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*Research article*

# On a parabolic partial differential equation and system modeling a production planning problem

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**Abstract:** We consider a parabolic partial differential equation and system derived from a production planning problem dependent on time. Our goal is to find a closed-form solution for the problem considered in our model. Our new theoretical results can be applied in the real world.

**Keywords:** numerical approximation; asymptotic behavior; closed solution; parabolic equation; parabolic system

## 1. Introduction

Several years ago, Bensoussan, Sethi, Vickson and Derzko [1] have been considered the case of a factory producing one type of economic goods and observed that it is necessary to solve the simple partial differential equation

$$\begin{cases} -\frac{\sigma^2}{2}\Delta z_s^\alpha + \frac{1}{4}\left|\nabla z_s^\alpha\right|^2 + \alpha z_s^\alpha = |x|^2 \text{ for } x \in \mathbb{R}^N, \\ z_s^\alpha = \infty \text{ as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $\sigma \in (0, \infty)$ denotes the diffusion coefficient, $\alpha \in [0, \infty)$ represents psychological rate of time discount, $x \in \mathbb{R}^N$ is the product vector, $z := z_s^\alpha(x)$ denotes the value function and $|x|^2$ is the loss function.

Regime switching refers to the situation when the characteristics of the state process are affected by several regimes (e.g., in finance bull and bear market with higher volatility in the bear market).

It is important to point out that, when dealing with regime switching, we can describe a wide variety of phenomena using partial differential equations. In [1], the authors Cadenillas, Lakner and Pinedo [2] adapted the model problem in [1] to study the optimal production management characterized by the two-state regime switching with limited/unlimited information and corresponding to the system

$$\begin{cases} -\frac{\sigma_1^2}{2}\Delta u_1^s + (a_{11}+\alpha_1)\, u_1^s - a_{11}u_2^s - \rho\frac{\sigma_1^2}{2}\sum_{i\neq j}\frac{\partial^2 u_1^s}{\partial x_i \partial x_j} - |x|^2 = -\frac{1}{4}\left|\nabla u_1^s\right|^2, x\in\mathbb{R}^N, \\ -\frac{\sigma_2^2}{2}\Delta u_2^s + (a_{22}+\alpha_2)\, u_2^s - a_{22}u_1^s - \rho\frac{\sigma_2^2}{2}\sum_{i\neq j}\frac{\partial^2 u_2^s}{\partial x_i \partial x_j} - |x|^2 = -\frac{1}{4}\left|\nabla u_2^s\right|^2, x\in\mathbb{R}^N, \\ u_1^s(x) = u_2^s(x) = \infty \text{ as } |x|\to\infty, \end{cases} \tag{1.2}$$

where $\sigma_1, \sigma_2 \in (0,\infty)$ denote the diffusion coefficients, $\alpha_1, \alpha_2 \in [0,\infty)$ represent the psychological rates of time discount from what place the exponential discounting, $x \in \mathbb{R}^N$ is the product vector, $u_r^s := u_r^s(x)$ $(r = 1,2)$ denotes the value functions, $|x|^2$ is the loss function, $\rho \in [-1,1]$ is the correlation coefficient and $a_{nm}$ $(n, m = 1,2)$ are the elements of the Markov chain's rate matrix, denoted by $G = [\vartheta_{nm}]_{2\times 2}$ with

$$\vartheta_{nn} = -a_{nn} \leq 0, \vartheta_{nm} = a_{nm} \geq 0 \text{ and } \vartheta_{nn}^2 + \vartheta_{nm}^2 \neq 0 \text{ for } n \neq m,$$

the diagonal elements $\vartheta_{nn}$ may be expressed as $\vartheta_{nn} = -\sum_{m\neq n}\vartheta_{nm}$.

Furthermore, in civil engineering, Dong, Malikopoulos, Djouadi and Kuruganti [3] applied the model described in [2] to the study of the optimal stochastic control problem for home energy systems with solar and energy storage devices; the two regimes switching are the peak and the peak energy demands.

After that, there have been numerous applications of regime switching in many important problems in economics and other fields, see the works of: Capponi and Figueroa-López [4], Elliott and Hamada [5], Gharbi and Kenne [6], Yao, Zhang and Zhou [7] and Wang, Chang and Fang [8] for more details. Other different research studies that explain the importance of regime switching in the real world are [9, 10].

In this paper, we focus on the following parabolic partial differential equation and system, corresponding to (1.1)

$$\begin{cases} \frac{\partial z}{\partial t}(x,t) - \frac{\sigma^2}{2}\Delta z(x,t) + \frac{1}{4}|\nabla z(x,t)|^2 + \alpha z(x,t) = |x|^2, \ (x,t)\in\mathbb{R}^N\times(0,\infty), \\ z(x,0) = c + z_s^\alpha(x), \text{ for all } x\in\mathbb{R}^N \text{ and fixed } c\in(0,\infty), \\ z(x,t) = \infty \text{ as } |x|\to\infty, \text{ for all } t\in[0,\infty), \end{cases} \tag{1.3}$$

and (1.2) respectively

$$\begin{cases} \frac{\partial u_1}{\partial t} - \frac{\sigma_1^2}{2}\Delta u_1 + (a_{11}+\alpha_1)\, u_1 - a_{11}u_2 - \rho\frac{\sigma_1^2}{2}\sum_{i\neq j}\frac{\partial^2 u_1}{\partial x_i\partial x_j} - |x|^2 = -\frac{1}{4}|\nabla u_1|^2, (x,t)\in\mathbb{R}^N\times(0,\infty), \\ \frac{\partial u_2}{\partial t} - \frac{\sigma_2^2}{2}\Delta u_2 + (a_{22}+\alpha_2)\, u_2 - a_{22}u_1 - \rho\frac{\sigma_2^2}{2}\sum_{i\neq j}\frac{\partial^2 u_2}{\partial x_i\partial x_j} - |x|^2 = -\frac{1}{4}|\nabla u_2|^2, (x,t)\in\mathbb{R}^N\times(0,\infty), \\ (u_1(x,0), u_2(x,0)) = \left(c_1 + u_1^s(x), c_2 + u_2^s(x)\right) \text{ for all } x\in\mathbb{R}^N \text{ and for fixed } c_1, c_2\in(0,\infty), \\ \frac{\partial u_1}{\partial t}(x,t) = \frac{\partial u_2}{\partial t}(x,t) = \infty \text{ as } |x|\to\infty \text{ for all } t\in[0,\infty), \end{cases} \tag{1.4}$$

where $z_s^\alpha$ is the solution of (1.1) and $\left(u_1^s(x), u_2^s(x)\right)$ is the solution of (1.2). The existence and the uniqueness for the case of (1.1) is proved by [10] and the existence for the system case of (1.2) by [11].

From the mathematical point of view the problem (1.3) has been extensively studied when the space $\mathbb{R}^N$ is replaced by a bounded domain and when $\alpha = 0$. In particular, some great results can be found in the old papers of Barles, Porretta [12] and Tchamba [13]. More recently, but again for the case of a bounded domain, $\alpha = 0$ and in the absence of the gradient term, the problem (1.3) has been also discussed by Alves and Boudjeriou [14]. The interest of these authors [12–14] is to give an asymptotic stable solution at infinity for the considered equation, i.e., a solution which tends to the stationary Dirichlet problem associated with (1.3) when the time go to infinity.

Next, we propose to find a similar result as of [12–14], for the case of equation (1.3) and system (1.4) that model some real phenomena. More that, our first interest is to provide a closed form solution for (1.3) and (1.4). Our second objective is inspired by the paper of [14, 15], and it is to solve the parabolic partial differential equation

$$\begin{cases} \frac{\partial z}{\partial t}(x,t) - \frac{\sigma^2}{2}\Delta z(x,t) + \frac{1}{4}|\nabla z(x,t)|^2 = |x|^2, & \text{in} \quad B_R \times [0,T), \\ z(x,T) = 0, & \text{for} \quad |x| = R, \end{cases} \tag{1.5}$$

where $T < \infty$ and $B_R$ is a ball of radius $R > 0$ with origin at the center of $\mathbb{R}^N$.

Let us finish our introduction and start with the main results.

## 2. The main results

We use the change of variable

$$u(x,t) = e^{-\frac{z(x,t)}{2\sigma^2}}, \tag{2.1}$$

in

$$\frac{\partial z}{\partial t}(x,t) - \frac{\sigma^2}{2}\Delta z(x,t) + \frac{1}{4}|\nabla z(x,t)|^2 + \alpha z(x,t) = |x|^2$$

to rewrite (1.3) and (1.5) in an equivalent form

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \frac{\sigma^2}{2}\Delta u(x,t) + \alpha u(x,t)\ln u(x,t) + \frac{1}{2\sigma^2}|x|^2 u(x,t) = 0, & \text{if} \quad (x,t) \in \Omega \times (0,T) \\ u(x,T) = u_{1,0}, & \text{on} \quad \partial\Omega, \\ u(x,0) = e^{-\frac{c+z_s^{\alpha}(x)}{2\sigma^2}}, & \text{for} \quad x \in \Omega = \mathbb{R}^N, c \in (0,\infty) \end{cases} \tag{2.2}$$

where

$$u_{1,0} = \begin{cases} 1 & \text{if} \quad \Omega = B_R, \text{ i.e., } |x| = R, T < \infty, \\ 0 & \text{if} \quad \Omega = \mathbb{R}^N, \text{ i.e., } |x| \to \infty, T = \infty. \end{cases}$$

Our first result is the following.

**Theorem 2.1.** *Assume $\Omega = B_R$, $N \geq 3$, $T < \infty$ and $\alpha = 0$. There exists a unique radially symmetric positive solution*

$$u(x,t) \in C^2(B_R \times [0,T)) \cap C\left(\overline{B}_R \times [0,T]\right),$$

*of (2.2) increasing in the time variable and such that*

$$\lim_{t \to T} u(x,t) = u_s(x), \tag{2.3}$$

*where $u_s \in C^2(B_R) \cap C\left(\overline{B}_R\right)$ is the unique positive radially symmetric solution of the Dirichlet problem*

$$\begin{cases} \frac{\sigma^2}{2}\Delta u_s = \left(\frac{1}{2\sigma^2}|x|^2 + 1\right)u_s, & \text{in} \quad B_R, \\ u_s = 1, & \text{on} \quad \partial B_R, \end{cases} \tag{2.4}$$

*which will be proved. In addition,*

$$z(x,t) = -2\sigma^2 (t-T) - 2\sigma^2 \ln u_s(|x|), (x,t) \in \overline{B}_R \times [0,T],$$

*is the unique radially symmetric solution of the problem (1.5).*

Instead of the existence results discussed in the papers of [12–14], in our proof of the Theorem 2.1 we give the numerical approximation of solution $u(x,t)$.

The next results refer to the entire Euclidean space $\mathbb{R}^N$ and present closed-form solutions.

**Theorem 2.2.** *Assume $\Omega = \mathbb{R}^N$, $N \geq 1$, $T = \infty$, $\alpha > 0$ and $c \in (0,\infty)$ is fixed. There exists a unique radially symmetric solution*

$$u(x,t) \in C^2\left(\mathbb{R}^N \times [0,\infty)\right),$$

*of (2.2), increasing in the time variable and such that*

$$u(x,t) \to u_s^{\alpha}(x) \text{ as } t \to \infty, \text{ for all } x \in \mathbb{R}^N, \tag{2.5}$$

*where $u_s^{\alpha} \in C^2\left(\mathbb{R}^N\right)$ is the unique radially symmetric solution of the stationary Dirichlet problem associated with (2.2)*

$$\begin{cases} \frac{\sigma^2}{2}\Delta u_s^{\alpha} = \alpha u_s^{\alpha} \ln u_s^{\alpha} + \frac{1}{2\sigma^2}|x|^2 u_s^{\alpha}, & \text{in} \quad \mathbb{R}^N, \\ u_s^{\alpha}(x) \to 0, & \text{as} \quad |x| \to \infty. \end{cases} \tag{2.6}$$

*Moreover, the closed-form radially symmetric solution of the problem (1.3) is*

$$z(x,t) = ce^{-\alpha t} + B|x|^2 + D, (x,t) \in \mathbb{R}^N \times [0,\infty), c \in (0,\infty), \tag{2.7}$$

*where*

$$B = \frac{1}{N\sigma^2}\left(\frac{1}{2}N\sigma^2\sqrt{\alpha^2+4} - \frac{1}{2}N\alpha\sigma^2\right), \quad D = \frac{1}{2\alpha}\left(N\sigma^2\sqrt{\alpha^2+4} - N\alpha\sigma^2\right). \tag{2.8}$$

The following theorem is our main result regarding the system (1.4).

**Theorem 2.3.** *Suppose that $N \geq 1$, $\alpha_1, \alpha_2 \in (0,\infty)$ and $a_{11}, a_{22} \in [0,\infty)$ with $a_{11}^2 + a_{22}^2 \neq 0$. Then, the system (1.4) has a unique radially symmetric convex solution*

$$(u_1(x,t), u_2(x,t)) \in C^2\left(\mathbb{R}^N \times [0,\infty)\right) \times C^2\left(\mathbb{R}^N \times [0,\infty)\right),$$

*of quadratic form in the x variable and such that*

$$(u_1(x,t), u_2(x,t)) \to (u_1^s(x), u_2^s(x)) \text{ as } t \to \infty \text{ uniformly for all } x \in \mathbb{R}^N, \tag{2.9}$$

*where*

$$(u_1^s(x), u_2^s(x)) \in C^2\left(\mathbb{R}^N\right) \times C^2\left(\mathbb{R}^N\right)$$

*is the radially symmetric convex solution of quadratic form in the x variable of the stationary system (1.2) which exists from the result of [11].*

Our results complete the following four main works: Bensoussan, Sethi, Vickson and Derzko [1], Cadenillas, Lakner and Pinedo [2], Canepa, Covei and Pirvu [15] and Covei [10], which deal with a stochastic control model problem with the corresponding impact for the parabolic case (see [13, 16] for details).

*2.1. **An auxiliary result***

To prove our Theorem 2.1, we use a lower and upper solution method and the comparison principle that can be found in [17].

**Lemma 2.1.** *If, there exist $\overline{u}(x)$, $\underline{u}(x) \in C^2(B_R) \cap C\left(\overline{B}_R\right)$ two positive functions satisfying*

$$\begin{cases} -\frac{\sigma^2}{2}\Delta\overline{u}(x) + \left(\frac{1}{2\sigma^2}|x|^2 + 1\right)\overline{u}(x) \geq 0 \geq -\frac{\sigma^2}{2}\Delta\underline{u}(x) + \left(\frac{1}{2\sigma^2}|x|^2 + 1\right)\underline{u}(x) & in \quad B_R, \\ \overline{u}(x) = 1 = \underline{u}(x) & on \quad \partial B_R, \end{cases}$$

*then*

$$\overline{u}(x) - \underline{u}(x) \geq 0 \, \textit{for all } x \in \overline{B}_R,$$

*and there exists*

$$u(x) \in C^2(B_R) \cap C\left(\overline{B}_R\right),$$

*a solution of (2.4) such that*

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x), \; x \in \overline{B}_R,$$

*where $\underline{u}(x)$ and $\overline{u}(x)$ are respectively, called a lower solution and an upper solution of (2.4).*

The corresponding result of Lemma 2.1 for the parabolic equations can be found in the work of Pao [18] and Amann [19]. To achieve our goal, complementary to the works [12–15] it can be used the well known books of Gilbarg and Trudinger [20], Sattinger [17], Pao [18] and a paper of Amann [19]. Further on, we can proceed to prove **Theorem 2.1**.

## 3. Proof of Theorem 2.1

By a direct calculation, if there exists and is unique, $u_s \in C^2(B_R) \cap C\left(\overline{B}_R\right)$, a positive solution of the stationary Dirichlet problem (2.4) then

$$u(x,t) = e^{t-T}u_s(x), (x,t) \in \overline{B}_R \times [0,T],$$

is the solution of the problem (2.2) and

$$z(x,t) = -2\sigma^2(t-T) - 2\sigma^2 \ln u_s(x), (x,t) \in \overline{B}_R \times [0,T],$$

is the solution of the problem (1.5) belonging to

$$C^2(B_R \times [0,T)) \cap C\left(\overline{B}_R \times [0,T]\right).$$

We prove that (2.4) has a unique radially symmetric solution. The existence of solution for (2.4) is obtained by a standard monotone iteration and the lower and the upper solution method, **Lemma 2.1**. Hence, starting from the initial iteration

$$u_s^0(x) = e^{-\frac{R^2-|x|^2}{2\sigma^2}},$$

we construct a sequence $\{u_s^k(x)\}_{k\geq 1}$ successively by

$$\begin{cases} \frac{\sigma^2}{2}\Delta u_s^k(x) = \left(\frac{1}{2\sigma^2}|x|^2+1\right)u_s^{k-1}(x), & \text{in} \quad B_R, \\ u_s^k(x) = 1, & \text{on} \quad \partial B_R, \end{cases} \tag{3.1}$$

and this sequence will be pointwise convergent to a solution $u_s(x)$ of (2.4).

Indeed, since for each $k$ the right-hand side of (3.1) is known, the existence theory for linear elliptic boundary-value problems implies that $\{u_s^k(x)\}_{k\geq 1}$ is well defined, see [20].

Let us prove that $\{u_s^k(x)\}_{k\geq 1}$ is a pointwise convergent sequence to a solution of (2.4) in $\overline{B}_R$. To do this, first we prove that $\{u_s^k(x)\}_{k\geq 1}$ is monotone nondecreasing of $k$. We apply the mathematical induction by verifying the first step, $k = 1$.

$$\begin{cases} \frac{\sigma^2}{2}\Delta u_s^1(x) \leq \frac{\sigma^2}{2}\Delta u_s^0(x), & \text{in} \quad B_R, \\ u_s^1(x) = 1 = u_s^0(x), & \text{on} \quad \partial B_R. \end{cases}$$

Now, by the standard comparison principle, **Lemma 2.1**, we have

$$u_s^0(x) \leq u_s^1(x) \text{ in } \overline{B}_R.$$

Moreover, the induction argument yields the following

$$u_s^0(x) = e^{-\frac{R^2-|x|^2}{2\sigma^2}} \leq \ldots \leq u_s^k(x) \leq u_s^{k+1}(x) \leq \ldots \text{ in } \overline{B}_R, \tag{3.2}$$

i.e., $\{u_s^k(x)\}_{k\geq 1}$ is a monotone nondecreasing sequence.

Next, using again **Lemma 2.1**, we find

$$\underline{u}_s(x) := u_s^0(x) = e^{-\frac{R^2-|x|^2}{2\sigma^2}} \leq \ldots \leq u_s^k(x) \leq u_s^{k+1}(x) \leq \ldots \leq \overline{u}_s(x) := 1 \text{ in } \overline{B}_R, \tag{3.3}$$

where we have used

$$\begin{aligned} \frac{\sigma^2}{2}\Delta \underline{u}_s(x) &= \underline{u}_s(x)\frac{\sigma^2}{2}\left(\frac{|x|^2+\sigma^2}{\sigma^4}+\frac{N-1}{\sigma^2}\right) \geq \underline{u}_s(x)\left(\frac{1}{2\sigma^2}|x|^2+1\right) \\ \frac{\sigma^2}{2}\Delta \overline{u}_s(x) &= \frac{\sigma^2}{2}\Delta 1 = 0 \leq \overline{u}_s(x)\left(\frac{1}{2\sigma^2}|x|^2+1\right) \end{aligned}$$

i.e., **Lemma 2.1 confirm.** Thus, in view of the monotone and bounded property in (3.3) the sequence $\{u_s^k(x)\}_{k\geq 1}$ converges. We may pass to the limit in (3.3) to get the existence of a solution

$$u_s(x) := \lim_{k\to\infty} u_s^k(x) \text{ in } \overline{B}_R,$$

associated to (2.4), which satisfies

$$\underline{u}_s(x) \leq u_s(x) \leq \overline{u}_s(x) \text{ in } \overline{B}_R.$$

Furthermore, the convergence of $\{u_s^k(x)\}$ is uniformly to $u_s(x)$ in $\overline{B}_R$ and $u_s(x)$ has a radial symmetry, see [15] for arguments of the proof. The regularity of solution $u_s(x)$ is a consequence of classical results from the theory of elliptic equations, see Gilbarg and Trudinger [20]. The uniqueness of $u_s(x)$ follows from a standard argument with the use of **Lemma 2.1** and we omit the details.

Clearly, $u(x,t)$ is increasing in the time variable. The regularity of $u(x,t)$ follows from the regularity of $u_s(x)$. Letting $t \to T$ we see that (2.3) holds. The solution of the initial problem (1.5) is saved from (2.1).

Finally, we prove the uniqueness for (2.2). Let

$$u(x,t), v(x,t) \in C^2(B_R \times [0,T)) \cap C\left(\overline{B}_R \times [0,T]\right),$$

be two solutions of the problem (2.2), i.e., its hold

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \frac{\sigma^2}{2}\Delta u(x,t) + \frac{1}{2\sigma^2}|x|^2 u(x,t) = 0, & \text{if} \quad (x,t) \in B_R \times [0,T), \\ u(x,T) = 1, & \text{on} \quad \partial B_R, \end{cases}$$

and

$$\begin{cases} \frac{\partial v}{\partial t}(x,t) - \frac{\sigma^2}{2}\Delta v(x,t) + \frac{1}{2\sigma^2}|x|^2 v(x,t) = 0, & \text{if} \quad (x,t) \in B_R \times [0,T), \\ v(x,T) = 1, & \text{on} \quad \partial B_R. \end{cases}$$

Setting

$$w(x,t) = u(x,t) - v(x,t), \text{ in } B_R \times [0,T],$$

and subtracting the two equations corresponding to $u$ and $v$ we find

$$\begin{cases} \frac{\partial w}{\partial t}(x,t) = \frac{\sigma^2}{2}\Delta w(x,t) - \frac{1}{2\sigma^2}|x|^2 w(x,t), & if \quad (x,t) \in B_R \times [0,T), \\ w(x,T) = 0, & on \quad \partial B_R. \end{cases}$$

Let us prove that $u(x,t) - v(x,t) \leq 0$ in $\overline{B}_R \times [0,T]$. If the conclusion were false, then the maximum of

$$w(x,t), \text{ in } B_R \times [0,T),$$

is positive. Assume that the maximum of $w$ in $\overline{B}_R \times [0,T]$ is achieved at $(x_0,t_0)$. Then, at the point $(x_0,t_0) \in B_R \times [0,T)$, where the maximum is attained, we have

$$\frac{\partial w}{\partial t}(x_0,t_0) \geq 0, \, \Delta w(x_0,t_0) \leq 0, \, \nabla w(x_0,t_0) = 0,$$

and

$$0 \leq \frac{\partial w}{\partial t}(x_0,t_0) = \frac{\sigma^2}{2}\Delta w(x_0,t_0) - \frac{1}{2\sigma^2}|x|^2 w(x_0,t_0) < 0$$

which is a contradiction. Reversing the role of $u$ and $v$ we obtain that $u(x,t) - v(x,t) \geq 0$ in $\overline{B}_R \times [0,T]$. Hence $u(x,t) = v(x,t)$ in $\overline{B}_R \times [0,T]$. The proof of Theorem 2.1 is completed.

Finally, our main result, Theorem 2.2 will be obtained by a direct computation.

## 4. Proof of Theorem 2.2

In view of the arguments used in the proof of **Theorem 2.1** and the real world phenomena, we use a purely intuitive strategy in order to prove **Theorem 2.2**.

Indeed, for the verification result in the production planning problem, we need $z(x,t)$ to be almost quadratic with respect to the variable $x$.

More exactly, we observe that there exists and is unique

$$u(x,t) = e^{-\frac{h(t)+B|x|^2+D}{2\sigma^2}}, (x,t) \in \mathbb{R}^N \times [0,\infty), \text{ with } B, D \in (0,\infty),$$

that solve (2.2), where

$$h(0) = c, \tag{4.1}$$

and $B$, $D$ are given in (2.8). The condition (4.1) is used to obtain the asymptotic behaviour of solution to the stationary Dirichlet problem associated with (2.2). Then our strategy is reduced to find $B, D \in (0,\infty)$ and the function $h$ which depends of time and $c \in (0,\infty)$ such that

$$-\frac{1}{2}\frac{h'(t)}{\sigma^2} - \frac{\sigma^2}{2}\left[-\frac{B}{\sigma^4}\left(\sigma^2 - B|x|^2\right) - (N-1)\frac{B}{\sigma^2}\right] + \alpha\left(-\frac{h(t) + B|x|^2 + D}{2\sigma^2}\right) + \frac{1}{2\sigma^2}|x|^2 = 0,$$

or, after rearranging the terms

$$|x|^2\left(1 - \alpha B - B^2\right) + N\sigma^2 B - \alpha D - h'(t) - \alpha h(t) = 0,$$

where (4.1) holds. Now, by a direct calculation we see that the system of equations

$$\begin{cases} 1 - \alpha B - B^2 = 0 \\ N\sigma^2 B - \alpha D = 0 \\ -h'(t) - \alpha h(t) = 0 \\ h(0) = c \end{cases}$$

has a unique solution that satisfies our expectations, namely,

$$u(x,t) = e^{-\frac{ce^{-\alpha t}+B|x|^2+D}{2\sigma^2}}, (x,t) \in \mathbb{R}^N \times [0,\infty), \tag{4.2}$$

where $B$ and $D$ are given in (2.8), is a radially symmetric solution of the problem (2.2). The uniqueness of the solution is followed by the arguments in [10] combined with the uniqueness proof in Theorem 2.1. The justification of the asymptotic behavior and regularity of the solution can be proved directly, once we have a closed-form solution. Finally, the closed-form solution in (2.7) is due to (2.1)–(4.2) and the proof of **Theorem 2.2** is completed.

## 5. Proof of Theorem 2.3

One way of solving this system of partial differential equation of parabolic type (1.4) is to show that the system (1.4) is solvable by

$$(u_1(x,t), u_2(x,t)) = \left(h_1(t) + \beta_1|x|^2 + \eta_1, h_2(t) + \beta_2|x|^2 + \eta_2\right), \tag{5.1}$$

for some unique $\beta_1, \beta_2, \eta_1, \eta_2 \in (0,\infty)$ and $h_1(t)$, $h_2(t)$ are suitable chosen such that

$$h_1(0) = c_1 \text{ and } h_2(0) = c_2. \tag{5.2}$$

The main task for the proof of existence of (5.1) is performed by proving that there exist

$$\beta_1,\beta_2,\eta_1,\eta_2,h_1,h_2,$$

such that

$$\begin{cases} h_1'(t)-\frac{2\beta_1 N\sigma_1^2}{2}+(a_{11}+\alpha_1)\left[h_1(t)+\beta_1|x|^2+\eta_1\right]-a_{11}\left[h_2(t)+\beta_2|x|^2+\eta_2\right]-|x|^2=-\frac{1}{4}(2\beta_1|x|)^2, \\ h_2'(t)-\frac{2\beta_2 N\sigma_2^2}{2}+(a_{22}+\alpha_2)\left[h_2(t)+\beta_2|x|^2+\eta_2\right]-a_{22}\left[h_1(t)+\beta_1|x|^2+\eta_1\right]-|x|^2=-\frac{1}{4}(2\beta_2|x|)^2, \end{cases}$$

or equivalently, after grouping the terms

$$\begin{cases} |x|^2\left[-a_{11}\beta_2+(a_{11}+\alpha_1)\beta_1+\beta_1^2-1\right]-\beta_1 N\sigma_1^2-a_{11}\eta_2+(a_{11}+\alpha_1)\eta_1 \\ \qquad +h_1'(t)+(a_{11}+\alpha_1)h_1(t)-a_{11}h_2(t)=0, \\ |x|^2\left[-a_{22}\beta_1+(a_{22}+\alpha_2)\beta_2+\beta_2^2-1\right]-\beta_2 N\sigma_2^2-a_{22}\eta_1+(a_{22}+\alpha_2)\eta_2 \\ \qquad +h_2'(t)+(a_{22}+\alpha_2)h_2(t)-a_{22}h_1(t)=0, \end{cases}$$

where $h_1(t)$, $h_2(t)$ must satisfy (5.2). Now, we consider the system of equations

$$\begin{cases} -a_{11}\beta_2+(a_{11}+\alpha_1)\beta_1+\beta_1^2-1=0 \\ -a_{22}\beta_1+(a_{22}+\alpha_2)\beta_2+\beta_2^2-1=0 \\ -\beta_1 N\sigma_1^2-a_{11}\eta_2+(a_{11}+\alpha_1)\eta_1=0 \\ -\beta_2 N\sigma_2^2-a_{22}\eta_1+(a_{22}+\alpha_2)\eta_2=0 \\ h_1'(t)+(a_{11}+\alpha_1)h_1(t)-a_{11}h_2(t)=0 \\ h_2'(t)+(a_{22}+\alpha_2)h_2(t)-a_{22}h_1(t)=0. \end{cases} \tag{5.3}$$

To solve (5.3), we can rearrange those equations 1, 2 in the following way

$$\begin{cases} -a_{11}\beta_2+(a_{11}+\alpha_1)\beta_1+\beta_1^2-1=0 \\ -a_{22}\beta_1+(a_{22}+\alpha_2)\beta_2+\beta_2^2-1=0 \end{cases}. \tag{5.4}$$

We distinguish three cases:

**1.** in the case $a_{22}=0$ we have an exact solution for (5.4) of the form

$$\begin{aligned} \beta_1 &= -\frac{1}{2}\alpha_1-\frac{1}{2}a_{11}+\frac{1}{2}\sqrt{\alpha_1^2+a_{11}^2-4a_{11}\left(\frac{1}{2}\alpha_2-\frac{1}{2}\sqrt{\alpha_2^2+4}\right)+2\alpha_1 a_{11}+4} \\ \beta_2 &= -\frac{1}{2}\alpha_2+\frac{1}{2}\sqrt{\alpha_2^2+4} \end{aligned}$$

**2.** in the case $a_{11}=0$ we have an exact solution for (5.4) of the form

$$\begin{aligned} \beta_1 &= -\frac{1}{2}\alpha_1+\frac{1}{2}\sqrt{\alpha_1^2+4} \\ \beta_2 &= -\frac{1}{2}\alpha_2-\frac{1}{2}a_{22}+\frac{1}{2}\sqrt{\alpha_2^2+a_{22}^2-4a_{22}\left(\frac{1}{2}\alpha_1-\frac{1}{2}\sqrt{\alpha_1^2+4}\right)+2\alpha_2 a_{22}+4} \end{aligned}$$

**3.** in the case $a_{11} \neq 0$ and $a_{22} \neq 0$, to prove the existence and uniqueness of solution for (5.4) we will proceed as follows. We retain from the first equation of (5.4)

$$\beta_1 = \frac{1}{2}\sqrt{\alpha_1^2 + 2\alpha_1 a_{11} + a_{11}^2 + 4\beta_2 a_{11} + 4} - \frac{1}{2}a_{11} - \frac{1}{2}\alpha_1.$$

and from the second equation

$$\beta_2 = \frac{1}{2}\sqrt{\alpha_2^2 + 2\alpha_2 a_{22} + a_{22}^2 + 4\beta_1 a_{22} + 4} - \frac{1}{2}a_{22} - \frac{1}{2}\alpha_2.$$

The existence of $\beta_1,\ \beta_2 \in (0,\infty)$ for (5.4) can be easily proved by observing that the continuous functions $f_1, f_2 : [0,\infty) \to \mathbb{R}$ defined by

$$\begin{aligned} f_1(\beta_1) &= -a_{11}\left(\tfrac{1}{2}\sqrt{\alpha_2^2 + 2\alpha_2 a_{22} + a_{22}^2 + 4\beta_1 a_{22} + 4} - \tfrac{1}{2}a_{22} - \tfrac{1}{2}\alpha_2\right) + (a_{11} + \alpha_1)\beta_1 + \beta_1^2 - 1, \\ f_2(\beta_2) &= -a_{22}\left(\tfrac{1}{2}\sqrt{\alpha_1^2 + 2\alpha_1 a_{11} + a_{11}^2 + 4\beta_2 a_{11} + 4} - \tfrac{1}{2}a_{11} - \tfrac{1}{2}\alpha_1\right) + (a_{22} + \alpha_2)\beta_2 + \beta_2^2 - 1, \end{aligned}$$

have the following properties

$$f_1(\infty) = \infty \text{ and } f_2(\infty) = \infty, \tag{5.5}$$

respectively

$$\begin{aligned} f_1(0) &= -a_{11}\left(\tfrac{1}{2}\sqrt{\alpha_2^2 + 2\alpha_2 a_{22} + a_{22}^2 + 4} - \tfrac{1}{2}a_{22} - \tfrac{1}{2}\alpha_2\right) - 1 < 0, \\ f_2(0) &= -a_{22}\left(\tfrac{1}{2}\sqrt{\alpha_1^2 + 2\alpha_1 a_{11} + a_{11}^2 + 4} - \tfrac{1}{2}a_{11} - \tfrac{1}{2}\alpha_1\right) - 1 < 0. \end{aligned} \tag{5.6}$$

The observations (5.5) and (5.6) imply

$$\begin{cases} f_1(\beta_1) = 0 \\ f_2(\beta_2) = 0 \end{cases}$$

has at least one solution $(\beta_1, \beta_2) \in (0,\infty) \times (0,\infty)$ and furthermore it is unique (see also, the references [21, 22] for the existence and the uniqueness of solutions).

The discussion from **cases 1–3** show that the system (5.4) has a unique positive solution. Next, letting

$$(\beta_1, \beta_2) \in (0,\infty) \times (0,\infty),$$

be the unique positive solution of (5.4), we observe that the equations 3, 4 of (5.3) can be written equivalently as a system of linear equations that is solvable and with a unique solution

$$\begin{pmatrix} a_{11} + \alpha_1 & -a_{11} \\ -a_{22} & a_{22} + \alpha_2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \beta_1 N \sigma_1^2 \\ \beta_2 N \sigma_2^2 \end{pmatrix}. \tag{5.7}$$

By defining

$$G_{a,\alpha} := \begin{pmatrix} a_{11} + \alpha_1 & -a_{11} \\ -a_{22} & a_{22} + \alpha_2 \end{pmatrix},$$

we observe that

$$G_{a,\alpha}^{-1} = \begin{pmatrix} \frac{\alpha_2 + a_{22}}{\alpha_1\alpha_2 + \alpha_2 a_{11} + \alpha_1 a_{22}} & \frac{a_{11}}{\alpha_1\alpha_2 + \alpha_2 a_{11} + \alpha_1 a_{22}} \\ \frac{a_{22}}{\alpha_1\alpha_2 + \alpha_2 a_{11} + \alpha_1 a_{22}} & \frac{\alpha_1 + a_{11}}{\alpha_1\alpha_2 + \alpha_2 a_{11} + \alpha_1 a_{22}} \end{pmatrix}.$$

Using the fact that $G_{a,\alpha}^{-1}$ has all ellements positive and rewriting (5.7) in the following way

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = G_{a,\alpha}^{-1} \begin{pmatrix} \beta_1 N\sigma_1^2 \\ \beta_2 N\sigma_2^2 \end{pmatrix},$$

we can see that there exist and are unique $\eta_1, \eta_2 \in (0, \infty)$ that solve (5.7). Finally, the equations 5, 6, 7 of (5.3) with initial condition (5.2) can be written equivalently as a solvable Cauchy problem for a first order system of differential equations

$$\begin{cases} \begin{pmatrix} h_1'(t) \\ h_2'(t) \end{pmatrix} + G_{a,\alpha} \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ h_1(0) = c_1 \text{ and } h_2(0) = c_2, \end{cases} \tag{5.8}$$

with a unique solution and then (5.1) solve (1.4). The rest of the conclusions are easily verified.

## 6. Application for the system case

Next, we present an application.

**Application 1.** Suppose there is one machine producing two products (see [23,24], for details). We consider a continuous time Markov chain generator

$$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

and the time-dependent production planning problem with diffusion $\sigma_1 = \sigma_2 = \frac{1}{\sqrt{2}}$ and let $\alpha_1 = \alpha_2 = \frac{1}{2}$ the discount factor. Under these assumptions, we can write the system (5.4) with our data

$$\begin{cases} \beta_1^2 + \beta_1 - \frac{1}{2}\beta_2 - 1 = 0 \\ \beta_2^2 - \frac{1}{2}\beta_1 + \beta_2 - 1 = 0 \end{cases}$$

which has a unique positive solution

$$\beta_1 = \frac{1}{4}\left(\sqrt{17} - 1\right),\ \beta_2 = \frac{1}{4}\left(\sqrt{17} - 1\right).$$

On the other hand, the system (5.7) becomes

$$\begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix},$$

which has a unique positive solution

$$\begin{aligned} \eta_1 &= \frac{4}{3}\beta_1 + \frac{2}{3}\beta_2 = \frac{1}{2}\left(\sqrt{17} - 1\right), \\ \eta_2 &= \frac{2}{3}\beta_1 + \frac{4}{3}\beta_2 = \frac{1}{2}\left(\sqrt{17} - 1\right). \end{aligned}$$

Finally, the system in (5.8) becomes

$$\begin{cases} \begin{pmatrix} h_1'(t) \\ h_2'(t) \end{pmatrix} + \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ h_1(0) = c_1 \text{ and } h_2(0) = c_2, \end{cases}$$

which has the solution

$$h_1(t) = s_1 e^{-\frac{1}{2}t} - s_2 e^{-\frac{3}{2}t},\ h_2(t) = s_1 e^{-\frac{1}{2}t} + s_2 e^{-\frac{3}{2}t}, \text{ with } s_1, s_2 \in \mathbb{R}.$$

Next, from

$$h_1(0) = c_1 \text{ and } h_2(0) = c_2,$$

we have

$$\begin{cases} s_1 - s_2 = c_1 \\ s_1 + s_2 = c_2 \end{cases} \Longrightarrow s_1 = \frac{1}{2}c_1 + \frac{1}{2}c_2,\ s_2 = \frac{1}{2}c_2 - \frac{1}{2}c_1,$$

and finally

$$\begin{cases} h_1(t) = \frac{1}{2}(c_1 + c_2) e^{-\frac{1}{2}t} - \frac{1}{2}(c_2 - c_1) e^{-\frac{3}{2}t}, \\ h_2(t) = \frac{1}{2}(c_1 + c_2) e^{-\frac{1}{2}t} + \frac{1}{2}(c_2 - c_1) e^{-\frac{3}{2}t}, \end{cases}$$

from where we can write the unique solution of the system (1.4) in the form (5.1).

## 7. Discussion

Let us point that in **Theorem 2.3** we have proved the existence and the uniqueness of a solution of quadratic form in the $x$ variable and then the existence of other different types of solutions remain an open problem.

## 8. Conclusions

Some closed-form solutions for equations and systems of parabolic type are presented. The form of the solutions is unique and tends to the solutions of the corresponding elliptic type problems that were considered.

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## Conflict of interest

The authors declare there is no conflict of interest.

## References

1. A. Bensoussan, S. P. Sethi, R. Vickson, N. Derzko, Stochastic production planning with production constraints, *SIAM J. Control Optim.*, **22** (1984), 920–935. https://doi.org/10.1137/0322060
2. A. Cadenillas, P. Lakner, M. Pinedo, Optimal production management when demand depends on the business cycle, *Oper. Res.*, **61** (2013), 1046–1062. https://doi.org/10.1287/opre.2013.1181

3. J. Dong, A. Malikopoulos, S. M. Djouadi, T. Kuruganti, Application of optimal production control theory for home energy management in a micro grid, *2016 American Control Conference (ACC)*, IEEE, (2016), 5014–5019.
4. A. Capponi, J. E. Figueroa-López, Dynamic Portfolio Optimization with a Defaultable Security and Regime-Switching, *Math. Finance*, **24** (2012), 207–249. https://doi.org/10.1111/j.1467-9965.2012.00522.x
5. R. Elliott, A. S. Hamada, Option pricing using a regime switching stochastic siscount factor, *Int. J. Theor. Appl. Finance*, **17** (2014), 1–26. https://doi.org/10.1142/S0219024914500204
6. A. Gharbi, J. P. Kenne, Optimal production control problem in stochastic multiple-product multiple-machine manufacturing systems, *IIE Trans.*, **35** (2003), 941–952. https://doi.org/10.1080/07408170309342346
7. D. D. Yao, Q. Zhang, X. Y. Zhou, A Regime-Switching Model for European Options, *Stochastic Processes. Optimization, and Control Theory: applications in Financial Engineering, Queueing Networks, and Manufacturing systems*, Springer, Boston, MA.
8. C. F. Wang, H. Chang, Z. M. Fang, Optimal Portfolio and Consumption Rule with a CIR Model Under HARA Utility, *J. Oper. Res. Soc. China*, **6** (2018), 107–137. https://doi.org/10.1007/s40305-017-0189-8
9. D. P. Covei, E. C. Canepa, T. A. Pirvu, Stochastic production planning with regime switching, *J. Ind. Manag. Optim.*, **2022** (2022), 1–17. https://doi.org/10.3934/jimo.2022013
10. D. P. Covei, An elliptic partial differential equation modeling the production planning problem, *J. Appl. Anal. Comput.*, **11** (2021), 903–910. https://doi.org/10.11948/20200112
11. D. P. Covei, T. A. Pirvu, A stochastic control problem with regime switching, *Carpathian J. Math.*, **37** (2021), 427–440. https://doi.org/10.37193/CJM.2021.03.06
12. G. Barles, A. Porretta, T. T. Tchamba, On the large time behavior of solutions of the Dirichlet problem for subquadratic viscous Hamilton–Jacobi equations, *J. Math. Pures Appl.*, **94** (2010), 497–519. https://doi.org/10.1016/j.matpur.2010.03.006
13. T. T. Tchamba, Large Time Behavior of Solutions of Viscous Hamilton-Jacobi Equations with Superquadratic Hamiltonian, *Asymptot. Anal.*, **66** (2010), 161–186. https://doi.org/10.3233/ASY-2009-0965
14. C. O. Alves, T. Boudjeriou, Existence of solution for a class of heat equation with double criticality, *J. Math. Anal. Appl.*, **504** (2021), 125–403. https://doi.org/10.1016/j.jmaa.2021.125403
15. E. C. Canepa, D. P. Covei, T. A. Pirvu, A Stochastic production planning problem, *Fixed Point Theory Appl.*, **23** (2022), 179–198. https://doi.org/10.24193/fpt-ro.2022.1.11
16. G. dos Reis, D. Šiška, *Stochastic Control and Dynamic Asset Allocation*, Partial lecture notes, School of Mathematics, University of Edinburgh, 2020. Available from: `https://www.maths.ed.ac.uk/~dsiska/LecNotesSCDAA.pdf`.
17. D. H. Sattinger, *Topics in Stability and Bifurcation Theory*, Lecture Notes in Mathematics, 309, Springer, Berlin, Heidelberg, 1973. https://doi.org/10.1007/BFb0060079
18. C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Springer, Boston, MA, 1992. https://doi.org/10.1007/978-1-4615-3034-3

19. H. Amann, Quasilinear parabolic systems under nonlinear boundary conditions, *Arch. Ration. Mech. Anal.*, **92** (1986), 153–192. https://doi.org/10.1007/BF00251255

20. D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Berlin: Springer-Verlag, 1983. https://doi.org/10.1007/978-3-642-61798-0

21. I. Győri, F. Hartung, N. A. Mohamady, Existence and uniqueness of positive solutions of a system of nonlinear algebraic equations, *Period. Math. Hung.*, **75** (2017), 114–127. https://doi.org/10.1007/s10998-016-0179-3

22. I. Győri, F. Hartung, N. A. Mohamady, Boundedness of positive solutions of a system of nonlinear delay differential equations, *Discrete Contin. Dyn. Syst. B*, **23** (2018), 809–836. https://doi.org/10.1007/s10998-016-0179-3

23. L. Sheng, Y. Zhu, K. Wang, Uncertain dynamical system-based decision making with application to production-inventory problems, *Appl. Math. Model.*, **56** (2018), 275–288. https://doi.org/10.1016/j.apm.2017.12.006

24. M. K. Ghosh, A. Arapostathis, S. I. Marcus, Optimal Control of Switching Diffusions with Application to Flexible Manufacturing Systems, *SIAM J. Control Optim.*, **31** (1992), 1183–1204. https://doi.org/10.1137/0331056