Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 30(4): 1236–1262.
DOI: 10.3934/era.2022065
Received: 21 September 2021
Revised: 10 December 2021
Accepted: 16 December 2021
Published: 15 March 2022

Research article

A dual-phase-lag porous-thermoelastic problem with microtemperatures

N. Bazarra[1], **J. R. Fernández**[1,*] **and R. Quintanilla**[2]

[1] Departamento de Matemática Aplicada I, Universidade de Vigo, Escola de Enxeñería de Telecomunicación, Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain

[2] Departamento de Matemáticas, E.S.E.I.A.A.T.-U.P.C., Colom 11, 08222 Terrassa, Barcelona, Spain

* **Correspondence:** Email: jose.fernandez@uvigo.es.

Abstract: In this work, we consider a multi-dimensional dual-phase-lag problem arising in porous-thermoelasticity with microtemperatures. An existence and uniqueness result is proved by applying the semigroup of linear operators theory. Then, by using the finite element method and the Euler scheme, a fully discrete approximation is numerically studied, proving a discrete stability property and a priori error estimates. Finally, we perform some numerical simulations to demonstrate the accuracy of the approximation and the behavior of the solution in one- and two-dimensional problems.

Keywords: dual-phase-lag; porous-thermoelasticity with microtemperatures; existence and uniqueness; finite elements; a priori error estimates; numerical simulations

1. Introduction

In the second part of the past century, there were developed different theories where the microstructure of elastic materials was taken into account. Two good books concerning these aspects were written by Eringen [1] and Ieşan [2]. In particular, Goodman and Cowin [3] established the theory of continuum for granular materials with interstician voids. They proposed the bulk density of the materials as a product of the matrix density and the volume fraction. Later, Cowin and Nunziato [4–6] stated a theory of porous elastic solids modeling the deformations of materials with small voids distributed within them. Of course, this theory was extended to include the thermal effects [7]. Contributions in this theory became huge and we can recall a few of them [8–19]. In general, the microstructure components of elastic materials have deserved much attention. One of the possibilities could be the so-called "microtemperatures" [20, 21].

If we revise the literature regarding the heat conduction we should agree that most of the contributions consider the Fourier's law. That is, the heat flux vector can be obtained as a linear expression of the gradient of temperature. However, it is also well-known that this assumption brings us to the fact that the thermal waves propagate instantaneously. Therefore, the causality principle is not satisfied. This paradox has motivated many scientists to look for alternative constitutive equations replacing Fourier's law in the heat flux vector. Perhaps the most known is the proposition of Cattaneo and Maxwell [22], who suggested to introduce a relaxation parameter to the Fourier law, bringing to a damped hyperbolic equation describing the heat conduction. Some other authors have proposed other alternatives, but in this paper we focus our attention in the Tzou theory [23]. In this case, the author introduced two relaxation parameters and the theory of Cattaneo and Maxwell can be seen as a particular case. Many contributions have been obtained in the context of the dual-phase-lag thermoelasticity [24–26]. In a very recent contribution, it was suggested a way to extend the dual-phase-lag theory to include the microtemperatures [27, 28]. In fact, several contributions have been obtained concerning the stability of the problem.

Here, we center our attention to the case of porous-thermo-elastic solids with microtemperatures within the context of the dual-phase-lag theory. For this reason, we consider the case where the heat and the microheat are determined by means of the dual-phase-lag. Therefore, in this work we first show the existence of solutions in the multi-dimensional setting and then, we propose a numerical study of the problem, proving a discrete stability property and a priori error estimates, and performing some numerical simulations.

The paper is structured in the following form. In the next section, we describe the model and the basic assumptions. Then, in Section 3 we prove that the thermomechanical problem has a unique solution in the general multi-dimensional setting by means of the semigroup of linear operators theory. The numerical approximation of this problem is presented in Section 4, by using the finite element method and the implicit Euler scheme to approximate the spatial variable and to discretize the time derivatives, respectively. A discrete stability property and a priori error estimates are shown. Finally, some numerical simulations are described in Section 5 to demonstrate the accuracy of the algorithm and the behavior of the solution.

2. The thermomechanical model

We recall in this section the basic equations and assumptions under we will work in this paper. We will consider a domain $B \subset \mathbb{R}^d$, $d = 1, 2, 3$, with a boundary $\Gamma = \partial B$ assumed to be smooth enough.

The basic equations for the dual-phase-lag thermoelastic problem with microtemperatures are given by the evolution equations (see [28]):

$$\begin{aligned}
\rho \ddot{u}_i &= t_{ij,j},\\
J\ddot{\phi} &= h_{i,i} + g,\\
\rho T_0 \dot{\eta} &= q_{i,i},\\
\rho \dot{\epsilon}_i &= P_{ij,j} + q_i - Q_i,
\end{aligned}$$

and the constitutive equations*:

$$t_{ij} = \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + \mu_0 \phi \delta_{ij} - \beta_0 \theta \delta_{ij},$$

*Though we could consider different relaxation parameters for the macrotemperatures and the microtemperatures, we assume here that they agree. The reason is basically the big difficulty to treat with the general case. We do recognize that we are not able to deal with it.

$$
\begin{aligned}
&h_i = a\phi_{,i} - \mu_2 T_i,\\
&g = -\mu_0 e_{ii} - \zeta\phi + \beta_1\theta,\\
&\rho\eta = \beta_0 e_{ii} + \beta_1\phi + a\theta,\\
&\rho\epsilon_i = -\mu_2\phi_{,i} - bT_i,\\
&q_i + \tau_1\dot{q}_i + \frac{\tau_1^2}{2}\ddot{q}_i = (k\theta_{,i} + \kappa_1 T_i) + \tau_2(k\dot{\theta}_{,i} + \kappa_1\dot{T}_i),\\
&P_{ij} + \tau_1\dot{P}_{ij} + \frac{\tau_1^2}{2}\ddot{P}_{ij} = -\kappa_4 T_{r,r}\delta_{ij} - \kappa_5 T_{j,i} - \kappa_6 T_{i,j}\\
&\qquad -\tau_2(\kappa_4\dot{T}_{r,r}\delta_{ij} + \kappa_5\dot{T}_{j,i} + \kappa_6\dot{T}_{i,j}),\\
&Q_i + \tau_1\dot{Q}_i + \frac{\tau_1^2}{2}\ddot{Q}_i = (\kappa - \kappa_1)\theta_{,i} + (\kappa_1 - \kappa_2)T_i\\
&\qquad +\tau_2((\kappa - \kappa_1)\dot{\theta}_{,i} + (\kappa_1 - \kappa_2)\dot{T}_i),
\end{aligned}
$$

where $\boldsymbol{u} = (u_i)$ is the displacement field, ϕ is the porosity function, θ is the temperature and $\boldsymbol{T} = (T_i)$ is the microtemperatures vector. Moreover, $\mathbf{e} = (e_{ij})$ denotes the linearized strain tensor given by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}),$$

t_{ij} is the stress tensor, h_i is the equilibrated stress, g is the equilibrated body force, η is the entropy, ϵ_i is the first moment of the energy, P_{ij} is the first heat flux moment, Q_i is the microheat flux average, τ_1 and τ_2 are the time relaxation parameters, and ρ, J, λ, μ, μ_0, β_0, a, μ_2, ζ, β_1, b, κ_1, κ_4, κ_5, κ_6, κ_2 and κ are constitutive parameters. From now on, we assume that $T_0 = 1$ to simplify the calculations. Anyway, this assumption is not restrictive.

After substitution of the constitutive equations into the evolution equations we obtain the following linear system:

$$
\begin{aligned}
&\rho\ddot{u}_i = \mu u_{i,jj} + (\lambda + \mu)u_{j,ji} + \mu_0\phi_{,i} - \beta_0\theta_{,i},\\
&J\ddot{\phi} = a_0\phi_{,ii} - \mu_2 T_{i,i} - \mu_0 u_{i,i} - \zeta\phi + \beta_1\theta,\\
&a(\dot{\theta} + \tau_1\ddot{\theta} + \frac{\tau_1^2}{2}\dddot{\theta}) = \kappa(\theta_{,ii} + \tau_2\dot{\theta}_{,ii}) + \kappa_1(T_{i,i} + \tau_2\dot{T}_{i,i})\\
&\qquad -\beta_0(\dot{u}_{i,i} + \tau_1\ddot{u}_{i,i} + \frac{\tau_1^2}{2}\dddot{u}_{i,i}) - \beta_1(\dot{\phi} + \tau_1\ddot{\phi} + \frac{\tau_1^2}{2}\dddot{\phi}),\\
&b(\dot{T}_i + \tau_1\ddot{T}_i + \frac{\tau_1^2}{2}\dddot{T}_i) = \kappa_6(T_{i,jj} + \tau_2\dot{T}_{i,jj}) + (\kappa_4 + \kappa_5)(T_{j,ji} + \tau_2\dot{T}_{j,ji})\\
&\qquad -\kappa_2(T_i + \tau_2\dot{T}_i) - \kappa_1(\theta_{,i} + \tau_2\dot{\theta}_{,i}) - \mu_2(\dot{\phi}_{,i} + \tau_1\ddot{\phi}_{,i} + \frac{\tau_1^2}{2}\dddot{\phi}_{,i}).
\end{aligned}
$$

Proceeding as in [28], defining $\hat{f} = f + \tau_1\dot{f} + \frac{\tau_1^2}{2}\ddot{f}$ we can write the previous system as follows:

$$\rho\ddot{\hat{u}}_i = \mu\hat{u}_{i,jj} + (\lambda + \mu)\hat{u}_{j,ji} + \mu_0\hat{\phi}_{,i} - \beta_0(\theta_{,i} + \tau_1\dot{\theta}_{,i} + \frac{\tau_1^2}{2}\ddot{\theta}_{,i}), \tag{2.1}$$

$$J\ddot{\hat{\phi}} = a_0\hat{\phi}_{,ii} - \mu_2(T_{i,i} + \tau_1\dot{T}_{i,i} + \frac{\tau_1^2}{2}\ddot{T}_{i,i}) - \mu_0\hat{u}_{i,i} - \zeta\hat{\phi} + \beta_1(\theta + \tau_1\dot{\theta} + \frac{\tau_1^2}{2}\ddot{\theta}), \tag{2.2}$$

$$a(\dot{\theta}+\tau_1\ddot{\theta}+\frac{\tau_1^2}{2}\dddot{\theta})=\kappa(\theta_{,ii}+\tau_2\dot{\theta}_{,ii})+\kappa_1(T_{i,i}+\tau_2\dot{T}_{i,i})-\beta_0\hat{\dot{u}}_{i,i}-\beta_1\hat{\dot{\phi}}, \tag{2.3}$$

$$b(\dot{T}_i+\tau_1\ddot{T}_i+\frac{\tau_1^2}{2}\dddot{T}_i)=\kappa_6(T_{i,jj}+\tau_2\dot{T}_{i,jj})+(\kappa_4+\kappa_5)(T_{j,ji}+\tau_2\dot{T}_{j,ji})$$
$$-\kappa_2(T_i+\tau_2\dot{T}_i)-\kappa_1(\theta_{,i}+\tau_2\dot{\theta}_{,i})-\mu_2\hat{\dot{\phi}}_{,i}. \tag{2.4}$$

However, in order to simplify the writing, we remove the superscript hat over all the variables.

Since we assume homogeneous Dirichlet boundary conditions, it follows that

$$u_i(\boldsymbol{x},t)=\phi(\boldsymbol{x},t)=\theta(\boldsymbol{x},t)=T_i(\boldsymbol{x},t)=0, \quad t\in[0,\infty), \quad \boldsymbol{x}\in\partial B, \tag{2.5}$$

and, to define a well posed problem, we will need to prescribe some initial conditions, for a.e. $\boldsymbol{x}\in B$,

$$\begin{array}{llll} u_i(\boldsymbol{x},0)=u_i^0(\boldsymbol{x}), & \dot{u}_i(\boldsymbol{x},0)=v_i^0(\boldsymbol{x}), & \phi(\boldsymbol{x},0)=\phi^0(\boldsymbol{x}), & \dot{\phi}(\boldsymbol{x},0)=\psi^0(\boldsymbol{x}), \\ \theta(\boldsymbol{x},0)=\theta^0(\boldsymbol{x}), & T_i(\boldsymbol{x},0)=T_i^0(\boldsymbol{x}), & \dot{\theta}(\boldsymbol{x},0)=\vartheta^0(\boldsymbol{x}), & \dot{T}_i(\boldsymbol{x},0)=S_i^0(\boldsymbol{x}), \\ \ddot{\theta}(\boldsymbol{x},0)=\xi^0(\boldsymbol{x}), & \ddot{T}_i(\boldsymbol{x},0)=R_i^0(\boldsymbol{x}). & & \end{array} \tag{2.6}$$

In this paper, we are going to assume that

$$\rho,\ J,\ a,\ b,\ \mu,\ a_0,\ \kappa,\ \kappa_6,\ \kappa_4+\kappa_5,\ \tau_1,\ \lambda+\mu,\ (\lambda+\mu)\zeta-\mu_0^2,\ \kappa\kappa_2-\kappa_1^2,\ \tau_2-\tau_1/2 \tag{2.7}$$

are positive numbers. These assumptions are the natural ones for this theory from the mechanical and thermal points of view.

Let us define the internal energy $\mathcal{E}$ and the dissipation $\mathbf{D}$ as follows:

$$\begin{aligned}
\mathcal{E}(t)=&\frac{1}{2}\int_B\Big[\rho\hat{\dot{u}}_i\hat{\dot{u}}_i+J|\hat{\dot{\phi}}|^2+2\mu_0\hat{u}_{i,i}\hat{\phi}+\zeta|\hat{\phi}|^2+a\hat{\phi}_{,i}\hat{\phi}_{,i}+\mu\hat{u}_{i,j}\hat{u}_{i,j}+(\lambda+\mu)\hat{u}_{i,i}\hat{u}_{j,j}\Big]\,d\boldsymbol{x}\\
&+\int_B\Big[a|\hat{\theta}|^2+b\hat{T}_i\hat{T}_i+\kappa(\tau_1+\tau_2)|\nabla\theta|^2+\frac{\kappa\tau_1^2\tau_2}{2}|\nabla\dot{\theta}|^2+\kappa_2(\tau_1+\tau_2)T_iT_i\\
&+\frac{\kappa_2\tau_1^2\tau_2}{2}\dot{T}_i\dot{T}_i+\kappa\tau_1^2\theta_{,i}\dot{\theta}_{,i}+\kappa_2\tau_1^2T_i\dot{T}_i+2(\tau_1+\tau_2)\kappa_1\theta_{,i}T_i+\kappa_1\tau_1^2\tau_2\dot{T}_i\dot{\theta}_{,i}\\
&+\kappa_1\tau_1^2(\theta_{,i}\dot{T}_i+\dot{\theta}_{,i}T_i)+\kappa_6(\tau_1+\tau_2)T_{i,j}T_{i,j}+\kappa_6\frac{\tau_1^2\tau_2}{2}\dot{T}_{i,j}\dot{T}_{i,j}+\kappa_6\tau_1^2T_{i,j}\dot{T}_{i,j}\\
&+(\kappa_4+\kappa_5)(\tau_1+\tau_2)T_{i,i}T_{j,j}+(\kappa_4+\kappa_5)\frac{\tau_1^2\tau_2}{2}\dot{T}_{i,i}\dot{T}_{j,j}+(\kappa_4+\kappa_5)\tau_1^2T_{i,i}\dot{T}_{j,j}\Big]\,d\boldsymbol{x},\\
\mathbf{D}(t)=&\int_B\Big[\kappa(|\nabla\theta|^2+(\tau_1\tau_2-\frac{\tau_1^2}{2})|\nabla\dot{\theta}|^2)+\kappa_2(T_iT_i+(\tau_1\tau_2-\frac{\tau_1^2}{2})\dot{T}_i\dot{T}_i)\\
&+2\kappa_1(\theta_{,i}T_i+(\tau_1\tau_2-\frac{\tau_1^2}{2})\dot{T}_i\dot{\theta}_{,i})+\kappa_6(T_{i,j}T_{i,j}+(\tau_1\tau_2-\frac{\tau_1^2}{2})\dot{T}_{i,j}\dot{T}_{i,j})\\
&+(\kappa_4+\kappa_5)(T_{i,i}T_{j,j}+(\tau_1\tau_2-\frac{\tau_1^2}{2})\dot{T}_{i,i}\dot{T}_{j,j})\Big]\,d\boldsymbol{x}.
\end{aligned}$$

It can be proved that, for a.e. $t\geq 0$,

$$\mathcal{E}(t)+\int_0^t\mathbf{D}(s)\,ds=\mathcal{E}(0).$$

In view of the previous assumptions, $\mathcal{E}(t)$ and $\mathbf{D}(t)$ are positive definite functions, and the last equality shows the stability of the solutions.

3. Existence and uniqueness

In this section, we give an existence and uniqueness result for the problem determined by system (2.1)-(2.4), boundary conditions (2.5) and initial conditions (2.6).

To this end we will work in the Hilbert space:

$$\mathcal{H} = \boldsymbol{W}_0^{1,2}(B) \times \boldsymbol{L}^2(B) \times W_0^{1,2}(B) \times L^2(B) \times W_0^{1,2}(B) \times W_0^{1,2}(B) \times L^2(B) \\ \times \boldsymbol{W}_0^{1,2}(B) \times \boldsymbol{W}_0^{1,2}(B) \times \boldsymbol{L}^2(B).$$

If we denote by

$$U = (\boldsymbol{u}, \boldsymbol{v}, \phi, \psi, \theta, \vartheta, \xi, \boldsymbol{T}, \boldsymbol{S}, \boldsymbol{R}), \quad U^* = (\boldsymbol{u}^*, \boldsymbol{v}^*, \phi^*, \psi^*, \theta^*, \vartheta^*, \xi^*, \boldsymbol{T}^*, \boldsymbol{S}^*, \boldsymbol{R}^*),$$

we define the inner product:

$$\begin{aligned}
< U, U^* >= \frac{1}{2} \int_B \Big[& \rho v_i v_i^* + J\psi\psi^* + \mu_0(u_{i,i}\phi^* + u_{i,i}^*\phi) + a\phi_{,i}\phi_{,i}^* + \mu u_{i,j}u_{i,j}^* \\
& + (\lambda + \mu) u_{i,i} u_{j,j}^* + a\hat{\theta}\hat{\theta}^* + b\hat{T}_i\hat{T}_i^* + \kappa(\tau_1 + \tau_2)\theta_{,i}\theta_{,i}^* + \frac{\kappa\tau_1^2\tau_2}{2}\vartheta_{,i}\vartheta_{,i}^* \\
& + \kappa_2(\tau_1 + \tau_2) T_i T_i^* + \frac{\kappa\tau_1^2\tau_2}{2} S_i S_i^* + \frac{\kappa\tau_1^2}{2}(\theta_{,i}\vartheta_{,i}^* + \theta_{,i}^*\vartheta_{,i}) + \frac{\kappa_2\tau_1^2}{2}(T_i S_i^* + T_i S_i^*) \\
& + (\tau_1 + \tau_2)\kappa_1(\theta_{,i}T_i^* + \theta_{,i}^* T_i) + \frac{\kappa_1\tau_1^2}{2}(\theta_{,i}S_i^* + \theta_{,i}^* S_i + \vartheta_{,i}T_i^* + \vartheta_{,i}^* T_i) \\
& + \frac{\kappa_1\tau_1^2\tau_2}{2}(\vartheta_{,i}S_i^* + \vartheta_{,i}^* S_i) + \kappa_6(\tau_1 + \tau_2) T_{i,j}T_{i,j}^* + \kappa_6 \frac{\tau_1^2}{2}(T_{i,j}S_{i,j}^* + T_{i,j}^* S_{i,j}) \\
& + \kappa_6 \frac{\tau_1^2\tau_2}{2} S_{ij} S_{ij}^* + (\kappa_4 + \kappa_5)(\tau_1 + \tau_2)(T_{i,i}S_{j,j}^* + T_{i,i}^* S_{j,j}) + (\kappa_4 + \kappa_5)\frac{\tau_1^2\tau_2}{2} S_{i,i}S_{j,j}^* \\
& + (\kappa_4 + \kappa_5)(\tau_1 + \tau_2) T_{i,i}T_{j,j}^* + (\kappa_4 + \kappa_5)\frac{\tau_1^2}{2}(T_{i,i}S_{j,j}^* + T_{i,i}^* S_{j,j}) \Big] d\boldsymbol{x},
\end{aligned}$$

where $\hat{\theta} = \theta + \tau_1\vartheta + \frac{\tau_1^2}{2}\xi$ and $\hat{\boldsymbol{T}} = \boldsymbol{T} + \tau_1\boldsymbol{S} + \frac{\tau_1^2}{2}\boldsymbol{R}$.

It is worth noting that this inner product defines a norm which is equivalent to the usual one in the Hilbert space because of the previous assumptions.

Our problem can be written as

$$\frac{dU}{dt}(t) = \mathcal{A}U(t), \quad U(0) = U^0,$$

where $U^0 = (\boldsymbol{u}^0, \boldsymbol{v}^0, \phi^0, \psi^0, \theta^0, \vartheta^0, \xi^0, \boldsymbol{T}^0, \boldsymbol{S}^0, \boldsymbol{R}^0)$ and

$$\mathcal{A} = \begin{pmatrix}
0 & \boldsymbol{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\boldsymbol{A} & 0 & \boldsymbol{B} & 0 & \boldsymbol{C} & \boldsymbol{D} & \boldsymbol{E} & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
G & 0 & F & 0 & H & J & K & L & M & N \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & O & 0 & P & Q & U & V & W & X & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\
0 & 0 & 0 & \boldsymbol{Y} & \boldsymbol{Z} & \boldsymbol{A}^* & 0 & \boldsymbol{B}^* & \boldsymbol{C}^* & \boldsymbol{D}^*
\end{pmatrix},$$

where I is the identity operator and

$$\begin{aligned}
&A_i\boldsymbol{u} = \frac{1}{\rho}\Big[\mu u_{i,jj} + (\lambda+\mu)u_{j,ji}\Big], \quad \boldsymbol{A} = (A_i),\\
&B_i\phi = \frac{1}{\rho}\mu_0\phi_{,i}, \quad \boldsymbol{B} = (B_i), \quad C_i\theta = -\frac{1}{\rho}\beta_0\theta_{,i}, \quad \boldsymbol{C} = (C_i),\\
&D_i\vartheta = -\frac{\tau_1}{\rho}\beta_0\vartheta_{,i}, \quad \boldsymbol{D} = (D_i), \quad E_i\xi = -\frac{\tau_1^2}{2}\beta_0\xi_{,i}, \quad \boldsymbol{E} = (E_i),\\
&F\phi = \frac{1}{J}[a_0\phi_{,ii} - \zeta\phi], \quad G\boldsymbol{u} = \frac{-1}{J}\mu_0 u_{i,i},\\
&H\theta = \frac{\beta_1}{J}\theta, \quad J\vartheta = \frac{\beta_1\tau_1}{J}\vartheta, \quad K\xi = \frac{\beta_1\tau_1^2}{2J}\xi,\\
&L\boldsymbol{T} = -\frac{\mu_2}{J}T_{i,i}, \quad M\boldsymbol{S} = \frac{-\mu_2\tau_1}{J}S_{i,i}, \quad N\boldsymbol{R} = -\frac{\mu_2\tau_1^2}{2J}R_{i,i},\\
&O\boldsymbol{v} = -\frac{2\beta_0}{a\tau_1^2}v_{i,i}, \quad P\psi = -\frac{2\beta_1}{a\tau_1^2}\psi,\\
&Q\theta = -\frac{2\kappa}{a\tau_1^2}\theta_{,ii}, \quad U\vartheta = -\frac{2}{a\tau_1^2}[\kappa\tau_2\vartheta_{,ii} - \frac{\tau_1^2}{2}\vartheta],\\
&V\xi = -\frac{2}{\tau_1}\xi, \quad W\boldsymbol{T} = \frac{2}{a\tau_1^2}\kappa_1 T_{i,i},\\
&X\boldsymbol{S} = \frac{2\tau_2}{a\tau_1^2}S_{i,i}, \quad Y_i\psi = -\frac{2\mu_2}{b\tau_1^2}\psi_{,i}, \quad \boldsymbol{Y} = (Y_i),\\
&Z_i\theta = -\frac{2\kappa_1}{b\tau_1^2}\theta_{,i}, \quad \boldsymbol{Z} = (Z_i), \quad A_i^*\vartheta = -\frac{2\kappa_1\tau_2}{b\tau_1^2}\vartheta_{,i}, \quad \boldsymbol{A}^* = (A_i^*),\\
&B_i^*\boldsymbol{T} = \frac{2}{b\tau_1^2}[\kappa_6 T_{i,jj} + (\kappa_4+\kappa_5)T_{j,ji}], \quad \boldsymbol{B}^* = (B_i^*),\\
&C_i^*\boldsymbol{S} = \frac{2}{b\tau_1^2}[\kappa_6\tau_2 S_{i,jj} + (\kappa_4+\kappa_5)\tau_2 T_{j,ji} - bS_i], \quad \boldsymbol{C}^* = (C_i^*),\\
&D_i^*\boldsymbol{R} = -\frac{2}{\tau_1}R_i, \quad \boldsymbol{D}^* = (D_i^*).
\end{aligned}$$

We note that the domain of our operator is the subset $(\boldsymbol{u}, \boldsymbol{v}, \phi, \psi, \theta, \vartheta, \xi, \boldsymbol{T}, \boldsymbol{S}, \boldsymbol{R}) \in \mathcal{H}$ such that

$$\begin{aligned}
&A\boldsymbol{u} \in \boldsymbol{L}^2(B), \quad \xi \in W_0^{1,2}(B), \quad F\phi \in L^2(B), \quad \boldsymbol{R} \in \boldsymbol{W}_0^{1,2}(B), \quad \boldsymbol{v} \in \boldsymbol{W}_0^{1,2}(B),\\
&P\theta + Q\vartheta \in L^2(B), \quad \boldsymbol{B}^*\boldsymbol{T} + \boldsymbol{C}^*\boldsymbol{S} \in \boldsymbol{L}^2(B), \quad \psi \in W_0^{1,2}(B).
\end{aligned}$$

It is clear that this is a dense subspace of our Hilbert space. On the other side, we have that

$$\begin{aligned}
Re < \mathcal{A}U, U >= -\frac{1}{2}\int_B \Big[&\kappa(\theta_{,i}\theta_{,i} + (\tau_1\tau_2 - \frac{\tau_1^2}{2})\vartheta_{,i}\vartheta_{,i})\\
&+\kappa_2(T_iT_i + (\tau_1\tau_2 - \frac{\tau_1^2}{2})S_iS_i) + 2\kappa_1(\theta_{,i}T_i + (\tau_1\tau_2 - \frac{\tau_1^2}{2})\vartheta_{,i}S_i)\\
&+\kappa_6(T_{i,j}T_{i,j} + (\tau_1\tau_2 - \frac{\tau_1^2}{2})S_{i,j}S_{i,j})\\
&+(\kappa_4+\kappa_5)(T_{i,i}T_{j,j} + (\tau_1\tau_2 - \frac{\tau_1^2}{2})S_{i,i}S_{j,j})\Big]\,d\boldsymbol{x} \le 0
\end{aligned}$$

for every U in the domain of the operator denoted by $\mathcal{D}(\mathcal{A})$.

To show that $\mathcal{A}$ generates a contractive semigroup, it is sufficient to prove that zero belongs to the resolvent of the operator.

To this end, we consider $F = (\boldsymbol{f}_1, \boldsymbol{f}_2, f_3, f_4, f_5, f_6, f_7, \boldsymbol{f}_8, \boldsymbol{f}_9, \boldsymbol{f}_{10}) \in \mathcal{H}$. We need to prove that the equation

$$\mathcal{A}U = F$$

has a solution in the domain of the operator. That is, we must solve the following system:

$$\begin{aligned}
&\boldsymbol{v} = \boldsymbol{f}_1,\\
&\boldsymbol{Au} + \boldsymbol{B}\phi + \boldsymbol{C}\theta + \boldsymbol{D}\vartheta + \boldsymbol{E}\xi = \boldsymbol{f}_2,\\
&\psi = f_3,\\
&\boldsymbol{Gu} + F\phi + H\theta + J\vartheta + K\xi + L\boldsymbol{T} + M\boldsymbol{S} + N\boldsymbol{R} = f_4,\\
&\vartheta = f_5,\\
&\xi = f_6,\\
&\boldsymbol{Ov} + P\psi + Q\theta + U\vartheta + V\xi + W\boldsymbol{T} + X\boldsymbol{S} = f_7,\\
&\boldsymbol{S} = \boldsymbol{f}_8,\\
&\boldsymbol{R} = \boldsymbol{f}_9,\\
&\boldsymbol{Y}\psi + \boldsymbol{Z}\theta + \boldsymbol{A}^*\vartheta + \boldsymbol{B}^*\boldsymbol{T} + \boldsymbol{C}^*\boldsymbol{S} + \boldsymbol{D}^*\boldsymbol{R} = \boldsymbol{f}_{10}.
\end{aligned}$$

We obtain the solution for $\boldsymbol{v}$, ψ, $\boldsymbol{R}$ and ξ satisfying the regularity conditions and we also find that

$$\begin{aligned}
&\boldsymbol{Au} + \boldsymbol{B}\phi + \boldsymbol{C}\theta = \boldsymbol{f}_2 - \boldsymbol{D}f_5 - \boldsymbol{E}f_6,\\
&\boldsymbol{Gu} + F\phi + H\theta + L\boldsymbol{T} = f_4 - Jf_5 - Kf_6 - M\boldsymbol{f}_8 - N\boldsymbol{f}_9,\\
&Q\theta + W\boldsymbol{T} = f_7 - \boldsymbol{Of}_1 - Pf_3 - Uf_5 - Vf_6 - X\boldsymbol{f}_8,\\
&\boldsymbol{Z}\theta + \boldsymbol{B}^*\boldsymbol{T} = \boldsymbol{f}_{10} - \boldsymbol{Y}f_3 - \boldsymbol{A}^*f_5 - \boldsymbol{C}^*\boldsymbol{f}_8 - \boldsymbol{D}^*\boldsymbol{f}_9.
\end{aligned}$$

If we consider the last two equations we see that the right-hand side belongs to $W^{-1,2}(B) \times \boldsymbol{W}^{-1,2}(B)$. On the other side, the bilinear form

$$\mathcal{B}[(\theta, \boldsymbol{T}), (\theta^*, \boldsymbol{T}^*)] =< (\theta, \boldsymbol{T}), (Q\theta^* + W\boldsymbol{T}^*, \boldsymbol{Z}\theta^* + \boldsymbol{B}^*\boldsymbol{T}^*) >$$

defines a bounded, and coercive, bilinear form in $W_0^{1,2}(B) \times \boldsymbol{W}_0^{1,2}(B)$. In view of the Lax-Milgram lemma, we conclude the existence of $(\theta, \boldsymbol{T}) \in W_0^{1,2}(B) \times \boldsymbol{W}_0^{1,2}(B)$ satisfying the last two equations. Now, if we introduce this solution into the first two ones we obtain

$$\begin{aligned}
&\boldsymbol{Au} + \boldsymbol{B}\phi = \boldsymbol{f}_2 - \boldsymbol{D}f_5 - \boldsymbol{E}f_6 - \boldsymbol{C}\theta,\\
&\boldsymbol{Gu} + F\phi = f_4 - Jf_5 - Kf_6 - M\boldsymbol{f}_8 - N\boldsymbol{f}_9 - H\theta.
\end{aligned}$$

A similar argument shows the existence of $(\boldsymbol{u}, \phi) \in \boldsymbol{W}_0^{1,2}(B) \times W_0^{1,2}(B)$.

In fact, we can also prove an inequality of the type

$$||U|| \leq C^*||F||$$

for a given positive constant $C^* > 0$, and so, we have proved the following.

Theorem 3.1. *The operator $\mathcal{A}$ generates a contractive semigroup.*

Therefore, we can conclude the existence and uniqueness of solution to problem (2.1)-(2.6), that we state as follows.

Theorem 3.2. *Assume that constitutive coefficients satisfy conditions (2.7) and that $U^0 \in \mathcal{D}(\mathcal{A})$. Then, there exists a unique solution $U \in C^1([0,\infty); \mathcal{H}) \cap C([0,\infty); \mathcal{D}(\mathcal{A}))$ to system (2.1)-(2.4), boundary conditions (2.5) and initial conditions (2.6).*

4. Fully discrete approximations: a priori error analysis

In this section, we consider a fully discrete aproximation to a variational problem of the above thermomechanical problem by using the finite element method and the implicit Euler scheme.

Let $Y = L^2(B)$, $H = [L^2(B)]^d$ and $Q = [L^2(B)]^{d\times d}$, and denote by $(\cdot,\cdot)_Y$, $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_Q$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_Y$, $\|\cdot\|_H$ and $\|\cdot\|_Q$. Moreover, let the variational spaces E and V be given as

$$E = \{z \in H^1(B)\,;\, z = 0 \quad \text{on} \quad \Gamma\},$$
$$V = \{\boldsymbol{w} \in [H^1(B)]^d\,;\, \boldsymbol{w} = \boldsymbol{0} \quad \text{on} \quad \Gamma\},$$

with respective scalar products $(\cdot,\cdot)_E$ and $(\cdot,\cdot)_V$, and norms $\|\cdot\|_E$ and $\|\cdot\|_V$.

By using Green's formula and taking into account the above boundary conditions, we write the variational formulation of the thermomechanical problem in terms of variables $\boldsymbol{v} = (v_i) = \dot{\boldsymbol{u}} = (\dot{u}_i)$ and $\psi = \dot{\phi}$, the temperature speed $\vartheta = \dot{\theta}$, the temperature acceleration $\xi = \dot{\vartheta}$, the microtemperature speed $\boldsymbol{S} = (S_i) = \dot{\boldsymbol{T}} = (\dot{T}_i)$ and the microtemperature acceleration $\boldsymbol{R} = (R_i) = \dot{\boldsymbol{S}} = (\dot{S}_i)$.

Problem VP. *Find the function $\boldsymbol{v} : [0,T_f] \to V$, the function $\psi : [0,T_f] \to E$, the temperature acceleration $\xi : [0,T_f] \to E$ and the microtemperatures acceleration $\boldsymbol{R} : [0,T_f] \to V$ such that $\boldsymbol{v}(0) = \boldsymbol{v}^0$, $\psi(0) = \psi^0$, $\xi(0) = \xi^0$ and $\boldsymbol{R}(0) = \boldsymbol{R}^0$, and, for a.e. $t \in (0,T_f)$ and $\boldsymbol{w}$, $\boldsymbol{\eta} \in V$, r, $z \in E$,*

$$\begin{aligned}&\rho(\dot{\boldsymbol{v}}(t),\boldsymbol{w})_H + \mu(\nabla\boldsymbol{u}(t),\nabla\boldsymbol{w})_Q + (\lambda+\mu)(\operatorname{div}\boldsymbol{u}(t),\operatorname{div}\boldsymbol{w})_Y\\ &\quad -\mu_0(\nabla\phi(t),\boldsymbol{w})_H + \beta_0(\nabla\Big(\frac{\tau_1^2}{2}\xi(t)+\tau_1\vartheta(t)+\theta(t)\Big),\boldsymbol{w})_H = 0,\end{aligned} \tag{4.1}$$

$$\begin{aligned}&J(\dot{\psi}(t),r)_Y + a_0(\nabla\phi(t),\nabla r)_H + \zeta(\phi(t),r)_Y = -\mu_0(\operatorname{div}\boldsymbol{u}(t),r)_Y\\ &\quad -\mu_2\Big(\frac{\tau_1^2}{2}\operatorname{div}\boldsymbol{R}(t)+\tau_1\operatorname{div}\boldsymbol{S}(t)+\operatorname{div}\boldsymbol{T}(t),r\Big)_Y\\ &\quad +\beta_1\Big(\frac{\tau_1^2}{2}\xi(t)+\tau_1\vartheta(t)+\theta(t),r\Big)_Y,\end{aligned} \tag{4.2}$$

$$\begin{aligned}&a\Big(\frac{\tau_1^2}{2}\dot{\xi}(t)+\tau_1\xi(t)+\vartheta(t),z\Big)_Y + \kappa(\nabla(\theta(t)+\tau_2\vartheta(t)),\nabla z)_H = -\beta_0(\operatorname{div}\boldsymbol{v}(t),z)_Y\\ &\quad -\beta_1(\psi(t),z)_Y + \kappa_1(\operatorname{div}\boldsymbol{T}(t)+\tau_2\operatorname{div}\boldsymbol{S}(t),z)_Y,\end{aligned} \tag{4.3}$$

$$\begin{aligned}&b\Big(\frac{\tau_1^2}{2}\dot{\boldsymbol{R}}(t)+\tau_1\boldsymbol{R}(t)+\boldsymbol{S}(t),\boldsymbol{\eta}\Big)_H + \kappa_6(\nabla(\boldsymbol{T}(t)+\tau_2\boldsymbol{S}(t)),\nabla\boldsymbol{\eta})_Q\\ &\quad +(\kappa_4+\kappa_5)(\operatorname{div}(\boldsymbol{T}(t)+\tau_2\boldsymbol{S}(t)),\operatorname{div}\boldsymbol{\eta})_Y + \kappa_2(\boldsymbol{T}(t)+\tau_2\boldsymbol{S}(t),\boldsymbol{\eta})_H\\ &\quad = -\kappa_1(\nabla\theta(t)+\tau_1\vartheta(t),\boldsymbol{\eta})_H - \mu_2(\nabla\psi(t),\boldsymbol{\eta})_H,\end{aligned} \tag{4.4}$$

where functions $\boldsymbol{u}$, ϕ, ϑ, θ, $\boldsymbol{S}$ and $\boldsymbol{T}$ are then recovered from the relations:

$$\begin{aligned}\boldsymbol{u}(t) &= \int_0^t \boldsymbol{v}(s)\,ds + \boldsymbol{u}^0, \quad & \phi(t) &= \int_0^t \psi(s)\,ds + \phi^0,\\ \vartheta(t) &= \int_0^t \xi(s)\,ds + \vartheta^0, \quad & \boldsymbol{S}(t) &= \int_0^t \boldsymbol{R}(s)\,ds + \boldsymbol{S}^0,\\ \theta(t) &= \int_0^t \vartheta(s)\,ds + \theta^0, \quad & \boldsymbol{T}(t) &= \int_0^t \boldsymbol{S}(s)\,ds + \boldsymbol{T}^0,\end{aligned} \tag{4.5}$$

and $T_f > 0$ denotes the final time of interest.

Thus, we construct the finite element spaces $V^h \subset V$and $E^h \subset E$ given by

$$V^h = \{\boldsymbol{w}^h \in [C(\overline{B})]^d \; ; \; \boldsymbol{w}^h_{|Tr} \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h, \quad \boldsymbol{w}^h = \boldsymbol{0} \text{ on } \Gamma\}, \tag{4.6}$$

$$E^h = \{r^h \in C(\overline{B}) \; ; \; r^h_{|Tr} \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h, \quad r^h = 0 \text{ on } \Gamma\}, \tag{4.7}$$

where B is assumed to be a polyhedral domain, $\mathcal{T}^h$ denotes a regular triangulation, in the sense of [29], of $\overline{B}$, and $P_1(Tr)$ represents the space of polynomials of global degree less or equal to 1 in element Tr. Here, $h > 0$ denotes the spatial discretization parameter.

In order to discretize the time derivatives, we use a uniform partition of the time interval $[0, T_f]$ denoted by $0 = t_0 < t_1 < \ldots < t_N = T_f$, with time step $k = T_f/N$. Moreover, for a continuous function $f(t)$ let $f_n = f(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Using the classical implicit Euler scheme, the fully discrete approximation of Problem VP is the following.

Problem VP^{hk}. *Find the discrete function $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$, the discrete function $\psi^{hk} = \{\psi_n^{hk}\}_{n=0}^N \subset E^h$, the discrete temperature acceleration $\xi^{hk} = \{\xi_n^{hk}\}_{n=0}^N \subset E^h$ and the discrete microtemperature acceleration $\boldsymbol{R}^{hk} = \{\boldsymbol{R}_n^{hk}\}_{n=0}^N \subset V^h$ such that $\boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h$, $\psi_0^{hk} = \psi_0^h$, $\xi_0^{hk} = \xi_0^h$ and $\boldsymbol{R}_0^{hk} = \boldsymbol{R}_0^h$, and, for all $n = 1, \ldots, N$ and $\boldsymbol{w}^h$, $\boldsymbol{\eta}^h \in V^h$, r^h, $z^h \in E^h$,*

$$\begin{aligned} &\rho(\delta \boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + \mu(\nabla \boldsymbol{u}_n^{hk}, \nabla \boldsymbol{w}^h)_Q + (\lambda + \mu)(\operatorname{div} \boldsymbol{u}_n^{hk}, \operatorname{div} \boldsymbol{w}^h)_Y \\ &\quad -\mu_0(\nabla \phi_n^{hk}, \boldsymbol{w}^h)_H + \beta_0(\nabla\Big(\frac{\tau_1^2}{2}\xi_n^{hk} + \tau_1 \vartheta_n^{hk} + \theta_n^{hk}\Big), \boldsymbol{w}^h)_H = 0, \end{aligned} \tag{4.8}$$

$$\begin{aligned} &J(\delta \psi_n^{hk}, r^h)_Y + a_0(\nabla \phi_n^{hk}, \nabla r^h)_H + \zeta(\phi_n^{hk}, r^h)_Y = -\mu_0(\operatorname{div} \boldsymbol{u}_n^{hk}, r^h)_Y \\ &\quad -\mu_2\Big(\frac{\tau_1^2}{2}\operatorname{div} \boldsymbol{R}_n^{hk} + \tau_1 \operatorname{div} \boldsymbol{S}_n^{hk} + \operatorname{div} \boldsymbol{T}_n^{hk}, r^h\Big)_Y \\ &\quad +\beta_1\Big(\frac{\tau_1^2}{2}\xi_n^{hk} + \tau_1 \vartheta_n^{hk} + \theta_n^{hk}, r^h\Big)_Y, \end{aligned} \tag{4.9}$$

$$\begin{aligned} &a\Big(\frac{\tau_1^2}{2}\delta\xi_n^{hk} + \tau_1 \xi_n^{hk} + \vartheta_n^{hk}, z^h\Big)_Y + \kappa(\nabla(\theta_n^{hk} + \tau_2 \vartheta_n^{hk}), \nabla z^h)_H = -\beta_0(\operatorname{div} \boldsymbol{v}_n^{hk}, z^h)_Y \\ &\quad -\beta_1(\psi_n^{hk}, z^h)_Y + \kappa_1(\operatorname{div} \boldsymbol{T}_n^{hk} + \tau_2 \operatorname{div} \boldsymbol{S}_n^{hk}, z^h)_Y, \end{aligned} \tag{4.10}$$

$$\begin{aligned} &b\Big(\frac{\tau_1^2}{2}\delta \boldsymbol{R}_n^{hk} + \tau_1 \boldsymbol{R}_n^{hk} + \boldsymbol{S}_n^{hk}, \boldsymbol{\eta}^h\Big)_H + \kappa_6(\nabla(\boldsymbol{T}_n^{hk} + \tau_2 \boldsymbol{S}_n^{hk}), \nabla \boldsymbol{\eta}^h)_Q \\ &\quad +(\kappa_4 + \kappa_5)(\operatorname{div} (\boldsymbol{T}_n^{hk} + \tau_2 \boldsymbol{S}_n^{hk}), \operatorname{div} \boldsymbol{\eta}^h)_Y + \kappa_2(\boldsymbol{T}_n^{hk} + \tau_2 \boldsymbol{S}_n^{hk}, \boldsymbol{\eta}^h)_H \\ &\quad = -\kappa_1(\nabla(\theta_n^{hk} + \tau_1 \vartheta_n^{hk}), \boldsymbol{\eta}^h)_H - \mu_2(\nabla \psi_n^{hk}, \boldsymbol{\eta}^h)_H, \end{aligned} \tag{4.11}$$

where discrete functions $\boldsymbol{u}_n^{hk}$, ϕ_n^{hk}, ϑ_n^{hk} θ_n^{hk}, $\boldsymbol{S}_n^{hk}$ and $\boldsymbol{T}_n^{hk}$ are then recovered from the relations:

$$\begin{aligned} &\boldsymbol{u}_n^{hk} = k\sum_{j=1}^n \boldsymbol{v}_j^{hk} + \boldsymbol{u}_0^h, \quad \phi_n^{hk} = k\sum_{j=1}^n \psi_j^{hk} + \phi_0^h, \\ &\vartheta_n^{hk} = k\sum_{j=1}^n \theta_j^{hk} + \vartheta_0^h, \quad \boldsymbol{S}_n^{hk} = k\sum_{j=1}^n \boldsymbol{R}_j^{hk} + \boldsymbol{S}_0^h, \\ &\theta_n^{hk} = k\sum_{j=1}^n \vartheta_j^{hk} + \theta_0^h, \quad \boldsymbol{T}_n^{hk} = k\sum_{j=1}^n \boldsymbol{S}_j^{hk} + \boldsymbol{T}_0^h, \end{aligned} \tag{4.12}$$

and the discrete initial conditions $\boldsymbol{u}_0^h$, $\boldsymbol{v}_0^h$, ϕ_0^h, ψ_0^h, θ_0^h, ϑ_0^h, ξ_0^h, $\boldsymbol{T}_0^h$, $\boldsymbol{S}_0^h$ *and* $\boldsymbol{R}_0^h$ *are defined as follows:*

$$\begin{array}{llll} \boldsymbol{u}_0^h = \mathcal{P}^{1h}\boldsymbol{u}^0, & \boldsymbol{v}_0^h = \mathcal{P}^{1h}\boldsymbol{v}^0, & \phi_0^h = \mathcal{P}^{2h}\phi^0, & \psi_0^h = \mathcal{P}^{2h}\psi^0, \\ \theta_0^h = \mathcal{P}^{2h}\theta^0, & \vartheta_0^h = \mathcal{P}^{2h}\vartheta^0, & \xi_0^h = \mathcal{P}^{2h}\xi^0, & \boldsymbol{T}_0^h = \mathcal{P}^{1h}\boldsymbol{T}^0, \\ \boldsymbol{S}_0^h = \mathcal{P}^{1h}\boldsymbol{S}^0, & \boldsymbol{R}_0^h = \mathcal{P}^{1h}\boldsymbol{R}^0. & & \end{array} \tag{4.13}$$

In the previous definitions, operators $\mathcal{P}^{1h}$ *and* $\mathcal{P}^{2h}$ *are the projection operators over the finite element spaces* V^h *and* E^h*, respectively (see, for instance, [30]).*

Using the previously given assumptions (2.7), applying the well-known Lax-Milgram lemma we can easily show that fully discrete problem VP^{hk} has a unique solution.

In this section, the objective is to prove a main error estimates result regarding the approximation of the solution to Problem VP by the solution to Problem VP^{hk}. So, first we have the following discrete stability property.

Lemma 4.1. *Under the assumptions of Theorem 3.2, it follows that the sequences*

$$\{\boldsymbol{u}^{hk}, \boldsymbol{v}^{hk}, \phi^{hk}, \psi^{hk}, \theta^{hk}, \vartheta^{hk}, \xi^{hk}, \boldsymbol{T}^{hk}, \boldsymbol{S}^{hk}, \boldsymbol{R}^{hk}\},$$

generated by Problem VP^{hk}*, satisfy the stability estimate:*

$$\begin{aligned} &\|\boldsymbol{v}_n^{hk}\|_H^2 + \|\boldsymbol{u}_n^{hk}\|_V^2 + \|\psi_n^{hk}\|_Y^2 + \|\phi_n^{hk}\|_V^2 + \|\theta_n^{hk}\|_E^2 + \|\vartheta_n^{hk}\|_E^2 + \|\xi_n^{hk}\|_Y^2 \\ &\quad + \|\boldsymbol{R}_n^{hk}\|_H^2 + \|\boldsymbol{S}_n^{hk}\|_V^2 + \|\boldsymbol{T}_n^{hk}\|_V^2 \le C, \end{aligned}$$

where C is a positive constant assumed to be independent of the discretization parameters h and k.

Proof. For the sake of simplicity, in this proof we need to assume that $\tau_1^2/2 = 1$. We note that we can extend the analysis provided below to the general case doing some simple modifications.

First, we estimate the terms on the discrete function $\boldsymbol{v}_n^{hk}$. Taking $\boldsymbol{w}^h = \boldsymbol{v}_n^{hk}$ as a test function in discrete variational equation (4.8) we obtain

$$\begin{aligned} &\rho(\delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n^{hk})_H + \mu(\nabla\boldsymbol{u}_n^{hk}, \nabla\boldsymbol{v}_n^{hk})_Q + (\lambda+\mu)(\operatorname{div}\boldsymbol{u}_n^{hk}, \operatorname{div}\boldsymbol{v}_n^{hk})_Y \\ &\quad -\mu_0(\nabla\phi_n^{hk}, \boldsymbol{v}_n^{hk})_H + \beta_0(\nabla(\xi_n^{hk} + \tau_1\vartheta_n^{hk} + \theta_n^{hk}), \boldsymbol{v}_n^{hk})_H = 0. \end{aligned}$$

From the estimates

$$\begin{aligned} &(\delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n^{hk})_H \ge \frac{1}{2k}\left\{\|\boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\}, \\ &(\nabla\boldsymbol{u}_n^{hk}, \nabla\boldsymbol{v}_n^{hk})_Q \ge \frac{1}{2k}\left\{\|\nabla\boldsymbol{u}_n^{hk}\|_Q^2 - \|\nabla\boldsymbol{u}_{n-1}^{hk}\|_Q^2\right\}, \\ &(\operatorname{div}\boldsymbol{u}_n^{hk}, \operatorname{div}\boldsymbol{v}_n^{hk})_Y \ge \frac{1}{2k}\left\{\|\operatorname{div}\boldsymbol{u}_n^{hk}\|_Y^2 - \|\operatorname{div}\boldsymbol{u}_{n-1}^{hk}\|_Y^2\right\}, \end{aligned}$$

applying the Cauchy's inequality:

$$ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2, \quad a, b, \epsilon \in \mathbb{R}, \ \epsilon > 0, \tag{4.14}$$

it follows that

$$\begin{aligned} &\frac{\rho}{2k}\left\{\|\boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\} + \frac{\mu}{2k}\left\{\|\nabla\boldsymbol{u}_n^{hk}\|_Q^2 - \|\nabla\boldsymbol{u}_{n-1}^{hk}\|_Q^2\right\} \\ &\quad + \frac{\lambda+\mu}{2k}\left\{\|\operatorname{div}\boldsymbol{u}_n^{hk}\|_Y^2 - \|\operatorname{div}\boldsymbol{u}_{n-1}^{hk}\|_Y^2 + \|\operatorname{div}(\boldsymbol{u}_n^{hk} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\right\} \\ &\quad + \beta_0(\nabla\xi_n^{hk}, \boldsymbol{v}_n^{hk})_H \le C\Big(\|\boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla\phi_n^{hk}\|_H^2 + \|\nabla\theta_n^{hk}\|_H^2 + \|\nabla\vartheta_n^{hk}\|_H^2\Big). \end{aligned}$$

Now, we estimate the terms on the discrete function ψ_n^{hk}. Taking $r^h = \psi_n^{hk}$ as a test function in discrete variational equation (4.9) we have

$$J(\delta\psi_n^{hk}, \psi_n^{hk})_Y + a_0(\nabla\phi_n^{hk}, \nabla\psi_n^{hk})_H + \zeta(\phi_n^{hk}, \psi_n^{hk})_Y = -\mu_0(\operatorname{div} \boldsymbol{u}_n^{hk}, \psi_n^{hk})_Y$$
$$-\mu_2(\operatorname{div} \boldsymbol{R}_n^{hk} + \tau_1 \operatorname{div} \boldsymbol{S}_n^{hk} + \operatorname{div} \boldsymbol{T}_n^{hk}, \psi_n^{hk})_Y + \beta_1(\xi_n^{hk} + \tau_1\vartheta_n^{hk} + \theta_n^{hk}, \psi_n^{hk})_Y,$$

and, using the estimates

$$(\delta\psi_n^{hk}, \psi_n^{hk})_Y \geq \frac{1}{2k}\left\{\|\psi_n^{hk}\|_Y^2 - \|\psi_{n-1}^{hk}\|_Y^2\right\},$$
$$(\nabla\phi_n^{hk}, \nabla\psi_n^{hk})_H \geq \frac{1}{2k}\left\{\|\nabla\phi_n^{hk}\|_H^2 - \|\nabla\phi_{n-1}^{hk}\|_Y^2\right\},$$
$$(\phi_n^{hk}, \psi_n^{hk})_Y \geq \frac{1}{2k}\left\{\|\phi_n^{hk}\|_Y^2 - \|\phi_{n-1}^{hk}\|_Y^2\right\},$$

we find that

$$\frac{1}{2k}\left\{\|\psi_n^{hk}\|_Y^2 - \|\psi_{n-1}^{hk}\|_Y^2\right\} + \frac{1}{2k}\left\{\|\nabla\phi_n^{hk}\|_H^2 - \|\nabla\phi_{n-1}^{hk}\|_Y^2\right\}$$
$$+\frac{1}{2k}\left\{\|\phi_n^{hk}\|_Y^2 - \|\phi_{n-1}^{hk}\|_Y^2\right\} + \mu_2(\operatorname{div} \boldsymbol{R}_n^{hk}, \psi_n^{hk})_Y$$
$$\leq C\Big(\|\psi_n^{hk}\|_Y^2 + \|\operatorname{div} \boldsymbol{u}_n^{hk}\|_Y^2 + \|\operatorname{div} \boldsymbol{S}_n^{hk}\|_Y^2 + \|\operatorname{div} \boldsymbol{T}_n^{hk}\|_Y^2 + \|\vartheta_n^{hk}\|_Y^2 + \|\theta_n^{hk}\|_Y^2 + \|\xi_n^{hk}\|_Y^2\Big).$$

Thirdly, we get the estimates on the discrete temperature acceleration ξ_n^{hk}. Thus, taking $z^h = \xi_n^{hk}$ as a test function in discrete variational equation (4.10) we obtain

$$a(\delta\xi_n^{hk} + \tau_1\xi_n^{hk} + \vartheta_n^{hk}, \xi_n^{hk})_Y + \kappa(\nabla(\theta_n^{hk} + \tau_2\vartheta_n^{hk}), \nabla\xi_n^{hk})_H = -\beta_0(\operatorname{div} \boldsymbol{v}_n^{hk}, \xi_n^{hk})_Y$$
$$-\beta_1(\psi_n^{hk}, \xi_n^{hk})_Y + \kappa_1(\operatorname{div} \boldsymbol{T}_n^{hk} + \tau_2 \operatorname{div} \boldsymbol{S}_n^{hk}, \xi_n^{hk})_Y,$$

and therefore, taking into account that

$$(\delta\xi_n^{hk}, \xi_n^{hk})_Y \geq \frac{1}{2k}\left\{\|\xi_n^{hk}\|_Y^2 - \|\xi_{n-1}^{hk}\|_Y^2\right\},$$
$$(\nabla\vartheta_n^{hk}, \nabla\xi_n^{hk})_H \geq \frac{1}{2k}\left\{\|\nabla\vartheta_n^{hk}\|_H^2 - \|\nabla\vartheta_{n-1}^{hk}\|_H^2\right\},$$
$$(\vartheta_n^{hk}, \xi_n^{hk})_Y \geq \frac{1}{2k}\left\{\|\vartheta_n^{hk}\|_Y^2 - \|\vartheta_{n-1}^{hk}\|_Y^2\right\},$$

we find that

$$\frac{a}{2k}\left\{\|\xi_n^{hk}\|_Y^2 - \|\xi_{n-1}^{hk}\|_Y^2\right\} + \frac{\tau_2}{2k}\left\{\|\nabla\vartheta_n^{hk}\|_H^2 - \|\nabla\vartheta_{n-1}^{hk}\|_H^2\right\} + \beta_0(\operatorname{div} \boldsymbol{v}_n^{hk}, \xi_n^{hk})_Y$$
$$+\frac{a}{2k}\left\{\|\vartheta_n^{hk}\|_Y^2 - \|\vartheta_{n-1}^{hk}\|_Y^2\right\} + \kappa(\nabla\theta_n^{hk}, \nabla\xi_n^{hk})_H$$
$$\leq C\Big(\|\xi_n^{hk}\|_Y^2 + \|\nabla\theta_n^{hk}\|_H^2 + \|\psi_n^{hk}\|_Y^2 + \|\operatorname{div} \boldsymbol{S}_n^{hk}\|_Y^2 + \|\operatorname{div} \boldsymbol{T}_n^{hk}\|_Y^2\Big).$$

Finally, we get the estimates on the discrete microtemperatures acceleration $\boldsymbol{R}_n^{hk}$. Then, taking $\boldsymbol{\eta}^h = \boldsymbol{R}_n^{hk}$ as a test function in discrete variational equation (4.11) it follows that

$$b(\delta\boldsymbol{R}_n^{hk} + \tau_1\boldsymbol{R}_n^{hk} + \boldsymbol{S}_n^{hk}, \boldsymbol{R}_n^{hk})_H + \kappa_6(\nabla(\boldsymbol{T}_n^{hk} + \tau_2\boldsymbol{S}_n^{hk}), \nabla\boldsymbol{R}_n^{hk})_Q$$
$$+(\kappa_4 + \kappa_5)(\operatorname{div}(\boldsymbol{T}_n^{hk} + \tau_2\boldsymbol{S}_n^{hk}), \operatorname{div} \boldsymbol{R}_n^{hk})_Y + \kappa_2(\boldsymbol{T}_n^{hk} + \tau_2\boldsymbol{S}_n^{hk}, \boldsymbol{R}_n^{hk})_H$$
$$= -\kappa_1(\nabla(\theta_n^{hk} + \tau_1\vartheta_n^{hk}), \boldsymbol{R}_n^{hk})_H - \mu_2(\nabla\psi_n^{hk}, \boldsymbol{R}_n^{hk})_H,$$

and so, keeping in mind that

$$
\begin{aligned}
&(\delta \boldsymbol{R}_n^{hk}, \boldsymbol{R}_n^{hk})_H \geq \frac{1}{2k}\left\{\|\boldsymbol{R}_n^{hk}\|_H^2 - \|\boldsymbol{R}_{n-1}^{hk}\|_H^2\right\},\\
&(\nabla \boldsymbol{S}_n^{hk}, \nabla \boldsymbol{R}_n^{hk})_Q \geq \frac{1}{2k}\left\{\|\nabla \boldsymbol{S}_n^{hk}\|_Q^2 - \|\nabla \boldsymbol{S}_{n-1}^{hk}\|_Q^2\right\},\\
&(\operatorname{div} \boldsymbol{S}_n^{hk}, \operatorname{div} \boldsymbol{R}_n^{hk})_Y \geq \frac{1}{2k}\left\{\|\operatorname{div} \boldsymbol{S}_n^{hk}\|_Y^2 - \|\operatorname{div} \boldsymbol{S}_{n-1}^{hk}\|_Y^2\right\},\\
&(\boldsymbol{S}_n^{hk}, \boldsymbol{R}_n^{hk})_H \geq \frac{1}{2k}\left\{\|\boldsymbol{S}_n^{hk}\|_H^2 - \|\boldsymbol{S}_{n-1}^{hk}\|_H^2\right\},\\
&(\nabla \psi_n^{hk}, \boldsymbol{R}_n^{hk})_H = -(\psi_n^{hk}, \operatorname{div} \boldsymbol{R}_n^{hk})_Y,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\frac{b}{2k}\left\{\|\boldsymbol{R}_n^{hk}\|_H^2 - \|\boldsymbol{R}_{n-1}^{hk}\|_H^2\right\} + \frac{\kappa_6}{2k}\left\{\|\nabla \boldsymbol{S}_n^{hk}\|_Q^2 - \|\nabla \boldsymbol{S}_{n-1}^{hk}\|_Q^2\right\} - \mu_2(\psi_n^{hk}, \operatorname{div} \boldsymbol{R}_n^{hk})_Y\\
&\quad + \frac{\kappa_4+\kappa_5}{2k}\left\{\|\operatorname{div} \boldsymbol{S}_n^{hk}\|_Y^2 - \|\operatorname{div} \boldsymbol{S}_{n-1}^{hk}\|_Y^2\right\} + \frac{\kappa_2\tau_2}{2k}\left\{\|\boldsymbol{S}_n^{hk}\|_H^2 - \|\boldsymbol{S}_{n-1}^{hk}\|_H^2\right\}\\
&\quad + (\kappa_4+\kappa_5)(\operatorname{div} \boldsymbol{T}_n^{hk}, \operatorname{div} \boldsymbol{R}_n^{hk})_Y + \kappa_6(\nabla \boldsymbol{T}_n^{hk}, \nabla \boldsymbol{R}_n^{hk})_Q\\
&\leq C\Big(\|\boldsymbol{R}_n^{hk}\|_H^2 + \|\nabla \theta_n^{hk}\|_H^2 + \|\nabla \vartheta_n^{hk}\|_H^2 + \|\boldsymbol{T}_n^{hk}\|_H^2\Big).
\end{aligned}
$$

Combining all these estimates, it leads to

$$
\begin{aligned}
&\frac{\rho}{2k}\left\{\|\boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1}^{hk}\|_H^2\right\} + \frac{\mu}{2k}\left\{\|\nabla \boldsymbol{u}_n^{hk}\|_Q^2 - \|\nabla \boldsymbol{u}_{n-1}^{hk}\|_Q^2\right\}\\
&\quad + \frac{\lambda+\mu}{2k}\left\{\|\operatorname{div} \boldsymbol{u}_n^{hk}\|_Y^2 - \|\operatorname{div} \boldsymbol{u}_{n-1}^{hk}\|_Y^2\right\}\\
&\quad + \frac{1}{2k}\left\{\|\psi_n^{hk}\|_Y^2 - \|\psi_{n-1}^{hk}\|_Y^2\right\} + \frac{1}{2k}\left\{\|\nabla \phi_n^{hk}\|_H^2 - \|\nabla \phi_{n-1}^{hk}\|_Y^2\right\}\\
&\quad + \frac{1}{2k}\left\{\|\phi_n^{hk}\|_Y^2 - \|\phi_{n-1}^{hk}\|_Y^2\right\} + \frac{1}{2k}\left\{\|\vartheta_n^{hk}\|_Y^2 - \|\vartheta_{n-1}^{hk}\|_Y^2\right\}\\
&\quad + \frac{1}{2k}\left\{\|\xi_n^{hk}\|_Y^2 - \|\xi_{n-1}^{hk}\|_Y^2\right\} + \frac{1}{2k}\left\{\|\nabla \vartheta_n^{hk}\|_H^2 - \|\nabla \vartheta_{n-1}^{hk}\|_H^2\right\}\\
&\quad + \kappa(\nabla \theta_n^{hk}, \nabla \xi_n^{hk})_H + (\kappa_4+\kappa_5)(\operatorname{div} \boldsymbol{T}_n^{hk}, \operatorname{div} \boldsymbol{R}_n^{hk})_Y + \kappa_6(\nabla \boldsymbol{T}_n^{hk}, \nabla \boldsymbol{R}_n^{hk})_Q\\
&\quad + \frac{1}{2k}\left\{\|\boldsymbol{R}_n^{hk}\|_H^2 - \|\boldsymbol{R}_{n-1}^{hk}\|_H^2\right\} + \frac{1}{2k}\left\{\|\nabla \boldsymbol{S}_n^{hk}\|_Q^2 - \|\nabla \boldsymbol{S}_{n-1}^{hk}\|_Q^2\right\}\\
&\quad + \frac{1}{2k}\left\{\|\operatorname{div} \boldsymbol{S}_n^{hk}\|_Y^2 - \|\operatorname{div} \boldsymbol{S}_{n-1}^{hk}\|_Y^2\right\} + \frac{1}{2k}\left\{\|\boldsymbol{S}_n^{hk}\|_H^2 - \|\boldsymbol{S}_{n-1}^{hk}\|_H^2\right\}\\
&\leq C\Big(\|\boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla \phi_n^{hk}\|_H^2 + \|\nabla \theta_n^{hk}\|_H^2 + \|\nabla \vartheta_n^{hk}\|_H^2 + \|\psi_n^{hk}\|_Y^2 + \|\operatorname{div} \boldsymbol{u}_n^{hk}\|_Y^2\\
&\quad + \|\operatorname{div} \boldsymbol{S}_n^{hk}\|_Y^2 + \|\operatorname{div} \boldsymbol{T}_n^{hk}\|_Y^2 + \|\vartheta_n^{hk}\|_Y^2 + \|\theta_n^{hk}\|_Y^2 + \|\xi_n^{hk}\|_Y^2 + \|\boldsymbol{R}_n^{hk}\|_H^2\\
&\quad + \|\boldsymbol{T}_n^{hk}\|_H^2 + \|\nabla \boldsymbol{T}_n^{hk}\|_Q^2\Big).
\end{aligned}
$$

Multiplying the above estimates by k and summing up to n we get

$$
\begin{aligned}
&\|\boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla \boldsymbol{u}_n^{hk}\|_Q^2 + \|\operatorname{div} \boldsymbol{u}_n^{hk}\|_Y^2 + \|\psi_n^{hk}\|_Y^2 + \|\nabla \phi_n^{hk}\|_H^2 + \|\phi_n^{hk}\|_Y^2 + \|\vartheta_n^{hk}\|_Y^2\\
&\quad + \|\xi_n^{hk}\|_Y^2 + \|\nabla \vartheta_n^{hk}\|_H^2 + \|\boldsymbol{R}_n^{hk}\|_H^2 + \|\nabla \boldsymbol{S}_n^{hk}\|_Q^2 + \|\operatorname{div} \boldsymbol{S}_n^{hk}\|_Y^2 + \|\boldsymbol{S}_n^{hk}\|_H^2\\
&\quad + k\sum_{j=1}^{n}\Big[\kappa(\nabla \theta_j^{hk}, \nabla \xi_j^{hk})_H + (\kappa_4+\kappa_5)(\operatorname{div} \boldsymbol{T}_j^{hk}, \operatorname{div} \boldsymbol{R}_j^{hk})_Y + \kappa_6(\nabla \boldsymbol{T}_j^{hk}, \nabla \boldsymbol{R}_j^{hk})_Q\Big]
\end{aligned}
$$

$$\leq Ck\sum_{j=1}^{n}\Big(\|\boldsymbol{v}_j^{hk}\|_H^2+\|\nabla\phi_j^{hk}\|_H^2+\|\nabla\theta_j^{hk}\|_H^2+\|\nabla\vartheta_j^{hk}\|_H^2+\|\psi_j^{hk}\|_Y^2+\|\mathrm{div}\,\boldsymbol{u}_j^{hk}\|_Y^2$$
$$+\|\mathrm{div}\,\boldsymbol{S}_j^{hk}\|_Y^2+\|\mathrm{div}\,\boldsymbol{T}_j^{hk}\|_Y^2+\|\vartheta_j^{hk}\|_Y^2+\|\theta_j^{hk}\|_Y^2+\|\xi_j^{hk}\|_Y^2+\|\boldsymbol{R}_j^{hk}\|_H^2$$
$$+\|\boldsymbol{T}_j^{hk}\|_H^2+\|\nabla\boldsymbol{T}_j^{hk}\|_Q^2\Big)+C\Big(\|\boldsymbol{v}_0^h\|_H^2+\|\boldsymbol{u}_n^{hk}\|_V^2+\|\psi_0^h\|_Y^2+\|\phi_0^h\|_E^2+\|\vartheta_0^h\|_E^2$$
$$+\|\xi_0^h\|_Y^2+\|\boldsymbol{R}_0^h\|_H^2+\|\boldsymbol{S}_0^h\|_V^2\Big).$$

From the definition of θ_n^{hk} and $\boldsymbol{T}_n^{hk}$ we can easily show that

$$\|\theta_n^{hk}\|_Y^2\leq Ck\sum_{j=1}^{n}\|\vartheta_j^{hk}\|_Y^2+\|\theta_0^h\|_Y^2,$$
$$\|\nabla\theta_n^{hk}\|_H^2\leq Ck\sum_{j=1}^{n}\|\nabla\vartheta_j^{hk}\|_H^2+\|\nabla\theta_0^h\|_H^2,$$
$$\|\boldsymbol{T}_n^{hk}\|_H^2\leq Ck\sum_{j=1}^{n}\|\boldsymbol{S}_j^{hk}\|_H^2+\|\boldsymbol{T}_0^h\|_H^2,$$
$$\|\nabla\boldsymbol{T}_n^{hk}\|_Q^2\leq Ck\sum_{j=1}^{n}\|\nabla\boldsymbol{T}_j^{hk}\|_Q^2+\|\nabla\boldsymbol{T}_0^h\|_Q^2,$$
$$\|\mathrm{div}\,\boldsymbol{T}_n^{hk}\|_Y^2\leq Ck\sum_{j=1}^{n}\|\mathrm{div}\,\boldsymbol{T}_j^{hk}\|_Y^2+\|\mathrm{div}\,\boldsymbol{T}_0^h\|_Y^2.$$

Observing that

$$k\sum_{j=1}^{n}(\nabla\theta_n^{hk},\nabla\xi_n^{hk})_H=\sum_{j=1}^{n}(\nabla\theta_j^{hk},\nabla(\vartheta_j^{hk}-\vartheta_{j-1}^{hk}))_H$$
$$=(\nabla\theta_n^{hk},\nabla\vartheta_n^{hk})_H+\sum_{j=1}^{n-1}(\nabla(\theta_j^{hk}-\theta_{j+1}^{hk}),\nabla\vartheta_j^{hk})_H+(\nabla\theta_1^{hk},\nabla\vartheta_0^h)_H,$$
$$\sum_{j=1}^{n-1}(\nabla(\theta_j^{hk}-\theta_{j+1}^{hk}),\nabla\vartheta_j^{hk})_H\leq Ck\sum_{j=1}^{n}\|\nabla\vartheta_j^{hk}\|_H^2+\frac{C}{k}\sum_{j=1}^{n-1}\|\nabla(\theta_j^{hk}-\theta_{j+1}^{hk})\|_H^2$$
$$\leq Ck\sum_{j=1}^{n}\|\nabla\vartheta_j^{hk}\|_H^2,$$
$$k\sum_{j=1}^{n}(\nabla\boldsymbol{T}_n^{hk},\nabla\boldsymbol{R}_n^{hk})_Q=\sum_{j=1}^{n}(\nabla\boldsymbol{T}_j^{hk},\nabla(\boldsymbol{S}_j^{hk}-\boldsymbol{S}_{j-1}^{hk}))_Q$$
$$=(\nabla\boldsymbol{T}_n^{hk},\nabla\boldsymbol{S}_n^{hk})_Q+\sum_{j=1}^{n-1}(\nabla(\boldsymbol{T}_j^{hk}-\boldsymbol{T}_{j+1}^{hk}),\nabla\boldsymbol{S}_j^{hk})_Q+(\nabla\boldsymbol{T}_1^{hk},\nabla\boldsymbol{S}_0^h)_Q,$$
$$\sum_{j=1}^{n-1}(\nabla(\boldsymbol{T}_j^{hk}-\boldsymbol{T}_{j+1}^{hk}),\nabla\boldsymbol{S}_j^{hk})_Q\leq Ck\sum_{j=1}^{n}\|\nabla\boldsymbol{S}_j^{hk}\|_Q^2+\frac{C}{k}\sum_{j=1}^{n-1}\|\nabla(\boldsymbol{T}_j^{hk}-\boldsymbol{T}_{j+1}^{hk})\|_Q^2$$
$$\leq Ck\sum_{j=1}^{n}\|\nabla\boldsymbol{S}_j^{hk}\|_Q^2,$$

$$k\sum_{j=1}^{n}(\mathrm{div}\,\boldsymbol{T}_n^{hk},\mathrm{div}\,\boldsymbol{R}_n^{hk})_Y = \sum_{j=1}^{n}(\mathrm{div}\,\boldsymbol{T}_j^{hk},\mathrm{div}\,(\boldsymbol{S}_j^{hk}-\boldsymbol{S}_{j-1}^{hk}))_Y$$
$$= (\mathrm{div}\,\boldsymbol{T}_n^{hk},\mathrm{div}\,\boldsymbol{S}_n^{hk})_Y + \sum_{j=1}^{n-1}(\mathrm{div}\,(\boldsymbol{T}_j^{hk}-\boldsymbol{T}_{j+1}^{hk}),\mathrm{div}\,\boldsymbol{S}_j^{hk})_Y + (\mathrm{div}\,\boldsymbol{T}_1^{hk},\mathrm{div}\,\boldsymbol{S}_0^{h})_Y,$$
$$\sum_{j=1}^{n-1}(\mathrm{div}\,(\boldsymbol{T}_j^{hk}-\boldsymbol{T}_{j+1}^{hk}),\mathrm{div}\,\boldsymbol{S}_j^{hk})_Y \le Ck\sum_{j=1}^{n}\|\mathrm{div}\,\boldsymbol{S}_j^{hk}\|_Y^2 + \frac{C}{k}\sum_{j=1}^{n-1}\|\mathrm{div}\,(\boldsymbol{T}_j^{hk}-\boldsymbol{T}_{j+1}^{hk})\|_Y^2$$
$$\le Ck\sum_{j=1}^{n}\|\mathrm{div}\,\boldsymbol{S}_j^{hk}\|_Y^2,$$

and, using a discrete version of Gronwall's inequality (see [31]), we conclude the discrete stability property. □

Now, we will derive a main error estimates result on the numerical errors $\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}$, $\psi_n - \psi_n^{hk}$, $\xi_n - \xi_n^{hk}$ and $\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}$. We have the following.

Theorem 4.2. *Under the assumptions of Lemma 4.1, if we denote by $(\boldsymbol{v},\psi,\xi,\boldsymbol{R})$ the solution to Problem VP and by $(\boldsymbol{v}^{hk},\psi^{hk},\xi^{hk},\boldsymbol{R}^{hk})$ the solution to Problem VP^{hk}, then we have the following a priori error estimates, for all $\{\boldsymbol{w}_j^h\}_{j=0}^N$, $\{\boldsymbol{\eta}_j^h\}_{j=0}^N \subset V^h$ and $\{r_j^h\}_{j=0}^N$, $\{z_j^h\}_{j=0}^N \subset E^h$,*

$$\max_{0\le n\le N}\Big\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2+\|\phi_n-\phi_n^{hk}\|_E^2+\|\psi_n-\psi_n^{hk}\|_Y^2+\|\xi_n-\xi_n^{hk}\|_Y^2$$
$$+\|\vartheta_n-\vartheta_n^{hk}\|_E^2+\|\theta_n-\theta_n^{hk}\|_E^2+\|\boldsymbol{R}_n-\boldsymbol{R}_n^{hk}\|_H^2+\|\boldsymbol{S}_n-\boldsymbol{S}_n^{hk}\|_V^2+\|\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}\|_V^2\Big\}$$
$$\le Ck\sum_{j=1}^{N}\Big(\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2+\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2+\|\dot{\psi}_j-\delta\psi_j\|_Y^2+\|\dot{\phi}_j-\delta\phi_j\|_E^2$$
$$+\|\psi_j-r_j^h\|_E^2+\|\dot{\xi}_j-\delta\xi_j\|_Y^2+\|\dot{\vartheta}_j-\delta\vartheta_j\|_E^2+\|\xi_j-z_j^h\|_E^2+\|\dot{\boldsymbol{R}}_j-\delta\boldsymbol{R}_j\|_H^2$$
$$+\|\boldsymbol{R}_j-\boldsymbol{\eta}_j^h\|_V^2+\|\dot{\boldsymbol{S}}_j-\delta\boldsymbol{S}_j\|_V^2+I_j^2+J_j^2\Big)$$
$$+C\max_{0\le n\le N}\Big\{\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_H^2+\|\psi_n-r_n^h\|_Y^2+\|\xi_n-z_n^h\|_Y^2+\|\boldsymbol{R}_n-\boldsymbol{\eta}_n^h\|_H^2\Big\}$$
$$+\frac{C}{k}\sum_{j=1}^{N-1}\Big[\|\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h)\|_H^2+\|\psi_j-r_j^h-(\psi_{j+1}-r_{j+1}^h)\|_Y^2$$
$$+\|\boldsymbol{R}_j-\boldsymbol{\eta}_j^h-(\boldsymbol{R}_{j+1}-\boldsymbol{\eta}_{j+1}^h)\|_H^2+\|\xi_j-z_j^h-(\xi_{j+1}-z_{j+1}^h)\|_Y^2\Big]$$
$$+C\Big(\|\boldsymbol{v}^0-\boldsymbol{v}_0^h\|_H^2+\|\phi^0-\phi_0^h\|_E^2+\|\boldsymbol{u}^0-\boldsymbol{u}_0^h\|_V^2+\|\psi^0-\psi_0^h\|_Y^2+\|\xi^0-\xi_0^h\|_Y^2$$
$$+\|\vartheta^0-\vartheta_0^h\|_E^2+\|\theta^0-\theta_0^h\|_E^2+\|\boldsymbol{R}^0-\boldsymbol{R}_0^h\|_H^2+\|\boldsymbol{S}^0-\boldsymbol{S}_0^h\|_V^2+\|\boldsymbol{T}^0-\boldsymbol{T}_0^h\|_V^2\Big),$$

where $C > 0$ is a positive constant which is be independent of the discretization parameters h and k, but depending on the continuous solution, $\delta\xi_j = (\xi_j-\xi_{j-1})/k$, $\delta\phi_j = (\phi_j-\phi_{j-1})/k$, $\delta\psi_j = (\psi_j-\psi_{j-1})/k$, $\delta\boldsymbol{v}_j = (\boldsymbol{v}_j-\boldsymbol{v}_{j-1})/k$, $\delta\boldsymbol{u}_j = (\boldsymbol{u}_j-\boldsymbol{u}_{j-1})/k$, $\delta\boldsymbol{R}_j = (\boldsymbol{R}_j-\boldsymbol{R}_{j-1})/k$, $\delta\boldsymbol{T}_j = (\boldsymbol{T}_j-\boldsymbol{T}_{j-1})/k$, $\delta\boldsymbol{S}_j = (\boldsymbol{S}_j-\boldsymbol{S}_{j-1})/k$, $\delta\vartheta_j = (\vartheta_j-\vartheta_{j-1})/k$ and $\delta\theta_j = (\theta_j-\theta_{j-1})/k$, and I_n and J_n are the integration errors defined as:

$$I_n = \Big\|\int_0^{t_n}\vartheta(s)\,ds - k\sum_{j=1}^{n}\theta_j\Big\|_E, \tag{4.15}$$

$$J_n = \Big\| \int_0^{t_n} \boldsymbol{S}(s)\, ds - k \sum_{j=1}^{n} \boldsymbol{S}_j \Big\|_V. \tag{4.16}$$

Proof. As we did in the proof of Lemma 4.1, we also assume that the time relaxation parameter $\frac{\tau_1^2}{2} = 1$.

First, we obtain a priori error estimates on function $\boldsymbol{v}$. Then, subtracting variational equation (4.1) at time $t = t_n$ for a test function $\boldsymbol{w} = \boldsymbol{w}^h \in V^h \subset V$ and discrete variational equation (4.8), we have, for all $\boldsymbol{w}^h \in V^h$,

$$\begin{aligned}&\rho(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + \mu(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla \boldsymbol{w}^h)_Q + (\lambda + \mu)(\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \operatorname{div} \boldsymbol{w}^h)_Y \\ &-\mu_0(\nabla(\phi_n - \phi_n^{hk}), \boldsymbol{w}^h)_H + \beta_0(\nabla(\xi_n - \xi_n^{hk} + \tau_1(\vartheta_n - \vartheta_n^{hk}) + \theta_n - \theta_n^{hk}), \boldsymbol{w}^h)_H = 0,\end{aligned}$$

and so, we obtain

$$\begin{aligned}&\rho(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + \mu(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \\ &\quad +(\lambda + \mu)(\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Y - \mu_0(\nabla(\phi_n - \phi_n^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H \\ &\quad +\beta_0(\nabla(\xi_n - \xi_n^{hk} + \tau_1(\vartheta_n - \vartheta_n^{hk}) + \theta_n - \theta_n^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H \\ &= \rho(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H + \mu(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{w}^h))_Q \\ &\quad +(\lambda + \mu)(\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \operatorname{div}(\boldsymbol{v}_n - \boldsymbol{w}^h))_Y - \mu_0(\nabla(\phi_n - \phi_n^{hk}), \boldsymbol{v}_n - \boldsymbol{w}^h)_H \\ &\quad +\beta_0(\nabla(\xi_n - \xi_n^{hk} + \tau_1(\vartheta_n - \vartheta_n^{hk}) + \theta_n - \theta_n^{hk}), \boldsymbol{v}_n - \boldsymbol{w}^h)_H \quad \forall \boldsymbol{w}^h \in V^h.\end{aligned}$$

Keeping in mind that

$$\begin{aligned}&(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H \geq (\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + \frac{1}{2k}\Big[\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\Big], \\ &(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \geq (\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n))_Q \\ &\qquad + \frac{1}{2k}\Big[\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Q^2\Big], \\ &(\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Y \geq (\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \operatorname{div}(\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n))_Y \\ &\qquad + \frac{1}{2k}\Big[\|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 - \|\operatorname{div}(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\Big], \\ &\beta_0(\nabla(\xi_n - \xi_n^{hk}), \boldsymbol{w})_H = -\beta_0(\xi_n - \xi_n^{hk}, \operatorname{div} \boldsymbol{w})_Y,\end{aligned}$$

using inequality (4.14) several times, after some easy calculations we get

$$\begin{aligned}&\frac{\rho}{2k}\Big[\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\Big] - \beta_0(\xi_n - \xi_n^{hk}, \operatorname{div}(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Y \\ &\quad + \frac{\mu}{2k}\Big[\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Q^2\Big] \\ &\quad + \frac{\lambda + \mu}{2k}\Big[\|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 - \|\operatorname{div}(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\Big] \\ &\leq C\Big(\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n\|_H^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 + \|\operatorname{div}(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 \\ &\quad + \|\boldsymbol{v}_n - \boldsymbol{w}^h\|_V^2 + \|\dot{\boldsymbol{u}}_n - \delta \boldsymbol{u}_n\|_V^2 + (\delta \boldsymbol{v}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{w}^h)_H + \|\xi_n - \xi_n^{hk}\|_Y^2 \\ &\quad + \|\nabla(\phi_n - \phi_n^{hk})\|_H^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2\Big) \quad \forall \boldsymbol{w}^h \in V^h.\end{aligned} \tag{4.17}$$

Secondly, we obtain the error estimates on function ψ. If we subtract variational equation (4.2) at time $t = t_n$ for a test function $r = r^h \in E^h \subset E$ and discrete variational equation (4.9), we find that, for all $r^h \in E^h$,

$$\begin{aligned}
&J(\dot{\psi}_n - \delta\psi_n^{hk}, r^h)_Y + a_0(\nabla(\phi_n - \phi_n^{hk}), \nabla r^h)_H + \zeta(\phi_n - \phi_n^{hk}, r^h)_Y \\
&\quad + \mu_0(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), r^h)_Y - \beta_1(\xi_n - \xi_n^{hk} + \tau_1(\vartheta_n - \vartheta_n^{hk}) + \theta_n - \theta_n^{hk}, r^h)_Y \\
&\quad + \mu_2(\mathrm{div}\,(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}) + \tau_1 \mathrm{div}\,(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk}) + \mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), r^h)_Y = 0.
\end{aligned}$$

Then, we have, for all $r^h \in E^h$,

$$\begin{aligned}
&J(\dot{\psi}_n - \delta\psi_n^{hk}, \psi_n - \psi_n^{hk})_Y + a_0(\nabla(\phi_n - \phi_n^{hk}), \nabla(\psi_n - \psi_n^{hk}))_H \\
&\quad + \zeta(\phi_n - \phi_n^{hk}, \psi_n - \psi_n^{hk})_Y + \mu_0(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \psi_n - \psi_n^{hk})_Y \\
&\quad + \mu_2(\mathrm{div}\,(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}) + \tau_1 \mathrm{div}\,(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk}) + \mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \psi_n - \psi_n^{hk})_Y \\
&\quad - \beta_1(\xi_n - \xi_n^{hk} + \tau_1(\vartheta_n - \vartheta_n^{hk}) + \theta_n - \theta_n^{hk}, \psi_n - \psi_n^{hk})_Y \\
&= J(\dot{\psi}_n - \delta\psi_n^{hk}, \psi_n - r^h)_Y + a_0(\nabla(\phi_n - \phi_n^{hk}), \nabla(\psi_n - r^h))_H \\
&\quad + \zeta(\phi_n - \phi_n^{hk}, \psi_n - r^h)_Y + \mu_0(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \psi_n - r^h)_Y \\
&\quad + \mu_2(\mathrm{div}\,(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}) + \tau_1 \mathrm{div}\,(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk}) + \mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \psi_n - r^h)_Y \\
&\quad - \beta_1(\xi_n - \xi_n^{hk} + \tau_1(\vartheta_n - \vartheta_n^{hk}) + \theta_n - \theta_n^{hk}, \psi_n - r^h)_Y,
\end{aligned}$$

and, taking into account that

$$\begin{aligned}
&(\dot{\psi}_n - \delta\psi_n^{hk}, \psi_n - \psi_n^{hk})_Y \geq (\dot{\psi}_n - \delta\psi_n, \psi_n - \psi_n^{hk})_Y + \frac{1}{2k}\left[\|\psi_n - \psi_n^{hk}\|_Y^2 - \|\psi_{n-1} - \psi_{n-1}^{hk}\|_Y^2\right], \\
&(\nabla(\phi_n - \phi_n^{hk}), \nabla(\dot{\phi}_n - \psi_n^{hk}))_H = (\nabla(\phi_n - \phi_n^{hk}), \nabla(\dot{\phi}_n - \delta\phi_n))_H \\
&\qquad + \frac{1}{2k}\Big[\|\nabla(\phi_n - \phi_n^{hk})\|_H^2 - \|\nabla(\phi_{n-1} - \phi_{n-1}^{hk})\|_H^2 + \|\nabla(\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk}))\|_H^2\Big], \\
&(\phi_n - \phi_n^{hk}, \dot{\phi}_n - \psi_n^{hk})_Y \geq (\phi_n - \phi_n^{hk}, \dot{\phi}_n - \delta\phi_n)_Y + \frac{1}{2k}\Big[\|\phi_n - \phi_n^{hk}\|_Y^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|_Y^2\Big], \\
&(\mathrm{div}\,(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}), \psi_n - r^h)_Y = -(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}, \nabla(\psi_n - r^h))_H,
\end{aligned}$$

using again several times inequality (4.14) we get, for all $r^h \in E^h$,

$$\begin{aligned}
&\frac{J}{2k}\left\{\|\psi_n - \psi_n^{hk}\|_Y^2 - \|\psi_{n-1} - \psi_{n-1}^{hk}\|_Y^2\right\} + \frac{\zeta}{2k}\Big\{\|\phi_n - \phi_n^{hk}\|_Y^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|_Y^2\Big\} \\
&\quad + \frac{a_0}{2k}\Big\{\|\nabla(\phi_n - \phi_n^{hk})\|_H^2 - \|\nabla(\phi_{n-1} - \phi_{n-1}^{hk})\|_H^2\Big\} \\
&\quad + \mu_2(\mathrm{div}\,(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}), \psi_n - \psi_n^{hk})_Y \\
&\leq C\Big(\|\dot{\psi}_n - \delta\psi_n\|_Y^2 + \|\dot{\phi}_n - \delta\phi_n\|_E^2 + \|\psi_n - r^h\|_E^2 + \|\vartheta_n - \vartheta_n^{hk}\|_Y^2 \\
&\quad + \|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 + \|\xi_n - \xi_n^{hk}\|^2 + \|\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Y^2 \\
&\quad + \|\phi_n - \phi_n^{hk}\|_Y^2 + \|\nabla(\phi_n - \phi_n^{hk})\|_H^2 + \|\mathrm{div}\,(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk})\|_Y^2 + \|\psi_n - \psi_n^{hk}\|_Y^2 \\
&\quad + \|\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}\|_H^2 + (\delta\psi_n - \delta\psi_n^{hk}, \psi_n - r^h)_Y\Big). \qquad (4.18)
\end{aligned}$$

Thirdly, we derive the estimates on the temperature acceleration ξ. Therefore, subtracting variational equation (4.3) at time $t = t_n$ for a test function $z = z^h \in E^h \subset E$ and discrete variational equation (4.10), for all $z^h \in E^h$ we find that

$$\begin{aligned}
&a(\dot{\xi}_n - \delta\xi_n^{hk} + \tau_1(\xi_n - \xi_n^{hk}) + \vartheta_n - \vartheta_n^{hk}, z^h)_Y + \kappa(\nabla(\theta_n - \theta_n^{hk} + \tau_2(\vartheta_n - \vartheta_n^{hk})), \nabla z^h)_H \\
&\quad + \beta_0(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), z^h)_Y + \beta_1(\psi_n - \psi_n^{hk}, z^h)_Y \\
&\quad - \kappa_1(\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}) + \tau_2 \mathrm{div}\,(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk}), z^h)_Y = 0.
\end{aligned}$$

Thus, it follows that

$$
\begin{aligned}
&a(\dot{\xi}_n-\delta\xi_n^{hk}+\tau_1(\xi_n-\xi_n^{hk})+\vartheta_n-\vartheta_n^{hk},\xi_n-\xi_n^{hk})_Y+\beta_0(\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),\xi_n-\xi_n^{hk})_Y\\
&\quad+\beta_1(\psi_n-\psi_n^{hk},\xi_n-\xi_n^{hk})_Y-\kappa_1(\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})+\tau_2\mathrm{div}\,(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk}),\xi_n-\xi_n^{hk})_Y\\
&\quad+\kappa(\nabla(\theta_n-\theta_n^{hk}+\tau_2(\vartheta_n-\vartheta_n^{hk})),\nabla(\xi_n-\xi_n^{hk}))_H\\
&=a(\dot{\xi}_n-\delta\xi_n^{hk}+\tau_1(\xi_n-\xi_n^{hk})+\vartheta_n-\vartheta_n^{hk},\xi_n-z^h)_Y+\beta_0(\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),\xi_n-z^h)_Y\\
&\quad+\beta_1(\psi_n-\psi_n^{hk},\xi_n-z^h)_Y-\kappa_1(\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})+\tau_2\mathrm{div}\,(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk}),\xi_n-z^h)_Y\\
&\quad+\kappa(\nabla(\theta_n-\theta_n^{hk}+\tau_2(\vartheta_n-\vartheta_n^{hk})),\nabla(\xi_n-z^h))_H\quad\forall z^h\in E^h.
\end{aligned}
$$

Using the following estimates

$$
\begin{aligned}
&(\dot{\xi}_n-\delta\xi_n^{hk},\xi_n-\xi_n^{hk})_Y\geq(\dot{\xi}_n-\delta\xi_n,\xi_n-\xi_n^{hk})_Y+\frac{1}{2k}\left[\|\xi_n-\xi_n^{hk}\|_Y^2-\|\xi_{n-1}-\xi_{n-1}^{hk}\|_Y^2\right],\\
&(\nabla(\vartheta_n-\vartheta_n^{hk}),\nabla(\dot{\vartheta}_n-\xi_n^{hk}))_H\geq(\nabla(\vartheta_n-\vartheta_n^{hk}),\nabla(\dot{\vartheta}_n-\delta\vartheta_n))_H\\
&\quad+\frac{1}{2k}\Big[\|\nabla(\vartheta_n-\vartheta_n^{hk})\|_H^2-\|\nabla(\vartheta_{n-1}-\vartheta_{n-1}^{hk})\|_H^2\Big],\\
&(\vartheta_n-\vartheta_n^{hk},\dot{\vartheta}_n-\xi_n^{hk})_Y\geq(\vartheta_n-\vartheta_n^{hk},\dot{\vartheta}_n-\delta\vartheta_n)_Y+\frac{1}{2k}\Big[\|\vartheta_n-\vartheta_n^{hk}\|_Y^2-\|\vartheta_{n-1}-\vartheta_{n-1}^{hk}\|_Y^2\Big],\\
&(\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),\xi_n-z^h)_Y=-(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\nabla(\xi_n-z^h))_H,
\end{aligned}
$$

applying several times inequality (4.14) we find that

$$
\begin{aligned}
&\frac{a}{2k}\left\{\|\xi_n-\xi_n^{hk}\|_Y^2-\|\xi_{n-1}-\xi_{n-1}^{hk}\|_Y^2\right\}+\frac{a}{2k}\Big\{\|\vartheta_n-\vartheta_n^{hk}\|_Y^2-\|\vartheta_{n-1}-\vartheta_{n-1}^{hk}\|_Y^2\Big\}\\
&\quad+\frac{\kappa}{2k}\Big\{\|\nabla(\vartheta_n-\vartheta_n^{hk})\|_H^2-\|\nabla(\vartheta_{n-1}-\vartheta_{n-1}^{hk})\|_H^2\Big\}\\
&\quad+\beta_0(\mathrm{div}\,(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),\xi_n-\xi_n^{hk})_Y+(\nabla(\theta_n-\theta_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H\\
&\leq C\Big(\|\dot{\xi}_n-\delta\xi_n\|_Y^2+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2+\|\dot{\vartheta}_n-\delta\vartheta_n\|_E^2+\|\xi_n-z^h\|_E^2\\
&\quad+\|\nabla(\vartheta_n-\vartheta_n^{hk})\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\xi_n-\xi_n^{hk}\|^2+\|\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})\|_Y^2\\
&\quad+\|\mathrm{div}\,(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk})\|_Y^2+\|\psi_n-\psi_n^{hk}\|_Y^2+(\delta\xi_n-\delta\xi_n^{hk},\xi_n-z^h)_Y\Big).
\end{aligned}
\tag{4.19}
$$

Finally, we get the error estimates for the microtemperature acceleration $\boldsymbol{R}$. Then, if we subtract variational equation (4.4) at time $t=t_n$ for a test function $\boldsymbol{\eta}=\boldsymbol{\eta}^h\in V^h\subset V$ and discrete variational equation (4.11), we have, for all $\psi^h\in V^h$,

$$
\begin{aligned}
&b(\dot{\boldsymbol{R}}_n-\delta\boldsymbol{R}_n^{hk}+\tau_1(\boldsymbol{R}_n-\boldsymbol{R}_n^{hk})+\boldsymbol{S}_n-\boldsymbol{S}_n^{hk},\boldsymbol{\eta}^h)_H\\
&\quad+(\kappa_4+\kappa_5)(\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}+\tau_2(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk})),\mathrm{div}\,\boldsymbol{\eta}^h)_Y\\
&\quad+\kappa_6(\nabla(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}+\tau_2(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk})),\nabla\boldsymbol{\eta}^h)_Q+\kappa_2(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}+\tau_2(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk}),\boldsymbol{\eta}^h)_H\\
&\quad+\kappa_1(\nabla(\theta_n-\theta_n^{hk}+\tau_1(\vartheta_n-\vartheta_n^{hk})),\boldsymbol{\eta}^h)_H+\mu_2(\nabla(\psi_n-\psi_n^{hk}),\boldsymbol{\eta}^h)_H=0,
\end{aligned}
$$

and therefore, we obtain

$$\begin{aligned}
& b(\dot{\boldsymbol{R}}_n - \delta\boldsymbol{R}_n^{hk} + \tau_1(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}) + \boldsymbol{S}_n - \boldsymbol{S}_n^{hk}, \boldsymbol{R}_n - \boldsymbol{R}_n^{hk})_H \\
& \quad + \kappa_6(\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk} + \tau_2(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk})), \nabla(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}))_Q \\
& \quad + (\kappa_4 + \kappa_5)(\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk} + \tau_2(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk})), \mathrm{div}\,(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}))_Y \\
& \quad + \kappa_2(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk} + \tau_2(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk}), \boldsymbol{R}_n - \boldsymbol{R}_n^{hk})_H + \mu_2(\nabla(\psi_n - \psi_n^{hk}), \boldsymbol{R}_n - \boldsymbol{R}_n^{hk})_H \\
& \quad + \kappa_1(\nabla(\theta_n - \theta_n^{hk} + \tau_1(\vartheta_n - \vartheta_n^{hk})), \boldsymbol{R}_n - \boldsymbol{R}_n^{hk})_H \\
& = b(\dot{\boldsymbol{R}}_n - \delta\boldsymbol{R}_n^{hk} + \tau_1(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}) + \boldsymbol{S}_n - \boldsymbol{S}_n^{hk}, \boldsymbol{R}_n - \boldsymbol{\eta}^h)_H \\
& \quad + \kappa_6(\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk} + \tau_2(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk})), \nabla(\boldsymbol{R}_n - \boldsymbol{\eta}^h))_Q \\
& \quad + (\kappa_4 + \kappa_5)(\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk} + \tau_2(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk})), \mathrm{div}\,(\boldsymbol{R}_n - \boldsymbol{\eta}^h))_Y \\
& \quad + \kappa_2(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk} + \tau_2(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk}), \boldsymbol{R}_n - \boldsymbol{\eta}^h)_H + \mu_2(\nabla(\psi_n - \psi_n^{hk}), \boldsymbol{R}_n - \boldsymbol{\eta}^h)_H \\
& \quad + \kappa_1(\nabla(\theta_n - \theta_n^{hk} + \tau_1(\vartheta_n - \vartheta_n^{hk})), \boldsymbol{R}_n - \boldsymbol{\eta}^h)_H \quad \forall \boldsymbol{\eta}^h \in V^h.
\end{aligned}$$

Keeping in mind that

$$\begin{aligned}
& (\dot{\boldsymbol{R}}_n - \delta\boldsymbol{R}_n^{hk}, \boldsymbol{R}_n - \boldsymbol{R}_n^{hk})_H \geq (\dot{\boldsymbol{R}}_n - \delta\boldsymbol{R}_n, \boldsymbol{R}_n - \boldsymbol{R}_n^{hk})_H + \frac{1}{2k}\left[\|\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}\|_H^2 - \|\boldsymbol{R}_{n-1} - \boldsymbol{R}_{n-1}^{hk}\|_H^2\right], \\
& (\nabla(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk}), \nabla(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}))_Q \geq (\nabla(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk}), \nabla(\dot{\boldsymbol{S}}_n - \delta\boldsymbol{S}_n))_Q \\
& \qquad + \frac{1}{2k}\left[\|\nabla(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{S}_{n-1} - \boldsymbol{S}_{n-1}^{hk})\|_Q^2\right], \\
& (\mathrm{div}\,(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk}), \mathrm{div}\,(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}))_Y \geq (\mathrm{div}\,(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk}), \mathrm{div}\,(\dot{\boldsymbol{S}}_n - \delta\boldsymbol{S}_n))_Y \\
& \qquad + \frac{1}{2k}\left[\|\mathrm{div}\,(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk})\|_Y^2 - \|\mathrm{div}\,(\boldsymbol{S}_{n-1} - \boldsymbol{S}_{n-1}^{hk})\|_Y^2\right], \\
& (\nabla(\psi_n - \psi_n^{hk}), \boldsymbol{\eta})_H = -(\psi_n - \psi_n^{hk}, \mathrm{div}\,\boldsymbol{\eta})_Y,
\end{aligned}$$

using again inequality (4.14), after easy calculations it leads to

$$\begin{aligned}
& \frac{b}{2k}\left[\|\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}\|_H^2 - \|\boldsymbol{R}_{n-1} - \boldsymbol{R}_{n-1}^{hk}\|_H^2\right] - \mu_2(\psi_n - \psi_n^{hk}, \mathrm{div}\,(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}))_Y \\
& \quad + \frac{\kappa_6}{2k}\left[\|\nabla(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{S}_{n-1} - \boldsymbol{S}_{n-1}^{hk})\|_Q^2\right] \\
& \quad + \frac{\kappa_4 + \kappa_5}{2k}\left[\|\mathrm{div}\,(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk})\|_Y^2 - \|\mathrm{div}\,(\boldsymbol{S}_{n-1} - \boldsymbol{S}_{n-1}^{hk})\|_Y^2\right] \\
& \quad + \kappa_6(\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \nabla(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}))_Q + (\kappa_4 + \kappa_5)(\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}), \mathrm{div}\,(\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}))_Y \\
& \leq C\Big(\|\boldsymbol{R}_n - \boldsymbol{R}_n^{hk}\|_H^2 + \|\dot{\boldsymbol{R}}_n - \delta\boldsymbol{R}_n\|_H^2 + \|\nabla(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk})\|_Q^2 + \|\mathrm{div}\,(\boldsymbol{S}_n - \boldsymbol{S}_n^{hk})\|_Y^2 \\
& \quad + \|\boldsymbol{R}_n - \boldsymbol{\eta}^h\|_V^2 + \|\dot{\boldsymbol{S}}_n - \delta\boldsymbol{S}_n\|_V^2 + (\delta\boldsymbol{R}_n - \delta\boldsymbol{R}_n^{hk}, \boldsymbol{R}_n - \boldsymbol{\eta}^h)_H + \|\boldsymbol{S}_n - \boldsymbol{S}_n^{hk}\|_H^2 \\
& \quad + \|\psi_n - \psi_n^{hk}\|_Y^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|\nabla(\vartheta_n - \vartheta_n^{hk})\|_H^2 + \|\boldsymbol{T}_n - \boldsymbol{T}_n^{hk}\|_H^2 \\
& \quad + \|\nabla(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Q^2 + \|\mathrm{div}\,(\boldsymbol{T}_n - \boldsymbol{T}_n^{hk})\|_Y^2\Big) \quad \forall \boldsymbol{\eta}^h \in V^h. \qquad (4.20)
\end{aligned}$$

Combining estimates (4.17)–(4.20) it follows that, for all $\boldsymbol{w}^h$, $\boldsymbol{\eta}^h \in V^h$ and r^h, $z^h \in E^h$,

$$\begin{aligned}
& \frac{\rho}{2k}\left[\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\right] + \frac{\zeta}{2k}\left\{\|\phi_n - \phi_n^{hk}\|_Y^2 - \|\phi_{n-1} - \phi_{n-1}^{hk}\|_Y^2\right\} \\
& \quad + \frac{\mu}{2k}\left[\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 - \|\nabla(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Q^2\right] \\
& \quad + \frac{\lambda + \mu}{2k}\left[\|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 - \|\mathrm{div}\,(\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})\|_Y^2\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{J}{2k}\left\{\|\psi_n-\psi_n^{hk}\|_Y^2-\|\psi_{n-1}-\psi_{n-1}^{hk}\|_Y^2\right\}+(\nabla(\theta_n-\theta_n^{hk}),\nabla(\xi_n-\xi_n^{hk}))_H\\
&+\frac{a_0}{2k}\left\{\|\nabla(\phi_n-\phi_n^{hk})\|_H^2-\|\nabla(\phi_{n-1}-\phi_{n-1}^{hk})\|_H^2\right\}\\
&+\frac{a}{2k}\left\{\|\xi_n-\xi_n^{hk}\|_Y^2-\|\xi_{n-1}-\xi_{n-1}^{hk}\|_Y^2\right\}+\frac{a}{2k}\left\{\|\vartheta_n-\vartheta_n^{hk}\|_Y^2-\|\vartheta_{n-1}-\vartheta_{n-1}^{hk}\|_Y^2\right\}\\
&+\frac{\kappa}{2k}\left\{\|\nabla(\vartheta_n-\vartheta_n^{hk})\|_H^2-\|\nabla(\vartheta_{n-1}-\vartheta_{n-1}^{hk})\|_H^2\right\}\\
&+\frac{b}{2k}\left[\|\boldsymbol{R}_n-\boldsymbol{R}_n^{hk}\|_H^2-\|\boldsymbol{R}_{n-1}-\boldsymbol{R}_{n-1}^{hk}\|_H^2\right]\\
&+\frac{\kappa_6}{2k}\left[\|\nabla(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{S}_{n-1}-\boldsymbol{S}_{n-1}^{hk})\|_Q^2\right]\\
&+\frac{\kappa_4+\kappa_5}{2k}\left[\|\operatorname{div}(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk})\|_Y^2-\|\operatorname{div}(\boldsymbol{S}_{n-1}-\boldsymbol{S}_{n-1}^{hk})\|_Y^2\right]\\
&+\kappa_6(\nabla(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}),\nabla(\boldsymbol{R}_n-\boldsymbol{R}_n^{hk}))_Q+(\kappa_4+\kappa_5)(\operatorname{div}(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}),\operatorname{div}(\boldsymbol{R}_n-\boldsymbol{R}_n^{hk}))_Y\\
\le\ & C\Big(\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n\|_H^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+\|\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2\\
&+\|\boldsymbol{v}_n-\boldsymbol{w}^h\|_V^2+\|\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n\|_V^2+(\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}^h)_H+\|\xi_n-\xi_n^{hk}\|_Y^2\\
&+\|\nabla(\phi_n-\phi_n^{hk})\|_H^2+\|\nabla(\theta_n-\theta_n^{hk})\|_H^2+\|\dot{\psi}_n-\delta\psi_n\|_Y^2+\|\dot{\phi}_n-\delta\phi_n\|_E^2\\
&+\|\psi_n-r^h\|_E^2+\|\vartheta_n-\vartheta_n^{hk}\|_Y^2+\|\theta_n-\theta_n^{hk}\|_Y^2+\|\operatorname{div}(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})\|_Y^2\\
&+\|\operatorname{div}(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk})\|_Y^2+\|\psi_n-\psi_n^{hk}\|_Y^2+\|\boldsymbol{R}_n-\boldsymbol{R}_n^{hk}\|_H^2+(\delta\psi_n-\delta\psi_n^{hk},\psi_n-r^h)_Y\\
&+\|\dot{\xi}_n-\delta\xi_n\|_Y^2+\|\dot{\vartheta}_n-\delta\vartheta_n\|_E^2+\|\xi_n-z^h\|_E^2+\|\nabla(\vartheta_n-\vartheta_n^{hk})\|_H^2\\
&+\|\psi_n-\psi_n^{hk}\|_Y^2+(\delta\xi_n-\delta\xi_n^{hk},\xi_n-z^h)_Y+\|\dot{\boldsymbol{R}}_n-\delta\boldsymbol{R}_n\|_H^2+\|\nabla(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk})\|_Q^2\\
&+\|\boldsymbol{R}_n-\boldsymbol{\eta}^h\|_V^2+\|\dot{\boldsymbol{S}}_n-\delta\boldsymbol{S}_n\|_V^2+(\delta\boldsymbol{R}_n-\delta\boldsymbol{R}_n^{hk},\boldsymbol{R}_n-\boldsymbol{\eta}^h)_H+\|\boldsymbol{S}_n-\boldsymbol{S}_n^{hk}\|_H^2\\
&+\|\psi_n-\psi_n^{hk}\|_Y^2+\|\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}\|_H^2+\|\nabla(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk})\|_Q^2\Big).
\end{aligned}
$$

Multiplying the above estimates by k and summing up to n it follows that

$$
\begin{aligned}
&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\phi_n-\phi_n^{hk}\|_Y^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+\|\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+\|\psi_n-\psi_n^{hk}\|_Y^2\\
&\quad+\|\nabla(\phi_n-\phi_n^{hk})\|_H^2+\|\xi_n-\xi_n^{hk}\|_Y^2+Ck\sum_{j=1}^{n}(\nabla(\theta_j-\theta_j^{hk}),\nabla(\xi_j-\xi_j^{hk}))_H\\
&\quad+\|\vartheta_n-\vartheta_n^{hk}\|_Y^2+\|\nabla(\vartheta_n-\vartheta_n^{hk})\|_H^2+\|\boldsymbol{R}_n-\boldsymbol{R}_n^{hk}\|_H^2+\|\nabla(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk})\|_Q^2\\
&\quad+\|\operatorname{div}(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk})\|_Y^2+Ck\sum_{j=1}^{n}(\nabla(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}),\nabla(\boldsymbol{R}_j-\boldsymbol{R}_j^{hk}))_Q\\
&\quad+Ck\sum_{j=1}^{n}(\operatorname{div}(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}),\operatorname{div}(\boldsymbol{R}_j-\boldsymbol{R}_j^{hk}))_Y\\
&\le Ck\sum_{j=1}^{n}\Big(\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_H^2+\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\nabla(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Q^2+\|\operatorname{div}(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Y^2\\
&\quad+\|\boldsymbol{v}_j-\boldsymbol{w}_j^h\|_V^2+\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2+(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H+\|\xi_j-\xi_j^{hk}\|_Y^2\\
&\quad+\|\nabla(\phi_j-\phi_j^{hk})\|_H^2+\|\nabla(\theta_j-\theta_j^{hk})\|_H^2+\|\dot{\psi}_j-\delta\psi_j\|_Y^2+\|\dot{\phi}_j-\delta\phi_j\|_E^2\\
&\quad+\|\psi_j-r_j^h\|_E^2+\|\vartheta_j-\vartheta_j^{hk}\|_Y^2+\|\theta_j-\theta_j^{hk}\|_Y^2+\|\operatorname{div}(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk})\|_Y^2
\end{aligned}
$$

$$
\begin{aligned}
&+\|\mathrm{div}\,(\boldsymbol{S}_j-\boldsymbol{S}_j^{hk})\|_Y^2+\|\psi_j-\psi_j^{hk}\|_Y^2+\|\boldsymbol{R}_j-\boldsymbol{R}_j^{hk}\|_H^2+(\delta\psi_j-\delta\psi_j^{hk},\psi_j-r_j^h)_Y\\
&+\|\dot{\xi}_j-\delta\xi_j\|_Y^2+\|\dot{\vartheta}_j-\delta\vartheta_j\|_E^2+\|\xi_j-z_j^h\|_E^2+\|\nabla(\vartheta_j-\vartheta_j^{hk})\|_H^2\\
&+\|\psi_j-\psi_j^{hk}\|_Y^2+(\delta\xi_j-\delta\xi_j^{hk},\xi_j-z_j^h)_Y+\|\dot{\boldsymbol{R}}_j-\delta\boldsymbol{R}_j\|_H^2+\|\nabla(\boldsymbol{S}_j-\boldsymbol{S}_j^{hk})\|_Q^2\\
&+\|\boldsymbol{R}_j-\boldsymbol{\eta}_j^h\|_V^2+\|\dot{\boldsymbol{S}}_j-\delta\boldsymbol{S}_j\|_V^2+(\delta\boldsymbol{R}_j-\delta\boldsymbol{R}_j^{hk},\boldsymbol{R}_j-\boldsymbol{\eta}_j^h)_H+\|\boldsymbol{S}_j-\boldsymbol{S}_j^{hk}\|_H^2\\
&+\|\psi_j-\psi_j^{hk}\|_Y^2+\|\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}\|_H^2+\|\nabla(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk})\|_Q^2\Big)\\
&+C\Big(\|\boldsymbol{v}^0-\boldsymbol{v}_0^h\|_H^2+\|\phi^0-\phi_0^h\|_E^2+\|\boldsymbol{u}^0-\boldsymbol{u}_0^h\|_V^2+\|\psi^0-\psi_0^h\|_Y^2+\|\xi^0-\xi_0^h\|_Y^2\\
&+\|\vartheta^0-\vartheta_0^h\|_E^2+\|\boldsymbol{R}^0-\boldsymbol{R}_0^h\|_H^2+\|\boldsymbol{S}^0-\boldsymbol{S}_0^h\|_V^2\Big).
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
k\sum_{j=1}^{n}(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H&=\sum_{j=1}^{n}(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}-(\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk}),\boldsymbol{v}_j-\boldsymbol{w}_j^h)_H\\
&=(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{w}_n^h)_H+(\boldsymbol{v}_0^h-\boldsymbol{v}_0,\boldsymbol{v}_1-\boldsymbol{w}_1^h)_H\\
&\quad+\sum_{j=1}^{n-1}(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{w}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{w}_{j+1}^h))_H,
\end{aligned}
$$

where similar estimates can be derived for the terms involving functions ψ, ξ and $\boldsymbol{R}$, and using the estimates:

$$
\begin{aligned}
&k\sum_{j=1}^{n}(\nabla(\theta_j-\theta_j^{hk}),\nabla(\delta\vartheta_j-\delta\vartheta_j^{hk}))_H=\sum_{j=1}^{n}(\nabla(\theta_j-\theta_j^{hk}),\nabla(\vartheta_j-\vartheta_j^{hk}-(\vartheta_{j-1}-\vartheta_{j-1}^{hk})))_H\\
&\quad=(\nabla(\theta_n-\theta_n^{hk}),\nabla(\vartheta_n-\vartheta_n^{hk}))_H+(\nabla(\theta_1^{hk}-\theta_1),\nabla(\vartheta^0-\vartheta_0^h))_H\\
&\qquad+\sum_{j=1}^{n-1}(\nabla(\theta_j-\theta_j^{hk}-(\theta_{j-1}-\theta_{j-1}^{hk})),\nabla(\vartheta_j-\vartheta_j^{hk}))_H,\\
&\sum_{j=1}^{n-1}(\nabla(\theta_j-\theta_j^{hk}-(\theta_{j-1}-\theta_{j-1}^{hk})),\nabla(\vartheta_j-\vartheta_j^{hk}))_H\\
&\qquad\le C\Big(k\sum_{j=1}^{n}\|\nabla(\dot{\theta}_j-\delta\theta_j)\|_H^2+k\sum_{j=1}^{n}\|\nabla(\vartheta_j-\vartheta_j^{hk})\|_H^2\Big),\\
&\|\theta_n-\theta_n^{hk}\|_E^2\le C\Big(\|\theta^0-\theta_0^h\|_E^2+I_n^2+k\sum_{j=1}^{n}\|\vartheta_j-\vartheta_j^{hk}\|_E^2\Big),\\
&k\sum_{j=1}^{n}(\nabla(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}),\nabla(\delta\boldsymbol{S}_j-\delta\boldsymbol{S}_j^{hk}))_Q=\sum_{j=1}^{n}(\nabla(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}),\nabla(\boldsymbol{S}_j-\boldsymbol{S}_j^{hk}-(\boldsymbol{S}_{j-1}-\boldsymbol{S}_{j-1}^{hk})))_Q\\
&\quad=(\nabla(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}),\nabla(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk}))_Q+(\nabla(\boldsymbol{T}_1^{hk}-\boldsymbol{T}_1),\nabla(\boldsymbol{S}^0-\boldsymbol{S}_0^h))_Q\\
&\qquad+\sum_{j=1}^{n-1}(\nabla(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}-(\boldsymbol{T}_{j-1}-\boldsymbol{T}_{j-1}^{hk})),\nabla(\boldsymbol{S}_j-\boldsymbol{S}_j^{hk}))_Q,
\end{aligned}
$$

$$\sum_{j=1}^{n-1}(\nabla(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}-(\boldsymbol{T}_{j-1}-\boldsymbol{T}_{j-1}^{hk})),\nabla(\boldsymbol{S}_j-\boldsymbol{S}_j^{hk}))_Q$$
$$\leq C\Big(k\sum_{j=1}^{n}\|\nabla(\dot{\boldsymbol{T}}_j-\delta\boldsymbol{T}_j)\|_Q^2+k\sum_{j=1}^{n}\|\nabla(\boldsymbol{S}_j-\boldsymbol{S}_j^{hk})\|_Q^2\Big),$$
$$\|\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}\|_V^2\leq C\Big(\|\boldsymbol{T}^0-\boldsymbol{T}_0^h\|_V^2+J_n^2+k\sum_{j=1}^{n}\|\boldsymbol{S}_j-\boldsymbol{S}_j^{hk}\|_V^2\Big),$$
$$k\sum_{j=1}^{n}(\mathrm{div}\,(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}),\mathrm{div}\,(\delta\boldsymbol{S}_j-\delta\boldsymbol{S}_j^{hk}))_Y=\sum_{j=1}^{n}(\mathrm{div}\,(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}),\mathrm{div}\,(\boldsymbol{S}_j-\boldsymbol{S}_j^{hk}-(\boldsymbol{S}_{j-1}-\boldsymbol{S}_{j-1}^{hk})))_Y$$
$$=(\mathrm{div}\,(\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}),\mathrm{div}\,(\boldsymbol{S}_n-\boldsymbol{S}_n^{hk}))_Y+(\mathrm{div}\,(\boldsymbol{T}_1^{hk}-\boldsymbol{T}_1),\mathrm{div}\,(\boldsymbol{S}^0-\boldsymbol{S}_0^h))_Y$$
$$+\sum_{j=1}^{n-1}(\mathrm{div}\,(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}-(\boldsymbol{T}_{j-1}-\boldsymbol{T}_{j-1}^{hk})),\mathrm{div}\,(\boldsymbol{S}_j-\boldsymbol{S}_j^{hk}))_Y,$$
$$\sum_{j=1}^{n-1}(\mathrm{div}\,(\boldsymbol{T}_j-\boldsymbol{T}_j^{hk}-(\boldsymbol{T}_{j-1}-\boldsymbol{T}_{j-1}^{hk})),\mathrm{div}\,(\boldsymbol{S}_j-\boldsymbol{S}_j^{hk}))_Y$$
$$\leq C\Big(k\sum_{j=1}^{n}\|\mathrm{div}\,(\dot{\boldsymbol{T}}_j-\delta\boldsymbol{T}_j)\|_Y^2+k\sum_{j=1}^{n}\|\mathrm{div}\,(\boldsymbol{S}_j-\boldsymbol{S}_j^{hk})\|_Y^2\Big),$$

where I_n and J_n are the integration errors defined in (4.15) and (4.16), respectively, applying a discrete version of Gronwall's inequality (see, again, [31]) we conclude the desired a priori error estimates. □

We note that the error estimates provided in Theorem 4.2 can be used to obtain the convergence order of the approximations provided by the fully discrete problem VP^{hk}. As a particular case, if we assume the additional regularity:

$$\begin{aligned}
&\phi\in H^3(0,T;Y)\cap W^{1,\infty}(0,T;H^2(B))\cap H^2(0,T;H^1(B)),\\
&\theta\in H^4(0,T;Y)\cap W^{2,\infty}(0,T;H^2(B))\cap H^3(0,T;H^1(B)),\\
&\boldsymbol{u}\in H^3(0,T;H)\cap W^{1,\infty}(0,T;[H^2(B)]^d)\cap H^2(0,T;[H^1(B)]^d),\\
&\boldsymbol{T}\in H^4(0,T;H)\cap W^{2,\infty}(0,T;[H^2(B)]^d)\cap H^3(0,T;[H^1(B)]^d),
\end{aligned}\tag{4.21}$$

we can conclude the following.

Theorem 4.3. *Under the assumptions of Theorem 4.2 and the additional regularity conditions (4.21), it follows the linear convergence of the approximations given by Problem VP^{hk}; that is, there exists a constant $C>0$, independent of parameters h and k, such that*

$$\max_{0\leq n\leq N}\Big\{\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\psi_n-\psi_n^{hk}\|_Y+\|\phi_n-\phi_n^{hk}\|_E+\|\theta_n-\theta_n^{hk}\|_E+\|\vartheta_n-\vartheta_n^{hk}\|_E$$
$$+\|\xi_n-\xi_n^{hk}\|_Y+\|\boldsymbol{T}_n-\boldsymbol{T}_n^{hk}\|_V+\|\boldsymbol{S}_n-\boldsymbol{S}_n^{hk}\|_V+\|\boldsymbol{R}_n-\boldsymbol{R}_n^{hk}\|_H\Big\}\leq C(h+k).$$

5. Numerical results

In this section, we describe some numerical results obtained solving one- and two-dimensional problems. First, the numerical convergence and the discrete energy decay are shown in the one-dimensional example. Secondly, the numerical dependence on the coupling parameter μ_2 is studied in a two-dimensional example.

5.1. First example: numerical convergence in a one-dimensional example

As an academical example, in order to show the accuracy of the approximations the following one-dimensional problem is considered:

$$
\begin{aligned}
&\rho\ddot{u} = \mu u_{xx} + \mu_0\phi_x - \beta_0(\theta_x + \tau_1\dot{\theta}_x + \frac{\tau_1^2}{2}\ddot{\theta}_x) + F_1 \quad \text{in } (0,\ell)\times(0,T_f),\\
&J\ddot{\phi} = a0\phi_{xx} - \mu_0 u_x - \mu_2(T_x + \tau_1\dot{T}_x + \frac{\tau_1^2}{2}\ddot{T}_x) + \beta_1(\theta + \tau_1\dot{\theta} + \frac{\tau_1^2}{2}\ddot{\theta}) - \xi\phi + F_2 \quad \text{in } (0,\ell)\times(0,T_f),\\
&a(\dot{\theta} + \tau_1\ddot{\theta} + \frac{\tau_1^2}{2}\dddot{\theta}) = -\beta_0\dot{u}_x - \beta_1\dot{\phi} + \kappa(\theta_{xx} + \tau_2\dot{\theta}_{xx})\\
&\qquad + \kappa_1(T_x + \tau_2\dot{T}_x) + F_3 \quad \text{in } (0,\ell)\times(0,T_f),\\
&b(\dot{T} + \tau_1\ddot{T} + \frac{\tau_1^2}{2}\dddot{T}) = \mu_1\dot{\phi}_x + \kappa_4 T_{xx} + \tau_2\kappa_6\dot{T}_{xx} - \kappa_2(T + \tau_2\dot{T})\\
&\qquad - \kappa_1(\theta_x + \tau_2\dot{\theta}_x) - \mu_2\dot{\phi} + F_4 \quad \text{in } (0,\ell)\times(0,T_f),\\
&u(0,t) = u(\ell,t) = \phi(0,t) = \phi(\ell,t) = 0 \quad \text{for a.e. } t\in(0,T_f),\\
&\theta(0,t) = \theta(\ell,t) = R(0,t) = R(\ell,t) = 0 \quad \text{for a.e. } t\in(0,T_f),\\
&u^0(x) = v^0(x) = \phi^0(x) = \psi^0(x) = \theta^0(x) = x(x-1) \quad \text{for a.e. } x\in(0,\ell),\\
&\vartheta^0(x) = \xi^0(x) = T^0(x) = S^0(x) = R^0(x) = x(x-1) \quad \text{for a.e. } x\in(0,\ell).
\end{aligned}
$$

The following data have been used:

$$
\begin{aligned}
&\ell = 1, \quad T_f = 1, \quad \rho = 1, \quad \mu = 2, \quad \mu_0 = 1, \quad \mu_2 = 1, \quad J = 1, \quad a_0 = 1,\\
&\beta_0 = 1, \quad \beta_1 = 1, \quad \xi = 2 \quad \kappa = 3, \quad a = 1, \quad b = 2, \quad \kappa_1 = 1, \quad \kappa_2 = 1,\\
&\kappa_4 = 2, \quad \tau_1 = 1, \quad \tau_2 = 2,
\end{aligned}
$$

and the artificial supply terms F_i, $i = 1, 2, 3, 4$, are given by, for a.e. $(x,t)\in(0,1)\times(0,1)$,

$$
\begin{aligned}
F_1(x,t) &= e^t(3x + x(x-1) - 11/2),\\
F_2(x,t) &= e^t(7x + x(x-1)/2 - 11/2),\\
F_3(x,t) &= -e^t(4x - 7x(x-1)/2 + 16),\\
F_4(x,t) &= e^t(8x + 8x(x-1) - 16).
\end{aligned}
$$

We note that the numerical scheme was implemented on a 3.2 Ghz PC using MATLAB, and a typical run ($h = k = 0.001$) took about 0.13 seconds of CPU time.

In this case, the exact solution to this one-dimensional problem can be easily calculated and it has the following form, for $(x,t)\in[0,1]\times[0,1]$:

$$u(x,t) = \phi(x,t) = \theta(x,t) = T(x,t) = e^t x(x-1).$$

Thus, the approximation errors estimated by

$$
\begin{aligned}
\max_{0\le n\le N} \Big\{ &\|v_n - v_n^{hk}\|_Y + \|u_n - u_n^{hk}\|_E + \|\phi_n - \phi_n^{hk}\|_E + \|\psi_n - \psi_n^{hk}\|_Y + \|\theta_n - \theta_n^{hk}\|_E\\
&+ \|\vartheta_n - \vartheta_n^{hk}\|_E + \|\xi_n - \xi_n^{hk}\|_Y + \|T_n - T_n^{hk}\|_E + \|S_n - S_n^{hk}\|_E + \|R_n - R_n^{hk}\|_Y \Big\}
\end{aligned}
$$

Table 1. Example 1: Numerical errors for some h and k.

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.574576	0.570780	0.568601	0.567893	0.567542	0.567332	0.567262
$1/2^4$	0.290868	0.286851	0.284646	0.283947	0.283604	0.283400	0.283332
$1/2^5$	0.149884	0.145325	0.142983	0.142270	0.141926	0.141724	0.141657
$1/2^6$	0.080538	0.074898	0.072236	0.071480	0.071127	0.070923	0.070856
$1/2^7$	0.047632	0.040251	0.036965	0.036114	0.035737	0.035528	0.035460
$1/2^8$	0.033167	0.023814	0.019518	0.018481	0.018056	0.017833	0.017764
$1/2^9$	0.027415	0.016591	0.011111	0.009758	0.009240	0.008989	0.008916
$1/2^{10}$	0.025348	0.013719	0.007310	0.005556	0.004879	0.004576	0.004495
$1/2^{11}$	0.024679	0.012688	0.005748	0.003656	0.002778	0.002384	0.002288
$1/2^{12}$	0.024485	0.012356	0.005169	0.002876	0.001829	0.001316	0.001192
$1/2^{13}$	0.024434	0.012260	0.004976	0.002586	0.001439	0.000821	0.000658

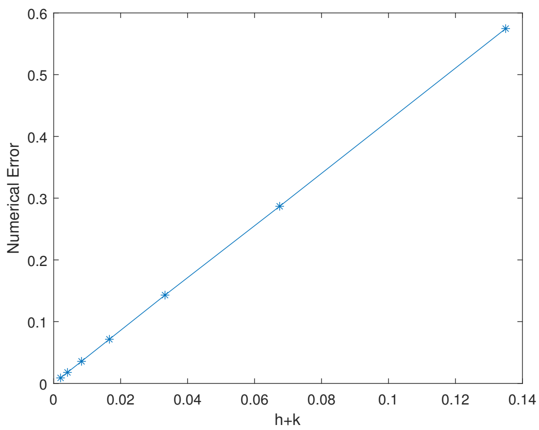

Figure 1. Example 1: Asymptotic constant error.

are presented in Table 1 for different values of h and k. Moreover, the evolution of the error depending on the parameter $h+k$ is plotted in Figure 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Theorem 4.3, is achieved.

If we assume now that there are not supply terms and we use the final time $T_f = 30$, the same data than in the previous example and the initial conditions:

$$\begin{aligned} &u^0 = v^0 = \phi^0 = \psi^0 = \theta^0 = \vartheta^0 = \xi^0 = 0, \\ &T^0(x) = R^0(x) = S^0(x) = x(x-1) \quad \text{for a.e. } x \in (0,1), \end{aligned}$$

taking the discretization parameters $h = k = 0.001$, the evolution in time of the discrete energy

$$\begin{aligned} E_n^{hk} = &\|v_n^{hk}\|_Y^2 + 2\|u_n^{hk}\|_E^2 + \|\psi_n^{hk}\|_Y^2 + \|\phi_n^{hk}\|_E^2 + \|\xi_n^{hk}\|_Y^2 + 3\|\vartheta_n^{hk}\|_E^2 + 3\|\theta_n^{hk}\|_E^2 \\ &+2\|R_n^{hk}\|_Y^2 + 2\|S_n^{hk}\|_E^2 + 2\|T_n^{hk}\|_E^2 \end{aligned}$$

is plotted in Figure 2 (in both natural and semi-log scales). As can be seen, it converges to zero and an exponential decay seems to be achieved.

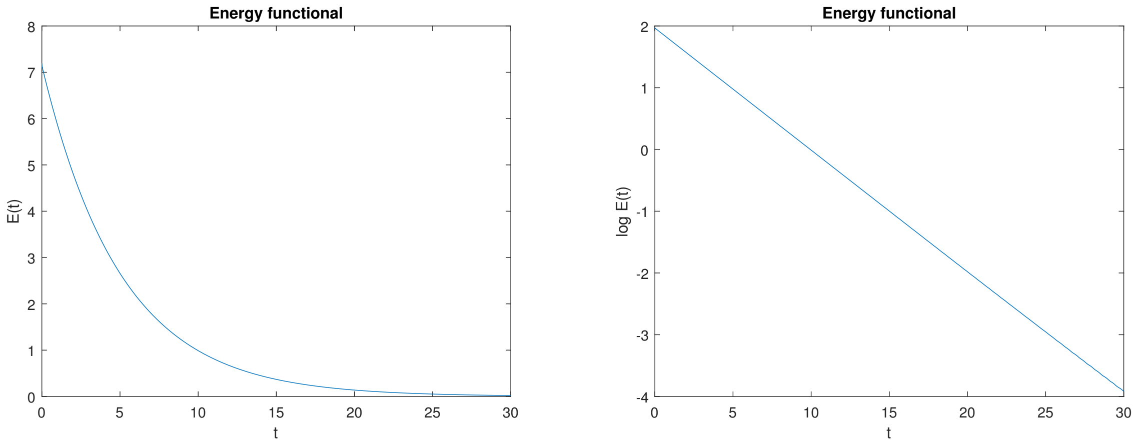

Figure 2. Example 1: Evolution in time of the discrete energy (natural and semi-log scales).

5.2. Second example: dependence on the coupling parameter μ_2

We consider now the two-dimensional domain $B = (0,1) \times (0,1)$. In this case, our aim is to study the dependence on the coupling parameter μ_2.

The following data have been used in the simulations of this example:

$$\begin{array}{l} \rho = 1, \quad T_f = 1, \quad \lambda = 1, \quad \mu = 2, \quad \mu_0 = 1, \quad J = 1, \quad a_0 = 1, \quad \beta_0 = 1, \quad \beta_1 = 1, \\ \kappa = 3, \quad a = 1, \quad b = 2, \quad \kappa_1 = 1, \quad \kappa_2 = 1, \quad \kappa_4 = 1, \quad \kappa_5 = 1, \quad \kappa_6 = 2, \quad \xi = 2, \\ \tau_1 = 1, \quad \tau_2 = 2, \end{array}$$

with the initial conditions:

$$\boldsymbol{u}^0 = \boldsymbol{v}^0 = \boldsymbol{T}^0 = \boldsymbol{R}^0 = \boldsymbol{S}^0 = \boldsymbol{0}, \quad \phi^0 = \psi^0 = \theta^0 = \vartheta^0 = \xi^0 = 0,$$

and null Dirichlet boundary conditions. We also add a supply term in the equation for function $\boldsymbol{u}$ given by $\boldsymbol{f}(x, y, t) = (tx(x-1)y(y-1), tx(x-1)y(y-1))$ for $(x, y) \in (0,1) \times (0,1)$ and $t \in [0,1]$.

Using the time discretization parameter $k = 0.01$ and the uniform finite element mesh obtained dividing the unit square into 2048 triangles, in Figure 3 we plot the norm of function $\boldsymbol{u}$ and the microtemperatures, at middle point $\boldsymbol{x} = (0.5, 0.5)$, for some values of parameter μ_2. It seems that, for the highest value $\mu_2 = 20$, an oscillation is produced.

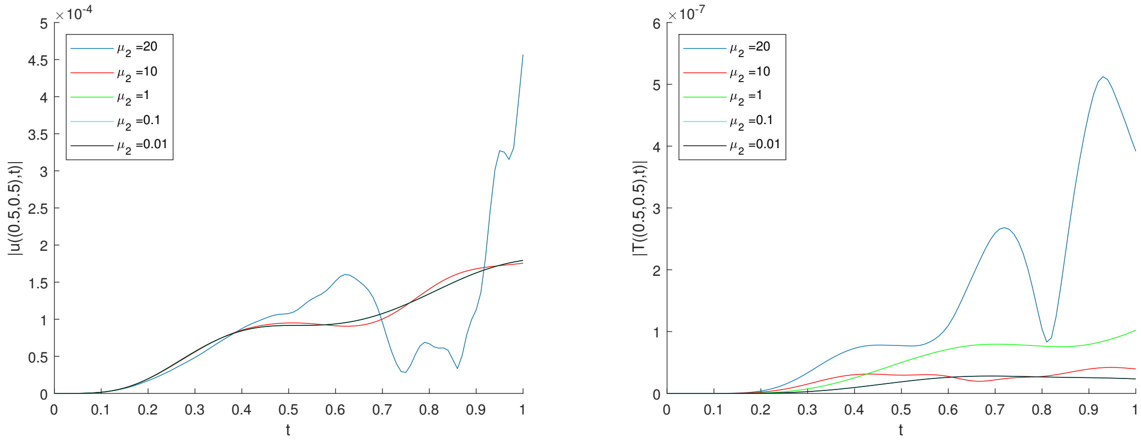

Figure 3. Example 2: Evolution in time of the norm of $\boldsymbol{u}$ and norm of the microtemperatures at the middle point for different values of parameter μ_2.

Finally, in Figure 4 we plot function ϕ and the temperature θ at final time for the value of parameter $\mu_2 = 20$. We can note that they have been generated by the resulting deformation.

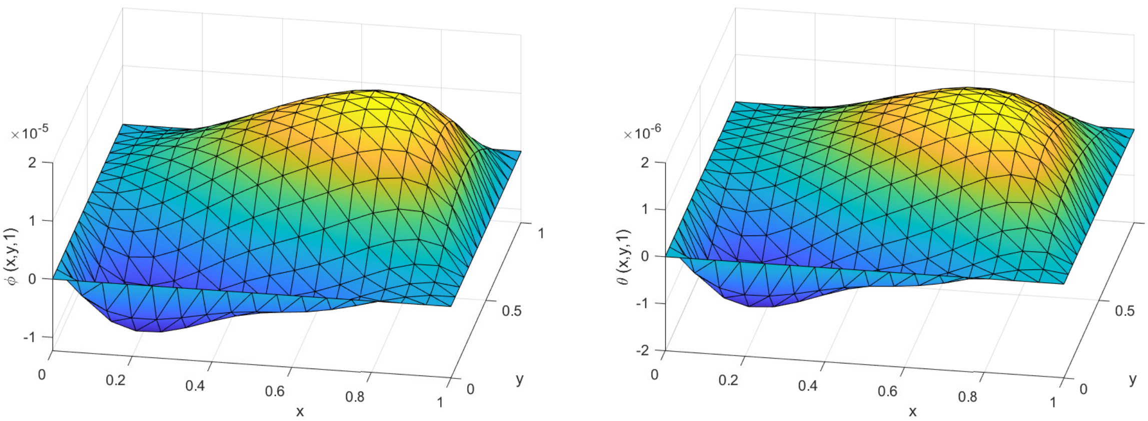

Figure 4. Example 2: Function ϕ and temperature θ at final time for parameter $\mu_2 = 20$.

Acknowledgments

The work of J.R. Fernández has been partially supported by Ministerio de Ciencia, Innovación y Universidades under the research project PGC2018-096696-B-I00 (FEDER, UE).

The work of R. Quintanilla has been supported by Ministerio de Ciencia, Innovación y Universidades under the research project "Análisis matemático aplicado a la termomecánica" (PID2019-105118GB-I00).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. A. C. Eringen, Microcontinuum Field Theories I. Foundations and Solids, Springer-Verlag, New York, 1999. https://doi.org/10.1007/978-1-4612-0555-5
2. D. Ieşan, Thermoelastic Models of Continua, Kluwer Academic, Dordrecht, 2004. https://doi.org/10.1007/978-1-4020-2310-1_5
3. M. A. Goodman, S. C. Cowin, A continuum theory for granular materials, *Arch. Rat. Mech. Anal.*, **44** (1972), 249–266. https://doi.org/10.1007/BF00284326
4. S. C. Cowin, J. W. Nunziato, Linear elastic materials with voids, *J. Elast.*, **13** (1983), 125–147. https://doi.org/10.1007/BF00041230
5. S. C. Cowin, The viscoelastic behavior of linear elastic materials with voids, *J. Elast.*, **15** (1985), 185–191. https://doi.org/10.1007/BF00041992
6. J. W. Nunziato, S. C. Cowin, A nonlinear theory of elastic materials with voids, *Arch. Ration. Mech. Anal.*, **72** (1979), 175–201. https://doi.org/10.1007/BF00249363
7. D. Ieşan, A theory of thermoelastic materials with voids, *Acta Mech.*, **60** (1986), 67–89. https://doi.org/10.1007/BF01302942
8. T. A. Apalara, Exponential decay in one-dimensional porous elasticity, *Quart. J. Mech. Appl. Math.*, **70** (2017), 363–372. https://doi.org/10.1093/qjmam/hbx012
9. T. A. Apalara, General decay of solutions in one-dimensional porous-elastic system with memory, *J. Math. Anal. Appl.*, **469** (2019), 457–471 https://doi.org/10.1016/j.jmaa.2017.08.007
10. N. Bazarra, J. R. Fernández, R. Quintanilla, An a priori error analysis of poro-thermoviscoelastic problems, *Appl. Math. Comput.*, **379** (2020), 125268. https://doi.org/10.1016/j.amc.2020.125268
11. B. Feng, Uniform decay of energy for porous thermoelastic systems with past history, *Appl. Anal.*, **97** (2018), 210–229. https://doi.org/10.1080/00036811.2016.1258116
12. B. Feng, T. A. Apalara, Optimal decay for a porous elasticity system with memory, *J. Math. Anal. Appl.*, **470** (2019), 1108–1128. https://doi.org/10.1016/j.jmaa.2018.10.052
13. B. Feng, L. Yan, D. S. Almeida Junior, Stabilization for an inhomogeneous porous-elastic system with temperatures and microtemperatures, *Z. Angew. Math. Mech.*, **101** (2021), e202000058. https://doi.org/10.1002/zamm.202000058
14. B. Feng, M. Yin, Decay of solutions for one-dimensional porous elasticity system with memory: the case of nonequal waves speed, *Math. Mech. Solids*, **24** (2019), 2361–2373. https://doi.org/10.1177/1081286518757299
15. J. R. Fernández, M. Masid, Analysis of a problem arising in porous thermoelasticity of type II, *J. Therm. Stress.*, **39** (2016), 513–531. https://doi.org/10.1080/01495739.2016.1158609
16. J. R. Fernández, M. Masid, Numerical analysis of a thermoelastic diffusion problem with voids, *Int. J. Numer. Anal. Model.*, **14** (2017), 153–174.

17. M. C. Leseduarte, A. Magaña, R. Quintanilla, On the time decay of solutions in porous-thermo-elasticity of type II, *Discrete Contin. Dyn. Syst. Ser. B*, **13** (2010), 375–391. https://doi.org/10.3934/dcdsb.2010.13.375
18. A. Magaña, R. Quintanilla, On the exponential decay of solutions in one-dimensional generalized porous-thermoelasticity, *Asymptot. Anal.*, **49** (2006), 173–187.
19. A. Miranville, R. Quintanilla, Exponential stability in type III thermoelasticity with voids, *Appl. Math. Lett.*, **94** (2019), 30–37. https://doi.org/10.1016/j.aml.2019.02.014
20. D. Ieşan, R. Quintanilla, On a theory of thermoelasticity with microtemperatures, *J. Therm. Stress.*, **23** (2000), 195–215. https://doi.org/10.1080/014957300280407
21. D. Ieşan, R. Quintanilla, On thermoelastic bodies with inner structure and microtemperatures, *J. Math. Anal. Appl.*, **354** (2009), 12–23. https://doi.org/10.1016/j.jmaa.2008.12.017
22. C. Cattaneo, On a form of heat equation which eliminates the paradox of instantaneous propagation, *C. R. Acad. Sci. Paris*, **247** (1958), 431–433.
23. D. Y. Tzou, A unified approach for heat conduction from macro to micro-scales, *ASME J. Heat Transfer*, **117** (1995), 8–16. https://doi.org/10.1115/1.2822329
24. Z. Liu, R. Quintanilla, Time decay in dual-phase-lag thermoelasticity: critical case, *Comm. Pure Appl. Anal.*, **17** (2018), 177–190. https://doi.org/10.3934/cpaa.2018011
25. F. Maes, K. Van Bockstal, Thermoelastic problem in the setting of dual-phase-lag heat conduction: Existence and uniqueness of a weak solution, *J. Math. Anal. Appl.*, **503** (2021), 125304. https://doi.org/10.1016/j.jmaa.2021.125304
26. R. Quintanilla, R. Racke, Qualitative aspects in dual-phase-lag thermoelasticity, *SIAM J. Appl. Math.*, **66** (2006), 977–1001. https://doi.org/10.1137/05062860X
27. N. Bazarra, M. I. M. Copetti, J. R. Fernández, R.Quintanilla, Numerical analysis of a dual-phase-lag model with microtemperatures, *Appl. Num. Math.*, **166** (2021), 1–25. https://doi.org/10.1016/j.apnum.2021.03.016
28. Z. Liu, R. Quintanilla, Y. Wang, Dual-phase-lag heat conduction with microtemperatures, *Z. Angew. Math. Mech.*, **101** (2021), e202000167. https://doi.org/10.1002/zamm.202000167
29. P. G. Ciarlet, Basic error estimates for elliptic problems. In: Handbook of Numerical Analysis, P.G. Ciarlet and J.L. Lions eds., vol II (1993), 17–351. https://doi.org/10.1016/S1570-8659(05)80039-0
30. Ph. Clement, Approximation by finite element functions using local regularization, *RAIRO Math. Model. Numer. Anal.*, **9** (1975), 77–84. https://doi.org/10.1051/m2an/197509R200771
31. M. Campo, J. R. Fernández, K. L. Kuttler, M. Shillor and J. M. Viaño, Numerical analysis and simulations of a dynamic frictionless contact problem with damage, *Comput. Methods Appl. Mech. Engrg.*, **196** (2006), 476–488. https://doi.org/10.1016/j.cma.2006.05.006