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Research article

Qualitative properties of stable solutions to some supercritical problems

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Abstract: In this paper, we study symmetry properties of stable solutions to the Lane-Emden equation

$$\Delta u + |u|^{p-1}u = 0 \quad \text{in} \quad \mathbb{R}^n$$

with $n \geq 11$, p in a suitable range and the Liouville equation

$$\Delta u + e^u = 0 \quad \text{in} \quad \mathbb{R}^n$$

with $n = 10$.

Keywords: Lane-Emden equation; Liouville equation; stable solution; rigidity results; symmetry

1. Introduction

In this paper, we consider the Lane-Emden equation

$$\Delta u + |u|^{p-1}u = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{1.1}$$

and the equation

$$\Delta u + e^u = 0 \quad \text{in} \quad \mathbb{R}^n. \tag{1.2}$$

The structures of the positive solutions of (1.1) and (1.2) have been studied intensively in the last several years. When $n = 3$, (1.1) arises in the stellar structure in astrophysics. When $n = 4$, (1.1) is relevant to the famous Yang-Mills equations. When $n = 2$, (1.2) is an interesting problem in differential geometry and is known as the "Prescribing Gaussian Curvature" problem.

For Eq (1.1), the Sobolev exponent

$$p_s(n) = \begin{cases} +\infty & \text{if} \quad 1 \le n \le 2, \\ \frac{n+2}{n-2} & \text{if} \quad n \ge 3 \end{cases}$$

plays a central role in the solvability question. In the subcritical case $1 < p < p_s(n)$, it was established by Gidas and Spruck in their celebrated work [1] that (1.1) has no positive solution. If $p = (n+2)/(n-2)$, then (1.1) is a special case of the Yamabe problem in conformal geometry. In [2], using the asymptotic symmetry technique, Caffarelli, Gidas and Spruck were able to classify all the positive solutions of (1.1) for $n \ge 3$. They showed that any positive solutions of (1.1) can be written in the form

$$u_{x_0,\lambda}(x) = \Big(\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + |x - x_0|^2}\Big)^{\frac{n-2}{2}},$$

where $\lambda > 0$ and x_0 is some point in $\mathbb{R}^n$. In [3], Chen and Li proved the same result for (1.1) by applying the moving plane method. In $n = 2$, Eq (1.2) is also classified in [3] under the additional assumption that

$$\int_{\mathbb{R}^2} e^u dx < \infty. \tag{1.3}$$

It is proved in [3] that if u is a solution of (1.2) such that (1.3) holds, then

$$u = \ln \frac{32\lambda^2}{(4 + \lambda^2|x - x_0|^2)^2}$$

for some $\lambda > 0$ and some point $x_0 \in \mathbb{R}^2$.

In the supercritical case $p > p_s(n)$, it is more difficult to classify the positive solutions of (1.1). The first result in this direction was given by Zou in [4]. It was proved in [4] that if $p_s(n) < p < p_s(n-1)$ and if u is a positive solution of (1.1) with algebraic decay rate $2/(p-1)$ at infinity, then u is radially symmetric about some point $x_0 \in \mathbb{R}^n$. In [5], Guo generalized Zou's result to $p \ge p_s(n-1)$ by assuming that

$$\lim_{|x|\to+\infty} |x|^{\frac{2}{p-1}} u(x) \equiv \Big[\frac{2}{p-1}\Big(n - 2 - \frac{2}{p-1}\Big)\Big]^{\frac{1}{p-1}}. \tag{1.4}$$

Moreover, it is showed in [5] that (1.4) is a necessary and sufficient condition for a positive solution of (1.1) to be radially symmetric about some point. The analogous result for second order equation (1.2) is considered in [6]. It is proved in [6] that if $n \ge 4$ and if $u \in C^2(\mathbb{R}^n)$ is an entire solution of (1.2), then u is radially symmetric about some point $x_0 \in \mathbb{R}^n$ if and only if

$$\lim_{|x|\to\infty} u(x) + 2\ln(|x|) - \ln(16) = 0.$$

If we focus on radial solutions, then the structure of positive solutions of (1.1) has been completely classified in [7]. They showed that for any $a > 0$, (1.1) admits a unique positive radial solution $u = u_a(r)$ with $u_a(0) = a$. Moreover, no two positive radial solutions of (1.1) can intersect each other when $p > p_{JL}(n)$, where $p_{JL}(n)$ is the exponent given by

$$p_{JL}(n) = \begin{cases} \infty & \text{if} \quad 3 \le n \le 10, \\ \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if} \quad n \ge 11. \end{cases}$$

Another important topic is the classification of stable solutions. In general, a solution of the semi-linear equation

$$\Delta u + f(u) = 0 \quad \text{in} \quad \mathbb{R}^n$$

with f be a Lipschitz function is called stable if

$$\int_{\mathbb{R}^n} |\nabla \psi|^2 dx - \int_{\mathbb{R}^n} f'(u)\psi^2 dx \geq 0 \quad \forall \psi \in C_0^\infty(\mathbb{R}^n).$$

One of the most interesting questions concerning stable solutions is the following De Giorgi's conjecture.

Conjecture: Let u be a bounded solution of the equation

$$\Delta u + u - u^3 = 0 \quad \text{in} \quad \mathbb{R}^n$$

such that $\frac{\partial u}{\partial x_n} > 0$. Then the level sets of u are hyperplanes, at least if $n \leq 8$.

De Giorgi's conjecture was proved in dimension $n = 2$ by Ghoussoub and Gui in [8]. For $n = 3$, this is proved by Ambrosio and Cabré in [9]. Savin proved in [10] that for $4 \leq n \leq 8$, the above conjecture is true under the additional limit condition that

$$u(x_1, ..., x_n) \to \pm 1 \quad \text{as} \quad x_n \to \pm\infty. \tag{1.5}$$

For $n > 9$, a counterexample is constructed in [11]. The conjecture is still open for dimensions $4 \leq n \leq 8$ without the additional assumption (1.5).

For Eq (1.1), there are also some results concerning stable solutions. In [12], Liouville type results for solutions with finite Morse index were established. By making a delicate use of the classical Moser iteration method, Farina was able to classify finite Morse index solutions in his seminal paper [13]. It was proved in [13] that if $u \in C^2(\mathbb{R}^n)$ is a stable solution of (1.1) with $1 < p < p_{JL}(n)$, then $u \equiv 0$. Moreover, (1.1) admits a smooth positive, bounded, stable and radial solution for $n \geq 11, p > p_{JL}(n)$. Actually, it was showed in [13] that the radial solutions considered in [7] are stable when $n \geq 11, p > p_{JL}(n)$. The results in [13] also have a lot of generalizations, we refer to [14, 15, 16, 17, 18, 19]. As for the classification of the stable solutions of (1.2), it was proved in [20] that for $1 \leq n \leq 9$, there is no stable solution $u \in C^2(\mathbb{R}^n)$ of (1.2).

In spite of the above mentioned results, there are some intriguing problems which are still open. In [21], the authors proposed the following conjecture, which is a natural extension of De Giorgi type conjecture:

Conjecture: Let $n \geq 11, p_{JL}(n) < p < p_{JL}(n-1)$, then all stable solutions to (1.1) must be radially symmetric around some point.

Remark 1.1. *When $p > p_{JL}(n-1)$,* (1.1) *has a positive stable solution which is not radially symmetric. Indeed, let u be a positive radial stable solution of the equation*

$$\Delta u + |u|^{p-1}u = 0 \quad \textit{in} \quad \mathbb{R}^{n-1}$$

for $p > p_{JL}(n-1)$ (see [13]), then u can also be viewed as a stable solution of the equation

$$\Delta u + |u|^{p-1}u = 0 \quad \textit{in} \quad \mathbb{R}^n.$$

But it is obvious that this solution is not radially symmetric in $\mathbb{R}^n$.

Our first objective in this paper is to give some partial results toward the above conjecture. To state our results in a more precise way, let us first introduce some new exponents. Let γ be a constant such that

$$1 \le \gamma < 2p + 2\sqrt{p(p-1)} - 1,$$
$$(p+\gamma)/(p-1) > (n-1)/2,$$

we define

$$\alpha(p,\gamma,n) = \{\frac{[\frac{(n-2)^2\gamma-(\gamma+1)^2\beta}{4p\gamma-(\gamma+1)^2}]^{\frac{p+\gamma}{p-1}}(n-1)\pi^{n-1}}{2^{n-2-(p+\gamma)}}\}^{\frac{2}{2(p+\gamma)-(p-1)(n-1)}}. \tag{1.6}$$

Let

$$\beta(p,n) = (\frac{2n+\beta}{p})^{\frac{1}{p-1}} \tag{1.7}$$

and let

$$c_s(p,n) = \frac{\omega_{n-1}(\frac{\frac{(n-2)^2}{4}-\beta}{p-1})^{\frac{2}{p-1}}}{\frac{\omega_{n-2}}{4}(\frac{2}{\pi})^{n-2}(\beta(p,n)-\beta^{\frac{1}{p-1}})^{\frac{n+3}{2}}(\alpha(p,n))^{-\frac{p(n-1)}{2}}}. \tag{1.8}$$

We consider

$$p-1 = (\frac{(n-2)^2}{4} - \frac{2p}{p-1}(n-2-\frac{2}{p-1}))c_s(p,n) \tag{1.9}$$

as a function with respect to the variable p. Let S be the set

$$S = \{p : p \text{ is a solution of (1.9) such that } p > p_{JL}(n)\}$$

and let p_* be the infimum of S. We define

$$p_{cs}(n) = \begin{cases} p_{JL}(n-1) & \text{if } p_* \ge p_{JL}(n-1), \\ p_* & \text{if } p_* < p_{JL}(n-1). \end{cases} \tag{1.10}$$

With the help of these numbers, we can give the statement of our first result.

Theorem 1.2. *Let $n \ge 11, p_{JL}(n) < p < p_{cs}(n)$, where $p_{cs}(n)$ is defined in* (1.10). *Let u be a positive stable solution of* (1.1) *such that u is even symmetric with respect to the planes $\{x_i = 0\}, i = 1, 2, \cdots n$, then u is radially symmetric with respect to the origin.*

In Theorem 1.2, we need the assumption that u is a positive solution of (1.1). But under suitable conditions, we can show that stable solutions of (1.1) do not change sign. For our purpose, we use $p_1(n) \le p_2(n) \le p_3(n)$ to denote the three numbers such that

$$(n-1)(p-1) = \frac{(n-2)^2}{4} - \frac{2p}{p-1}(n-2-\frac{2}{p-1}). \tag{1.11}$$

We define

$$p_{si}(n) = \begin{cases} p_{JL}(n-1) & \text{if } p_3(n) \le p_{JL}(n), \\ p_{JL}(n-1) & \text{if } p_2(n) \ge p_{JL}(n-1), \\ p_2(n) & \text{if } p_3(n) > p_{JL}(n) \text{ and } p_2(n) < p_{JL}(n-1). \end{cases} \tag{1.12}$$

Theorem 1.3. *Let $n \ge 11, p_{JL}(n) < p < p_{si}(n)$, where $p_{si}(n)$ is defined in* (1.12). *Let u be a stable solution of Eq* (1.1), *then u does not change sign. If u is a axially symmetric stable solution of* (1.1), *then u does not change sign when $p_{JL}(n) < p < p_{JL}(n-1)$.*

Remark 1.4. *By using MATLAB, we can give some examples for $p_{cs}(n)$ and $p_{si}(n)$.*

n	$p_{JL}(n)$	$p_{cs}(n)$	$p_{si}(n)$	$p_{JL}(n-1)$
n=12	3.9266499161	4.1229824119	6.9220245868	6.9220245868
n=13	2.9306913006	3.0772258656	3.9266499161	3.9266499161
n=14	2.4342585459	2.5559714732	2.9306913006	2.9306913006
n=15	2.1374347552	2.2443064930	2.4342585459	2.4342585459

For Eq (1.2), we have the following result.

Theorem 1.5. *Let u be a smooth stable solution of Eq* (1.2) *for $n = 10$, then u is radially symmetric with respect to some point in $\mathbb{R}^n$.*

Remark 1.6. *If $n \geq 11$, then Eq* (1.2) *has a smooth stable solution which is not radially symmetric. For more discussions, we refer to [22].*

The rest of the paper will be organized as follows. In section 2, we consider rigidity results on the unit sphere for some second order equations. Rigidity results on compact manifolds have been considered by several authors, see for instance [1, 23, 24, 25, 26, 27]. We point out that in the above papers, the proof of the rigidity results depends heavily on the classical Bochner formula. In section 3, we first use a monotonicity formula to study the qualitative properties of solutions. Then, by combing the rigidity results and the qualitative properties of solutions, we can verify the assumption of Theorem 1.1 in [5] and obtain the symmetry properties of stable solutions. In section 4, we give the prove of Theorem 1.5.

Notation. In some situations, we will write a point $x \in \mathbb{R}^n$ as $x = (r, \theta)$, where (r, θ) is the spherical coordinates and $S^{n-1} \subset \mathbb{R}^n$ is the unit sphere. In the rest of the paper, c will denote a positive constant which may vary from line to line.

2. A second order equation on the unit sphere

In this section, we consider the equation

$$\Delta_{S^{n-1}}\phi - \beta\phi + |\phi|^{p-1}\phi = 0, \tag{2.1}$$

where

$$\beta = \frac{2}{p-1}(n - 2 - \frac{2}{p-1})$$

and $\Delta_{S^{n-1}}$ is the Laplace-Beltrimi operator on the unit sphere. In the rest of this section, we will always assume that $n \geq 11$. The main result in this section is the following.

Theorem 2.1. *Let γ be a constant such that*

$$1 \leq \gamma < 2p + 2\sqrt{p(p-1)} - 1,$$
$$(p + \gamma)/(p - 1) > (n - 1)/2.$$

If $p_{JL}(n) < p < p_{cs}(n)$ with $p_{cs}(n)$ be the number defined in (1.9) *and if ϕ is a positive solution of* (2.1) *such that* (2.2) *holds. Suppose*

$$\int_{S^{n-1}} \phi\Phi_i d\theta = 0, \quad i = 1, 2 \cdots, n$$

where $\Phi_i, i = 1, 2, \cdots, n$ are the eigenfunctions of the operator $-\Delta_{S^{n-1}}$ corresponding to the eigenvalue $n-1$, then ϕ is a constant solution of Eq (2.1)*. If $\{\phi - \beta^{\frac{1}{p-1}}\}$ has at least three connected components, then the same result holds.*

In order to prove Theorem 2.1, we need several lemmas.

Lemma 2.2. *Let $p_{JL}(n) < p < p_{JL}(n-1)$ and let $\phi \in H^1(S^{n-1})$ be a weak solution of* (2.1) *such that*

$$\int_{S^{n-1}} |\nabla_{S^{n-1}}\psi|^2 d\theta + \frac{(n-2)^2}{4}\int_{S^{n-1}} \psi^2 d\theta \geq p\int_{S^{n-1}} |\phi|^{p-1}\psi^2 d\theta \tag{2.2}$$

for every $\psi \in H^1(S^{n-1})$, then $\phi \in C^2(S^{n-1})$.

Proof. We take $\psi = |\phi|^{\frac{\gamma-1}{2}}\phi$ into (2.2), where γ is a positive constant which will be chosen later. Then

$$p\int_{S^{n-1}} |\phi|^{p+\gamma} d\theta \leq \frac{(n-2)^2}{4}\int_{S^{n-1}} |\phi|^{\gamma+1} d\theta + \int_{S^{n-1}} |\nabla_{S^{n-1}}(|\phi|^{\frac{\gamma-1}{2}}\phi)|^2 d\theta. \tag{2.3}$$

Multiplying the both sides of (2.1) by $|\phi|^{\gamma-1}\phi$ and integrating over S^{n-1}, we can get that

$$\int_{S^{n-1}} \nabla_{S^{n-1}}\phi \cdot \nabla_{S^{n-1}}(|\phi|^{\gamma-1}\phi) d\theta + \beta\int_{S^{n-1}} |\phi|^{\gamma+1} d\theta = \int_{S^{n-1}} |\phi|^{p+\gamma} d\theta. \tag{2.4}$$

(2.4) is equivalent to

$$\frac{4\gamma}{(\gamma+1)^2}\int_{S^{n-1}} |\nabla_{S^{n-1}}(|\phi|^{\frac{\gamma-1}{2}}\phi)|^2 d\theta + \beta\int_{S^{n-1}} |\phi|^{\gamma+1} d\theta = \int_{S^{n-1}} |\phi|^{p+\gamma} d\theta. \tag{2.5}$$

By combining (2.3) and (2.5) together, we can obtain that

$$[p - \frac{(\gamma+1)2}{4\gamma}]\int_{S^{n-1}} |\phi|^{p+\gamma} d\theta \leq [\frac{(n-2)^2}{4} - \frac{(\gamma+1)^2\beta}{4\gamma}]\int_{S^{n-1}} |\phi|^{\gamma+1} d\theta. \tag{2.6}$$

It is easy to check that

$$p - \frac{(\gamma+1)^2}{4\gamma} > 0$$

when $1 \leq \gamma < 2p + 2\sqrt{p(p-1)} - 1$. By applying Hölder's inequality, we have

$$[p - \frac{(\gamma+1)^2}{4\gamma}]\int_{S^{n-1}} |\phi|^{p+\gamma} d\theta \leq c(\int_{S^{n-1}} |\phi|^{p+\gamma} d\theta)^{\frac{\gamma+1}{p+\gamma}}. \tag{2.7}$$

It follows from (2.7) that $\int_{S^{n-1}} |\phi|^{p+\gamma} d\theta$ is finite. By formula (5.10) in [13], we know that there exists a constant γ such that $(p+\gamma)/(p-1) > (n-1)/2$. Therefore, $|\phi|^{p-1} \in L^q(S^{n-1})$ for some $q > (n-1)/2$. The standard regularity results in [28] imply that $\phi \in C^2(S^{n-1})$. □

Lemma 2.3. *Let $p_{JL}(n) < p < p_{JL}(n-1)$ and let $\phi \in C^2(S^{n-1})$ be a positive solution of* (2.1) *such that* (2.2) *holds, then*

$$||\phi||_{L^\infty(S^{n-1})} \leq \alpha(p, \gamma, n), \tag{2.8}$$

where γ is the constant used in the proof of Lemma 2.2 and $\alpha(p, \gamma, n)$ is the constant defined in (1.6).

Proof. Let $\theta_0 \in S^{n-1}$ be a point such that $\phi(\theta_0) = \|\phi\|_{L^\infty(S^{n-1})} = \eta$. By taking suitable orthogonal transformation, we may assume that θ_0 is the south pole. Let us introduce the following coordinates on S^{n-1},

$$\begin{cases}\theta_1 = \sin\xi\sin\xi_{n-2}\cdots\sin\xi_2\sin\xi_1,\\ \theta_2 = \sin\xi\sin\xi_{n-2}\cdots\sin\xi_2\cos\xi_1,\\ \theta_3 = \sin\xi\sin\xi_{n-2}\cdots\cos\xi_2,\\ \quad\cdots,\\ \theta_{n-1} = \cos\xi,\end{cases}$$

where $\xi \in [0,\pi), \xi_1 \in [0,2\pi), \xi_k \in [0,\pi)$ for $k = 2,3,\cdots n-2$. The coordinate of the point θ_0 is given by $(0,0,\cdots,0)$. By (2.1), we know that ϕ satisfies the equation

$$\frac{1}{\sin^{n-2}\xi}\frac{d}{d\xi}(\sin^{n-2}\xi\frac{d\phi}{d\xi}(\xi)) + \frac{1}{\sin^2\xi}\Delta_{S^{n-2}}\phi - \beta\phi + \phi^p = 0, \tag{2.9}$$

where S^{n-2} is the unit sphere in $\mathbb{R}^{n-1}$ and $\Delta_{S^{n-2}}$ is the Laplace -Beltrami operator on S^{n-2}. We define

$$\hat{\phi}(\xi) = \frac{1}{\omega_{n-2}}\int_{S^{n-2}}\phi(\xi,\theta')d\theta',$$

where ω_{n-2} is the area of S^{n-2}. It follows from (2.9) that $\hat{\phi}$ satisfies

$$\frac{1}{\sin^{n-2}\xi}\frac{d}{d\xi}(\sin^{n-2}\xi\frac{d\hat{\phi}}{d\xi}(\xi)) - \beta\hat{\phi} + \frac{1}{\omega_{n-2}}\int_{S^{n-2}}\phi^p(\xi,\theta')d\theta' = 0. \tag{2.10}$$

By the Jensen's inequality, we can get that

$$\frac{1}{\sin^{n-2}\xi}\frac{d}{d\xi}(\sin^{n-2}\xi\frac{d\hat{\phi}}{d\xi}(\xi)) - \beta\hat{\phi} + \hat{\phi}^p \le 0, \quad \text{in} \quad (0,\pi). \tag{2.11}$$

Let ξ_1 be the first point such that $\hat{\phi}(\xi_1) = \frac{\eta}{2}$. It follows from (2.11) that $\hat{\phi}$ is strictly decreasing in $(0,\xi_1)$. We will focus on the case $\xi_1 < \frac{\pi}{2}$ since the case $\xi_1 \ge \frac{\pi}{2}$ can be dealt with similarly. Let γ be the constant used in the proof of Lemma 2.2. By (2.10), we can obtain that

$$\begin{aligned}\hat{\phi}(\xi_1) - \hat{\phi}(0) &= \int_0^{\xi_1}\frac{1}{\sin^{n-2}\xi}\int_0^{\xi}\sin^{n-2}\tau[\beta\hat{\phi}(\tau) - \frac{1}{\omega_{n-2}}\int_{S^{n-2}}\phi^p(\tau,\theta')d\theta']d\tau d\xi\\ &\ge -\eta^p\int_0^{\xi_1}\frac{1}{\sin^{n-2}\xi}\int_0^{\xi}\sin^{n-2}\tau d\tau d\xi\\ &\ge -\frac{\xi_1^2}{2}\eta^p.\end{aligned}$$

This implies $\xi_1 \ge \eta^{\frac{1-p}{2}}$. By the above analysis, we can get that

$$\begin{aligned}\int_{\{\xi\le\xi_1\}}\phi^{p+\gamma}d\theta &= \int_0^{\xi_1}\int_{S^{n-2}}\sin^{n-2}\xi\phi^{p+\gamma}(\xi,\theta')d\theta' d\xi\\ &\ge \omega_{n-2}\int_0^{\xi_1}\sin^{n-2}\xi\hat{\phi}^{p+\gamma}(\xi)d\xi\\ &\ge \omega_{n-2}\frac{2^{n-2-(p+\gamma)}}{(n-1)\pi^{n-2}}\eta^{p+\gamma+\frac{(1-p)(n-1)}{2}}.\end{aligned} \tag{2.12}$$

By Lemma 2.2, we know that

$$\int_{S^{n-1}} \phi^{p+\gamma} d\theta \leq [\frac{\frac{(n-2)^2}{4} - \frac{(\gamma+1)^2}{4\gamma}\beta}{p - \frac{(\gamma+1)^2}{4\gamma}}]^{\frac{p+\gamma}{p-1}} \omega_{n-1}. \tag{2.13}$$

We get from (2.12) and (2.13) that

$$\begin{aligned} \eta &\leq \{\frac{[\frac{\frac{(n-2)^2}{4} - \frac{(\gamma+1)^2}{4\gamma}\beta}{p - \frac{(\gamma+1)^2}{4\gamma}}]^{\frac{p+\gamma}{p-1}} \omega_{n-1}(n-1)\pi^{n-2}}{\omega_{n-2} 2^{n-2-(p+\gamma)}}\}^{\frac{1}{p+\gamma+\frac{(1-p)(n-1)}{2}}} \\ &\leq \{\frac{[\frac{(n-2)^2\gamma-(\gamma+1)^2\beta}{4p\gamma-(\gamma+1)^2}]^{\frac{p+\gamma}{p-1}}(n-1)\pi^{n-1}}{2^{n-2-(p+\gamma)}}\}^{\frac{2}{2(p+\gamma)-(p-1)(n-1)}}. \end{aligned} \tag{2.14}$$

Hence (2.8) holds. □

Lemma 2.4. *Let ϕ be a positive solution of* (2.1) *such that*

$$\int_{S^{n-1}} \phi \Phi_i d\theta = 0 \quad \text{for} \quad i = 1, 2, \cdots, n, \tag{2.15}$$

where $\Phi_i, i = 1, 2, \cdots, n$ are the eigenfunctions of the operator $-\Delta_{S^{n-1}}$ corresponding to the eigenvalue $n-1$, then

$$\|\phi\|_{L^\infty(S^{n-1})} \geq (\frac{2n+\beta}{p})^{\frac{1}{p-1}}. \tag{2.16}$$

Proof. We define

$$\tilde{\phi} = \phi - \overline{\phi},$$

where $\overline{\phi}$ is given by

$$\overline{\phi} = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \phi d\theta.$$

Then $\tilde{\phi}$ satisfies the equation

$$\Delta_{S^{n-1}} \tilde{\phi} - \beta\phi + \phi^p = 0. \tag{2.17}$$

Multiplying the both sides of (2.17) by $\tilde{\phi}$ and using integration by part, we can get that

$$\int_{S^{n-1}} |\nabla_{S^{n-1}} \tilde{\phi}|^2 d\theta + \beta \int_{S^{n-1}} \tilde{\phi}^2 d\theta - \int_{S^{n-1}} (\phi^p - \overline{\phi}^p)(\phi - \overline{\phi}) d\theta = 0. \tag{2.18}$$

By (2.15) and the definition of $\tilde{\phi}$, we know that

$$\int_{S^{n-1}} \tilde{\phi} d\theta = 0,$$

$$\int_{S^{n-1}} \tilde{\phi} \Phi_i d\theta = 0, \quad i = 1, 2 \cdots, n.$$

By (2.18) and the Poincaré's inequality, we have

$$2n \int_{S^{n-1}} \tilde{\phi}^2 d\theta + \beta \int_{S^{n-1}} \tilde{\phi}^2 d\theta - p\|\phi\|^{p-1}_{L^\infty(S^{n-1})} \int_{S^{n-1}} \tilde{\phi}^2 d\theta \leq 0. \tag{2.19}$$

If $\tilde{\phi} \neq 0$, then

$$2n + \beta - p\|\phi\|^{p-1}_{L^\infty(S^{n-1})} \leq 0.$$

It follows that

$$\|\phi\|_{L^\infty(S^{n-1})} \geq (\frac{2n+\beta}{p})^{\frac{1}{p-1}}, \tag{2.20}$$

Hence (2.16) holds. □

Lemma 2.5. *If ϕ is a positive solution of* (2.1) *such that $\{\phi - \beta^{\frac{1}{p-1}} \neq 0\}$ has at least three connected components, then*

$$\|\phi\|_{L^\infty(S^{n-1})} \geq (\frac{2n+\beta}{p})^{\frac{1}{p-1}}. \tag{2.21}$$

Proof. The equation (2.1) can be written as

$$\Delta_{S^{n-1}}\phi - \beta(\phi - \beta^{\frac{1}{p-1}}) + \phi^p - \beta^{\frac{p}{p-1}} = 0. \tag{2.22}$$

Since $\{\phi - \beta^{\frac{1}{p-1}} \neq 0\}$ has at least three connected components, then there is a connected component Ω_0 of $\{\phi - \beta^{\frac{1}{p-1}} \neq 0\}$ such that the area of Ω_0 is less than $\frac{1}{3}\omega_{n-1}$. let 1_{Ω_0} be the function defined by

$$1_{\Omega_0} = \begin{cases} 1 & \text{in} \quad \Omega_0, \\ 0 & \text{on} \quad S^{n-1}\backslash\Omega_0. \end{cases}$$

Multiplying the both sides of (2.22) by $(\phi - \beta^{\frac{1}{p-1}})1_{\Omega_0}$ and using integration by part, we can get that

$$-\int_{\Omega_1} |\nabla_{S^{n-1}}\phi|^2 d\theta - \beta\int_{\Omega_1}(\phi - \beta^{\frac{1}{p-1}})^2 d\theta + \int_{\Omega_1}(\phi^p - \beta^{\frac{p}{p-1}})(\phi - \beta^{\frac{1}{p-1}})d\theta = 0. \tag{2.23}$$

Let $\lambda_1(\Omega_0)$ be the first eigenvalue of the eigenvalue problem

$$\begin{cases} \Delta_{S^{n-1}}\Phi + \lambda\Phi = 0 & \text{in} \quad \Omega_0, \\ \Phi = 0 & \text{on} \quad \partial\Omega_0. \end{cases}$$

By (2.23) and the mean value theorem, we can get that

$$(-\lambda_1(\Omega_0) - \beta + p\|\phi\|^{p-1}_{L^\infty(S^{n-1})})\int_{\Omega_1}(\phi - \beta^{\frac{1}{p-1}})^2 d\theta \geq 0. \tag{2.24}$$

It follows from (2.24) that

$$-\lambda_1(\Omega_0) - \beta + p\|\phi\|^{p-1}_{L^\infty(S^{n-1})} \geq 0. \tag{2.25}$$

Since the area of Ω_0 is less than $\frac{1}{3}\omega_{n-1}$, where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$. By using Schwartz symmetrization, we can get that

$$\lambda_1(\Omega_0) \geq 2n. \tag{2.26}$$

It follows from (2.25) and (2.26) that

$$\|\phi\|_{L^\infty(S^{n-1})} \geq (\frac{2n+\beta}{p})^{\frac{1}{p-1}}, \tag{2.27}$$

Hence (2.21) holds. □

Remark 2.6. *We notice that*

$$(p-1)\beta = 2(n-2-\frac{2}{p-1}) < 2n,$$

then

$$\|\phi\|_{L^\infty(S^{n-1})} \geq (\frac{2n+\beta}{p})^{\frac{1}{p-1}} > \beta^{\frac{1}{p-1}}.$$

Lemma 2.7. *Let $\overline{p}$ be a constant such that $p_{JL}(n) < \overline{p} < p_{JL}(n-1)$. There exists a positive constant c such that if $\phi \in C^2(S^{n-1})$ is a nonconstant solution of* (2.1) *for $p_{JL}(n) < p < \overline{p}$, then*

$$\int_{S^{n-1}} \phi^2 d\theta \leq c \int_{S^{n-1}} |\nabla_{S^{n-1}}\phi|^2 d\theta. \tag{2.28}$$

Proof. Suppose (2.28) does not hold, then there exists a sequence $\{\phi_m\}$ such that ϕ_m satisfies

$$\Delta_{S^{n-1}}\phi_m - \frac{2}{p_m-1}(n-2-\frac{2}{p_m-1})\phi_m + |\phi|^{p_m-1}\phi_m = 0 \tag{2.29}$$

and

$$\int_{S_{n-1}} \phi_m^2 d\theta \geq m \int_{S^{n-1}} |\nabla_{S^{n-1}}\phi_m|^2 d\theta. \tag{2.30}$$

Since $-\phi_m$ is also a solution of (2.29), without loss of generality, we can assume that

$$\phi_m(\theta_m) = \max_{\theta\in S^{n-1}} \phi_m(\theta) > 0. \tag{2.31}$$

It follows from the proof of Lemma 2.2 that $\int_{S^{n-1}} \phi_m^2 d\theta$ remains bounded. So (2.30) implies

$$\lim_{m\to+\infty} \int_{S^{n-1}} |\nabla_{S^{n-1}}\phi_m|^2 d\theta = 0. \tag{2.32}$$

By (2.8) and (2.32), we can get that there exist two constants p_0 and c_0 such that

$$\lim_{m\to+\infty} p_m = p_0, \quad \lim_{m\to+\infty} \phi_m = c_0.$$

Moreover, c_0 is a constant solution of (2.1) for $p = p_0$. Therefore,

$$c_0 = 0 \quad \text{or} \quad c_0 = [\frac{1}{p_0-1}(n-2-\frac{2}{p_0-1})]^{\frac{1}{p_0-1}}.$$

We get from (2.31) that

$$\Delta\phi_m(\theta_m) = (\beta_m - \phi_m^{p_m-1}(\theta_m))\phi_m(\theta_m) \leq 0. \tag{2.33}$$

Therefore,

$$\phi_m(\theta_m) \geq (\beta_m)^{\frac{1}{p_m-1}}. \tag{2.34}$$

It follows from (2.34) that c_0 is not zero. Let

$$\phi_m = \beta_m^{\frac{1}{p_m-1}} + \psi_m,$$

then $\lim_{m\to+\infty}\psi_m = 0$ and ψ_m satisfies the equation

$$\Delta_{S^{n-1}}\psi_m + (p_m-1)\beta_m\psi_m + (\psi_m+\beta_m^{\frac{1}{p_m-1}})^{p_m} - \beta_m^{\frac{p_m}{p_m-1}} - p_m\beta_m\psi_m = 0. \tag{2.35}$$

It is easy to verify that

$$(\psi_m+\beta_m^{\frac{1}{p_m-1}})^{p_m} - \beta_m^{\frac{p_m}{p_m-1}} - p_m\beta_m\psi_m \le c\|\psi_m\|^2_{L^\infty(S^{n-1})}$$

for some positive constant c independent of m. We define

$$v_m = \frac{\psi_m}{\|\psi_m\|_{L^\infty(S^{n-1})}},$$

then v_m satisfies

$$\Delta_{S^{n-1}}v_m + (p_m-1)\beta_m v_m + \frac{(\psi_m+\beta_m^{\frac{1}{p_m-1}})^{p_m} - \beta_m^{\frac{p_m}{p_m-1}} - p_m\beta_m\psi_m}{\|\psi_m\|_{L^\infty(S^{n-1})}} = 0. \tag{2.36}$$

Since

$$\|v_m\|_{L^\infty(S^{n-1})} = 1$$

and

$$\lim_{m\to+\infty}\left\|\frac{(\psi_m+\beta_m^{\frac{1}{p_m-1}})^{p_m} - \beta_m^{\frac{p_m}{p_m-1}} - p_m\beta_m\psi_m}{\|\psi_m\|_{L^\infty(S^{n-1})}}\right\|_{L^\infty(S^{n-1})} = 0.$$

By standard elliptic estimates, we know that there exists a nontrivial function v_∞ such that $v_m \to v_\infty$ in $H^1(S^{n-1})$. Moreover, v_∞ satisfies the equation

$$\Delta_{S^{n-1}}v_\infty + (p_0-1)\beta_0 v_\infty = 0. \tag{2.37}$$

Then we deduce that v_∞ is a nontrivial eigenfunction of $-\Delta_{S^{n-1}}$ corresponding to the eigenvalue $(p_0-1)\beta_0$. On the other hand, it is easy to see that

$$(p_0-1)\beta_0 = 2(n-2-\frac{2}{p_0-1}) < 2n$$

and $p_{JL}(n) > (n+1)/(n-3)$ when $n \ge 11$. Therefore, $(p_0-1)\beta_0$ can not be an eigenvalue of $-\Delta_{S^{n-1}}$. By combining these two facts together, we obtain a contradiction. □

Next, we can give some estimates about the constant c in Lemma 2.7.

Lemma 2.8. *Let $p_{JL}(n) < p < p_{JL}(n-1)$ and let ϕ be a positive solution of* (2.1) *such that*

$$\|\phi\|_{L^\infty(S^{n-1})} \ge (\frac{2n+\beta}{p})^{\frac{1}{p-1}},$$

then the constant c in Lemma 2.7 can be estimated by $c_s(p,n)$, where $c_s(p,n)$ is defined by (1.8).

Proof. Multiplying the both sides of (2.1) by ϕ and integrating over S^{n-1}, we can get that

$$\int_{S^{n-1}} |\nabla_{S^{n-1}}\phi|^2 d\theta + \beta \int_{S^{n-1}} \phi^2 d\theta = \int_{S^{n-1}} |\phi|^{p+1} d\theta. \tag{2.38}$$

We take $\psi = \phi$ into (2.2), then

$$\int_{S^{n-1}} |\nabla_{S^{n-1}}\phi|^2 d\theta + \frac{(n-2)^2}{4} \int_{S^{n-1}} \phi^2 d\theta \geq p \int_{S^{n-1}} |\phi|^{p+1} d\theta. \tag{2.39}$$

By (2.38) and (2.39), we can get that

$$\int_{S^{n-1}} \phi^{p+1} d\theta \leq \frac{\frac{(n-2)^2}{4} - \beta}{p-1} \int_{S^{n-1}} \phi^2 d\theta. \tag{2.40}$$

By the Poincaré's inequality, we know that

$$\int_{S^{n-1}} \phi^2 d\theta \leq \omega_{n-1}^{\frac{p-1}{p+1}} (\int_{S^{n-1}} \phi^{p+1} d\theta)^{\frac{2}{p+1}} \tag{2.41}$$

It follows from (2.40) and (2.41) that

$$\int_{S^{n-1}} \phi^2 d\theta \leq \omega_{n-1} (\frac{\frac{(n-2)^2}{4} - \beta}{p-1})^{\frac{2}{p-1}}. \tag{2.42}$$

In order to estimate the constant c in Lemma 2.7, we need to give a lower bound for $\int_{S^{n-1}} |\nabla_{S^{n-1}}\phi|^2 d\theta$. Since we have assumed that

$$||\phi||_{L^\infty(S^{n-1})} \geq (\frac{2n+\beta}{p})^{\frac{1}{p-1}} = \beta(p,n),$$

then there exists a point θ_0 such that $\phi(\theta_0) = \beta(p,n)$. By taking suitable orthogonal transformation, we may assume that θ_0 is the south pole. We use the coordinates used in the proof of Lemma 2.2. By (2.1), we know that ϕ satisfies Eq (2.9). We define

$$\hat{\phi}(\xi) = \frac{1}{\omega_{n-2}} \int_{S^{n-2}} \phi(\xi, \theta') d\theta',$$

then $\hat{\phi}$ satisfies (2.10) and (2.11). Let ξ_1 be the first point such that

$$\hat{\phi}(\xi_1) = \frac{\beta(p,n) + \beta^{\frac{1}{p-1}}}{2}.$$

We know from (2.11) that

$$\hat{\phi}(\xi) > \frac{\beta(p,n) + \beta^{\frac{1}{p-1}}}{2} \quad \text{in} \quad (0, \xi_1).$$

We will assume that $\xi_1 < \frac{\pi}{2}$ since the case $\xi_1 < \frac{\pi}{2}$ can be dealt with similarly. By (2.10), we can get that

$$\begin{aligned}
\hat{\phi}(\xi_1) - \hat{\phi}(0) &= \int_0^{\xi_1} \frac{1}{\sin^{n-2}\xi} \int_0^{\xi} \sin^{n-2}\tau [\beta\hat{\phi}(\tau) - \frac{1}{\omega_{n-2}} \int_{S^{n-2}} \phi^p(\tau, \theta') d\theta'] d\tau d\xi \\
&\geq -(\alpha(p,n))^p \int_0^{\xi_1} \frac{1}{\sin^{n-2}\xi} \int_0^{\xi} \sin^{n-2}\tau d\tau d\xi \\
&\geq -\frac{\xi_1^2}{2} (\alpha(p,n))^p.
\end{aligned}$$

We deduce that

$$\xi_1 > (\beta(p,n) - \beta^{\frac{1}{p-1}})^{\frac{1}{2}}(\alpha(p,n))^{-\frac{p}{2}}. \tag{2.43}$$

Let

$$\overline{\phi} = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \phi d\theta.$$

By (2.1) and the Jensen's inequality, we can get that $\overline{\phi} \leq \beta^{\frac{1}{p-1}}$. Therefore,

$$\begin{aligned}
&\int_{S^{n-1}} (\phi - \overline{\phi})^2 d\theta \\
&= \int_0^{\pi} \int_{S^{n-2}} \sin^{n-2} \xi (\phi - \overline{\phi})^2 d\theta' d\xi \\
&\geq \omega_{n-2} \int_0^{\xi_1} \sin^{n-2} \xi (\hat{\phi} - \overline{\phi})^2 d\xi \\
&\geq \frac{\omega_{n-2}}{4(n-1)} (\frac{2}{\pi})^{n-2} (\beta(p,n) - \beta^{\frac{1}{p-1}})^{\frac{n+3}{2}} (\alpha(p,n))^{-\frac{p(n-1)}{2}}.
\end{aligned} \tag{2.44}$$

It follows from the Poincaré's inequality that

$$\int_{S^{n-1}} |\nabla_{S^{n-1}} \phi|^2 d\theta \geq \frac{\omega_{n-2}}{4} (\frac{2}{\pi})^{n-2} (\beta(p,n) - \beta^{\frac{1}{p-1}})^{\frac{n+3}{2}} (\alpha(p,n))^{-\frac{p(n-1)}{2}}.$$

Therefore,

$$\frac{\int_{S^{n-1}} |\nabla_{S^{n-1}} \phi|^2 d\theta}{\int_{S^{n-1}} \phi^2 d\theta} \geq \frac{\omega_{n-1} (\frac{\frac{(n-2)^2}{4} - \beta}{p-1})^{\frac{2}{p-1}}}{\frac{\omega_{n-2}}{4} (\frac{2}{\pi})^{n-2} (\beta(p,n) - \beta^{\frac{1}{p-1}})^{\frac{n+3}{2}} (\alpha(p,n))^{-\frac{p(n-1)}{2}}}.$$

By the above analysis, we know that (1.8) holds. □

Lemma 2.9. *Let ϕ be a positive solution of* (2.1) *such that* (2.2) *holds. If*

$$\|\phi\|_{L^\infty(S^{n-1})} \geq (\frac{2n+\beta}{p})^{\frac{1}{p-1}},$$

then ϕ is a constant when $p_{JL}(n) < p < p_{cs}(n)$, where $p_{cs}(n)$ is defined by (1.9).

Proof. By (2.38) and (2.39), we have

$$(p-1) \int_{S^{n-1}} |\nabla_{S^{n-1}} \phi|^2 d\theta \leq \int_{S^{n-1}} (\frac{(n-2)^2}{4} - p\beta) \phi^2 d\theta. \tag{2.45}$$

Let ϕ be a nonconstant solution of (2.1) satisfying (2.2), we know from Lemma 2.7 that ϕ satisfies (2.28). By combining (2.28) and (2.45) together, we can get that

$$(p-1) \int_{S^{n-1}} |\nabla_{S^{n-1}} \phi|^2 d\theta \leq (\frac{(n-2)^2}{4} - p\beta) c_s(p,n) \int_{S^{n-1}} |\nabla_{S^{n-1}} \phi|^2 d\theta. \tag{2.46}$$

It follows from (2.46) that

$$\int_{S^{n-1}} |\nabla_{S^{n-1}} \phi|^2 d\theta = 0$$

when $p_{JL}(n) < p < p_{cs}(n)$. Since we have assumed that ϕ is a nonconstant solution of (2.1), this is a contradiction. □

Proof of Theorem 2.1: This result follows from Lemma 2.4, Lemma 2.5 and Lemma 2.9. □

Remark 2.10. *In this section, we always assume that ϕ is positive. But we can prove that if ϕ is a solution of* (2.1) *depends only on the variable ξ, then ϕ does not change sign. The proof of this fact will be given in the appendix.*

Remark 2.11. *It is proved in [29] that if $n \geq 4$ and $(n+1)/(n-3) < p < p_{JL}(n-1)$, then* (2.1) *has a nonconstant positive solution.*

Remark 2.12. *By Lemma 1 in [30], we have the following Hardy type inequality,*

$$\int_{S^{n-1}} |\nabla_{S^{n-1}}\phi|^2 d\theta + \frac{(n-2)^2}{4}\int_{S^{n-1}} \phi^2 d\theta \geq \frac{(n-3)^2}{4}\int_{S^{n-1}} \frac{\phi^2}{\sin^2\xi} d\theta. \tag{2.47}$$

The equation (2.1) *has a singular solution which is given by*

$$\phi_*(\xi) = [\frac{2}{p-1}(n-3-\frac{2}{p-1})]^{\frac{1}{p-1}}(\sin\xi)^{-\frac{2}{p-1}} = \beta_*(\sin\xi)^{-\frac{2}{p-1}}.$$

Suppose ϕ_* satisfies (2.2)*, then*

$$\int_{S^{n-1}} |\nabla_{S^{n-1}}\phi|^2 d\theta + \frac{(n-2)^2}{4}\int_{S^{n-1}} \phi^2 d\theta \geq p\beta_*^{p-1}\int_{S^{n-1}} \frac{\phi^2}{\sin^2\xi} d\theta. \tag{2.48}$$

If $p = p_{JL}(n-1)$, then

$$\frac{2p}{p-1}(n-3-\frac{2}{p-1}) = \frac{(n-3)^2}{4}.$$

Let us define

$$g(p) = \frac{2p}{p-1}(n-3-\frac{2}{p-1}),$$

then

$$g'(p) = \frac{-2}{(p-1)^2}(n-5-\frac{4}{p-1}).$$

If $p > (n-1)/(n-5)$, then $g'(p) < 0$. Therefore, the singular solution ϕ_* satisfies (2.2) *if $p \geq p_{JL}(n-1)$.*

3. Qualitative properties of stable solutions

In this section, we give the proof of Theorem 1.2 and Theorem 1.3.

Lemma 3.1. *Let $n \geq 11, p_{JL}(n) < p < p_{si}(n)$, where $p_{si}(n)$ is defined in* (1.12)*. If ϕ is a nontrivial solution of* (2.1) *such that* (2.2) *holds, then ϕ does not change sign.*

Proof. We assume that ϕ change sign. Without loss of generality, we can assume that there exists a connected component Ω_1 of $\{\phi > 0\}$ such that $\lambda_1(\Omega_1) \geq n-1$, where $\lambda_1(\Omega_1)$ is the first eigenvalue of the eigenvalue problem

$$\begin{cases} \Delta_{S^{n-1}}\Phi + \lambda\Phi = 0 & \text{in} \quad \Omega_1, \\ \Phi = 0 & \text{on} \quad \partial\Omega_1. \end{cases}$$

Multiplying the both sides of (2.1) by ϕ and integrating over Ω_1, we can get that

$$\int_{\Omega_1} |\nabla_{S^{n-1}}\phi|^2 d\theta + \beta \int_{\Omega_1} \phi^2 d\theta = \int_{\Omega_1} |\phi|^{p+1} d\theta. \tag{3.1}$$

We take $\psi = u 1_{\Omega_1}$ into (2.2), where 1_{Ω_1} is the function defined by

$$1_{\Omega_1} = \begin{cases} 1 & \text{in} \quad \Omega_1 \\ 0 & \text{on} \quad S^{n-1}\backslash\Omega_1. \end{cases}$$

Then

$$\int_{\Omega_1} |\nabla_{S^{n-1}}\phi|^2 d\theta + \frac{(n-2)^2}{4} \int_{\Omega_1} \phi^2 d\theta \geq p \int_{\Omega_1} |\phi|^{p+1} d\theta. \tag{3.2}$$

By (3.1) and (3.2), we know that

$$(p-1)\int_{\Omega_1} |\nabla_{S^{n-1}}\phi|^2 d\theta \leq \frac{1}{\lambda_1(\Omega_1)} \int_{\Omega_1} (\frac{(n-2)^2}{4} - p\beta)|\nabla_{S^{n-1}}\phi|^2 d\theta. \tag{3.3}$$

It follows that if $p_{JL}(n) < p < p_{si}(n)$, then ϕ vanishes identically on Ω_1. Since we have assumed that $\phi > 0$ on Ω_1, this is a contradiction. □

Proof of Theorem 1.3: We consider the transform

$$u(r,\theta) = r^{-\frac{2}{p-1}} w(t,\theta), \quad t = \ln r.$$

Since u satisfies (1.1), then w is a bounded solution of the equation

$$\partial_{tt} w + (n-2-\frac{4}{p-1})\partial_t w + \Delta_{S^{n-1}} w - \frac{2}{p-1}(n-2-\frac{2}{p-1})w + |w|^{p-1}w = 0. \tag{3.4}$$

We set

$$\begin{aligned} A &= n-2-\frac{4}{p-1}, \\ B &= -\frac{2}{p-1}(n-2-\frac{2}{p-1}), \\ E(w) &= \int_{S^{n-1}} \frac{1}{2}|\nabla_{S^{n-1}} w|^2 - \frac{B}{2}w^2 - \frac{1}{p+1}|w|^{p+1} d\theta. \end{aligned} \tag{3.5}$$

By (3.4), we get that

$$A\int_{S^{n-1}} (\partial_t w)^2 d\theta = \frac{d}{dt}[E(w)(t) - \frac{1}{2}\int_{S^{n-1}} (\partial_t w)^2 d\theta]. \tag{3.6}$$

By the estimates in [19], we can get that $\partial_t w, \partial_{tt} w, |\nabla_{S^{n-1}}|$ are uniformly bounded. Integrating (3.6) from $-s$ to s, we find

$$A\int_{-s}^{s}\int_{S^{n-1}} (\partial_t w)^2 d\theta dt < c \tag{3.7}$$

for some constant c independent of s. Let s tend to $+\infty$ in (3.7), then

$$A\int_{-\infty}^{+\infty}\int_{S^{n-1}} (\partial_t w)^2 d\theta dt = 0.$$

Similar to the proof of Theorem 1.4 in [1], we can obtain that

$$\lim_{t\to+\infty}\int_{S^{n-1}}(\partial_t w)^2 d\theta = 0. \tag{3.8}$$

For any sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$, we consider the translation of w defined by $w_k(t,\theta) = w(t + t_k, \theta)$. Then there exist a subsequence $\{w_{l_k}(t,\theta)\}$ and a function $w_\infty(t,\theta)$ such that $w_{l_k}(t,\theta) \to w_\infty(t,\theta)$ in $C^2([-1,1]\times S^{n-1})$. By (3.8) and the dominated convergence theorem, we know that there exists a function $\phi(\theta)$ such that $w_\infty(t,\theta) = \phi(\theta)$. Moreover, ϕ is a solution of (2.1) such that (2.2) holds. If $\phi = 0$, then $\lim_{t\to+\infty} E(w)(t) = 0$. But we also have $\lim_{t\to-\infty} E(w)(t) = 0$ since u is regular at the origin. It follows easily that $w \equiv 0$. Since we have assumed that u is a nontrivial solution, this is a contradiction. Therefore ϕ is not zero. If $\phi \neq 0$, we know from Lemma 3.1 that ϕ does not change sign. Suppose there exist two sequences $\{t_k\}$ and $\{\tilde{t}_k\}$ such that

$$\lim_{k\to\infty} w(t_k,\theta) < 0$$

and

$$\lim_{k\to\infty} w(\tilde{t}_k,\theta) > 0,$$

then $\{u \neq 0\}$ has a bounded connected component. Without loss of generality, we can assume there exists a bounded connected component Ω_- such that $u < 0$ on Ω_-. Then u satisfies the equation

$$\begin{cases} \Delta u + |u|^{p-1}u = 0 & \text{in} \quad \Omega_-, \\ u = 0 & \text{on} \quad \partial\Omega_-. \end{cases} \tag{3.9}$$

Since u is a stable solution of (1.1), then $L = \Delta + p|u|^{p-1}$ satisfies the refined maximum principle (see [31]). Since

$$\begin{cases} Lu = (p-1)|u|^{p-1}u \leq 0 & \text{in} \quad \Omega_-, \\ u = 0 & \text{on} \quad \partial\Omega_-, \end{cases} \tag{3.10}$$

we get from the refined maximum principle that $u \geq 0$ on Ω_-. In view of the definition of Ω_-, we get a contradiction. By the above arguments, we know that there exits a positive constant R_0 such that u doesn't change sign on $\mathbb{R}^n \backslash B_{R_0}$. By applying the refined maximum principle again, we know that u does not change sign.

If u is axially symmetric, then the proof is essentially the same as the arguments above. The only difference is that we need to use remark 2.10 rather than Lemma 3.1 to show that there exits a positive constant R_0 such that u doesn't change sign on $\mathbb{R}^n \backslash B_{R_0}$. □

Proof of Theorem 1.2. Let $n \geq 11$, $p_{JL}(n) < p < p_{cs}(n)$ and let u be a positive stable solution of (1.1). Let us consider the transform

$$u(r,\theta) = r^{-\frac{2}{p-1}} w(t,\theta), \quad t = \ln r.$$

Since u satisfies (1.1), then w is a bounded solution of Eq (3.4). Let $\{t_k\}$ be a sequence such that $t_k \to \infty$ as $k \to \infty$. Similar to the arguments used in the proof of Theorem 1.3, we know that there exists a function $\phi \in H^1(S^{n-1})$ such that

$$\lim_{|k|\to\infty} w(t_k,\theta) = \phi.$$

Moreover, ϕ is a solution of (2.1) such that (2.2) holds. By Lemma 2.2, we have $\phi \in C^2(S^{n-1})$. If u is even symmetric with respect to the $\{x_i = 0\}, i = 1, 2, \cdots n$, then

$$\int_{S^{n-1}} \phi \Phi_i d\theta = 0 \quad \text{for} \quad i = 1, 2, \cdots, n.$$

Since $p_{JL}(n) < p < p_{cs}(n)$, we know from Theorem 2.1 that ϕ is a constant function. In particular, we have

$$\phi = [\frac{2}{p-1}(n-2-\frac{2}{p-1})]^{\frac{1}{p-1}}.$$

Since the sequence $\{t_k\}$ can be arbitrary, we conclude that

$$\lim_{|x|\to\infty} |x|^{\frac{2}{p-1}} u(x) = [\frac{2}{p-1}(n-2-\frac{2}{p-1})]^{\frac{1}{p-1}}.$$

Since $p > p_{JL}(n)$, then $p > n/(n-4)$. By Theorem 4.4 in [5], we can get that

$$u(x) = r^{-\frac{2}{p-1}}((-B)^{\frac{1}{p-1}} + \xi(r) + \frac{v(r,\theta)}{r}), \tag{3.11}$$

where

$$\xi(r) = r^{\frac{2}{p-1}} \overline{u}(r) - (-B)^{\frac{1}{p-1}} \tag{3.12}$$

and

$$\overline{u}(r) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} u(r,\theta) d\theta.$$

Moreover, for any integer $\tau \geq 0$, we have $v(r, \theta)$ satisfies

$$v(r,\theta) \to V(\theta) \quad \text{as} \quad r \to 0 \tag{3.13}$$

uniformly in $C^\tau(S^{n-1})$, where V equals either zero or a first eigenfunctions of the operator $-\Delta_{S^{n-1}}$. Since we have obtained the asymptotic expansion (3.11) which is good enough to apply the moving plane method, then the rest of the proof is essentially the same as the proof of Theorem 1.1 in [4]. □

4. The proof of Theorem 1.5

In this section, we give the proof of Theorem 1.5, the proof is mainly based on the following observation.

Proposition 4.1. *Let $n = 10$ and let u be a smooth stable solution of Eq* (1.2)*, then*

$$\lim_{|x|\to\infty} u(x) + 2\ln(|x|) - \ln(16) = 0. \tag{4.1}$$

In order to prove Proposition 4.1, we first recall a monotonicity formula.

Lemma 4.2. *If u is a solution of the equation* (1.2)*, then*

$$\frac{dE}{d\rho} = \rho^{2-n} \int_{\partial B_\rho} (\frac{\partial u}{\partial \rho} + \frac{2}{\rho})^2 d\theta, \tag{4.2}$$

where

$$E(\rho, u) = \rho^{2-n} \int_{B_\rho} (\frac{1}{2}|\nabla u|^2 - e^u)dx - 2\rho^{1-n} \int_{\partial B_\rho} (u + 2\ln(\rho))d\theta.$$

Moreover, if u is a smooth stable solution of (1.1)*, then*

$$\lim_{\rho\to+\infty} E(\rho, u) < +\infty. \tag{4.3}$$

Proof. The proof of (4.2) follows from a scaling argument which is similar to the proof Proposition 5.1 in [32]. The proof of (4.3) follows easily from the capacity estimates in [33]. □

With the help of Lemma 4.2, we can give the proof of Proposition 4.1.

proof of Proposition 4.1. The proof of Proposition 4.1 will consist of the following four steps.

Step 1: Let $\{\lambda_k\}$ be a sequence such that $\lim_{k\to+\infty} \lambda_k = +\infty$. For any λ_k, we define $u^{\lambda_k}(x) = u(\lambda_k x) + 2\ln(\lambda_k)$. It is easy to check that $u^{\lambda_k}(x)$ is also a stable solution of (1.1). By the capacity estimates (see for instance [33]), we know that $u^{\lambda_k} \to u^\infty$ for some function $u^\infty \in H^1_{loc}(\mathbb{R}^n)$. Moreover, u^∞ is a stable solution of (1.1).

Step 2: For any $0 < R_1 < R_2 < +\infty$, by Lemma 4.2,

$$\lim_{k\to+\infty} E(\lambda_k R_2; 0, u) - E(\lambda_k R_1; 0, u) = 0. \tag{4.4}$$

By the scaling invariance of E, we have

$$\lim_{k\to+\infty} E(R_2; 0, u^{\lambda_k}) - E(R_1; 0, u^{\lambda_k}) = 0. \tag{4.5}$$

We use Lemma 4.2 again, then

$$\begin{aligned} 0 &= \lim_{k\to+\infty} E(R_2; 0, u^{\lambda_k}) - E(R_1; 0, u^{\lambda_k}) \\ &= \lim_{k\to+\infty} \int_{B_{R_2}\setminus B_{R_1}} |x|^{2-n}(\frac{\partial u^{\lambda_k}}{\partial r} + \frac{2}{|x|})^2 dx \\ &\geq \int_{B_{R_2}\setminus B_{R_1}} |x|^{2-n}(\frac{\partial u^{\lambda_\infty}}{\partial r} + \frac{2}{|x|})^2 dx. \end{aligned} \tag{4.6}$$

Therefore,

$$\frac{2}{r} + \frac{\partial u^\infty}{\partial r} = 0 \quad a.e. \quad \text{in} \quad \mathbb{R}^N. \tag{4.7}$$

It follows that there exists a function $\phi \in H^1(S^{n-1})$ such that $u^\infty = \phi - 2\ln(r)$. Moreover, ϕ satisfies the equation

$$\Delta_{S^{n-1}}\phi - 2(n-2) + e^\phi = 0. \tag{4.8}$$

Step 3: For every $\delta > 0$, we choose a function $\eta_\delta \in C_0^\infty((\frac{\delta}{2}, \frac{2}{\delta}))$ such that $\eta_\delta \equiv 1$ in $(\delta, \frac{1}{\delta})$, and $r|\eta'_\delta(r)| \leq 4$. For every $\psi \in H^1(S^{n-1})$, we define $\psi_\delta = r^{-\frac{n-2}{2}}\psi(\theta)\eta_\delta(r)$. For every $\psi \in H^1(S^{n-1})$, we define $\psi_\delta =$

$r^{-\frac{n-2}{2}}\psi(\theta)\eta_\delta(r)$. Since u^∞ is stable, we have

$$\begin{aligned}&\int_{S^{n-1}} e^\phi\psi^2 d\theta\int_0^{+\infty} r^{-1}\eta_\delta^2 dr\\ \leq &\int_{S^{n-1}}\psi^2 d\theta\int_0^\infty r^{n-1}(\eta_\delta' r^{-\frac{n-2}{2}}-\frac{n-2}{2}r^{-\frac{n}{2}}\eta_\delta)^2 dr\\ &+\int_{S^{n-1}}|\nabla_{S^{n-1}}\psi|^2 d\theta\int_0^\infty r^{n-1}(\eta_\delta r^{-\frac{n}{2}})^2 dr\end{aligned}$$

Therefore, ϕ satisfies

$$\int_{S^{n-1}}|\nabla_{S^{n-1}}\psi|^2 d\theta+\frac{(n-2)^2}{4}\int_{S^{n-1}}\psi^2 d\theta\geq\int_{S^{n-1}}e^\phi\psi^2 d\theta \tag{4.9}$$

for every $\psi\in H^1(S^{n-1})$.

Step 4: We take $\psi=e^{\frac{\phi}{2}}$ into (4.9), then

$$\frac{1}{4}\int_{S^{n-1}}e^\phi|\nabla_{S^{n-1}}\phi|^2 d\theta+\frac{(n-2)^2}{4}\int_{S^{n-1}}e^\phi d\theta\geq\int_{S^{n-1}}e^{2\phi}d\theta. \tag{4.10}$$

Multiplying the both sides of (4.8) by e^ϕ and using integration by part, we have

$$\frac{1}{2}\int_{S^{n-1}}e^\phi|\nabla_{S^{n-1}}\phi|^2 d\theta+2(n-2)\int_{S^{n-1}}e^\phi d\theta=\int_{S^{n-1}}e^{2\phi}d\theta. \tag{4.11}$$

If $n=10$, then $(n-2)^2/4=2(n-2)$. By (4.10) and (4.11), we can get that

$$\int_{S^{n-1}}e^\phi|\nabla_{S^{n-1}}\phi|^2 d\theta\leq 0. \tag{4.12}$$

It follows from (4.12) that $\phi=\ln(16)$ is a constant. Since $\{\lambda_k\}$ can be arbitrary, we can obtain that proposition 4.1 holds. □

Proof of Theorem 1.5. It follows from proposition 4.1 and Theorem 1.3 in [6]. □

Appendix 1: A Liouville type result

In this appendix, we prove the claim in remark 2.10. The proof is based on the the following result.

Proposition 4.3. *Let $p\geq\frac{n+1}{n-3}$ and $(p-1)\mu\geq n-1$. If ϕ is a solution of the equation*

$$\begin{cases}(\frac{1+|x|^2}{2})^{n-1}\mathrm{div}((\frac{2}{1+|x|^2})^{n-3}\nabla\phi)-\mu\phi+|\phi|^{p-1}\phi=0 & in\ B_r,\\ \phi=0 & on\ \partial B_r,\end{cases} \tag{4.13}$$

where $B_r\subset\mathbb{R}^{n-1}$ is a ball and $0<r<1$, then $\phi=0$.

Proof. Multiplying the both sides of (4.13) by $(\frac{2}{1+|x|^2})^{n-1}\phi$ and using integration by part, we can get that

$$\int_{B_r}|\nabla\phi|^2(\frac{2}{1+|x|^2})^{n-3}+\mu\int_{B_r}\phi^2(\frac{2}{1+|x|^2})^{n-1}=\int_{B_r}|\phi|^{p+1}(\frac{2}{1+|x|^2})^{n-1}. \tag{4.14}$$

Multiplying the both sides of (4.13) by $(\frac{2}{1+|x|^2})^{n-1}(x\cdot\nabla\phi)$ and using integration by part, we can get that

$$\begin{aligned}h(r)\int_{\partial B_r}|\nabla\phi|^2 &= \int_{B_r}(\frac{2}{1+|x|^2})^{n-3}\nabla\phi\nabla(x\cdot\nabla\phi)+\mu\int_{B_r}(\frac{2}{1+|x|^2})^{n-1}\phi(x\cdot\nabla\phi)\\ &\quad-\int_{B_r}(\frac{2}{1+|x|^2})^{n-1}|\phi|^{p-1}\phi(x\cdot\nabla\phi)\\ &=\frac{h(r)}{2}\int_{\partial B_r}|\nabla\phi|^2+\frac{3-n}{2}\int_{B_r}(\frac{2}{1+|x|^2})^{n-3}|\nabla\phi|^2\\ &\quad-\frac{(n-1)\mu}{2}\int_{B_r}(\frac{2}{1+|x|^2})^{n-1}\phi^2+\frac{n-1}{p+1}\int_{B_r}(\frac{2}{1+|x|^2})^{n-1}|\phi|^{p+1}\\ &\quad-\frac{1}{2}\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-3}|\nabla\phi|^2-\frac{\mu}{2}\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-1}\phi^2\\ &\quad+\frac{1}{p+1}\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-1}|\phi|^{p+1},\end{aligned}$$

where

$$h(r)=r(\frac{2}{1+r^2})^{n-3}.$$

It follows that

$$\begin{aligned}&\frac{3-n}{2}\int_{B_r}(\frac{2}{1+|x|^2})^{n-3}|\nabla\phi|^2-\frac{(n-1)\mu}{2}\int_{B_r}(\frac{2}{1+|x|^2})^{n-1}\phi^2\\ &+\frac{n-1}{p+1}\int_{B_r}(\frac{2}{1+|x|^2})^{n-1}|\phi|^{p+1}-\frac{1}{2}\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-3}|\nabla\phi|^2\\ &-\frac{\mu}{2}\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-1}\phi^2+\frac{1}{p+1}\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-1}|\phi|^{p+1}\\ &=\frac{h(r)}{2}\int_{\partial B_r}|\nabla\phi|^2.\end{aligned} \tag{4.15}$$

Multiplying the both sides of (4.13) by $x\cdot\nabla(\frac{2}{1+|x|^2})^{n-1}\phi$ and using integration by part, we can get that

$$\begin{aligned}0&=-(n-1)\int_{B_r}(\frac{1+|x|^2}{2})^{n-1}\mathrm{div}((\frac{2}{1+|x|^2})^{n-3}\nabla\phi)(|x|^2(\frac{2}{1+|x|^2})^{n})\\ &\quad-\mu\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-1}\phi^2+\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-1}|\phi|^{p+1}\\ &=-(n-1)\int_{B_r}\frac{2|x|^2}{1+|x|^2}\phi\,\mathrm{div}((\frac{2}{1+|x|^2})^{n-3}\nabla\phi)\\ &\quad-\mu\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-1}\phi^2+\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-1}|\phi|^{p+1}.\end{aligned} \tag{4.16}$$

By some computations, we can get that

$$
\begin{aligned}
&-(n-1)\int_{B_r}\frac{2|x|^2}{1+|x|^2}\phi\mathrm{div}((\frac{2}{1+|x|^2})^{n-3}\nabla\phi)\\
&=(n-1)\int_{B_r}(\frac{2}{1+|x|^2})^{n-3}(\nabla(\frac{2|x|^2}{1+|x|^2}\phi))\nabla\phi \qquad (4.17)\\
&=-\frac{n-1}{n-3}\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-3}|\nabla\phi|^2+\frac{n-1}{2(n-2)}\int_{B_r}\Delta(\frac{2}{1+|x|^2})^{n-2}\phi^2,
\end{aligned}
$$

By (4.16) and (4.17), we have

$$
\begin{aligned}
0=&-\frac{n-1}{2}\int_{B_r}[x\cdot\nabla(\frac{2}{1+|x|^2})^{n-1}+(n-1)(\frac{2}{1+|x|^2})^{n-1}]\phi^2\\
&-\mu\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-1}\phi^2+\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-1}|\phi|^{p+1} \qquad (4.18)\\
&-\frac{n-1}{n-3}\int_{B_r}x\cdot\nabla(\frac{2}{1+|x|^2})^{n-3}|\nabla\phi|^2.
\end{aligned}
$$

We combine (4.14), (4.15) and (4.18) in the following way:

$$
(4.14)\times\frac{n-1}{p+1}+(4.15)-\frac{1}{p+1}\times(4.18),
$$

then

$$
\begin{aligned}
\frac{h(r)}{2}\int_{\partial B_r}|\nabla\phi|^2=&(\frac{n-1}{p+1}-\frac{n-3}{2})\int_{B_r}(\frac{2}{1+|x|^2})^{n-3}\frac{1-|x|^2}{1+|x|^2}|\nabla\phi|^2\\
&+\frac{n-1}{2(p+1)}(n-1-(p-1)\mu)\int_{B_r}(\frac{2}{1+|x|^2})^{n-1}\frac{1-|x|^2}{1+|x|^2}\phi^2.
\end{aligned}
$$

If $p\geq\frac{n+1}{n-3}$ and $(p-1)\mu\geq(n-1)$, then the left hand side of the last identity will become non-positive, therefore, Eq (4.13) has only trivial solution. □

Corollary 4.4. *If* $p\geq\frac{n+1}{n-3}$ *and if* ϕ *is a nontrivial solution of Eq* (2.1) *depends only on the variable* ξ, *here we use the coordinates in the proof of Lemma 2.3, then* ϕ *does not change sign.*

Proof. If ϕ change sign, then there exists $0<r<1$ such that (4.13) has a nontrivial solution, this is a contradiction. □

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Conflict of interest

We declare that we have no financial and personal relationships with other people or organizations that can inappropriately influence our work.

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