Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 30(4): 1209–1235.
DOI: 10.3934/era.2022064
Received: 07 July 2021
Revised: 30 December 2021
Accepted: 30 December 2021
Published: 14 March 2022

Research article

Compactness and blow up results for doubly perturbed Yamabe problems on manifolds with non umbilic boundary

Marco G. Ghimenti and Anna Maria Micheletti*

Dipartimento di Matematica Università di Pisa Largo B. Pontecorvo 5, 56126 Pisa, Italy

* **Correspondence:** Email: a.micheletti@dma.unipi.it.

Abstract: We study the stability of compactness of solutions for the Yamabe boundary problem on a compact Riemannian manifold with non umbilic boundary. We prove that the set of solutions of Yamabe boundary problem is a compact set when perturbing the mean curvature of the boundary from below and the scalar curvature with a function whose maximum is not too positive. In addition, we prove the counterpart of the stability result: there exists a blowing up sequence of solutions when we perturb the mean curvature from above or the mean curvature from below and the scalar curvature with a function with a large positive maximum.

Keywords: non umbilic boundary; Yamabe problem; compactness; blow up analysis

1. Introduction

Let (M,g), a smooth, compact Riemannian manifold of dimension $n \geq 7$ with non umbilic boundary. We recall that the boundary of M is called non umbilic if the trace-free second fundamental form of ∂M is everywhere different from zero. Here we study the linearly perturbed problem

$$\begin{cases} -\Delta_g u + \frac{n-2}{4(n-1)} R_g u + \varepsilon_1 \alpha u = 0 & \text{in } M \\ \frac{\partial u}{\partial \nu} + \frac{n-2}{2} h_g u + \varepsilon_2 \beta u = (n-2) u^{\frac{n}{n-2}} & \text{on } \partial M \end{cases}. \tag{1.1}$$

Where Δ_g is the Laplace-Beltrami operator and ν denotes the outer normal. Also, $\varepsilon_1, \varepsilon_2$ are positive parameters and $\alpha, \beta : M \to \mathbb{R}$ are smooth functions. We can restate Problem (1.1) in the more compact form

$$\begin{cases} L_g u - \varepsilon_1 \alpha u = 0 & \text{in } M \\ B_g u - \varepsilon_2 \beta u + (n-2) u^{\frac{n}{n-2}} = 0 & \text{on } \partial M \end{cases},$$

where $L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g$ and $B_g = -\frac{\partial}{\partial \nu} - \frac{n-2}{2} h_g$.

Problem (1.1) is the perturbed version of the Yamabe boundary problem when the target metric has zero scalar curvature, that is, given a compact Riemannian manifold with boundary, finding a

Riemannian metric, conformal to the original one, with zero scalar curvature and constant boundary mean curvature. This represents an extension of the Yamabe problem on manifold with boundary and, since the target metric is conformally flat, also a generalization of the Riemann mapping theorem to higher dimensions. Solving this problem is equivalent to find a positive solution of the equation

$$\begin{cases} -\Delta_g u + \frac{n-2}{4(n-1)} R_g u = 0 & \text{in } M \\ \frac{\partial u}{\partial \nu} + \frac{n-2}{2} h_g u = (n-2) u^{\frac{n}{n-2}} & \text{on } \partial M \end{cases} \tag{1.2}$$

which is, as noticed before, the unperturbed version of (1.1). In this paper we study if the perturbation term affects the property of solutions. In particular we want to investigate if the compactness of the set of the solution of the problem holds true for the perturbed problem. Our main results are the following.

Theorem 1. *Let (M, g) a smooth, n-dimensional Riemannian manifold of positive type with regular boundary ∂M. Suppose that $n \geq 7$ and that $\pi(x)$, the trace free second fundamental form of ∂M, is non zero everywhere.*

Let $\alpha, \beta : M \to \mathbb{R}$ smooth functions such that $\beta < 0$ on ∂M and $\max_{q \in \partial M} \{\alpha(q) - \frac{n-6}{4(n-1)(n-2)^2} \|\pi(q)\|^2\} < 0$. Then, there exist two constants $C > 0$ and $0 < \bar{\varepsilon} < 1$ such that, for any $0 \leq \varepsilon_1, \varepsilon_2 \leq \bar{\varepsilon}$ and for any $u > 0$ solution of (1.1), it holds

$$C^{-1} \leq u \leq C \text{ and } \|u\|_{C^{2,\eta}(M)} \leq C$$

for some $0 < \eta < 1$. The constant C does not depend on $u, \varepsilon_1, \varepsilon_2$.

Theorem 2. *Let (M, g) a smooth, n-dimensional Riemannian manifold of positive type with regular boundary ∂M. Suppose that $n \geq 7$ and that the trace free second fundamental form of ∂M, is non zero everywhere. Let $\alpha, \beta : M \to \mathbb{R}$ smooth functions.*

- *If $\beta > 0$ on ∂M then for $\varepsilon_1, \varepsilon_2 > 0$ small enough there exists a sequence of solutions $u_{\varepsilon_1, \varepsilon_2}$ of (1.1) which blows up at a suitable point of ∂M as $(\varepsilon_1, \varepsilon_2) \to (0, 0)$.*
- *If $\beta < 0$ on ∂M, $\varepsilon_1 = 1$, $\alpha > 0$ on M and $\inf_{q \in \partial M} \alpha(q) + \frac{1}{B} \varphi(q) > 0$, then for $\varepsilon_2 > 0$ small enough there exists a sequence of solutions u_{ε_2} of (1.1) which blows up at a suitable point of ∂M as $\varepsilon_2 \to 0$.*

Here B and $\varphi(q)$ are defined in Lemma 24.

We remark that in the above Theorem 2, B is strictly positive, $\varphi(q)$ is strictly negative, and both are completely determined by (M, g).

The result of Theorem 1 (and its counterpart Theorem 2) is somewhat unexpected: in classical Yamabe problem [1] the compactness of solution is guaranteed as soon as α is negative. In a forthcoming paper we prove that also for boundary Yamabe problem on manifold with umbilic boundary compactness is granted when α is negative while for α positive everywhere there exists a blowing up sequence of solutions. So, this is an example in which the strong analogy between classical Yamabe problem and boundary Yamabe problem breaks down.

The boundary Yamabe problem was firstly introduced by Escobar in [2]. Existence results for (1.2) were proved by Escobar [2], Marques [3], Almaraz [4], Brendle and Chen [5], Mayer and Ndiaye [6].

Solutions of (1.2) could be found by minimization of the quotient

$$Q(M,\partial M) := \inf_{u\in H^1\smallsetminus 0} \frac{\int\limits_M \left(|\nabla u|^2 + \frac{n-2}{4(n-1)}R_g u^2\right) dv_g + \int\limits_{\partial M} \frac{n-2}{2} h_g u^2 d\sigma_g}{\left(\int\limits_{\partial M} |u|^{\frac{2(n-1)}{n-2}} d\sigma_g\right)^{\frac{n-2}{n-1}}}.$$

In particular, the solution of is unique, up to symmetries, when $-\infty < Q \leq 0$ while multiplicity results hold when $Q > 0$. Manifolds for which $Q > 0$ are called of *positive type,* and it is natural to ask, in that case, when the full set of the solutions of (1.2) forms a C^2-compact set. This is in complete analogy with classical Yamabe problem. In addition, the celebrated strategy of Khuri, Marques and Schoen [7] to prove compactness of solutions of Yamabe problem up to dimension $n = 24$ can be succesfully adapted to Problem (1.2). Indeed, with this method compactness has been proved firstly in the case of locally flat manifolds not conformally equivalent to euclidean balls in [8], then for manifolds with non umbilic boundary in [9], and, recently, for manifolds with umbilic boundary on which the Weyl tensor does not vanish, in [10, 11]. These results have been successively extended, but an exhaustive list of reference of compactness results is beyond the scope of this introduction. In [12], Almaraz proved that, for $n \geq 25$, it is possible to construct umbilic boundary manifolds, not conformally equivalent to euclidean balls, for which Problem (1.2) admits a non compact set of solutions. It is conjectured that also for boundary Yamabe the critical dimension is $n = 24$, but compactness for dimension $n \leq 24$ is not yet proved in all generality.

As mentioned before, a parallelism arises studying stability of Yamabe problem with respect of small perturbations of curvatures. For classical Yamabe problem, Druet, in the second claim of the main theorem of [1], proves that the set of solutions of $-\Delta_g u + \frac{n-2}{4(n-1)}a(x)u = cu^{\frac{n+2}{n-2}}$ in M is still compact if $a(x) \leq R_g(x)$ on a manifold M which is not conformally the round sphere and which dimension is $n = 3, 4, 5$. Thus, he claims that in this case the Yamabe problem is *stable* with respect of perturbation of scalar curvature from below. Also he claims that these results could be extended to higher dimensions. On the other hand, in [13, 14], Druet, Hebey and Robert found counterexamples to compactness, and so *instability,* when $a(x)$ is greater than $R_g(x)$. In [15] the same problem is studied in the case of boundary Yamabe equation by perturbing the mean curvature term, and the same compactness versus blow up phenomenon appeared. So, a first analogy between the role of scalar curvature in classical case and mean curvature in boundary case is established. An analogy between the role of scalar curvature in classical and boundary Yamabe problem when the boundary is umbilic will be investigated by the authors in a forthcoming paper.

As far as we know, Theorem 1 is the first case in which stability is possible when pertrubing a curvature from above, and, therefore, in which the parallelism between classical and boundary Yamabe problem is lost. The result of Theorem 1 is strictly related to non umbilicity of the boundary. In fact, the trace-free second fundamental form competes with the perturbation of the scalar curvature. Thus, when the tensor does not vanish, it could compensate a small positive perturbation. This is clearly observable in Proposition 17, which is a key tool to prove the compactness result (and for the blow-up counterpart, in Lemma 24).

The paper is organized as follows. Hereafter we recall basic definitions and preliminary notions useful to achieve the result. Section 2 is devoted to the proof of the compactness theorem, while in Section 3 we prove the non compactness result.

1.1. Notations and preliminary definitions

Remark 3 (Notations). We will use the indices $1 \le i, j, k, m, p, r, s \le n-1$ and $1 \le a, b, c, d \le n$. Moreover we use the Einstein convention on repeated indices. We denote by g the Riemannian metric, by R_{abcd} the full Riemannian curvature tensor, by R_{ab} the Ricci tensor and by R_g and h_g respectively the scalar curvature of (M, g) and the mean curvature of ∂M. The bar over an object (e.g., $\bar{R}_g$) will means the restriction to this object to the metric of ∂M

On the half space $\mathbb{R}^n_+ = \{y = (y_1, \dots, y_{n-1}, y_n) \in \mathbb{R}^n, y_n \ge 0\}$ we set $B_r(y_0) = \{y \in \mathbb{R}^n, |y - y_0| \le r\}$ and $B^+_r(y_0) = B_r(y_0) \cap \{y_n > 0\}$. When $y_0 = 0$ we will use simply $B_r = B_r(y_0)$ and $B^+_r = B^+_r(y_0)$. On the half ball B^+_r we set $\partial' B^+_r = B^+_r \cap \partial\mathbb{R}^n_+ = B^+_r \cap \{y_n = 0\}$ and $\partial^+ B^+_r = \partial B^+_r \cap \{y_n > 0\}$. On $\mathbb{R}^n_+$ we will use the following decomposition of coordinates: $(y_1, \dots, y_{n-1}, y_n) = (\bar{y}, y_n) = (z, t)$ where $\bar{y}, z \in \mathbb{R}^{n-1}$ and $y_n, t \ge 0$.

Fixed a point $q \in \partial M$, we denote by $\psi_q : B^+_r \to M$ the Fermi coordinates centered at q. We denote by $B^+_g(q, r)$ the image of B^+_r. When no ambiguity is possible, we will denote $B^+_g(q, r)$ simply by B^+_r, omitting the chart ψ_q.

We recall also that ω_{n-2} denotes the volume of the $n-1$ dimensional unit sphere $\mathbb{S}^{n-1}$.

At last we introduce here the standard bubble $U(y) := \dfrac{1}{\left[(1 + y_n)^2 + |\bar{y}|^2\right]^{\frac{n-2}{2}}}$ which is the unique solution, up to translations and rescaling, of the nonlinear critical problem .

$$\begin{cases} -\Delta U = 0 & \text{on } \mathbb{R}^n_+; \\ \frac{\partial U}{\partial y_n} = -(n-2) U^{\frac{n}{n-2}} & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{1.3}$$

Set

$$j_l := \partial_l U = -(n-2) \frac{y_l}{\left[(1 + y_n)^2 + |\bar{y}|^2\right]^{\frac{n}{2}}} \tag{1.4}$$

$$\partial_k \partial_l U = (n-2) \left\{ \frac{n y_l y_k}{\left[(1 + y_n)^2 + |\bar{y}|^2\right]^{\frac{n+2}{2}}} - \frac{\delta^{kl}}{\left[(1 + y_n)^2 + |\bar{y}|^2\right]^{\frac{n}{2}}} \right\}$$

$$j_n := y^b \partial_b U + \frac{n-2}{2} U = -\frac{n-2}{2} \frac{|y|^2 - 1}{\left[(1 + y_n)^2 + |\bar{y}|^2\right]^{\frac{n}{2}}}, \tag{1.5}$$

we recall that $j_1, \dots, j_n$ are a base of the space of the H^1 solutions of the linearized problem

$$\begin{cases} -\Delta \phi = 0 & \text{on } \mathbb{R}^n_+, \\ \frac{\partial \phi}{\partial t} + n U^{\frac{2}{n-2}} \phi = 0 & \text{on } \partial\mathbb{R}^n_+, \\ \phi \in H^1(\mathbb{R}^n_+). \end{cases} \tag{1.6}$$

Given a point $q \in \partial M$, we introduce now the function γ_q which arises from the second order term of the expansion of the metric g on M (see 1.14). The choice of this function plays a twofold role in this paper. On the one hand, using the function γ_q we are able to perform the estimates of Lemmas 13, 14 and Proposition 15. On the other hand, it gives the correct correction to the standard bubble in order to perform finite dimensional reduction.

For the proof of the following Lemma we refer to [9, Prop 5.1] and [16, Proposition 7]

Lemma 4. *Assume $n \geq 3$. Given a point $q \in \partial M$, there exists a unique $\gamma_q : \mathbb{R}^n_+ \to \mathbb{R}$ a solution of the linear problem*

$$\begin{cases} -\Delta\gamma = 2h_{ij}(q)t\partial^2_{ij}U & \text{on } \mathbb{R}^n_+; \\ \frac{\partial\gamma}{\partial t} + nU^{\frac{2}{n-2}}\gamma = 0 & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{1.7}$$

which is $L^2(\mathbb{R}^n_+)$-orthogonal to the functions $j_1, \dots, j_n$ defined in (1.4) and (1.5).

In addition it holds

$$|\nabla^\tau\gamma_q(y)| \leq C(1 + |y|)^{3-\tau-n} \text{ for } \tau = 0, 1, 2. \tag{1.8}$$

$$\int_{\mathbb{R}^n_+} \gamma_q\Delta\gamma_q dy \leq 0, \tag{1.9}$$

$$\int_{\partial\mathbb{R}^n_+} U^{\frac{n}{n-2}}(t, z)\gamma_q(t, z)dz = 0 \tag{1.10}$$

$$\gamma_q(0) = \frac{\partial\gamma_q}{\partial y_1}(0) = \cdots = \frac{\partial\gamma_q}{\partial y_{n-1}}(0) = 0. \tag{1.11}$$

Finally the map $q \mapsto \gamma_q$ is $C^2(\partial M)$.

1.2. Expansion of the metric

It is well known that there exists a metric $\tilde{g}$, conformal to g, such that $h_{\tilde{g}} \equiv 0$ (see [2, Lemma 3.3]). So, up to a global conformal change of coordinates Problem (1.1) becomes

$$\begin{cases} -\Delta_g u + \frac{n-2}{4(n-1)}R_g u + \varepsilon_1\alpha u = 0 & \text{in } M \\ \frac{\partial u}{\partial\nu} + \varepsilon_2\beta u = (n-2)u^{\frac{n}{n-2}} & \text{on } \partial M \end{cases}. \tag{1.12}$$

With this change of coordinates the expansion of the metric is

$$|g(y)|^{1/2} = 1 - \frac{1}{2}\left[\|\pi\|^2 + \mathrm{Ric}(0)\right]y_n^2 - \frac{1}{6}\bar{R}_{ij}(0)y_iy_j + O(|y|^3) \tag{1.13}$$

$$\begin{aligned} g^{ij}(y) =& \delta_{ij} + 2h_{ij}(0)y_n + \frac{1}{3}\bar{R}_{ikjl}(0)y_ky_l + 2\frac{\partial h_{ij}}{\partial y_k}(0)ty_k \\ &+ \left[R_{injn}(0) + 3h_{ik}(0)h_{kj}(0)\right]y_n^2 + O(|y|^3) \end{aligned} \tag{1.14}$$

$$g^{an}(y) = \delta_{an} \tag{1.15}$$

where π is the second fundamental form and $h_{ij}(0)$ are its coefficients, and $\mathrm{Ric}(0) = R_{nini}(0) = R_{nn}(0)$ (see [2]).

2. The compactness result

We start this section by recalling a Pohozaev type identity. This indentity gives us a fundamental sign condition to rule out the possibility of blowing up sequence, as shown in subsection 2.5. A recall of preliminary results on blow up points is collected in subsection 2.2, while a careful analysis of blow up sequences is performed in subsection 2.4. This allows us to conclude the section with the proof of Theorem 1. Throughout this section we work in $\tilde{g}$ metric. For the sake of readability we will omit the tilde symbol in all this section.

2.1. A Pohozaev type identity

A Pohozaev type identity is often used in Yamabe boundary problem. Here we use the same local version which is introduced in [9, 10].

Theorem 5 (Pohozaev Identity)**.** *Let u a C^2-solution of the following problem*

$$\begin{cases} -\Delta_g u + \frac{n-2}{4(n-1)} R_g u + \varepsilon_1 \alpha u = 0 = 0 & \text{in } B_r^+ \\ \frac{\partial u}{\partial \nu} + \varepsilon_2 \beta u = (n-2) u^{\frac{n}{n-2}} & \text{on } \partial' B_r^+ \end{cases}$$

for $B_r^+ = \psi_q^{-1}(B_g^+(q,r))$ for $q \in \partial M$. Let us define

$$P(u,r) := \int\limits_{\partial^+ B_r^+} \left(\frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{r}{2} |\nabla u|^2 + r \left| \frac{\partial u}{\partial r} \right|^2 \right) d\sigma_r + \frac{r(n-2)^2}{2(n-1)} \int\limits_{\partial(\partial' B_r^+)} u^{\frac{2(n-1)}{n-2}} d\bar{\sigma}_g,$$

and

$$\hat{P}(u,r) := -\int\limits_{B_r^+} \left(y^a \partial_a u + \frac{n-2}{2} u \right) [(L_g - \Delta) u] dy +$$
$$+ \varepsilon_1 \int\limits_{B_r^+} \left(y^a \partial_a u + \frac{n-2}{2} u \right) \alpha u dy$$
$$+ \frac{n-2}{2} \varepsilon_2 \int\limits_{\partial' B_r^+} \left(\bar{y}^k \partial_k u + \frac{n-2}{2} u \right) \beta u d\bar{y}.$$

Then $P(u,r) = \hat{P}(u,r)$.

Here $a = 1, \dots, n$, $k = 1, \dots, n-1$ and $y = (\bar{y}, y_n)$, where $\bar{y} \in \mathbb{R}^{n-1}$ and $y_n \geq 0$.

2.2. Isolated and isolated simple blow up points

We collect here the definitions of some type of blow up points, and the basic properties about the behavior of these blow up points (see [8, 9, 19, 20]).

Let $\{u_i\}_i$ be a sequence of positive solution to

$$\begin{cases} L_{g_i} u - \varepsilon_{1,i} \alpha u = 0 & \text{in } M \\ B_{g_i} u + (n-2) u^{\frac{n}{n-2}} - \varepsilon_{2,i} \beta u = 0 & \text{on } \partial M \end{cases} . \tag{2.1}$$

where $g_i \to g_0$ in the C^3_{loc} topology and $0 < \varepsilon_{1,i}, \varepsilon_{2,i} < \bar{\varepsilon}$ for some $0 < \bar{\varepsilon} \leq 1$. As before, we suppose without loss of generality that $h_{g_0} \equiv 0$ and $h_{g_i} \equiv 0$ for all i.

Definition 6. 1) We say that $x_0 \in \partial M$ is a blow up point for the sequence u_i of solutions of (2.1) if there is a sequence $x_i \in \partial M$ of local maxima of $u_i|_{\partial M}$ such that $x_i \to x_0$ and $u_i(x_i) \to +\infty$.

Shortly we say that $x_i \to x_0$ is a blow up point for $\{u_i\}_i$.

2) We say that $x_i \to x_0$ is an isolated blow up point for $\{u_i\}_i$ if $x_i \to x_0$ is a blow up point for $\{u_i\}_i$ and there exist two constants $\rho, C > 0$ such that

$$u_i(x) \leq C d_{\bar{g}}(x, x_i)^{\frac{2-n}{2}} \text{ for all } x \in \partial M \smallsetminus \{x_i\}, \ d_{\bar{g}}(x, x_i) < \rho.$$

Given $x_i \to x_0$ an isolated blow up point for $\{u_i\}_i$, and given $\psi_i : B^+_\rho(0) \to M$ the Fermi coordinates centered at x_i, we define the spherical average of u_i as

$$\bar{u}_i(r) = \frac{2}{\omega_{n-1} r^{n-1}} \int_{\partial^+ B_r^+} u_i \circ \psi_i d\sigma_r$$

and

$$w_i(r) := r^{\frac{2-n}{2}} \bar{u}_i(r)$$

for $0 < r < \rho$.

3) We say that $x_i \to x_0$ is an isolated simple blow up point for $\{u_i\}_i$ solutions of (2.1) if $x_i \to x_0$ is an isolated blow up point for $\{u_i\}_i$ and there exists ρ such that w_i has exactly one critical point in the interval $(0, \rho)$.

Remark 7. Notice that blow up for elliptic equation with neumann boundary condition often occurs at a point of the boundary (e.g., in the pioneering paper of Ni and Takagi [25]). Concerning boundary Yamabe problem, this fact was at first explicitly proved, as in [8]. Later on, it was assumed, without loss of generality, that the blow up point x_0 as well as the whole sequence $x_i \to x_0$ belongs to the boundary (see [9, Definition 4.1]),

Given $x_i \to x_0$ a blow up point for $\{u_i\}_i$, we set

$$M_i := u_i(x_i) \text{ and } \delta_i := M_i^{\frac{2}{2-n}}.$$

Obviously $M_i \to +\infty$ and $\delta_i \to 0$.

The proofs of the following propositions can be found in [4] and in [8].

Proposition 8. *Let $x_i \to x_0$ is an isolated blow up point for $\{u_i\}_i$ and ρ as in Definition 6. We set*

$$v_i(y) = M_i^{-1}(u_i \circ \psi_i)(M_i^{\frac{2}{2-n}} y), \text{ for } y \in B^+_{\rho M_i^{\frac{n-2}{2}}}(0).$$

Then, given $R_i \to \infty$ and $c_i \to 0$, up to subsequences, we have

1. $|v_i - U|_{C^2\left(B^+_{R_i}(0)\right)} < c_i$;
2. $\lim_{i\to\infty} \dfrac{R_i}{\log M_i} = 0.$

Proposition 9. *Let $x_i \to x_0$ be an isolated simple blow-up point for $\{u_i\}_i$. Let η small. If $0 < \bar{\varepsilon} \le 1$ is small enough and $0 < \varepsilon_1, \varepsilon_2 < \bar{\varepsilon}$, then there exist $C, \rho > 0$ such that*

$$M_i^{\lambda_i} |\nabla^k u_i(\psi_i(y))| \le C|y|^{2-k-n+\eta}$$

for $y \in B^+_\rho(0) \setminus \{0\}$ and $k = 0, 1, 2$. Here $\lambda_i = \left(\frac{2}{n-2}\right)(n - 2 - \eta) - 1$.

2.3. A splitting lemma

Here we summarize a result which proves that only a finite number of blow up points may occur to a blowing up sequence of solution. For its proof we refer to [21, Proposition 5.1], [22, Lemma 3.1], [19, Proposition 1.1], [9, Propositions 4.2 and 8.2].

Proposition 10. *Given $K > 0$ and $R > 0$ there exist two constants $C_0, C_1 > 0$ (depending on K, R and (M, g)) such that if u is a solution of*

$$\begin{cases} L_g u - \varepsilon_1 \alpha = 0 & \text{in } M \\ B_g u - \varepsilon_2 \beta u + (n-2) u^{\frac{n}{n-2}} = 0 & \text{on } \partial M \end{cases} \tag{2.2}$$

and $\max_{\partial M} u > C_0$, then there exist $q_1, \ldots, q_N \in \partial M$, with $N = N(u) \geq 1$ with the following properties: for $j = 1, \ldots, N$

1. *set $r_j := Ru(q_j)^{1-p}$, then $\left\{B_{r_j} \cap \partial M\right\}_j$ are a disjoint collection;*
2. *we have $\left|u(q_j)^{-1} u(\psi_j(y)) - U(u(q_j)^{p-1} y)\right|_{C^2(B^+_{2r_j})} < K$ (here ψ_j are the Fermi coordinates at point q_j;*
3. *we have*

$$u(x) d_{\bar{g}} \left(x, \{q_1, \ldots, q_n\}\right)^{\frac{1}{p-1}} \leq C_1 \text{ for all } x \in \partial M$$
$$u(q_j) d_{\bar{g}} \left(q_j, q_k\right)^{\frac{1}{p-1}} \geq C_0 \text{ for any } j \neq k.$$

In addition, if $n \geq 7$ and $|\pi(x)| \neq 0$ for any $x \in \partial M$, there exists $d = d(K, R)$ such that

$$\min_{\substack{i \neq j \\ 1 \leq i, j \leq N(u)}} d_{\bar{g}}(q_i(u), q_j(u)) \geq d.$$

Here $\bar{g}$ is the geodesic distance on ∂M.

2.4. Blowup estimates

In this section we provide a fine estimate for the approximation of the rescaled solution near an isolated simple blow up point.

Proposition 11. *Let $x_i \to x_0$ be an isolated simple blow-up point for $\{u_i\}_i$ and $\beta < 0$. Then $\varepsilon_{2,i} \to 0$.*

Proof. We compute the Pohozaev identity in a ball of radius r and we set $\frac{r}{\delta_i} =: R_i \to \infty$.

By Proposition 9 we have that

$$P(u_i, r) \leq \delta_i^{\lambda_i(n-2)}. \tag{2.3}$$

We estimate now $\hat{P}(u_i, r)$. By comparing this term with $P(u_i, r)$ we will get the proof.

$$\begin{aligned} \hat{P}(u_i, r) := & - \int_{B_r^+} \left(y^a \partial_a u_i + \frac{n-2}{2} u_i\right) [(L_g - \Delta) u_i] dy + \varepsilon_{1,i} \int_{B_r^+} \left(y^a \partial_a u_i + \frac{n-2}{2} u_i\right) \alpha u_i dy \\ & + \frac{n-2}{2} \varepsilon_{2,i} \int_{\partial' B_r^+} \left(\bar{y}^k \partial_k u_i + \frac{n-2}{2} u_i\right) \beta u_i d\bar{y} =: I_1(u_i, r) + I_2(u_i, r) + I_3(u_i, r). \end{aligned}$$

The terms I_3 has been estimated in [15, Proposition 8] and it holds

$$I_3(u_i, r) = \varepsilon_{2,i} \delta_i (B + o(1)), \tag{2.4}$$

where B is a positive real constant.

For $I_2(u_i, r)$ we have, by change of variables,

$$I_2(u_i, r) = \varepsilon_{1,i}\delta_i^2 \frac{n-2}{2}\alpha(x_i) \int\limits_{\mathbb{R}^n_+} \frac{1-|y|^2}{\left[(1+y_n)^2+|\bar{y}|^2\right]^{n-1}} dy + \varepsilon_{1,i}\delta_i^2 O(\delta_i^2).$$

Now, set $I_m^\alpha := \int_0^\infty \frac{s^\alpha ds}{(1+s^2)^m}$ we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n_+} \frac{1-|y|^2}{\left[(1+y_n)^2+|\bar{y}|^2\right]^{n-1}} dy \\
&\qquad = \omega_{n-2}\left[I_{n-1}^{n-2} \int_0^\infty \frac{1-t^2}{(1+t)^{n-2}} dt - I_{n-1}^n \int_0^\infty \frac{1}{(1+t)^{n-2}} dt \right] \\
&\qquad = \omega_{n-2}\left[I_{n-1}^{n-2} \frac{n-5}{(n-3)(n-4)} - I_{n-1}^n \frac{1}{n-4} \right]
\end{aligned}$$

using the identities $\int_0^\infty \frac{t^k dt}{(1+t)^m} = \frac{k!}{(m-1)(m-2)\cdots(m-1-k)}$ and $\int_0^\infty \frac{dt}{(1+t)^m} = \frac{1}{m-1}$. At this point, since $I_m^\alpha = \frac{2m}{2m-\alpha-1} I_{m+1}^\alpha$ and $I_m^\alpha = \frac{2m-\alpha-3}{\alpha+1} I_m^{\alpha+2}$ (see [9, Lemma 9.4]) we have

$$\frac{(n-5)I_{n-1}^{n-2}}{(n-3)(n-4)} - \frac{I_{n-1}^n}{n-4} = -\frac{4I_{n-1}^n}{(n-1)(n-4)} = -\frac{8I_n^n}{(n-3)(n-4)},$$

thus

$$\begin{aligned}
\int\limits_{B_r^+} \left(y^a \partial_a u_i + \frac{n-2}{2} u_i \right) \varepsilon_{1,i} \alpha u_i dy &= -\frac{4(n-2) I_n^n \omega_{n-2}}{(n-3)(n-4)} \varepsilon_{1,i}\delta_i^2 \alpha(x_i) + o(\delta_i^2) \\
&= \varepsilon_{1,i}\delta_i^2 (A + o(1)) \qquad (2.5)
\end{aligned}$$

where A is a real constant.

For the term $I_1(u_i, r)$ we slightly improve the estimate provided by Almaraz in [9]. By the expansion of the metric (1.13), (1.14) and (1.15) we have

$$I_1(u_i, r) \le -\delta_i \int\limits_{B^+_{r/\delta_i}} \left(y^a \partial_a v_i + \frac{n-2}{2} v_i \right) v_i h_{kl}(0) y_n \partial_k \partial_l v_i dy + O(\delta_i^2)$$

By simmetry reasons we have that

$$\begin{aligned}
&\lim_{i\to\infty} \int\limits_{B^+_{r/\delta_i}} \left(y^a \partial_a v_i + \frac{n-2}{2} v_i \right) v_i h_{kl}(0) y_n \partial_k \partial_l v_i dy \\
&\qquad = \int_{\mathbb{R}^n_+} \left(y^a \partial_a U + \frac{n-2}{2} U \right) U h_{kl}(0) y_n \partial_k \partial_l U dy \\
&\qquad = h_g(0) \int_{\mathbb{R}^n_+} \left(y^a \partial_a U + \frac{n-2}{2} U \right) U y_n \partial_1 \partial_1 U dy = 0
\end{aligned}$$

since we choose a metric for which the mean curvature of the boundary is zero. So

$$|I_1(u_i, r)| \leq \delta_i o^+(1) \tag{2.6}$$

where $o^+(1)$ is a nonnegative constant that vanishes when $i \to \infty$.

Comparing $\hat{P}(u_i, r)$ and $P(u_i, r)$, by (2.3), (2.4), (2.5) and (2.6) we get

$$-c\delta_i o^+(1) + (A + o(1))\varepsilon_{1,i}\delta_i^2 + (B + o(1))\varepsilon_{2,i}\delta_i \leq \delta_i^{\lambda_i(n-2)},$$

so

$$-co^+(1) + (A + o(1))\varepsilon_{1,i}\delta_i + (B + o(1))\varepsilon_{2,i} \leq \delta_i^{\lambda_i(n-2)-1}.$$

Being $\varepsilon_{1,i} < \bar{\varepsilon} < 1$, the above inequality holds only if $\varepsilon_{2,i} \to 0$. □

Since $\varepsilon_{2,i} \to 0$, $\delta_i \to 0$ and $\varepsilon_{1,i} < \bar{\varepsilon} < 1$, the proof of the next proposition is analogous to Prop. 4.3 of [9].

Proposition 12. *Let $x_i \to x_0$ be an isolated simple blow-up point for $\{u_i\}_i$. Then there exist $C, \rho > 0$ such that*

1. $M_i u_i(\psi_i(y)) \leq C|y|^{2-n}$ *for all* $y \in B_\rho^+(0) \smallsetminus \{0\}$*;*
2. $M_i u_i(\psi_i(y)) \geq C^{-1} G_i(y)$ *for all* $y \in B_\rho^+(0) \smallsetminus B_{r_i}^+(0)$ *where* $r_i := R_i M_i^{\frac{2}{2-n}}$ *and* G_i *is the Green's function which solves*

$$\begin{cases} L_{g_i} G_i = 0 & \text{in } B_\rho^+(0) \smallsetminus \{0\} \\ G_i = 0 & \text{on } \partial^+ B_\rho^+(0) \\ B_{g_i} G_i = 0 & \text{on } \partial' B_\rho^+(0) \smallsetminus \{0\} \end{cases}$$

and $|y|^{n-2} G_i(y) \to 1$ *as* $|z| \to 0$.

By Proposition 8 and Proposition 12 we have that, if $x_i \to x_0$ is an isolated simple blow-up point for $\{u_i\}_i$, then it holds

$$v_i \leq CU \text{ in } B^+_{\rho M_i^{\frac{2}{2-n}}}(0).$$

Which follows is the core of the compacntess claim: we provide the estimates of the blowup profile of an isolated simple blow up point $x_i \to x_0$ for a sequence $\{u_i\}_i$ of solutions of (2.1). The strategy to achieve these results is similar to the one contained in [9, Lemma 6.1] and in [15, Section 5], so we will give only the general scheme and emphasize the main differences, while we refer to the cited papers for detailed proofs. Set

$$\delta_i := u_i^{\frac{2}{2-n}}(x_i) = M_i^{\frac{2}{2-n}} \quad v_i(y) := \delta_i^{\frac{n-2}{2}} u_i(\delta_i y) \text{ for } y \in B^+_{\frac{R}{\delta_i}}(0), \tag{2.7}$$

we have that v_i satisfies

$$\begin{cases} L_{\hat{g}_i} v_i - \varepsilon_{1,i}\alpha(\delta_i y) v_i = 0 & \text{in } B^+_{\frac{R}{\delta_i}}(0) \\ B_{\hat{g}_i} v_i + (n-2) v_i^{\frac{n}{n-2}} - \varepsilon_{2,i}\beta(\delta_i y) v_i = 0 & \text{on } \partial' B^+_{\frac{R}{\delta_i}}(0) \end{cases} \tag{2.8}$$

where $\hat{g}_i := g_i(\delta_i y)$.

Lemma 13. *Assume $n \geq 7$. Let γ_{x_i} be defined in (1.7). There exist $R, C > 0$ such that*

$$|v_i(y) - U(y) - \delta_i \gamma_{x_i}(y)| \leq C\left(\delta_i^2 + \varepsilon_{1,i}\delta_i^2 + \varepsilon_{2,i}\delta_i\right)$$

for $|y| \leq R/\delta_i$.

Proof. Let y_i such that

$$\mu_i := \max_{|y| \leq R/\delta_i} |v_i(y) - U(y) - \delta_i \gamma_{x_i}(y)| = |v_i(y_i) - U(y_i) - \delta_i \gamma_{x_i}(y_i)|.$$

We can assume, without loss of generality, that $|y_i| \leq \frac{R}{2\delta_i}$. This will be useful in the next.

By contradiction, suppose that

$$\max\left\{\mu_i^{-1}\delta_i^2, \mu_i^{-1}\varepsilon_{1,i}\delta_i^2, \mu_i^{-1}\varepsilon_{2,i}\delta_i\right\} \to 0 \text{ when } i \to \infty. \tag{2.9}$$

Defined

$$w_i(y) := \mu_i^{-1}\left(v_i(y) - U(y) - \delta_i \gamma_{x_i}(y)\right) \text{ for } |y| \leq R/\delta_i,$$

we have, by direct computation, that

$$\begin{cases} L_{\hat{g}_i} w_i = A_i & \text{in } B^+_{\frac{R}{\delta_i}}(0) \\ B_{\hat{g}_i} w_i + b_i w_i = F_i & \text{on } \partial' B^+_{\frac{R}{\delta_i}}(0) \end{cases} \tag{2.10}$$

where

$$\begin{aligned} b_i =&(n-2)\frac{v_i^{\frac{n}{n-2}} - (U + \delta_i \gamma_{x_i})^{\frac{n}{n-2}}}{v_i - U - \delta_i \gamma_{x_i}}, \\ Q_i =&-\frac{1}{\mu_i}\left\{\left(L_{\hat{g}_i} - \Delta\right)(U + \delta_i \gamma_{x_i}) + \delta_i \Delta \gamma_{x_i}\right\}, \\ A_i =&Q_i + \frac{\varepsilon_{1,i}\delta_i^2}{\mu_i}\alpha_i(\delta_i y) v_i(y), \\ \bar{Q}_i =&-\frac{1}{\mu_i}\left\{(n-2)(U + \delta_i \gamma_{x_i})^{\frac{n}{n-2}} - (n-2)U^{\frac{n}{n-2}} - n\delta_i U^{\frac{2}{n-2}}\gamma_{x_i}\right\}, \\ F_i =&\bar{Q}_i + \frac{\varepsilon_{2,i}\delta_i}{\mu_i}\beta_i(\delta_i y) v_i(y). \end{aligned}$$

Since $v_i \to U$ in $C^2_{\text{loc}}(\mathbb{R}^n_+)$ we have, at once,

$$b_i \to nU^{\frac{2}{n-2}} \text{ in } C^2_{\text{loc}}(\mathbb{R}^n_+) \tag{2.11}$$

$$|b_i(y)| \leq (1 + |y|)^{-2} \text{ for } |y| \leq R/\delta_i. \tag{2.12}$$

We proceed now by estimating Q_i and $\bar{Q}_i$. As in [9, Lemma 6.1], using the expansion of the metric and the decays properties of U and γ_{x_i} we obtain

$$Q_i = O(\mu_i^{-1}\delta_i^2(1 + |y|)^{2-n}) \text{ and } \bar{Q}_i = O(\mu_i^{-1}\delta_i^2(1 + |y|)^{3-n}) \tag{2.13}$$

from which we get

$$A_i = O(\mu_i^{-1}\delta_i^2\,(1+|y|)^{2-n}) + O(\mu_i^{-1}\varepsilon_{1,i}\delta_i^2\,(1+|y|)^{2-n}) \tag{2.14}$$
$$F_i = O(\mu_i^{-1}\delta_i^2\,(1+|y|)^{3-n}) + O(\mu_i^{-1}\varepsilon_{2,i}\delta_i\,(1+|y|)^{2-n}).$$

In light of (2.9) we also have $A_i \in L^p(B^+_{R/\delta_i})$ and $F_i \in L^p(\partial' B^+_{R/\delta_i})$ for all $p \geq 2$. Since $|w_i(y)| \leq 1$, by (2.9) (2.11), (2.12), (2.14) and by standard elliptic estimates we conclude that $\{w_i\}_i$, up to subesequences, converges in $C^2_{\text{loc}}(\mathbb{R}^n_+)$ to some w solution of

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^n_+ \\ \frac{\partial}{\partial \nu} w + nU^{\frac{n}{n-2}} w = 0 & \text{on } \partial\mathbb{R}^n_+ \end{cases}. \tag{2.15}$$

Now we prove that $|w(y)| \leq C(1+|y|^{-1})$ for $y \in \mathbb{R}^n_+$. Consider G_i the Green function for the conformal Laplacian $L_{\hat{g}_i}$ defined on B^+_{r/δ_i} with boundary conditions $B_{\hat{g}_i} G_i = 0$ on $\partial' B^+_{r/\delta_i}$ and $G_i = 0$ on $\partial^+ B^+_{r/\delta_i}$. It is well known that $G_i = O(|\xi - y|^{2-n})$. By the Green formula and by (2.14) we have

$$\begin{aligned} w_i(y) = &- \int_{B^+_{\frac{R}{\delta_i}}} G_i(\xi, y) A_i(\xi) d\mu_{\hat{g}_i}(\xi) - \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} \frac{\partial G_i}{\partial \nu}(\xi, y) w_i(\xi) d\sigma_{\hat{g}_i}(\xi) \\ &+ \int_{\partial' B^+_{\frac{R}{\delta_i}}} G_i(\xi, y)\,(b_i(\xi) w_i(\xi) - F_i(\xi))\, d\sigma_{\hat{g}_i}(\xi), \end{aligned}$$

so

$$\begin{aligned} |w_i(y)| \leq &\frac{\delta_i^2}{\mu_i} \int_{B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n} (1+|\xi|)^{2-n} d\xi + \frac{\varepsilon_{1,i}\delta_i^2}{\mu_i} \int_{B^+_{\frac{R}{\delta_i}}} |\xi - y|^{2-n}(1+|\xi|)^{2-n} d\xi \\ &+ \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} |\xi - y|^{1-n} w_i(\xi) d\sigma(\xi) \\ &+ \int_{\partial' B^+_{\frac{R}{\delta_i}}} |\bar{\xi} - y|^{2-n} \left((1+|\bar{\xi}|)^{-2} + \frac{\delta_i^2}{\mu_i}(1+|\bar{\xi}|)^{3-n} + \frac{\varepsilon_{2,i}\delta_i}{\mu_i}(1+|\bar{\xi}|)^{2-n} \right) d\bar{\xi}, \end{aligned}$$

Notice that in the third integral we used that $|y| \leq \frac{R}{2\delta_i}$ to estimate $|\xi - y| \geq |\xi| - |y| \geq \frac{R}{2\delta_i}$ on $\partial^+ B^+_{R/\delta_i}$. Moreover, since $v_i(\xi) \leq CU(\xi)$, we get $|w_i(\xi)| \leq C\frac{\delta_i^{n-2}}{\mu_i}$ on $\partial^+ B^+_{R/\delta_i}$. Hence

$$\int_{\partial^+ B^+_{\frac{R}{\delta_i}}} |\xi - y|^{1-n} w_i(\xi) d\sigma(\xi) \leq C \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} \frac{\delta_i^{2n-3}}{\mu_i} d\sigma_{\hat{g}_i}(\xi) \leq C\frac{\delta_i^{n-2}}{\mu_i}. \tag{2.16}$$

For the other terms we use the formula

$$\int_{\mathbb{R}^m} |\xi - y|^{l-m}(1+|\xi|)^{-\eta} d\xi \leq C(1+|y|)^{l-\eta} \tag{2.17}$$

where $y \in \mathbb{R}^{m+k} \supseteq \mathbb{R}^m$, $\eta, l \in \mathbb{N}$, $0 < l < \eta < m$ (see [9, Lemma 9.2] and [23,24]), obtaining at last

$$|w_i(y)| \leq C\left((1+|y|)^{-1} + \frac{\delta_i^2}{\mu_i}(1+|y|)^{4-n} + \frac{\varepsilon_{1,i}\delta_i^2}{\mu_i}(1+|y|)^{4-n} + \frac{\varepsilon_{2,i}\delta_i}{\mu_i}(1+|y|)^{3-n} \right)$$

for $|y| \le \frac{R}{2\delta_i}$. By assumption (2.9) we prove

$$|w(y)| \le C(1+|y|)^{-1} \text{ for } y \in \mathbb{R}^n_+ \tag{2.18}$$

as claimed.

Now we can derive a contradiction. Notice that, since $v_i \to U$ near 0, and by (1.11) we have $w_i(0) \to 0$ as well as $\frac{\partial w_i}{\partial y_j}(0) \to 0$ for $j = 1, \dots, n-1$. This implies that

$$w(0) = \frac{\partial w}{\partial y_1}(0) = \cdots = \frac{\partial w}{\partial y_{n-1}}(0) = 0. \tag{2.19}$$

It is known (see [9, Lemma 2]) that any solution of (2.15) that decays as (2.18) is a linear combination of $\frac{\partial U}{\partial y_1}, \dots, \frac{\partial U}{\partial y_{n-1}}, \frac{n-2}{2}U + y^b \frac{\partial U}{\partial y_b}$. This, combined with (2.19), implies that $w \equiv 0$.

Now, on one hand $|y_i| \le \frac{R}{2\delta_i}$, so estimate (2.18) holds; on the other hand, since $w_i(y_i) = 1$ and $w \equiv 0$, we get $|y_i| \to \infty$, obtaining

$$1 = w_i(y_i) \le C(1+|y_i|)^{-1} \to 0$$

which gives us the contradiction. □

Lemma 14. *Assume $n \ge 7$ and $\beta < 0$. There exists $R, C > 0$ such that*

$$\varepsilon_{2,i} \le C\delta_i$$

for $|y| \le R/\delta_i$.

Proof. We proceed by contradiction, supposing that

$$\varepsilon_{2,i}^{-1}\delta_i = (\varepsilon_{2,i}\delta_i)^{-1}\,\delta_i^2 \to 0 \text{ when } i \to \infty. \tag{2.20}$$

Thus, by Lemma 13, we have

$$|v_i(y) - U(y) - \delta_i \gamma_{x_i}(y)| \le C\varepsilon_{2,i}\delta_i \text{ for } |y| \le R/\delta_i.$$

We define, similarly to Lemma 13,

$$w_i(y) := \frac{1}{\varepsilon_{2,i}\delta_i}\left(v_i(y) - U(y) - \delta_i\gamma_{x_i}(y)\right) \text{ for } |y| \le R/\delta_i,$$

and we have that w_i satisfies (2.10), where b_i is as in Lemma 13, and

$$\begin{aligned}
Q_i &= -\frac{1}{\varepsilon_{2,i}\delta_i}\left\{\left(L_{\hat{g}_i} - \Delta\right)(U + \delta_i\gamma_{x_i}) + \delta_i \Delta\gamma_{x_i}\right\},\\
A_i &= Q_i + \frac{\varepsilon_{1,i}\delta_i^2}{\varepsilon_{2,i}\delta_i}\alpha_i(\delta_i y)v_i(y),\\
\bar{Q}_i &= -\frac{1}{\varepsilon_{2,i}\delta_i}\left\{(n-2)(U+\delta_i\gamma_{x_i})^{\frac{n}{n-2}} - (n-2)U^{\frac{n}{n-2}} - n\delta_i U^{\frac{2}{n-2}}\gamma_{x_i}\right\},\\
F_i &= \bar{Q}_i + \beta_i(\delta_i y)v_i(y).
\end{aligned}$$

As before, b_i satisfies inequality (2.12) while

$$A_i = O\left(\frac{\delta_i^2}{\varepsilon_{2,i}\delta_i}(1+|y|)^{2-n}\right) \tag{2.21}$$
$$F_i = O\left(\frac{\delta_i^2}{\varepsilon_{2,i}\delta_i}(1+|y|)^{3-n}\right) + O\left((1+|y|)^{2-n}\right),$$

so by classic elliptic estimates we can prove that the sequence w_i converges in $C^2_{\text{loc}}(\mathbb{R}^n_+)$ to some w.

We proceed as in Lemma 13 to deduce that, by (2.20)

$$\begin{aligned} |w_i(y)| &\le C\left((1+|y|)^{-1} + \frac{\delta_i^2(1+|y|)^{4-n}}{\varepsilon_{2,i}\delta_i} + (1+|y|)^{3-n}\right) \\ &\le C\left((1+|y|)^{-1}\right) \text{ for } |y| \le \frac{R}{2\delta_i}. \end{aligned} \tag{2.22}$$

Now let j_n be defined as in (1.5). Similarly to [15, Lemma 12], since w_i satisfies (2.10), integrating by parts we obtain

$$\begin{aligned} \int_{\partial' B^+_{\frac{R}{\delta_i}}} j_n F_i d\sigma_{\hat{g}_i} &= \int_{\partial' B^+_{\frac{R}{\delta_i}}} j_n \left[B_{\hat{g}_i} w_i + b_i w_i\right] d\sigma_{\hat{g}_i} \\ &= \int_{\partial' B^+_{\frac{R}{\delta_i}}} w_i \left[B_{\hat{g}_i} j_n + b_i j_n\right] d\sigma_{\hat{g}_i} + \int_{\partial^+ B^+_{\frac{R}{\delta_i}}} \left[\frac{\partial j_n}{\partial \eta_i} w_i - \frac{\partial w_i}{\partial \eta_i} j_n\right] d\sigma_{\hat{g}_i} \\ &\quad + \int_{B^+_{\frac{R}{\delta_i}}} \left[w_i L_{\hat{g}_i} j_n - j_n L_{\hat{g}_i} w_i\right] d\mu_{\hat{g}_i} \end{aligned} \tag{2.23}$$

where η_i is the inward unit normal vector to $\partial^+ B^+_{\frac{R}{\delta_i}}$. One can check easily that

$$\lim_{i\to+\infty} \int_{\partial' B^+_{\frac{R}{\delta_i}}} j_n \bar{Q}_i d\sigma_{\hat{g}_i} = 0.$$

Also, since $\beta < 0$, by Proposition 8, we have

$$\beta(\delta_i y) v_i(y) \to \beta(x_0) U(y) \text{ for } i \to +\infty.$$

and thus

$$\lim_{i\to+\infty} \int_{\partial' B^+_{\frac{R}{\delta_i}}} \beta(\delta_i y) v_i(y) j_n(y) = \frac{n-2}{2}\beta(x_0) \int_{\mathbb{R}^{n-1}} \frac{1-|\bar{y}|^2}{(1+|\bar{y}|^2)^{n-1}} =: B > 0 \tag{2.24}$$

so

$$\int_{\partial' B^+_{\frac{R}{\delta_i}}} j_n F_i d\sigma_{\hat{g}_i} = B + o(1). \tag{2.25}$$

By (2.23) and (2.25) we derive a contradiction. Indeed, by the decay of j_n and by the decay of w_i, given by (2.22) and by (2.20), we have

$$\lim_{i\to+\infty}\int_{\partial^+ B^+_{\frac{R}{\delta_i}}}\left[\frac{\partial j_n}{\partial\eta_i}w_i-\frac{\partial w_i}{\partial\eta_i}j_n\right]d\sigma_{\hat{g}_i}=0 \tag{2.26}$$

Since $\Delta j_n=0$, one can check that

$$\lim_{i\to+\infty}\int_{B^+_{\frac{R}{\delta_i}}} w_i L_{\hat{g}_i} j_n d\mu_{\hat{g}_i}=0. \tag{2.27}$$

Also, we can prove that

$$\lim_{i\to+\infty}\int_{B^+_{\frac{R}{\delta_i}}} j_n Q_i d\mu_{\hat{g}_i}=0. \tag{2.28}$$

Finally

$$\lim_{i\to+\infty}\int_{\partial' B^+_{\frac{R}{\delta_i}}} w_i\left[B_{\hat{g}_i}j_n+b_i j_n\right]d\sigma_{\hat{g}_i}=\int_{\partial\mathbb{R}^n_+} w\left[\frac{\partial j_n}{\partial y_n}+nU^{\frac{2}{n-2}}j_n\right]d\sigma_{\hat{g}_i}=0 \tag{2.29}$$

since $\frac{\partial j_n}{\partial y_n}+nU^{\frac{2}{n-2}}j_n=0$ when $y_n=0$.

In light of (2.26) (2.28) and (2.27) we infer, by (2.23), that

$$\int_{\partial' B^+_{\frac{R}{\delta_i}}} j_n F_i d\sigma_{\hat{g}_i}=-\int_{B^+_{\frac{R}{\delta_i}}}\left[j_n A_i w_i\right]d\mu_{\hat{g}_i}+o(1). \tag{2.30}$$

Again we have $\alpha(\delta_i y)v_i(y)\to\alpha(x_0)U(y)$ for $i\to+\infty$, so,

$$\lim_{i\to\infty}\int_{B^+_{\frac{R}{\delta_i}}} j_n(y)\alpha(\delta_i y)v_i(y)d\mu_{\hat{g}_i}=\alpha(x_0)\lim_{i\to\infty}\int_{\mathbb{R}^n_+}\left(s^a\partial_a v_i+\frac{n-2}{2}v_i\right)v_i ds=:A\in\mathbb{R}. \tag{2.31}$$

Thus

$$\int_{B^+_{\frac{R}{\delta_i}}}\left[j_n A_i w_i\right]=-\frac{\delta_i^2}{\varepsilon_{2,i}\delta_i}(A+o(1))=o(1) \tag{2.32}$$

by (2.20). At this point, by (2.25), (2.30) and (2.32), we get

$$B+o(1)=o(1). \tag{2.33}$$

which gives us a contradiction since $B>0$. □

The following proposition is the main result of this section. The proof can be obtained with a first estimate in the spirit of the previous Lemmas, which is iterated until we get the final result. In fact, once one have the result of Lemma 14, the proof of the Proposition is very similar to the one of [9, Proposition 6.1]. For the sake of brevity we prefer to omit it.

Proposition 15. *Assume $n \geq 7$ and $\beta < 0$. Let γ_{x_i} be defined in (1.7). There exist $R, C > 0$ such that*

$$\left|\nabla_{\bar{y}}^{\tau}\left(v_i(y) - U(y) - \delta_i\gamma_{x_i}(y)\right)\right| \leq C\delta_i^2(1+|y|)^{4-\tau-n}$$
$$\left|y_n\frac{\partial}{\partial_n}\left(v_i(y) - U(y) - \delta_i\gamma_{x_i}(y)\right)\right| \leq C\delta_i^2(1+|y|)^{4-n}$$

for $|y| \leq \frac{R}{2\delta_i}$. Here $\tau = 0, 1, 2$ and $\nabla_{\bar{y}}^{\tau}$ is the differential operator of order τ with respect the first $n-1$ variables.

2.5. Sign estimates of Pohozaev identity terms

We estimate now $P(u_i, r)$, where $\{u_i\}_i$ is a family of solutions of (2.1) which has an isolated simple blow up point $x_i \to x_0$. This estimate, contained in Proposition 17, is a crucial point when proving the vanishing of the second fundamental form at an isolated simple blow up point.

The leading term of $P(u_i, r)$ will be $-\int_{B^+_{r/\delta_i}}\left(y^b\partial_b u + \frac{n-2}{2}u\right)\left[(L_{\hat{g}_i} - \Delta)v\right]dy$, so we set

$$R(u, v) = -\int_{B^+_{r/\delta_i}}\left(y^b\partial_b u + \frac{n-2}{2}u\right)\left[(L_{\hat{g}_i} - \Delta)v\right]dy, \tag{2.34}$$

and we recall the following result by Almaraz [9, Propositions 5.2 and 7.1].

Lemma 16. *For $n \geq 7$ we have*

$$\begin{aligned}R(U+\delta^2\gamma_q, U+\delta^2\gamma_q) =& \delta^2\frac{(n-6)\omega_{n-2}I_n^n}{(n-1)(n-2)(n-3)(n-4)}\left[\|\pi\|^2\right]\\ &-\frac{1}{2}\delta^2\int_{\mathbb{R}^n_+}\gamma_q\Delta\gamma_q dy + o(\delta^2)\end{aligned}$$

where $I_n^n :=:= \int_0^\infty \frac{s^n ds}{(1+s^2)^n}$.

Proposition 17. *Let $x_i \to x_0$ be an isolated simple blow-up point for u_i solutions of (2.1). Let $\beta < 0$ and $n \geq 7$. Fixed r, we have, for i large*

$$\begin{aligned}\hat{P}(u_i, r) \geq& \delta_i^2\frac{(n-6)\omega_{n-2}I_n^n}{(n-1)(n-2)(n-3)(n-4)}\left[\|\pi\|^2\right]\\ &-\varepsilon_{1,i}\delta_i^2\frac{4(n-2)I_n^n\omega_{n-2}}{(n-3)(n-4)}\alpha(x_i) + o(\delta_i^2).\end{aligned}$$

Proof. We remind that the definition of $\hat{P}$ is given in Theorem 5 and we take $v_i(y)$ as in (2.7). By Proposition 15 and by (1.8) of Lemma 4, for $|y| < R/\delta_i$ we have

$$|v_i(y) - U(y)| = O(\delta_i^2(1+|y|^{4-n}) + O(\delta_i(1+|y|^{3-n}) = O(\delta_i(1+|y|^{3-n})$$
$$|y_a\partial_a v_i(y) - y_a\partial_a U(y)| = O(\delta_i^2(1+|y|^{4-n}) + O(\delta_i(1+|y|^{3-n}) = O(\delta_i(1+|y|^{3-n}),$$

so, recalling (2.5) we have

$$\int_{B_r^+}\left(y^a\partial_a u_i + \frac{n-2}{2}u_i\right)\varepsilon_{1,i}\alpha_i u_i dy = -\frac{4(n-2)I_n^n\omega_{n-2}}{(n-3)(n-4)}\varepsilon_{1,i}\delta_i^2\alpha(x_i) + o(\delta_i^2).$$

Analogously we obtain

$$\int_{\partial' B_r^+} \left(\bar{y}^k \partial_k u_i + \frac{n-2}{2} u_i\right) \varepsilon_{2,i} \beta_i u_i d\bar{y}$$
$$= \varepsilon_{2,i} \delta_i \frac{n-2}{2} \beta(x_i) \int_{\mathbb{R}^{n-1}} \frac{1-|\bar{y}|^2}{[1+|\bar{y}|^2]^{n-1}} d\bar{y} + \varepsilon_{2,i} \delta_i O(\delta_i^2) > 0.$$

So, for i sufficiently large we obtain

$$\begin{aligned}\hat{P}(u_i, r) \geq &- \int_{B^+_{r/\delta_i}} \left(y^b \partial_b v_i + \frac{n-2}{2} v_i\right) \left[(L_{\hat{g}_i} - \Delta) v_i\right] dy \\ &- \frac{4(n-2) I_n^n \omega_{n-2}}{(n-3)(n-4)} \varepsilon_{1,i} \delta_i^2 \alpha(x_i) + o(\delta_i^2).\end{aligned}$$

Then, by the estimates on v_i obtained in the previous section, using Lemma 16, and recalling that, by inequality (1.9), $\int_{\mathbb{R}^n_+} \gamma_q \Delta \gamma_q dy \leq 0$, we get

$$\begin{aligned}\hat{P}(u_i, r) \geq & R(U + \delta^2 \gamma_q, U + \delta^2 \gamma_q) \\ &- \frac{4(n-2) I_n^n \omega_{n-2}}{(n-3)(n-4)} \varepsilon_{1,i} \delta_i^2 \alpha(x_i) + o(\delta_i^2) \\ \geq & \delta_i^2 \frac{(n-6)\omega_{n-2} I_n^n}{(n-1)(n-2)(n-3)(n-4)} \left[|h_{kl}(x_i)|^2\right] \\ &- \frac{4(n-2) I_n^n \omega_{n-2}}{(n-3)(n-4)} \varepsilon_{1,i} \delta_i^2 \alpha(x_i) + o(\delta_i^2)\end{aligned}$$

which gives the proof. □

Proposition 18. *Assume $n \geq 7$, $0 \leq \varepsilon_{1,i}, \varepsilon_{2,1} \leq \bar{\varepsilon} < 1$, $\beta < 0$ and*

$$\max_{q \in \partial M} \left\{ \alpha(q) - \frac{n-6}{4(n-1)(n-2)^2} \|\pi(q)\|^2 \right\} < 0.$$

Let $x_i \to x_0$ be an isolated simple blow-up point for u_i solutions of (2.1). Then

1. *For i large, $\hat{P}(u_i, r) \geq \delta_i^2 C_1 \left[\|\pi(x_i)\|^2\right] + o(\delta_i^2)$ for some $C_1 > 0$;*
2. *$\|\pi(x_0)\| = 0$.*

Proof. By Proposition 11 and Proposition 9 we have

$$P(u_i, r) \leq C \delta_i^{n-2}.$$

On the other hand recalling Proposition 17, Theorem 5, the assumption on α, and that $\varepsilon_{1,i} < 1$, we have

$$P(u_i, r) = \hat{P}(u_i, r) \geq \delta_i^2 C_1 \left[\|\pi(x_i)\|^2\right] + o(\delta_i^2),$$

with $C_1 > 0$. In addition, we get $\|\pi(x_i)\|^2 \leq \delta_i^{n-4}$, which gives the proof. □

Once we have the result of Proposition 17, with strategy similar to 18, we can prove the following Proposition. For a detailed proof we refer to [9, Proposition 8.1].

Proposition 19. *Let $x_i \to x_0$ be an isolated blow up point for u_i solutions of (2.1). Assume $n \geq 7$, $0 \leq \varepsilon_{1,i}, \varepsilon_{2,1} \leq \bar{\varepsilon} < 1$, $\beta < 0$, $\max_{q\in\partial M}\left\{\alpha(q) - \frac{n-6}{4(n-1)(n-2)^2}\|\pi(q)\|^2\right\} < 0$ and $\|\pi(x_0)\| \neq 0$. Then x_0 is isolated simple.*

2.6. Proof of Theorem 1

Using what we have obtained throughout this section, we can now prove the compactness result.

Proof of Theorem 1. . By contradiction, suppose that $x_i \to x_0$ is a blowup point for u_i solutions of (1.1). Let $q_1^i, \dots q_{N(u_i)}^i$ the sequence of points given by Proposition 10. By Claim 3 of Proposition 10 there exists a sequence of indices $k_i \in 1, \dots N$ such that $d_{\bar{g}}\left(x_i, q_{k_i}^i\right) \to 0$. Up to relabeling, we say $k_i = 1$ for all i. Then also $q_1^i \to x_0$ is a blow up point for u_i. By Proposition 10 and Proposition 19 we have that $q_1^i \to x_0$ is an isolated simple blow up point for u_i. Then by Proposition 18 we deduce that $\|\pi(x_0)\| = 0$, contradicting the assumption of the theorem. This concludes the proof. □

3. The non compactness result

In this section we perform the Ljapunov-Schmidt finite dimensional reduction, which relies on three steps. First, we start finding a solution of the infinite dimensional problem (3.4) with a ansatz $u = W_{\delta,q} + \delta V_{\delta,q} + \phi$ where $W_{\delta,q} + \delta V_{\delta,q}$ is a model solution and $\phi = \phi_{\delta,q}$ is a small remainder. Then, we study a finite dimensional reduced problem which depends only on δ, q. Finally, we give the proof of Theorem 2.

In the Ljapunov-Schmidt procedure, it will be necessary that $-L_g + \varepsilon_1\alpha$ is a positive definite operator. Since $-L_g$ is positive definite, in the case $\alpha < 0$, we choose ε_1 small enough in order to ensure the positivity of $-L_g + \varepsilon_1\alpha$.

3.1. The finite dimensional reduction

Since $-L_g + \varepsilon_1\alpha$ is a positive definite operator, we define an equivalent scalar product on H^1 as

$$\langle\langle u, v\rangle\rangle_g = \int_M (\nabla_g u \nabla_g v + \frac{n-2}{4(n-1)} R_g uv + \varepsilon_1 \alpha uv) d\mu_g \tag{3.1}$$

which leads to the norm $\|\cdot\|_g$ equivalent to the usual one.

Given $1 \leq t \leq \frac{2(n-1)}{n-2}$ we have the well known embedding

$$i : H^1(M) \to L^t(\partial M),$$

and we define, by the scalar product $\langle\langle\cdot,\cdot\rangle\rangle_g$,

$$i_\alpha^* : L^{t'}(\partial M) \to H^1(M)$$

in the following sense: given $f \in L^{\frac{2(n-1)}{n-2}}(\partial M)$ there exists a unique $v \in H^1(M)$ such that

$$v = i_\alpha^*(f) \iff \langle\langle v, \varphi\rangle\rangle_g = \int_{\partial M} f\varphi d\sigma \text{ for all } \varphi \tag{3.2}$$

$$\iff \begin{cases} -\Delta_g v + \frac{n-2}{4(n-1)} R_g v + \varepsilon_1 \alpha = 0 & \text{on } M; \\ \frac{\partial v}{\partial \nu} = f & \text{on } \partial M. \end{cases}$$

At this point Problem (1.1) is equivalent to find $v \in H^1(M)$ such that

$$v = i_\alpha^*(f(v) - \varepsilon_2 \beta v)$$

where

$$f(v) = (n-2)\,(v^+)^{\frac{n}{n-2}}\,.$$

Notice that, if $v \in H_g^1$, then $f(v) \in L^{\frac{2(n-1)}{n}}(\partial M)$.

Also, problem (1.1) has a variational structure and a positive solution for (1.1) is a critical point for the following functional defined on $H^1(M)$

$$\begin{aligned} J_{\varepsilon_1,\varepsilon_2,g}(v) = J_g(v) : = & \frac{1}{2}\int_M |\nabla_g v|^2 + \frac{n-2}{4(n-1)} R_g v^2 + \varepsilon_1 \alpha v^2 d\mu_g \\ & + \frac{1}{2}\int_{\partial M} \varepsilon_2 \beta v^2 d\sigma_g - \frac{(n-2)^2}{2(n-1)} \int_{\partial M} (v^+)^{\frac{2(n-1)}{n-2}} d\sigma_g. \end{aligned}$$

We define a model solution of (1.1) by means of the standard bubble U and of the function γ_q introduced in Lemma 4

Given $q \in \partial M$ and $\psi_q^\partial : \mathbb{R}_+^n \to M$ the Fermi coordinates in a neighborhood of q, we define

$$\begin{aligned} W_{\delta,q}(\xi) &= U_\delta\left(\left(\psi_q^\partial\right)^{-1}(\xi)\right)\chi\left(\left(\psi_q^\partial\right)^{-1}(\xi)\right) = \\ &= \frac{1}{\delta^{\frac{n-2}{2}}} U\left(\frac{y}{\delta}\right)\chi(y) = \frac{1}{\delta^{\frac{n-2}{2}}} U(x)\chi(\delta x) \end{aligned}$$

where $y = (z, t)$, with $z \in \mathbb{R}^{n-1}$ and $t \geq 0$, $\delta x = y = \left(\psi_q^\partial\right)^{-1}(\xi)$ and χ is a radial cut off function, with support in ball centered in 0, of radius R. In an analogous way, we define

$$V_{\delta,q}(\xi) = \frac{1}{\delta^{\frac{n-2}{2}}} \gamma_q\left(\frac{1}{\delta}\left(\psi_q^\partial\right)^{-1}(\xi)\right)\chi\left(\left(\psi_q^\partial\right)^{-1}(\xi)\right).$$

Finally, given j_a defined in (1.4) and (1.5) we define

$$Z_{\delta,q}^b(\xi) = \frac{1}{\delta^{\frac{n-2}{2}}} j_b\left(\frac{1}{\delta}\left(\psi_q^\partial\right)^{-1}(\xi)\right)\chi\left(\left(\psi_q^\partial\right)^{-1}(\xi)\right).$$

By means of $\langle\langle\cdot,\cdot\rangle\rangle_g$ it is possible to decompose H^1 in the direct sum of the following two subspaces

$$\begin{aligned} K_{\delta,q} &= \operatorname{Span}\left\langle Z_{\delta,q}^1, \dots, Z_{\delta,q}^n \right\rangle \\ K_{\delta,q}^\perp &= \left\{\varphi \in H^1(M) \ : \ \left\langle\left\langle \varphi, Z_{\delta,q}^b \right\rangle\right\rangle_g = 0,\ b = 1, \dots, n\right\} \end{aligned}$$

and to define the projections

$$\Pi = H^1(M) \to K_{\delta,q} \text{ and } \Pi^\perp = H^1(M) \to K_{\delta,q}^\perp.$$

As claimed before, we look for a solution u_q of (1.1) having the form

$$u_q = W_{\delta,q} + \delta V_{\delta,q} + \phi$$

where $\phi \in K^{\perp}_{\delta,q}$. Using i^*_α, (1.1) is equivalent to the following pair of equations

$$\Pi\left\{W_{\delta,q} + \delta V_{\delta,q} + \phi - i^*_\alpha\left[f(W_{\delta,q} + \delta V_{\delta,q} + \phi) - \varepsilon_2\beta(W_{\delta,q} + \delta V_{\delta,q} + \phi)\right]\right\} = 0 \tag{3.3}$$

$$\Pi^{\perp}\left\{W_{\delta,q} + \delta V_{\delta,q} + \phi - i^*_\alpha\left[f(W_{\delta,q} + \delta V_{\delta,q} + \phi) - \varepsilon_2\beta(W_{\delta,q} + \delta V_{\delta,q} + \phi)\right]\right\} = 0. \tag{3.4}$$

Let us define the linear operator $L : K^{\perp}_{\delta,q} \to K^{\perp}_{\delta,q}$ as

$$L(\phi) = \Pi^{\perp}\left\{\phi - i^*_\alpha\left(f'(W_{\delta,q} + \delta V_{\delta,q})[\phi]\right)\right\}, \tag{3.5}$$

and a nonlinear term $N(\phi)$ and a remainder term R as

$$N(\phi) = \Pi^{\perp}\left\{i^*_\alpha\left(f(W_{\delta,q} + \delta V_{\delta,q} + \phi) - f(W_{\delta,q} + \delta V_{\delta,q}) - f'(W_{\delta,q} + \delta V_{\delta,q})[\phi]\right)\right\} \tag{3.6}$$

$$R = \Pi^{\perp}\left\{i^*_\alpha\left(f(W_{\delta,q} + \delta V_{\delta,q})\right) - W_{\delta,q} - \delta V_{\delta,q}\right\}, \tag{3.7}$$

With these operators, the infinite dimensional equation (3.4) becomes

$$L(\phi) = N(\phi) + R - \Pi^{\perp}\left\{i^*_\alpha\left(\varepsilon_2\beta(W_{\delta,q} + \delta V_{\delta,q} + \phi)\right)\right\}.$$

In this subsection we find, for any δ, q given, a function ϕ which solves equation (3.4). Many of the estimates which follow are contained in [16], which we refer to for further details. Here we describe only the main steps of each proof.

Lemma 20. *Assume $n \geq 7$. It holds*

$$\|R\|_g = O(\delta^2)$$

Proof. Take the unique Γ such that

$$\Gamma = i^*_\alpha\left(f(W_{\delta,q} + \delta V_{\delta,q})\right),$$

that is the function solving

$$\begin{cases} -\Delta_g\Gamma + \frac{n-2}{4(n-1)}R_g\Gamma + \varepsilon_1\alpha\Gamma = 0 & \text{on } M; \\ \frac{\partial\Gamma}{\partial\nu} = (n-2)\left((W_{\delta,q} + \delta V_{\delta,q})^+\right)^{\frac{n}{n-2}} & \text{on } \partial M. \end{cases}$$

We have, by (3.1) that

$$\begin{aligned} \|R\|^2_g &\leq \left\|i^*_g\left(f(W_{\delta,q} + \delta V_{\delta,q}\right) - W_{\delta,q} - \delta V_{\delta,q}\right\|^2_g = \|\Gamma - W_{\delta,q} - \delta V_{\delta,q}\|^2_g \\ &= \int_M\left[\Delta_g(W_{\delta,q} + \delta V_{\delta,q}) - \frac{n-2}{4(n-1)}R_g(W_{\delta,q} + \delta V_{\delta,q})\right](\Gamma - W_{\delta,q} - \delta V_{\delta,q})d\mu_g \\ &\quad - \int_M \varepsilon_1\alpha(W_{\delta,q} + \delta V_{\delta,q})(\Gamma - W_{\delta,q} - \delta V_{\delta,q})d\mu_g \end{aligned}$$

$$+ \int_{\partial M} \left[f(W_{\delta,q} + \delta V_{\delta,q}) - \frac{\partial}{\partial \nu}(W_{\delta,q} + \delta V_{\delta,q}) \right] (\Gamma - W_{\delta,q} - \delta V_{\delta,q}) d\sigma_g$$
$$=: I_1 + I_2 + I_3.$$

For I_1 we have

$$I_1 \leq \left| \Delta_g(W_{\delta,q} + \delta V_{\delta,q}) - \frac{n-2}{4(n-1)} R_g(W_{\delta,q} + \delta V_{\delta,q}) \right|_{L_g^{\frac{2n}{n+2}}(M)} \|R\|_g$$

and direct computation and by the expansions of the metric (1.13) (1.14) we have (see [16, Lemma 9])

$$\left| W_{\delta,q} + \delta^2 V_{\delta,q} \right|_{L_{\tilde g}^{\frac{2n}{n+2}}(M)} = O(\delta^2),$$

$$\left| \Delta_{\tilde g}(W_{\delta,q} + \delta^2 V_{\delta,q}) \right|_{L_{\tilde g}^{\frac{2n}{n+2}}(M)} = O(\delta^2).$$

Similarly

$$I_2 \leq \varepsilon_1 O(\delta^2) \|R\|_g = O(\delta^2) \|R\|_g .$$

The proof of estimate for I_3 is more delicate, and uses in a crucial way that γ_q solves (1.7). As shown in [16, Lemma 9] we have indeed

$$I_3 \leq O(\delta^2) \|R\|_g$$

which completes the proof. □

Lemma 21. *Given $(\varepsilon_1, \varepsilon_2)$, for any pair (δ, q) there exists a positive constant $C = C(\delta, q)$ such that for any $\varphi \in K_{\delta,q}^{\perp}$ it holds*

$$\|L(\varphi)\|_g \geq C \|\varphi\|_g.$$

This lemma is a standard tool in finite dimensional reduction, so we refer to [17, 18] for the proof.

Proving that N is a contraction it is also standard. In fact there exists $\eta < 1$ such that, for any $\varphi_1, \varphi_2 \in K_{\delta,q}^{\perp}$ it holds

$$\|N(\varphi)\|_g \leq \eta \|\varphi\|_g \text{ and } \|N(\varphi_1) - N(\varphi_2)\|_g \leq \eta \|\varphi_1 - \varphi_2\|_g \tag{3.8}$$

By Lemma 20, Lemma 21, and by (3.8) we get the last result of this subsection.

Proposition 22. *Given $(\varepsilon_1, \varepsilon_2)$, for any pair (δ, q) there exists a unique $\phi = \phi_{\delta,q} \in K_{\delta,q}^{\perp}$ which solves (3.4) such that*

$$\|\phi\|_g = O(\delta^2 + \varepsilon_2 \delta).$$

In addition the map $q \mapsto \phi$ is C^1.

Proof. Lemma 21 and (3.8) and by the properties of i_α, there exists $C > 0$ such that

$$\left\| L^{-1} \left(N(\phi) + R - \Pi^{\perp} \left\{ i_\alpha^* \left(\varepsilon_2 \beta (W_{\delta,q} + \delta V_{\delta,q} + \phi) \right) \right\} \right) \right\|_g$$
$$\leq C \left((\eta \|\phi\|_g + \|R\|_g + \left\| i_\alpha^* \left(\varepsilon_2 \beta (W_{\delta,q} + \delta V_{\delta,q} + \phi) \right) \right\|_g \right).$$

Now, it is easy to estimate that

$$\begin{aligned}\left\| i_\alpha^* \left(\varepsilon_2 \beta (W_{\delta,q} + \delta V_{\delta,q} + \phi)\right)\right\|_g &\le \varepsilon_2 \left(\left\| W_{\delta,q} + \delta V_{\delta,q} \right\|_{L_g^{\frac{2(n-1)}{n}}(\partial M)} + \|\phi\|_g \right) \\ &\le C\left(\varepsilon_2 \delta + \varepsilon_2 \|\phi\|_g\right). \end{aligned} \tag{3.9}$$

By Lemma 20 and by the previous estimates, for the map

$$T(\tilde{\phi}) := L^{-1}\left(N(\tilde{\phi}) + R - \Pi^\perp \left\{ i_\alpha^* \left(\varepsilon_2 \beta (W_{\delta,q} + \delta V_{\delta,q} + \phi)\right)\right\}\right)$$

it holds

$$\|T(\phi)\|_g \le C\left((\eta + \varepsilon_2)\|\phi\|_g + \varepsilon_2 \delta + \delta^2\right).$$

So, it is possible to choose $\rho > 0$ such that T is a contraction from the ball $\|\phi\|_g \le \rho(\varepsilon_2 \delta + \delta^2)$ in itself. Hence, by the fixed point Theorem, we have the first claim. The second claim is proved by the implicit function Theorem. □

3.2. The reduced functional

Once a solution of Problem (3.4) is found, it is possible to look for a critical point of $J_g\left(W_{\delta,q} + \delta V_{\delta,q} + \phi\right)$, solving a finite dimensional problem which depends only on (δ, q).

Lemma 23. *Assume $n \ge 7$. It holds*

$$\left| J_g\left(W_{\delta,q} + \delta V_{\delta,q} + \phi\right) - J_g\left(W_{\delta,q} + \delta V_{\delta,q}\right)\right| = o(1)\,\|\phi\|_g$$

C^0-uniformly for $q \in \partial M$.

Proof. We have, for some $\theta \in (0, 1)$

$$\begin{aligned} J_g(W_{\delta,q} + \delta V_{\delta,q} + \phi) - J_g(W_{\delta,q} + \delta V_{\delta,q}) &= J_g'(W_{\delta,q} + \delta V_{\delta,q})[\phi] \\ &\quad + \frac{1}{2} J_g''(W_{\delta,q} + \delta V_{\delta,q} + \theta\phi)[\phi, \phi] \\ = \int_M \left(\nabla_g W_{\delta,q} + \delta \nabla_g V_{\delta,q}\right) \nabla_g \phi &+ \left(\frac{n-2}{4(n-1)} R_g + \varepsilon_1 \alpha\right)\left(W_{\delta,q} + \delta V_{\delta,q}\right)\phi d\mu_g \\ &- (n-2)\int_{\partial M} \left(\left(W_{\delta,q} + \delta V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}} \phi d\sigma_g \\ &+ \int_{\partial M} \varepsilon_2 \beta \left(W_{\delta,q} + \delta V_{\delta,q}\right)\phi d\sigma_g + \frac{1}{2}\|\phi\|_g^2 \\ &- \frac{n}{2}\int_{\partial M} \left(\left(W_{\delta,q} + \delta V_{\delta,q} + \theta \phi_{\delta,q}\right)^+\right)^{\frac{2}{n-2}} \phi_{\delta,q}^2 d\sigma_g + \frac{1}{2}\int_{\partial M} \varepsilon_2 \beta |\phi|^2 d\sigma_g. \end{aligned} \tag{3.10}$$

All the terms but $\int_M \varepsilon_1 \alpha \left(W_{\delta,q} + \delta V_{\delta,q}\right)\phi d\mu_g$ have been estimated in [16, Lemma 12], so we summarize only the key steps. As in Lemma 20, the most delicate term is the nonlinear term on the boundary. In particular we have that

$$\int_{\partial M}\left[(n-2)\left(\left(W_{\delta,q}+\delta V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}}-\frac{\partial}{\partial\nu}W_{\delta,q}\right]\phi d\sigma_g$$
$$=\left|(n-2)\left(\left(W_{\delta,q}+\delta V_{\delta,q}\right)^+\right)^{\frac{n}{n-2}}-\frac{\partial}{\partial\nu}W_{\delta,q}\right|_{L^{\frac{2(n-1)}{n}}(\partial M)}\|\phi\|_g$$
$$=o(|\delta\log\delta|)\|\phi\|_g=o(1)\|\phi\|_g.$$

The other terms in (3.10) are easier to estimate and lead to higher order terms.

At last, by Holder inequality we have

$$\left|\int_M W_{\delta,q}\phi d\mu_g\right|\le C|W_{\delta,q}|_{L_g^{\frac{2n}{n+2}}}|\phi|_{L_g^{\frac{2n}{n-2}}}\le C\delta^2\|\phi\|_g$$
$$\delta\left|\int_M V_{\delta,q}\phi d\mu_g\right|\le C\delta|V_{\delta,q}|_{L_g^2}|\phi|_{L_g^2}\le C\delta\|\phi\|_g.$$

so

$$\int_M\varepsilon_1\alpha\left(W_{\delta,q}+\delta V_{\delta,q}\right)\phi d\mu_g=O(\delta)\|\phi\|_g$$

and we are in position to prove the result. □

Lemma 24. *Let $n\ge 7$. It holds*

$$J_g(W_{\delta,q}+\delta V_{\delta,q})=A+\varepsilon_1\delta^2\alpha(q)B+\varepsilon_2\delta\beta(q)C+\delta^2\varphi(q)+o(\varepsilon_1\delta^2)+o(\varepsilon_2\delta)+o(\delta^2)$$

where

$$A=\frac{(n-2)(n-3)}{2(n-1)^2}\omega_{n-2}I_{n-1}^n>0;$$
$$B=\frac{n-2}{(n-1)(n-4)}\omega_{n-2}I_{n-1}^n>0;$$
$$C=\frac{n-2}{n-1}\omega_{n-2}I_{n-1}^n>0;$$
$$\varphi(q)=\frac{1}{2}\int_{\mathbb{R}_+^n}\gamma_q\Delta\gamma_q dy-\frac{(n-6)(n-2)\omega_{n-1}I_{n-1}^n}{4(n-1)^2(n-4)}\|\pi(q)\|^2\le 0.$$

Here $I_{n-1}^n:=\int_0^\infty\frac{s^n}{(1+s^2)^{n-1}}ds$ and $\pi(q)$ is the trace free tensor of the second fundamental form.

Proof. The main estimates of this proof are proved in [16, Proposition 13], which we refer for to for a detailed proof. Here, we limit ourselves to estimate the perturbation terms. We have

$$\begin{aligned}J_g(W_{\delta,q}+\delta V_{\delta,q})=&\frac{1}{2}\int_M|\nabla_g(W_{\delta,q}+\delta V_{\delta,q})|^2d\mu_g+\frac{n-2}{8(n-1)}\int_M R_g(W_{\delta,q}+\delta V_{\delta,q})^2d\mu_g\\&+\frac{1}{2}\varepsilon_1\int_M\alpha(W_{\delta,q}+\delta V_{\delta,q})^2d\mu_g\\&+\frac{1}{2}\varepsilon_2\int_{\partial M}\beta(W_{\delta,q}+\delta V_{\delta,q})^2d\sigma_g\\&-\frac{(n-2)^2}{2(n-1)}\int_{\partial M}\left(W_{\delta,q}+\delta V_{\delta,q}\right)^{\frac{2(n-1)}{n-2}}.\end{aligned}$$

We easily compute the terms involving ε_1 and ε_2 taking in account the expansion of the volume form (1.13), getting

$$\frac{1}{2}\varepsilon_1\int_M \alpha(W_{\delta,q}+\delta V_{\delta,q})^2 d\mu_g = \frac{1}{2}\varepsilon_1\delta^2\alpha(q)\int_{\mathbb{R}^n} U(y)^2dy + o(\varepsilon_1\delta^2)$$

and

$$\frac{1}{2}\varepsilon_2\int_{\partial M} \beta(W_{\delta,q}+\delta V_{\delta,q})^2 d\mu_g = \frac{1}{2}\varepsilon_2\delta\beta(q)\int_{\mathbb{R}^{n-1}} U(\bar{y},0)^2d\bar{y} + o(\varepsilon_2\delta).$$

By direct computation, and by Remark 18 in [16] (see also [16, page 1332]) we have that $\int_{\mathbb{R}^n} U(y)^2dy = \frac{2(n-2)}{(n-1)(n-4)}\omega_{n-2}I^n_{n-1}$ and $\int_{\mathbb{R}^{n-1}} U(\bar{y},0)^2d\bar{y} = \frac{2(n-2)}{n-1}\omega_{n-2}I^n_{n-1}$, getting the value of the positive constants B and C.

For the remaining terms we refer to [16, Proposition 13] in which is proved that

$$\frac{1}{2}\int_M |\nabla_{\tilde{g}_q}(W_{\delta,q}+\delta V_{\delta,q})|^2 d\mu_g + \frac{n-2}{8(n-1)}\int_M R_g(W_{\delta,q}+\delta V_{\delta,q})^2 d\mu_g \\ - \frac{(n-2)^2}{2(n-1)}\int_{\partial M}\left(W_{\delta,q}+\delta V_{\delta,q}\right)^{\frac{2(n-1)}{n-2}} d\sigma_{g_q} = A + \delta^2\varphi(q) + o(\delta^2).$$

We conclude by noticing that $\varphi(q)\le 0$ by (1.9). □

3.3. *Proof of Theorem 2*

At first we recall that, in the hypotheses of Theorem 2, we have that the function φ defined in Lemma 24 is strictly negative on ∂M. Infact $\|\pi(q)\|^2$ is non zero by assumption, and $\int_{\mathbb{R}^n_+}\gamma_q\Delta\gamma_q$ is non positive by (1.9). With this in mind, we are in position to prove the result.

Proof of Theorem 2. . We start with the first case, $\beta>0$. We choose

$$\varepsilon_1 = o(1)$$
$$\delta = \lambda\varepsilon_2$$

where $\lambda\in\mathbb{R}^+$. With this choice, by Lemma 23 and Proposition 22 we have that

$$\left|J_g\left(W_{\lambda\varepsilon_2,q}+\lambda\varepsilon_2 V_{\lambda\varepsilon_2,q}+\phi\right) - J_g\left(W_{\lambda\varepsilon_2,q}+\lambda\varepsilon_2 V_{\lambda\varepsilon_2,q}\right)\right| = o(\varepsilon_2^2)$$

and that, by Lemma 24,

$$J_g\left(W_{\lambda\varepsilon_2,q}+\lambda\varepsilon_2 V_{\lambda\varepsilon_2,q}\right) = A + \varepsilon_2^2\left(\lambda\beta(q)C+\lambda^2\varphi(q)\right) + o(\varepsilon_2^2).$$

We recall a result which is a key tool in Ljapunov-Schmidt procedure, and which is proved, for instance, in [16, Lemma 15]) and which relies on the estimates of Lemma 23.

Remark. Given $(\varepsilon_1,\varepsilon_2)$, if $(\bar\lambda,\bar q)\in(0,+\infty)\times\partial M$ is a critical point for the reduced functional $I_{\varepsilon_1,\varepsilon_2}(\lambda,q) := J_g\left(W_{\lambda\varepsilon_2,q}+\lambda\varepsilon_2 V_{\lambda\varepsilon_2,q}+\phi\right)$, then the function $W_{\bar\lambda\varepsilon_2,\bar q}+\bar\lambda\varepsilon_2 V_{\bar\lambda\varepsilon_2,\bar q}+\phi$ is a solution of (1.1).

To conclude the proof it lasts to find a pair $(\bar\lambda,\bar q)$ which is a critical point for $I_{\varepsilon_1,\varepsilon_2}(\lambda,q)$.

In this first case we consider $G(\lambda,q) := \lambda\beta(q)C+\lambda^2\varphi(q)$. We have that $\beta(q)C$ is strictly positive on ∂M, by our assumptions, while, as recalled before, φ is strictly negative on ∂M. At this point there exists a compact set $[a, b] \subset \mathbb{R}^+$ such that the function G admits an absolute maximum in $(a, b) \times \partial M$, which also is the absolute maximum value of G on $\mathbb{R}^+ \times \partial M$. This maximum is also C^0-stable, in the sense that, if (λ_0, q_0) is the maximum point for G, for any function $f \in C^1([a, b] \times \partial M)$ with $\|f\|_{C^0}$ sufficiently small, then the function $G + f$ on $[a, b] \times \partial M$ admits a maximum point $(\bar{\lambda}, \bar{q})$ close to (λ_0, q_0). By the C_0 stability of this maximum point (λ_0, q_0), and by Lemma 24, given ε_2 sufficiently small (and $\varepsilon_1 = o(1)$), there exists a pair $(\lambda_{\varepsilon_1,\varepsilon_2}, q_{\varepsilon_1,\varepsilon_2})$ which is a maximum point for $I_{\varepsilon_1,\varepsilon_2}(\lambda, q)$. This implies, in light of the above Remark, that there exists a pair $\left(\bar{\lambda}_{\varepsilon_1,\varepsilon_2}, \bar{q}_{\varepsilon_1,\varepsilon_2}\right)$ such that $W_{\bar{\lambda}_{\varepsilon_1,\varepsilon_2}\varepsilon_2,\bar{q}_{\varepsilon_1,\varepsilon_2}} + \bar{\lambda}_{\varepsilon_1,\varepsilon_2}\varepsilon_2 V_{\bar{\lambda}_{\varepsilon_1,\varepsilon_2}\varepsilon_2,\bar{q}_{\varepsilon_1,\varepsilon_2}} + \phi$ is a solution of (1.1), and the proof for the case $\beta > 0$ is complete.

The proof in the second case is similar. In this case, by assumption, we have that $B\alpha(q) + \varphi(q) > 0$ on ∂M. Then we choose

$$\varepsilon_1 = 1$$
$$\delta = \lambda\varepsilon_2$$

and we obtain that

$$I_{\varepsilon_1,\varepsilon_2}(\lambda, q) = A + \varepsilon_2^2\left[\lambda\beta(q)C + \lambda^2\left(\alpha(q)B + \varphi(q)\right)\right] + o(\varepsilon_2^2).$$

In this case we define the function $G(\lambda, q)$ as

$$G(\lambda, q) := \lambda\beta(q)C + \lambda^2\left(\alpha(q)B + \varphi(q)\right).$$

and, by our assumptions, the coefficient of λ is strictly negative on ∂M while the coefficient of λ^2 is strictly positive on ∂M, so we can conclude the proof follows in a similar way. □

Conflict of interest

The authors declare there is no conflict of interest.

References

1. O. Druet, Compactness for Yamabe metrics in low dimensions, *Int. Math. Res. Not.*, **23** (2004), 1143–1191. https://doi.org/10.1155/S1073792804133278
2. J. Escobar, Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary, *Ann. Math.*, **136** (1992), 1–50. https://doi.org/10.2307/2946545
3. F. Marques, Existence results for the Yamabe problem on manifolds with boundary, *Indiana Univ. Math. J.*, **54** (2005), 1599–1620. https://doi.org/10.1512/iumj.2005.54.2590
4. S. Almaraz, An existence theorem of conformal scalar-flat metrics on manifolds with boundary, *Pacific J. Math.*, **248** (2010), 1–22. https://doi.org/10.2140/pjm.2010.248.1
5. S. Brendle, S. S. Chen, An existence theorem for the Yamabe problem on manifolds with boundary, *J. Eur. Math. Soc.*, **16** (2014), 991–1016. https://doi.org/10.4171/JEMS/453
6. M. Mayer, C. B. Ndiaye, Barycenter technique and the Riemann mapping problem of Cherrier-Escobar, *J. Differ. Geom.*, **107** (2017), 519–560. https://doi.org/10.4310/jdg/1508551224

7. M. Khuri, F. Marques, R. Schoen, A compactness theorem for the Yamabe problem, *J. Differ. Geom.*, **81** (2009), 143–196. https://doi.org/10.4310/jdg/1228400630
8. V. Felli, M. Ould Ahmedou, Compactness results in conformal deformations of Riemannian metrics on manifolds with boundaries, *Math. Z.*, **244** (2003), 175–210. https://doi.org/10.1007/s00209-002-0486-7
9. S. Almaraz, A compactness theorem for scalar-flat metrics on manifolds with boundary, *Calc. Var.*, **41** (2011), 341–386. https://doi.org/10.1007/s00526-010-0365-8
10. M. G. Ghimenti, A. M. Micheletti, Compactness for conformal scalar-flat metrics on umbilic boundary manifolds, *Nonlinear Anal.*, **200** (2020). https://doi.org/10.1016/j.na.2020.111992
11. M. Ghimenti, A. M. Micheletti, *A compactness result for scalar-flat metrics on low dimensional manifolds with umbilic boundary*, in press on Calc Var PDE.
12. S. Almaraz, Blow-up phenomena for scalar-flat metrics on manifolds with boundary, *J. Differ. Equ.*, **251** (2011), 1813–1840. https://doi.org/10.1016/j.jde.2011.04.013
13. O. Druet, E. Hebey, Sharp asymptotics and compactness for local low energy solutions of critical elliptic systems in potential form, *Calc. Var. Partial Differ. Equ.*, **31** (2008), 205–230. https://doi.org/10.1007/s00526-007-0111-z
14. O. Druet, E. Hebey, F. Robert, A C0-theory for the blow-up of second order elliptic equations of critical Sobolev growth, *Elect. Res. Ann. A.M.S*, **9** (2003), 19–25. https://doi.org/10.1090/S1079-6762-03-00108-2
15. M. G. Ghimenti, A. M. Micheletti, *Compactness results for linearly perturbed Yamabe problem on manifolds with boundary*, *Discrete Contin Dyn Syst, series S*, **14** (2021), 1757–1778 . https://doi.org/10.3934/dcdss.2020453
16. M. G. Ghimenti, A. M. Micheletti, A. Pistoia, Linear Perturbation of the Yamabe Problem on Manifolds with Boundary, *J. Geom. Anal.*, **28** (2018), 1315–1340 https://doi.org/10.1007/s12220-017-9864-6
17. P. Esposito, A. Pistoia, J. Vetois, The effect of linear perturbations on the Yamabe problem, *Math. Ann.*, **358** (2014), 511–560. https://doi.org/10.1007/s00208-013-0971-9
18. A. M. Micheletti, A. Pistoia, The role of the scalar curvature in a nonlinear elliptic problem on Riemannian manifolds, *Calc. Var. Partial Differ. Equ.*, **34** (2009), 233–265. https://doi.org/10.1007/s00526-008-0183-4
19. Z. C. Han, Y. Li, The Yamabe problem on manifolds with boundary: existence and compactness results, *Duke Math, J.*, **99** (1999), 489–542. https://doi.org/10.1215/S0012-7094-99-09916-7
20. F. Marques, A priori estimates for the Yamabe problem in the non-locally conformally flat case, *J. Differ. Geom.*, **71** (2005), 315–346. https://doi.org/10.4310/jdg/1143651772
21. Y. Li, M. Zhu, Yamabe type equations on three dimensional Riemannian manifolds, *Commun. Contemp. Math.*, **1** (1999), 1–50. https://doi.org/10.1142/S021919979900002X
22. R. Schoen, D. Zhang, Prescribed scalar curvature on the n-sphere, *Calc. Var. Partial Differ. Equ.* , **4** (1996), 1–25. https://doi.org/10.1007/BF01322307
23. T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*. Springer Monographs in Mathematics. Springer, Berlin, 1998. https://doi.org/10.1007/978-3-662-13006-3

24. G. Giraud, Sur la problème de Dirichlet généralisé, *Ann. Sci.École Norm. Sup.*, **46** (1929), 131–145. https://doi.org/10.24033/asens.793

25. W. M. Ni, I. Takagi, On the Shape of Least-Energy Solutions to a Semilinear Neumann Problem, *Comm. Pure Appl. Math.*, **44** (1991), 819–851. https://doi.org/10.1002/cpa.3160440705