpha-1}(1-z^\tau)^{-\lambda-n};y\right] t^n = (1-t)^{-\lambda} D_z^{\alpha-\beta,p}\left[z^{\alpha-1}\left[1-\frac{z^\tau}{1-t}\right]^{-\lambda};y\right].$$

Using Proposition 6, the result follows. □

Theorem 4.2. *The extended incomplete τ-hypergeometric function can be represented by a linear generating relation as follows:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \, {}_2R_1^{(\tau,p)} \left(\rho-n, [\alpha,\beta;y]\,; z^\tau\right) t^n = (1-t)^{-\lambda} \; F_1^{(\tau,p)}\left[\alpha,\rho,\lambda;\beta;z^\tau, \frac{-z^\tau t}{1-t};y\right]$$

where $\tau \in \mathbb{N}, |t| < \frac{1}{1+|z|}$.

Proof. By expanding as a binomial series, we have for $|t| < |1-z^\tau|$ that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}(1-z^\tau)^n t^n = [1-(1-z^\tau)t]^{-\lambda} = (1-t)^{-\lambda}\left[1+\frac{z^\tau t}{1-t}\right]^{-\lambda}.$$

Multiplying by $z^{\alpha-1}(1-z^\tau)^{-\rho}$ on both sides and applying the extended incomplete fractional derivative operator $D_z^{\alpha-\beta,p}[f(z);y]$ on both sides, we have

$$D_z^{\alpha-\beta,p}\left[\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^{\alpha-1}(1-z^\tau)^{-\rho+n} t^n;y\right] = (1-t)^{-\lambda} D_z^{\alpha-\beta,p}\left[z^{\alpha-1}(1-z^\tau)^{-\rho}\left[1-\frac{-z^\tau t}{1-t}\right]^{-\lambda};y\right].$$

Interchanging the order, we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_z^{\alpha-\beta,p}\left[z^{\alpha-1}(1-z^\tau)^{-(\rho-n)};y\right]t^n=(1-t)^{-\lambda}D_z^{\alpha-\beta,p}\left[z^{\alpha-1}(1-z^\tau)^{-\rho}\left[1-\frac{-z^\tau t}{1-t}\right]^{-\lambda};y\right].$$

To get the desired result, we use Propositions 6 and 7. □

Theorem 4.3. *The extended incomplete τ-hypergeometric function can be represented by a bilinear generating relation as follows:*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2R_1^{(\tau,p)}\left(-n,[\gamma,\delta;y];x^\tau\right){}_2R_1^{(\tau,p)}\left(\lambda+n,[\alpha,\beta;y];z^\tau\right)t^n=(1-t)^{-\lambda}F_2^{(\tau,p)}\left[\lambda,\alpha,\gamma;\beta,\delta;\frac{z^\tau}{1-t},\frac{-x^\tau t}{1-t};y\right]$$

where $\tau\in\mathbb{N}, |t|<\frac{1-|z|}{1+|x|}$ *and* $|z|<1$.

Proof. Starting from (4.2), we replace t with $(1-x^\tau)t$, introduce a factor of $x^{\gamma-1}$, and apply $D_x^{\gamma-\delta,p}[f(x);y]$. Then we have

$$\begin{aligned}&D_x^{\gamma-\delta,p}\left[\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}x^{\gamma-1}\,{}_2R_1^{(\tau,p)}\left(\lambda+n,[\alpha,\beta;y];z^\tau\right)\,(1-x^\tau)^n t^n;y\right]\\&=D_x^{\gamma-\delta,p}\left[(1-(1-x^\tau)t)^{-\lambda}x^{\gamma-1}\,{}_2R_1^{(\tau,p)}\left(\lambda,[\alpha,\beta;y];\frac{z^\tau}{1-(1-x^\tau)t}\right);y\right].\end{aligned}$$

Provided that $|z|<1,\left|\frac{1-x}{1-z}t\right|<1$ and $\left|\frac{z}{1-t}\right|+\left|\frac{xt}{1-t}\right|<1$, we can interchange the order to obtain:

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_x^{\gamma-\delta,p}\left[x^{\gamma-1}(1-x^\tau)^n;y\right]\,{}_2R_1^{(\tau,p)}\left(\lambda+n,[\alpha,\beta;y];z^\tau\right)\\&=(1-t)^{-\lambda}D_x^{\gamma-\delta,p}\left[x^{\gamma-1}\left(1-\frac{-x^\tau t}{1-t}\right)^{-\lambda}\,{}_2R_1^{(\tau,p)}(\lambda,[\alpha,\beta;y];\frac{\frac{z^\tau}{1-t}}{1-\frac{-x^\tau t}{1-t}});y\right].\end{aligned}$$

To get the result, we use Propositions 6 and 8. □

In a similar manner, the linear and bilinear type generating relation can be given for incomplete τ-hypergeometric function.

Corollary 4.4. *The incomplete τ-hypergeometric function can be represented by a linear generating relation as follows:*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2R_1^{\tau}\left(\lambda+n,[\alpha,\beta;y];z^\tau\right)t^n=(1-t)^{-\lambda}\;{}_2R_1^{\tau}\left(\lambda,[\alpha,\beta;y];\frac{z^\tau}{1-t}\right).$$

Corollary 4.5. *The incomplete τ-hypergeometric function can be represented by a linear generating relation as follows:*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2R_1^{\tau}\left(\rho-n,[\alpha,\beta;y];z^\tau\right)t^n=(1-t)^{-\lambda}\;F_1^{\tau}\left[\alpha,\rho,\lambda;\beta;z^\tau,\frac{-z^\tau t}{1-t};y\right].$$

Corollary 4.6. *The incomplete τ-hypergeometric function can be represented by a bilinear generating relation as follows:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_2R_1^{\tau}\left(-n,[\gamma,\delta;y];x^{\tau}\right){}_2R_1^{\tau}\left(\lambda+n,[\alpha,\beta;y];z^{\tau}\right)t^n = (1-t)^{-\lambda}F_2^{\tau}\left[\lambda,\alpha,\gamma;\beta,\delta;\frac{z^{\tau}}{1-t},\frac{-x^{\tau}t}{1-t};y\right].$$

Here we shall obtain linear and bilinear type generating relations for the extended incomplete τ-hypergeometric functions.

Theorem 4.7. *The extended incomplete τ-hypergeometric function can be represented by a linear generating relation as follows:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_2R_1^{(\tau,p)}\left(\lambda+n,[\alpha,\beta;y];z^{\tau}\right)t^n = (1-t)^{-\lambda}\ {}_2R_1^{(\tau,p)}\left(\lambda,[\alpha,\beta;y];\frac{z^{\tau}}{1-t}\right) \tag{4.2}$$

where $\tau\in\mathbb{N}, |z|<\min\{1,|1-t|\}$.

Proof. By expanding as a binomial series, we have for $|t|<|1-z^{\tau}|$ that

$$(1-z^{\tau})^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\left(\frac{t}{1-z^{\tau}}\right)^n = [(1-z^{\tau})-t]^{-\lambda} = (1-t)^{-\lambda}\left[1-\frac{z^{\tau}}{1-t}\right]^{-\lambda}.$$

Multiplying by $z^{\alpha-1}$ on both sides and applying the extended incomplete fractional derivative operator $D_z^{\alpha-\beta,p}[f(z);y]$ on both sides, we have

$$D_z^{\alpha-\beta,p}\left[\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-z^{\tau})^{-\lambda}\left(\frac{t}{1-z^{\tau}}\right)^n z^{\alpha-1};y\right] = (1-t)^{-\lambda}D_z^{\alpha-\beta,p}\left[z^{\alpha-1}\left[1-\frac{z^{\tau}}{1-t}\right]^{-\lambda};y\right].$$

Swapping the summation and integration, we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_z^{\alpha-\beta,p}\left[z^{\alpha-1}(1-z^{\tau})^{-\lambda-n};y\right]t^n = (1-t)^{-\lambda}D_z^{\alpha-\beta,p}\left[z^{\alpha-1}\left[1-\frac{z^{\tau}}{1-t}\right]^{-\lambda};y\right].$$

Using Proposition 6, the result follows. □

Theorem 4.8. *The extended incomplete τ-hypergeometric function can be represented by a linear generating relation as follows:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_2R_1^{(\tau,p)}\left(\rho-n,[\alpha,\beta;y];z^{\tau}\right)t^n = (1-t)^{-\lambda}\ F_1^{(\tau,p)}\left[\alpha,\rho,\lambda;\beta;z^{\tau},\frac{-z^{\tau}t}{1-t};y\right]$$

where $\tau\in\mathbb{N}, |t|<\frac{1}{1+|z|}$.

Proof. By expanding as a binomial series, we have for $|t|<|1-z^{\tau}|$ that

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-z^{\tau})^n t^n = [1-(1-z^{\tau})t]^{-\lambda} = (1-t)^{-\lambda}\left[1+\frac{z^{\tau}t}{1-t}\right]^{-\lambda}.$$

Multiplying by $z^{\alpha-1}(1-z^\tau)^{-\rho}$ on both sides and applying the extended incomplete fractional derivative operator $D_z^{\alpha-\beta,p}[f(z);y]$ on both sides, we have

$$D_z^{\alpha-\beta,p}\left[\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}z^{\alpha-1}(1-z^\tau)^{-\rho+n}t^n;y\right]=(1-t)^{-\lambda}D_z^{\alpha-\beta,p}\left[z^{\alpha-1}(1-z^\tau)^{-\rho}\left[1-\frac{-z^\tau t}{1-t}\right]^{-\lambda};y\right].$$

Interchanging the order, we get

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_z^{\alpha-\beta,p}\left[z^{\alpha-1}(1-z^\tau)^{-(\rho-n)};y\right]t^n=(1-t)^{-\lambda}D_z^{\alpha-\beta,p}\left[z^{\alpha-1}(1-z^\tau)^{-\rho}\left[1-\frac{-z^\tau t}{1-t}\right]^{-\lambda};y\right].$$

To get the desired result, we use Propositions 6 and 7. □

Theorem 4.9. *The extended incomplete τ-hypergeometric function can be represented by a bilinear generating relation as follows:*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2R_1^{(\tau,p)}\left(-n,[\gamma,\delta;y];x^\tau\right){}_2R_1^{(\tau,p)}\left(\lambda+n,[\alpha,\beta;y];z^\tau\right)t^n=(1-t)^{-\lambda}F_2^{(\tau,p)}\left[\lambda,\alpha,\gamma;\beta,\delta;\frac{z^\tau}{1-t},\frac{-x^\tau t}{1-t};y\right]$$

where $\tau\in\mathbb{N}$, $|t|<\frac{1-|z|}{1+|x|}$ and $|z|<1$.

Proof. Starting from (4.2), we replace t with $(1-x^\tau)t$, introduce a factor of $x^{\gamma-1}$, and apply $D_x^{\gamma-\delta,p}[f(x);y]$. Then we have

$$\begin{aligned}&D_x^{\gamma-\delta,p}\left[\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}x^{\gamma-1}\,{}_2R_1^{(\tau,p)}\left(\lambda+n,[\alpha,\beta;y];z^\tau\right)(1-x^\tau)^n t^n;y\right]\\&=D_x^{\gamma-\delta,p}\left[(1-(1-x^\tau)t)^{-\lambda}x^{\gamma-1}\,{}_2R_1^{(\tau,p)}\left(\lambda,[\alpha,\beta;y];\frac{z^\tau}{1-(1-x^\tau)t}\right);y\right].\end{aligned}$$

Provided that $|z|<1$, $\left|\frac{1-x}{1-z}t\right|<1$ and $\left|\frac{z}{1-t}\right|+\left|\frac{xt}{1-t}\right|<1$, we can interchange the order to obtain:

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}D_x^{\gamma-\delta,p}\left[x^{\gamma-1}(1-x^\tau)^n;y\right]\,{}_2R_1^{(\tau,p)}\left(\lambda+n,[\alpha,\beta;y];z^\tau\right)\\&=(1-t)^{-\lambda}D_x^{\gamma-\delta,p}\left[x^{\gamma-1}\left(1-\frac{-x^\tau t}{1-t}\right)^{-\lambda}\,{}_2R_1^{(\tau,p)}(\lambda,[\alpha,\beta;y];\frac{\frac{z^\tau}{1-t}}{1-\frac{-x^\tau t}{1-t}});y\right].\end{aligned}$$

To get the result, we use Propositions 6 and 8. □

In a similar manner, the linear and bilinear type generating relation can be given for incomplete τ-hypergeometric function.

Corollary 4.10. *The incomplete τ-hypergeometric function can be represented by a linear generating relation as follows:*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2R_1^\tau\left(\lambda+n,[\alpha,\beta;y];z^\tau\right)t^n=(1-t)^{-\lambda}\,{}_2R_1^\tau\left(\lambda,[\alpha,\beta;y];\frac{z^\tau}{1-t}\right).$$

Corollary 4.11. *The incomplete τ-hypergeometric function can be represented by a linear generating relation as follows:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_2R_1^{\tau}\left(\rho - n, [\alpha,\beta; y]\,; z^{\tau}\right) t^n = (1-t)^{-\lambda}\ F_1^{\tau}\left[\alpha, \rho, \lambda; \beta; z^{\tau}, \frac{-z^{\tau} t}{1-t}; y\right].$$

Corollary 4.12. *The incomplete τ-hypergeometric function can be represented by a bilinear generating relation as follows:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\ {}_2R_1^{\tau}\left(-n, [\gamma,\delta; y]\,; x^{\tau}\right){}_2 R_1^{\tau}\left(\lambda + n, [\alpha,\beta; y]\,; z^{\tau}\right) t^n = (1-t)^{-\lambda} F_2^{\tau}\left[\lambda, \alpha, \gamma; \beta, \delta; \frac{z^{\tau}}{1-t}, \frac{-x^{\tau} t}{1-t}; y\right].$$

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. P. Agarwal, J. Choi, Fractional calculus operators and their image formulas, *J. Korean Math. Soci.*, **53** (2016), 1183–1210. https://doi.org/10.4134/JKMS.j150458
2. A. Çetinkaya , A comperative study on generating function relations for generalized hypergeometric functions via generalized fractional operators, *Adv. Differ. Equations 2018*, **156** (2018). https://doi.org/10.1186/s13662-018-1612-0
3. R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000. https://doi.org/10.1142/3779
4. R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House Publishers, Connecticut, 2006.
5. F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, 2010. https://doi.org/10.1142/p614
6. R. Srivastava, Some classes of generating functions associated with a certain family of extended and generalized hypergeometric functions, *Appl. Math. Comput.*, **243** (2014), 132–137. https://doi.org/10.1016/j.amc.2014.05.074
7. R. Srivastava, N. E. Cho, Generating functions for a certain class of hypergeometric polynomials, *Appl. Math. Comput.*, **219** (2012), 3219–3225. https://doi.org/10.1016/j.amc.2012.09.059
8. H. M. Srivastava, R. K. Parmar, P. Chopra, A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions, *Axioms*, **1** (2012), 238–258. https://doi.org/10.3390/axioms1030238
9. H. M. Srivastava, H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press, New York, 1984.
10. H. G. Sun, Y. Zhang, D. Baleanu, W. Chen, Y. Q. Chen, A new collection of real world applications of fractional calculus in science and engineering, *Commun. Nonlinear Sci. Numer. Simul.*, **64** (2018), 213–231. https://doi.org/10.1016/j.cnsns.2018.04.019

11. M. A. Özarslan, E. Özergin, Some generating relations for extended hypergeometric functions via generalized fractional derivative operator, *Math. Comput. Model.*, **52** (2010), 1825–1833. https://doi.org/10.1016/j.mcm.2010.07.011

12. M. A. Chaudhry, A. Qadir, M. Rafique, S. M. Zubair, Extension of Euler's beta function, *J. Comput. Appl. Math.*, **78** (2014), 19–32. https://doi.org/10.1016/S0377-0427(96)00102-1

13. M. A. Chaudhry, A. Qadir, H. M. Srivastava, R. B. Paris, Extended hypergeometric and confluent hypergeometric functions, *Appl. Math. Comput.*, **159** (2004), 589–602. https://doi.org/10.1016/j.amc.2003.09.017

14. M. A. Chaudhry, S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, CRC Press (Chapman and Hall), Boca Raton, FL, USA, 2002. https://doi.org/10.1201/9781420036046

15. M. A. Özarslan, C. Ustaoğlu, Some incomplete hypergeometric functions and incomplete Riemann–Liouville fractional integral operators, *Mathematics*, **7** (2019), 483. https://doi.org/10.3390/math7050483

16. M. A. Özarslan, C. Ustaoğlu, Incomplete Caputo fractional derivative operators, *Adv. Differ. Equations*, **1** (2018), 209. https://doi.org/10.1186/s13662-018-1656-1

17. A. Fernandez, C. Ustaoğlu, M. A. Özarslan, On the analytical development of incomplete Riemann–Liouville fractional calculus, *Turkish J. Math.*, **45** (2021), 1418–1443. https://doi.org/10.3906/mat-2101-64

18. H. M. Srivastava, Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformation, *J. Nonlinear Convex Anal.*, **22** (2021), 1501–1520.

19. A. Çetinkaya, The incomplete second Appell hypergeometric functions, *Appl. Math. Comput.*, **219** (2013), 8332–8337. https://doi.org/10.1016/j.amc.2012.11.050

20. S. A. Dar, R. B. Paris, A (p,q)- extension of Srivastava's triple hypergeometric function H_B and its properties, *J. Comput. Appl. Math.*, **348** (2019), 237–245. https://doi.org/10.1016/j.cam.2018.08.045

21. S. D. Lin, H. M. Srivastava, M. M. Wong, Some applications of Srivastava's theorem involving a certain family of generalized and extended hypergeometric polynomials, *Filomat*, **29** (2015), 1811–1819. https://doi.org/10.2298/FIL1508811L

22. M. A. Özarslan, C. Ustaoğlu, Extension of incomplete gamma, beta and hypergeometric functions. *Prog. Fractional Differ. Appl.*, **5** (2019), 1–15. https://doi.org/10.18576/pfda/050101

23. E. Özergin, M. A. Özarslan, A. Altin, Extension of gamma, beta and hypergeometric functions. *J. Comput. Appl. Math.*, **235** (2011), 4601–4610. https://doi.org/10.1016/j.cam.2010.04.019

24. E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960 ; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

25. R. Srivastava, Some generalizations of Pochhammer's symbol and their associated families of hypergeometric functions and hypergeometric polynomials, *Appl. Math. Inform. Sci.*, **7** (2013), 2195–2206. https://doi.org/10.12785/amis/070609

26. R. Srivastava, N. E. Cho, Some extended Pochhammer symbols and their applications involving generalized hypergeometric polynomials, *Appl. Math. and Comput.*, **234** (2014), 277–285. https://doi.org/10.1016/j.amc.2014.02.036

27. H. M. Srivastava, M. A. Chaudry, R. P. Agarwal, The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, *Integr. Transf. Spec. Funct.*, **23** (2012), 659–683. https://doi.org/10.1080/10652469.2011.623350

28. H. M. Srivastava, A. Çetinkaya, O. I. Kıymaz, A certain generalized Pochhammer symbol and its applications to hypergeometric functions, *Appl. Math. Comput.*, **226** (2014), 484–491. https://doi.org/10.1016/j.amc.2013.10.032

29. H. M. Srivastava, A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics, *Symmetry*, **13** (2021), 1–22. https://doi.org/10.3390/sym13122294

30. H. M. Srivastava, An introductory overview of fractional calculus operators based upon the Fox-Wright and related higher transcendental functions, *J. Adv. Eng. Comput.*, **5** (2021) , 135–166. https://doi.org/10.55579/jaec.202153.340

31. S. Hussain, J. Khalid, Y. M. Chu, Some generalized fractional integral Simpson's type inequalities with applications, *AIMS Math.*, **5** (2020), 5859–5883. https://doi.org/10.3934/math.2020375

32. N. M. Temme, Incomplete Laplace integrals: Uniform asymptotic expansion with application to the incomplete beta function, *SIAM J. Math. Anal.*, **18** (1987), 1637–1663. https://doi.org/10.1137/0518118