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Research article

Dynamics of a nonlinear differential advertising model with single parameter sales promotion strategy

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Abstract: Advertising and sales promotion are two important specific marketing communications tools. In this paper, nonlinear differential equation and single parameter sales promotion strategy are introduced into an advertising model and investigated quantitatively. The existence and stability of period-*nT* (n=1,2,4,8) solutions are investigated. Interestingly, both period doubling bifurcation and inverse flip bifurcation occur at different parameter values in the same advertising model. The results show that the system enters into chaos from stable state through flip bifurcation and enters into stable state from chaos through inverse flip bifurcation. An effective control strategy, which suppresses flip bifurcation and promotes inverse flip bifurcation, is proposed to eliminate chaos. These results have some significant theoretical and practical value in related markets.

Keywords: advertising model; sales promotion strategy; flip bifurcation; periodic solution

1. Introduction

For firms, advertising is a form of promoting goods or providing services and disseminating goods or service information to consumers or users through advertising media, such as newspaper, magazines, television commercial, radio advertisement, outdoor advertising, direct mail, websites, text messages, etc. [1]. As one of the indispensable means in enterprise competitive strategy, advertising's role in developing market, competing for share and creating economic benefits can not be ignored and underestimated. In advertising research, qualitative analysis and quantitative analysis are two main research methods [2]. Qualitative insights can aid empirical analysis in promising new directions, and the gap between the qualitative specialists and the quantitative gurus really needs to be bridged for the most beneficial research to be developed.

Advertising research has been growing and has been enriched in recent decades. Zhao and Ma [3] discussed dynamics and implications on a cooperative advertising model. By using the theory of non-linear dynamic system, the authors found four possible Nash equilibrium points and provide the conditions for their existence, analyzed the impacts of key parameters on the stability of the positive Nash equilibrium point and prove that two kinds of bifurcation may occur when this positive Nash equilibrium point becomes unstable. A differential game theory approach is suggested to seek equilibrium trajectories of price and advertising over time. In paper [4], based on chance theory, the optimal control for uncertain stochastic dynamic systems was considered and the principle of optimality is presented by drawing on the dynamic programming method. As an application, an advertising problem is analyzed, the corresponding optimal pricing policies and advertising strategies are provided. Chenavaz, Feichtinger and Hartl et al. [5] investigated the interplay between price, advertising, and quality in an optimal control model. The obtained results generalize the condition of Dorfman-Steiner in a dynamic context.

There are also many literatures on the models and methods of advertising research, such as artificial neural network approach [6], tourism advertising effects model [7], advertising capital model with analytical and numerical studies [8], etc.

In the past few decades, the dynamic advertising literature contains a large number of continuous time dynamic optimization models and differential game models in different mathematical forms for diverse advertising problems. Many classical models and their extensions are used to investigate advertising problems, such as Nerlove-Arrow model [9], Vidale-Wolfe model[10], and so on. Most of these continuous time differential game models are described by linear differential equations. Theoretical analysis is easy in a linear differential advertising model, which is an ideal description of advertising in reality.

As a pulse control strategy [11], sales promotion is a promotional mix variable primarily used to bolster sales in the short run. Sales promotion can be targeted either at the consumer (consumer sales promotion), the distribution channel members (trade promotion), or the sales staff (sales force promotion). Typical examples of consumer sales promotion tools include contests and sweepstakes, branded give-away merchandise, bonus-size packaging, limited-time discounts, rebates, coupons, free trials, demonstrations, and point-accumulation systems.

Because it can increase the sales of products in a short time, sales promotion has been valued and applied by the company's managers. In recent years, a lot of theoretical research results on sales promotion have been obtained. Previous studies mainly used the methods of data-analysis and quality analysis to discuss the role of sales promotion. For example, from the perspective of empirical research, paper [12] investigated whether various types of sales promotions together with hedonic shopping motivation (value shopping) and positive affect drive impulse buying, and further explored the moderation impact of trait constructs viz deal proneness and impulsive buying tendency in impulse buying. By using three years of super-market scanner data and sales promotions for pound cake, McColl et al. [13] estimated cannibalization effects for two common price reductions, across large, medium and small supermarkets. Almendros [14] assessed which type of online promotional incentive is the most effective at achieving purchase intention for airline tickets, depending on the user's level of Internet experience (characterized as novice or expert user).

At present, there are few quantitative research results on sales promotion. A linear advertising competition model with sales promotion was constructed and investigated in paper [15]. It is difficult for this sales promotion strategy, which contains two parameters, to ensure the change in sales volume is positive. As mentioned above, most of advertising models are described by linear differential equations. To investigate the complex change process in advertising, nonlinear differential equations should be introduced into advertising models. Motivated by these facts, a nonlinear differential equation and single parameter sales promotion strategy are introduced into an advertising model in this study. This research focuses on the complex dynamics of this model and the effect of sales promotion on sales level and profit.

The paper is organized as follows. In Section 2, we build an advertising model by using nonlinear differential equation and single parameter sales promotion strategy. The existence and stability of periodic solutions, and bifurcation of periodic solutions are investigated in Section 3. In Section 4, control measures are taken to eliminate chaos by suppressing flip bifurcation and promoting inverse flip bifurcation. Finally, some conclusions are drawn in section 5.

2. Model description

Vidale and Wolfe [10] built the following model to investigate the impact of advertising on sales of firm's product.

$$\dot{S}(t) = \frac{r}{M}u(t)(M - S(t)) - \lambda S(t), \tag{1}$$

where $S(t)$ is sales level at time t, M is the size of the potential market or saturation level, $u(t)$ is advertising expenditure at time t, r is response rate to advertising, λ is decay constant.

Bass [16] developed a growth model for the timing of initial purchase of new products, which is given by the following nonlinear differential equation.

$$\dot{S}(t) = (p + qS(t))(M - S(t)), \tag{1*}$$

where $S(t)$ is the sales of one new product, p and q are positive parameters, referred to as innovation and imitation parameter, respectively, and M is the fixed market potential of the product.

The Vidale and Wolfe model (1), which is built by using linear differential equations, simplifies the change of product sales. The influence of advertising on sales is complex, and we consider using nonlinear equation to describe the change of sales. In view of nonlinear differential equation (1*), the advertising effect $\frac{r}{M}u(t)(M - S(t))$ is replaced by $\frac{r}{M}u(t)S(t)(M - S(t))$ in (1).

In some cases, it is difficult to estimate the size of the potential market or saturation level M. For example, the size of the potential market or saturation level will change with the change of consumer preferences and the development of emerging industries. The size of the potential market or saturation level is not considered in this paper, and $M - S(t)$ is replaced by $S(t)$ in (1). We also assume that the firm advertises at a constant level, that is $u(t) = U$. So the advertising effect $\frac{r}{M}u(t)S(t)(M - S(t))$ is replaced by $rUS(t)S(t) = \gamma S^2(t)$ in (1), where $\gamma = rU$.

In order to try to discuss the complex process of product sales, the following nonlinear differential equation is used in this paper.

$$\frac{dS(t)}{dt} = -\rho S(t) + \gamma S^2(t), \tag{2}$$

where $S(t)$ is sales level at time t, ρ is decay constant, and γ is response rate to advertising.

System (2) can be rewritten as

$$\frac{dS(t)}{dt} = -\rho S(t)\left(1 - \frac{S(t)}{\frac{\rho}{\gamma}}\right).$$

It's seen that $\frac{dS(t)}{dt} < 0$ for $S(t) < \frac{\rho}{\gamma}$ while $\frac{dS(t)}{dt} > 0$ for $S(t) > \frac{\rho}{\gamma}$. $\frac{\rho}{\gamma}$ is considered as the promotion threshold. When the sales level $S(t)$ is greater than $\frac{\rho}{\gamma}$, the sales level will continue to increase and no promotion strategy is considered. In the case of $S(t) < \frac{\rho}{\gamma}$, the sales level decreases with the increase of time t. So measures are discussed to improve sales of products under the condition $S(t) < \frac{\rho}{\gamma}$ in this paper.

In view of the fact sales promotion is an incentive provided to consumers to motivate them to buy immediately, sales promotion strategy is applied to improve sales here. In paper [15], the impulsive promotion effect is $\Delta S(t) = S(t^+) - S(t) = (b - cS(t))S(t)$, which contains two parameters b, c, and cannot guarantee that value of $\Delta S(t)$ is always greater than 0. To avoid these shortcomings, we consider the following single parameter sales promotion strategy.

$$\Delta S(t) = S(t^+) - S(t) = \frac{\alpha}{S(t)}, \quad t = nT, \tag{3}$$

where α is a promotion coefficient and $a > 0$, $n = 1, 2, \cdots$, $S(t^+) = \lim_{\tau \to 0^+} S(t + \tau)$. The measures to promote sales are supposed to be taken at moments $t = nT$ and the sales level $S(t)$ turns from $S(nT)$ to $S(nT^+)$, where $S(nT^+) = S(nT) + \frac{\alpha}{S(nT)}$ and $\frac{\alpha}{S(nT)}$ is the increment of sales. It follows from (3) that the increment $\Delta S(t) > 0$ for $S(t) > 0$ and $a > 0$, and hence the single-parameter promotion strategy (3) can guarantee that the increment $\Delta S(t)$ is always greater than 0.

When the product sales level of a firm is low, the firm will increase the promotion. The sales level is S_1 and the increment is $\Delta S(t_1)$ at time $t = t_1$ while the sales level is S_2 and the increment is $\Delta S(t_2)$ at time $t = t_2$, where $S_1 = S(t_1)$ and $S_2 = S(t_2)$. If the sales level S_2 is less than S_1, the firm will increase the promotion at time $t = t_2$ to make the increment $\Delta S(t_2)$ larger than the increment $\Delta S(t_1)$, which can be realized through the single parameter sales promotion strategy (3). Figure 1(a) provides a schematic illustration of this sales promotion strategy. It's seen that $\frac{\alpha}{S_2} > \frac{\alpha}{S_1}$ for $S_2 < S_1$.

Now build the following nonlinear advertising model with single parameter sales promotion strategy.

$$\begin{cases} \dot{S} = -\rho S(t)\left(1 - \frac{S(t)}{\frac{\rho}{\gamma}}\right), & t \neq nT, \\ S(t^+) = S(t) + \frac{\alpha}{S(t)}, & t = nT. \end{cases} \tag{4}$$

Figure 1(b) shows one solution to (4). The trajectory originating from the initial point $(0, S(0))$ reaches the point $(T, S(T))$ at $t = T$, next jumps to the point $(T, S(T^+))$ due to the effect of sales promotion, and so on. Hence,

$$S(t) = \frac{\rho S(0)}{\gamma S(0) - (\gamma S(0) - \rho)\exp(\rho t)}, \quad 0 \le t \le T,$$

$$S(T^+) = S(T) + \frac{\alpha}{S(T)},$$

$$S(t) = \frac{\rho S(T^+)}{\gamma S(T^+) - (\gamma S(T^+) - \rho)\exp(\rho(t - T))}, \quad T < t \le 2T.$$

3. Periodic solutions and bifurcation

Although product sales will change continuously, the company does not want this change to be disorderly. Therefore, we discuss periodic change of firm's product sales under sales promotion strategy in this section.

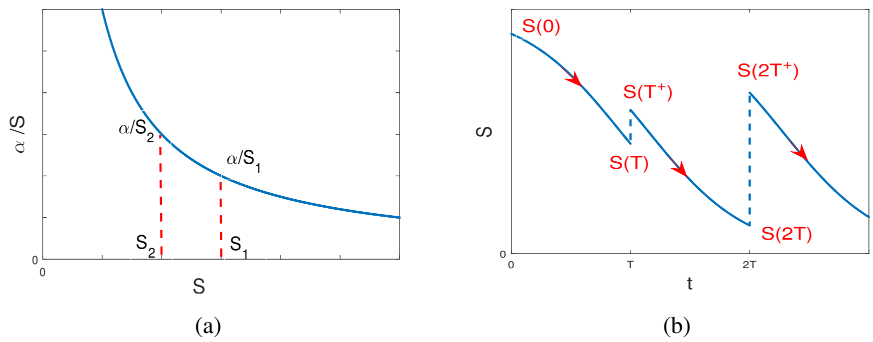


Figure 1. For system (4), (a) sketch of sales promotion strategy, (b) the solution from the initial point (0, S(0)).

Suppose the solution of (4) arrives at the point $(kT,\ S_k)$ at moment $t = kT$, then jumps to point $(kT,\ S_k^+)$ due to the effect of sales promotion, reaches the point $((k+1)T,\ S_{k+1})$ at moment $t = (k+1)T$, where $S_k = S(kT)$, $S_k^+ = S(kT^+) = S_k + \frac{\alpha}{S_k}$, $S_{k+1} = S((k+1)T)$. It follows from (4) that

$$S((k+1)T) = \frac{\rho S(kT^+)}{\gamma S(kT^+) - (\gamma S(kT^+) - \rho)\exp(\rho T)},$$

and a discrete map

$$S_{k+1} = \frac{\rho(S_k + \frac{\alpha}{S_k})}{\gamma(S_k + \frac{\alpha}{S_k}) - (\gamma(S_k + \frac{\alpha}{S_k}) - \rho)\exp(\rho T)},$$

namely,

$$S_{k+1} = \frac{\rho(S_k + \frac{\alpha}{S_k})}{(1 - \exp(\rho T))\gamma(S_k + \frac{\alpha}{S_k}) + \rho\exp(\rho T)}. \tag{5}$$

If $S(kT) = S((k+1)T)$, then sales level $S(t)$ will change periodically. For this to hold, there must be a fixed point S_0 in discrete map (5), that is,

$$S_0 = \frac{\rho(S_0 + \frac{\alpha}{S_0})}{(1 - \exp(\rho T))\gamma(S_0 + \frac{\alpha}{S_0}) + \rho\exp(\rho T)}.$$

To avoid the tedious calculation, we set $\gamma = 0.1$ and

$$T = \frac{1}{\rho}\ln\left(1 + \frac{\rho}{\gamma}\right). \tag{6}$$

It follows that $(1 - \exp(\rho T))\gamma = -\rho$. Hence, the discrete map (5) can be written as

$$S_{k+1} = \frac{S_k + \dfrac{\alpha}{S_k}}{1 + 10\rho - (S_k + \frac{\alpha}{S_k})} = f(\alpha,\ S_k). \tag{7}$$

The fixed point S_0 of map (7) is the solution of the following equation.

$$S = \frac{S + \frac{\alpha}{S}}{1 + 10\rho - (S + \frac{\alpha}{S})},$$

that is

$$S^3 - 10\rho S^2 + aS + a = 0. \tag{8}$$

Consider function $h(S) = S^3 - 10\rho S^2 + aS + a$. For $S \in (-\infty, 0)$, $h'(S) = 3S^2 - 20\rho S + a > 0$. Since $h(0) = \alpha > 0$ and $\lim_{S \to -\infty} h(S) = -\infty$, Eq (8) has a unique root for $S \in (-\infty, 0)$.

In Eq (8), the coefficients are $a = 1$, $b = -10\rho$, $c = \alpha$, $d = \alpha$. Now set

$$A = b^2 - 3ac = 100\rho^2 - 3\alpha,\ B = bc - 9ad = -10\rho\alpha - 9\alpha,\ C = c^2 - 3bd = \alpha^2 + 30\rho\alpha.$$

The discriminant is

$$\Delta = B^2 - 4AC = \alpha(12\alpha^2 + (-300\rho^2 + 540\rho + 81)\alpha - 12000\rho^3).$$

It follows from

$$\Delta = B^2 - 4AC = \alpha(12\alpha^2 + (-300\rho^2 + 540\rho + 81)\alpha - 12000\rho^3) = 0,$$

that $\alpha_1 = 0$,

$$\alpha_2 = \frac{100\rho^2 - 180\rho - 27 - \sqrt{(100\rho^2 - 180\rho - 27)^2 + 64000\rho^3}}{8} < 0,$$

and

$$\alpha_3 = \frac{100\rho^2 - 180\rho - 27 + \sqrt{(100\rho^2 - 180\rho - 27)^2 + 64000\rho^3}}{8} > 0. \tag{9}$$

Hence $\Delta = B^2 - 4AC < 0$ for $0 < \alpha < \alpha_3$ and $\Delta = B^2 - 4AC > 0$ for $\alpha > \alpha_3$, where α_3 is shown in (9).

For $\alpha > \alpha_3$, the discriminant $\Delta = B^2 - 4AC > 0$ and Eq (8) has one real root and two imaginary roots. Together with the fact that Eq (8) has a unique root for $S \in (-\infty, 0)$, we obtain Eq (8) has no positive real roots. So map (7) has no positive real fixed points for $\alpha > \alpha_3$, and system (4) has no period-T solutions. The results is given as follows.

Proposition 3.1. *System (4) has no period-T solutions for $\alpha > \alpha_3$, where α_3 is shown in (9).*

For $0 < \alpha < \alpha_3$, the discriminant $\Delta = B^2 - 4AC < 0$ and Eq (8) has the following three different real roots.

$$S_{01} = \frac{10\rho + \sqrt{100\rho^2 - 3\alpha}\left(cos\left(\frac{\theta}{3}\right) + \sqrt{3}sin\left(\frac{\theta}{3}\right)\right)}{3}, \tag{10}$$

$$S_{02} = \frac{10\rho + \sqrt{100\rho^2 - 3\alpha}\left(cos\left(\frac{\theta}{3}\right) - \sqrt{3}sin\left(\frac{\theta}{3}\right)\right)}{3}, \tag{11}$$

$$S_{03} = \frac{10\rho - 2\sqrt{100\rho^2 - 3\alpha}cos\left(\frac{\theta}{3}\right)}{3},$$

where $\theta = \arccos\left(\dfrac{-2000\rho^3 + 90\rho^2\alpha + 27\alpha}{2\sqrt{(100\rho^2 - 3\alpha)^3}}\right)$.

The root S_{03} is negative while S_{01} and S_{02} are positive. Map (7) has two positive real fixed points S_{01} and S_{02} for $0 < \alpha < \alpha_3$, and system (4) has the following two period-T solutions, which correspond to fixed points S_{01} and S_{02}.

$$S_1(t) = \frac{\rho(S_{02} + \frac{\alpha}{S_{02}})}{\gamma(S_{02} + \frac{\alpha}{S_{02}}) - (\gamma(S_{02} + \frac{\alpha}{S_{02}}) - \rho)\exp(\rho(t - kT))}, \quad kT < t \leq (k+1)T, \tag{12}$$

$$S_2(t) = \frac{\rho(S_{03} + \frac{\alpha}{S_{03}})}{\gamma(S_{03} + \frac{\alpha}{S_{03}}) - (\gamma(S_{03} + \frac{\alpha}{S_{03}}) - \rho)\exp(\rho(t - kT))}, \quad kT < t \leq (k+1)T, \tag{13}$$

where $\gamma = 0.1$ and $T = \frac{1}{\rho}\ln(1 + 10\rho)$.

The eigenvalue of the fixed point S_{0i} is

$$\begin{aligned}\lambda_i = \frac{\partial f(\alpha,\ S_{0i})}{\partial S_{0i}} &= \frac{(1 - \frac{\alpha}{S_{0i}^2})\left(1 + 10\rho - (S_{0i} + \frac{\alpha}{S_{0i}})\right) - (S_{0i} + \frac{\alpha}{S_{0i}})(-(1 - \frac{\alpha}{S_{0i}^2}))}{\left(1 + 10\rho - (S_{0i} + \frac{\alpha}{S_{0i}})\right)^2} \\ &= 1 + \frac{10\rho S_{0i}^2 - 2\alpha S_{0i} - 3\alpha}{S_{0i}^2 + \alpha}, \quad i = 1, 2. \end{aligned}\tag{14}$$

If the initial sales values are $S_{01}+\frac{\alpha}{S_{01}}$ and $S_{02}+\frac{\alpha}{S_{02}}$, the sales level $S(t)$ changes periodically according to (12) and (13). The eigenvalue λ_i of the fixed point S_{0i} is used to judge the stability of the solution $S_i(t)$. The following results are obtained.

Proposition 3.2. *For $0 < \alpha < \alpha_3$, system (4) has two period-T solutions $S_i(t)(i = 1, 2)$, which are shown in (12) and (13). The period-T solution $S_i(t)$ is stable for $|\lambda_i| < 1$ and unstable for $|\lambda_i| > 1$, where λ_i is shown in (14).*

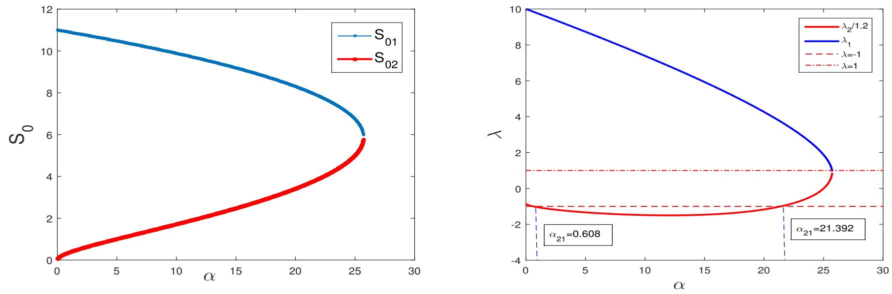
Set $\rho = 1.1$, one can obtain $\alpha_3 \approx 25.7298$ from (9). Map (7) has two fixed points, which are shown in Figure 2(a), for $\alpha \in (0,\ 25.7298)$ and hence system (4) has two periodic solutions. The eigenvalues of these two fixed points $S_{0i}(i = 1, 2)$ are shown in Figure 2(b). For $\alpha \in (0,\ 25.7298)$, $\lambda_1 > 1$ and the fixed point S_{01} is unstable, and hence the corresponding periodic solution $S_1(t)$ is unstable. Since $|\lambda_2| < 1$ for $\alpha \in (0,\ 0.6081) \cup (21.3922,\ 25.7298)$ and $|\lambda_2| > 1$ for $\alpha \in (0.6081,\ 21.3922)$, the corresponding periodic solution $S_2(t)$ of system (4) with $\rho = 1.1$ is unstable for $\alpha \in (0,\ 0.6081) \cup (21.3922,\ 25.7298)$ and unstable for $\alpha \in (0.6081,\ 21.3922)$.

Now set $\alpha = 0.6$, it follows from (10) and (11) that $S_{01} \approx 10.9401$, and $S_{02} \approx 0.2660$. Note that $S_{01} + \frac{\alpha}{S_{01}} = 10.9949$ and $S_{02} + \frac{\alpha}{S_{02}} = 2.5216$, there exist the following two periodic solutions in system (4) with $\rho = 1.1$ and $\alpha = 0.6$.

$$S_1(t) = \frac{11 * 10.9949}{10.9949 - (10.9949 - 11)\exp(1.1(t - kT))}, \quad kT < t \leq (k+1)T,$$

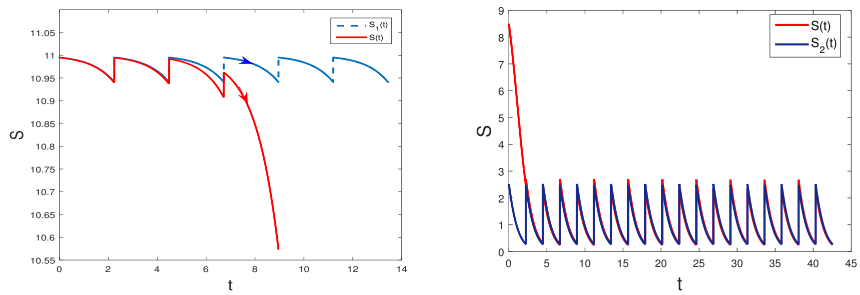
$$S_2(t) = \frac{11 * 2.5216}{2.5216 - (2.5216 - 11)\exp(1.1(t - kT))}, \quad kT < t \leq (k+1)T,$$

where $T = \frac{1}{\rho}\ln(1 + 10\rho) = 2.2590$. These two periodic solutions and their stability are shown in Figure 3. With time increasing, the solution $S(t)$ with the initial point $(0,\ 10.9940)$ moves away from the periodic solution $S_1(t)$ and the solution $S(t)$ with the initial point $(0,\ 8.5)$ tends to the periodic



(a) (b)

Figure 2. For system (4) with $\rho = 1.1$, (a) two fixed points S_{01} and S_{02}, (b) the eigenvalues of fixed points S_{01} and S_{02}.



(a) (b)

Figure 3. For system (4) with $\rho = 1.1$ and $\alpha = 0.6$, (a) the periodic solution $S_1(t)$ and the solution $S(t)$ with the initial point $(0,\ 10.9940)$, (b) the periodic solution $S_2(t)$ and the solution $S(t)$ with the initial point $(0,\ 8.5)$.

solution $S_2(t)$. So the periodic solution $S_1(t)$ is unstable and $S_2(t)$ is stable for system (4) with $\rho = 1.1$ and $\alpha = 0.6$.

It's seen from Figure 2(b) that the eigenvalue λ_2 of the fixed point S_{02} is equal to -1 at $\alpha = 0.6081$ and $\alpha = 21.3922$ for $\rho = 1.1$. An eigenvalue with -1 is associated with a flip bifurcation (or flip bifurcation). So $\alpha = 0.6081$ and $\alpha = 21.3922$ are candidates for flip bifurcation, and a stable period$-2T$ solution may occur in the system. Now use the following lemma [17] to discuss the stability and direction of bifurcation of period$-2T$ solutions in the case of $\alpha = 0.6081$.

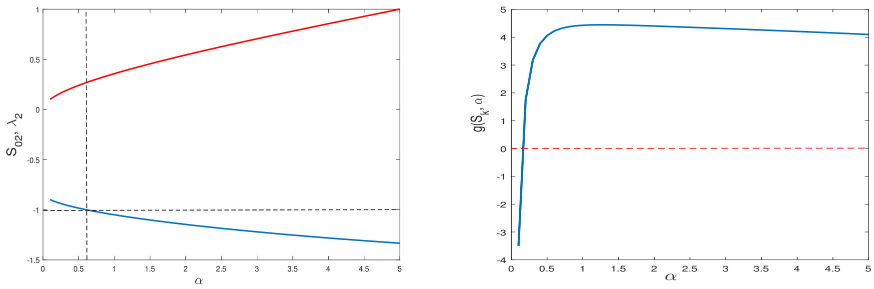
Lemma 3.1. *Let $f_\mu : \mathbb{R} \to \mathbb{R}$ be a one-parameter family of map such that f_{μ_0} has a fixed point x_0 with eigenvalue -1. Assume the following conditions:*

(C_1) $\left(\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial \mu}\right) \neq 0$ at $(x_0,\ \mu_0)$;

(C_2) $g(x,\ \mu) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + \frac{1}{3}\left(\frac{\partial^3 f}{\partial x^3}\right) \neq 0$ at $(x_0,\ \mu_0)$.

Then, there is a smooth curve *of fixed points of f_μ passing through $(x_0,\ \mu_0)$, the stability of which changes at $(x_0,\ \mu_0)$. There is also a smooth curve γ passing through $(x_0,\ \mu_0)$ so that $\gamma \backslash (x_0,\ \mu_0)$ is a union of hyperbolic period-2 orbits.*

It follows from (11) and Figure 4(a) that $S_{02} = 0.2681$ for $\alpha = 0.6081$. One can calculate that $\left(\frac{\partial f(\alpha,\ S_k)}{\partial \alpha}\frac{\partial^2 f(\alpha,\ S_k)}{\partial S_k^2} + 2\frac{\partial^2 f(\alpha,\ S_k)}{\partial S_k \partial \alpha}\right) \neq 0$ at $(S_{02},\ \alpha) = (0.2681,\ 0.6081)$. In (C_2) the sign of $g(x_0,\ \mu_0)$



(a) (b)

Figure 4. For system (4) with $\rho = 1.1$ and $\alpha \in (0,\ 5)$, (a) the fixed point S_{02} and its eigenvalue λ_2, (b) the function $g(S_k,\ \alpha)$.

determines the stability and the direction of bifurcation of the orbits of period-2. If $g(x_0,\ \mu_0)$ is positive, the orbits are stable; if $g(x_0,\ \mu_0)$ is negative they are unstable. In our case,

$$\frac{\partial^2 f(\alpha,\ S_k)}{\partial S_k^2} = 2(1+10\rho)\frac{\frac{\alpha}{S_k^3} + (1-\frac{\alpha}{S_k^2})^2}{\left(1+10\rho-(S_k+\frac{\alpha}{S_k})\right)^3},$$

$$\frac{\partial^3 f(\alpha,\ S_k)}{\partial S_k^3} = 2(1+10\rho)\frac{(3-\frac{3\alpha}{S_k^2})(\frac{\alpha}{S_k^3}+(1-\frac{\alpha}{S_k^2})^2) - (\frac{3\alpha}{S_k^4}+\frac{4\alpha}{S_k^3}-\frac{4\alpha^2}{S_k^5})(1+10\rho-S_k-\frac{\alpha}{S_k})}{\left(1+10\rho-(S_k+\frac{\alpha}{S_k})\right)^4},$$

and $g(S_k,\ \alpha) = \frac{1}{2}(\frac{\partial^2 f(\alpha,\ S_k)}{\partial S_k^2})^2 + \frac{1}{3}\frac{\partial^3 f(\alpha,\ S_k)}{\partial S_k^3} = 4.2326 > 0$ at $(S_{02},\ \alpha) = (0.2681,\ 0.6081)$.

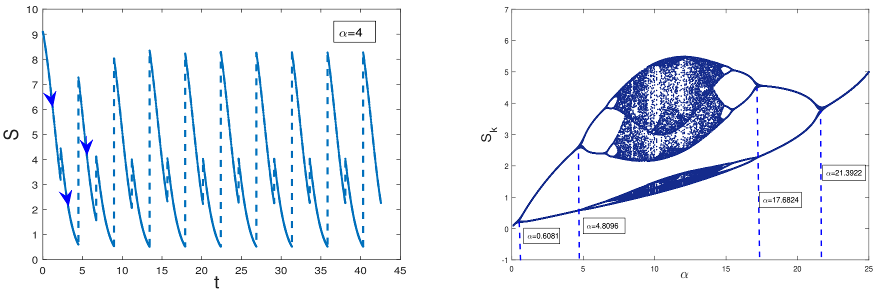
Hence a flip bifurcation occurs at $\alpha = 0.6081$ and system (4) has a stable period$-2T$ solution for $\alpha \in (0.6081,\ 0.6081+\epsilon)$, where $\epsilon > 0$.

To get the expression of the period-$2T$ solution $S_3(t)$, consider the following quadratic iterative map of (7).

$$S_{k+1} = \frac{\frac{S_k+\frac{\alpha}{S_k}}{1+10\rho-(S_k+\frac{\alpha}{S_k})} + \frac{\alpha}{\frac{S_k+\frac{\alpha}{S_k}}{1+10\rho-(S_k+\frac{\alpha}{S_k})}}}{1+10\rho-\left(\frac{S_k+\frac{\alpha}{S_k}}{1+10\rho-(S_k+\frac{\alpha}{S_k})} + \frac{\alpha}{\frac{S_k+\frac{\alpha}{S_k}}{1+10\rho-(S_k+\frac{\alpha}{S_k})}}\right)} = f^2(\alpha,\ S_k). \tag{15}$$

Map (15) has 6 fixed points $S_{0ij},\ i = 1,2,3,\ j = 1,2$, which meet conditions $f(\alpha,\ S_{0i1}) = S_{0i2}$ and $f(\alpha,\ S_{0i2}) = S_{0i1}$. Similar to the case of map (7), two fixed points, which marked as S_{021} and S_{022} here, are stable for $\alpha \in (0.6081,\ 0.6081+\epsilon)$. Hence system (4) has the following stable period-$2T$ solution.

$$S_3(t) = \begin{cases} \frac{\rho(S_{021}+\frac{\alpha}{S_{021}})}{\gamma(S_{021}+\frac{\alpha}{S_{021}})-(\gamma(S_{021}+\frac{\alpha}{S_{021}})-\rho)\exp(\rho(t-kT))}, & kT < t \leq (k+1)T, \\ \frac{\rho(S_{022}+\frac{\alpha}{S_{022}})}{\gamma(S_{022}+\frac{\alpha}{S_{022}})-(\gamma(S_{022}+\frac{\alpha}{S_{022}})-\rho)\exp(\rho(t-(k+1)T))}, & (k+1)T < t \leq (k+2)T. \end{cases} \tag{16}$$



(a) (b)

Figure 5. For system (4) with $\rho = 1.1$, (a) the solution from the initial points $(0,\ 9.1)$ for $\alpha = 4$, (b) bifurcation diagram for $\alpha \in (0,\ 25.7298)$.

Set $\alpha = 4$, the solution from the initial points $(0,\ 9.1)$ is shown in Figure 5(a), which tends to a stable period-$2T$ solution $S_3(t)$. Figure 5(b) shows bifurcation of periodic solutions of system (4) with $\rho = 1.1$, $\alpha \in (0,\ 25.7298)$, and the initial points $(0,\ 10)$. A period-T solution $S_2(t)$ is stable for $\alpha \in (0,\ 0.6081)$ and a flip bifurcation occurs at $\alpha = 0.6081$. A period-$2T$ solution $S_3(t)$ is bifurcated from the the period-T solution $S_2(t)$ through flip bifurcation at $\alpha = 0.6081$. The period-$2T$ solution $S_3(t)$ solution is stable for $\alpha \in (0.6081,\ 4.8096)$ in system (4) with $\rho = 1.1$.

It's also seen from Figure 5(b) that an inverse flip bifurcation occurs at $\alpha = 21.392$ in system (4) with $\rho = 1.1$. There exists a stable period-$2T$ solution for $\alpha \in (17.6824,\ 21.3922)$. A stable period-$T$ solution, which bifurcates from the stable period-$2T$ solution through inverse flip bifurcation, is stable for $\alpha \in (21.3922,\ 25.7298)$.

Now we change the value of ρ to discuss periodic solutions and their bifurcation in system (4). Set $\rho = 0.6,\ 0.8,\ 1.26$ in system (4). The eigenvalues of the fixed point S_{02} are shown in Figure 6(a). In the case of $\rho = 0.6$, $\alpha_3 = 6.8380$ and $0 < \lambda_2 < 1$ for $\alpha \in (0,\ 6.8380)$, so the fixed point S_{02} is stable for $\alpha \in (0,\ 6.8380)$ (see Figure 6(b)) and hence system (4) has only one stable period-T solution for $\rho = 0.6$.

In the case of $\rho = 0.8$, $\alpha_3 = 12.9098$, $|\lambda_2| < 1$ for $\alpha \in (0,\ 1.2501)$ and $\alpha \in (8.001,\ 12.9098)$. The bifurcation of fixed points is shown in Figure 6(c). The exist a stable period-T solution for $\alpha \in (0,\ 1.2501) \cup (8.001,\ 12.9098)$ and a stable period-$2T$ solution for $\alpha \in (1.2501,\ 8.001)$. Except for stable period-T and period-$2T$ solutions, other types of periodic solutions do not exist in system (4) with $\rho = 0.8$.

In the case of $\rho = 1.26$, $\alpha_3 = 34.3965$ and $|\lambda_2| < 1$ for $\alpha \in (0,\ 0.4841) \cup (30.1510,\ 34.3965)$. The bifurcation of fixed points is shown in Figure 6(d). A flip bifurcation occurs at $\alpha = 0.4810$ and an inverse flip bifurcation occurs at $\alpha = 30.1510$. It's seen that system (4) with $\rho = 1.26$ has a stable period-$4T$ solution for $\alpha = 5$ and a period-$8T$ solution for $\alpha = 25.2$, which are shown in Figure 7.

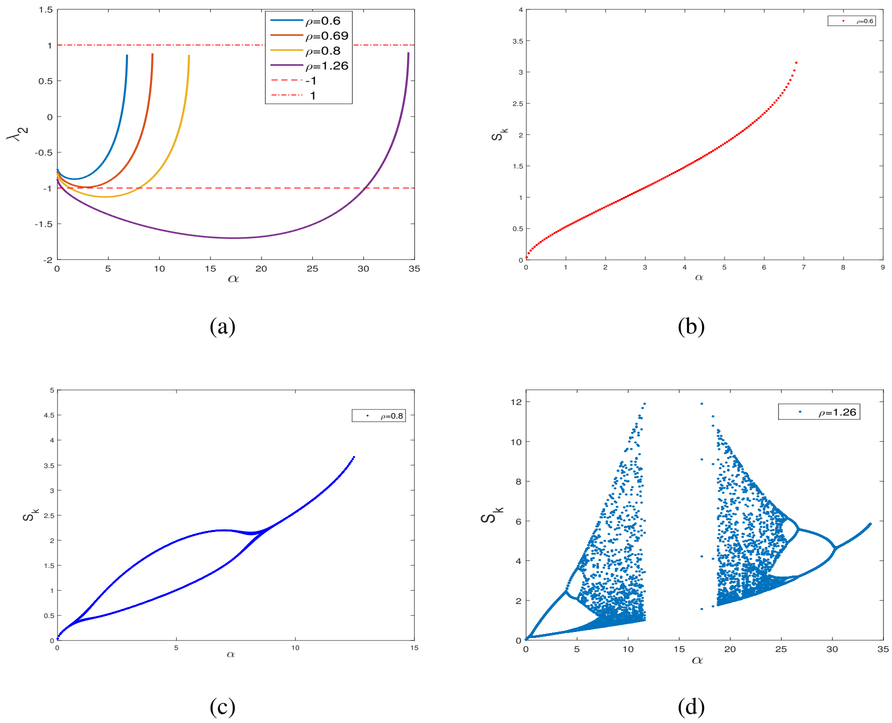


Figure 6. For system (4), (a) eigenvalue λ_2 with $\rho = 0.6,\ 0.8.\ 1.26$, (b) the stable fixed point S_{02} with $\rho = 0.6$, (c) bifurcation diagram with $\rho = 0.8$, (d) bifurcation diagram with $\rho = 1.26$.

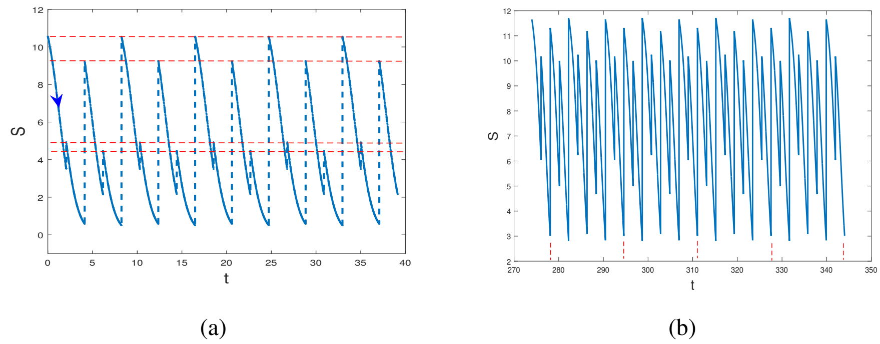
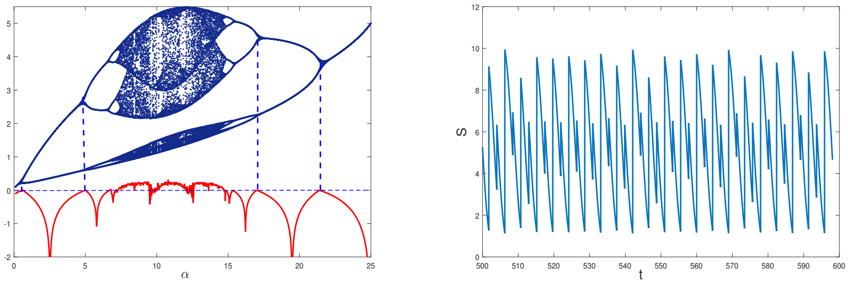


Figure 7. For system (4) with $\rho = 1.26$, (a) period-4T solution for $\alpha = 5$, (b) period-8T solution for $\alpha = 25.2$.

4. Bifurcation control

In the above section, the existence of flip bifurcation is discussed. As it well known, one path to chaos is the cascade of flip bifurcations. Figure 8(a) shows the cascade of flip bifurcations and the positive Lyapunov exponent in system (4) with $\rho = 1.1$, which illustrate the existence of chaos. A chaotic solution of system (4) with $\rho = 1.1$ and $\alpha = 10$ is shown in Figure 8(b).



(a) (b)

Figure 8. For system (4) with $\rho = 1.1$, (a) bifurcation diagram and Lyapunov exponent for $\alpha \in (0,\ 25.7298)$, (b) a chaotic solution for $\alpha = 10$.

In the state of chaos, the product sales change disorderly and is out of control. To stabilize the market, we should take measures to suppress bifurcation and eliminate chaos. Here, a small constant is introduced into sales promotion strategy, that is,

$$\Delta S(t) = S(t^+) - S(t) = \frac{\alpha}{S(t)} + \beta, \quad t = nT, \tag{17}$$

where β is a small increase in sales at promotion time $t = nT$. So the following bifurcation control system is obtained.

$$\begin{cases} \dot{S} = -\rho S(t)\left(1 - \frac{S(t)}{\frac{\rho}{\gamma}}\right), & t \neq nT, \\ S(t^+) = S(t) + \frac{\alpha}{S(t)} + \beta, & t = nT. \end{cases} \tag{18}$$

Similar to the previous section, we obtain the following map under conditions $\gamma = 0.1$ and $(1 - \exp(\rho T))\gamma = -\rho$.

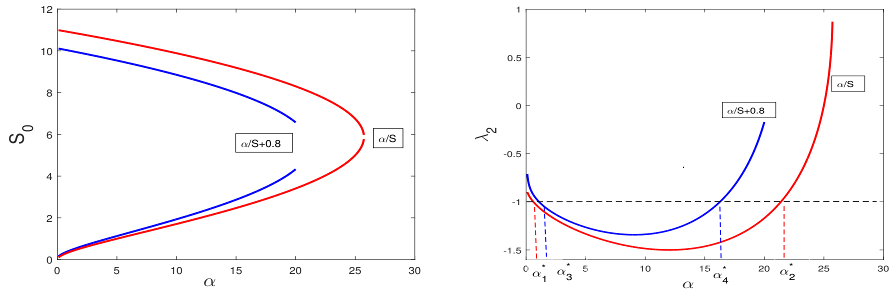
$$S_{k+1} = \frac{S_k + \dfrac{\alpha}{S_k} + \beta}{1 + 10\rho - (S_k + \frac{\alpha}{S_k} + \beta)} = f_1(S_k). \tag{19}$$

Suppose the positive fixed point of (19) is $\bar{S}_{0i}$, the eigenvalue of the fixed point is

$$\bar{\lambda}_{0i} = \frac{1 + \beta + \bar{S}_0 + \frac{\beta}{\bar{S}_0} - \frac{\alpha + 2\alpha\beta}{\bar{S}_0^2} - \frac{\alpha}{\bar{S}_0^3} - \frac{\alpha^2}{\bar{S}_0^4}}{(1 + \frac{\beta}{\bar{S}_0} + \frac{\alpha}{\bar{S}_0^2})^2} \tag{20}$$

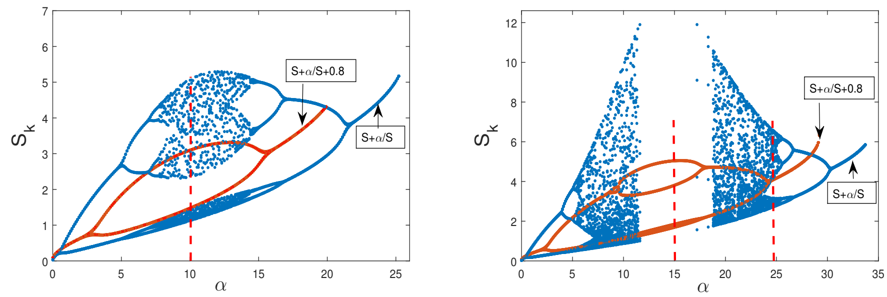
The fixed points of maps (7) and (19) are shown in Figure 9(a) for $\rho = 1.1$, $\Delta S(t) = \frac{\alpha}{S(t)}$, $\Delta S(t) = \frac{\alpha}{S(t)} + 0.8$. Figure 9(b) shows that $\alpha_1^* < \alpha_3^* < \alpha_4^* < \alpha_2^*$, where $\lambda_2(\alpha_1^*)) = \lambda_2(\alpha_2^*)) = -1$ and $\bar{\lambda}_{02}(\alpha_3^*) = \bar{\lambda}_{02}(\alpha_4^*) = -1$. After taking control measures $\Delta S(t) = \frac{\alpha}{S(t)} + 0.8$ at $t = \frac{n}{\rho}\ln(1 + 10\rho)$, the flip bifurcation is suppressed from $\alpha = \alpha_1^*$ to $\alpha = \alpha_3^*$ while the inverse flip bifurcation occurs from $\alpha = \alpha_1^*$ to $\alpha = \alpha_3^*$ in advance. So chaos in system (4) with $\Delta S(t) = \frac{\alpha}{S(t)}$ may be eliminated.

The effects of bifurcation control on system (4) with $\rho = 1.1$ and $\rho = 1.26$ are shown in Figure 10. Chaos in system (4) with $\Delta S(t) = \frac{\alpha}{S(t)}$ are controlled into period-$2T$ state for $\rho = 1.1$ (see Figure 10(a)) and into period-T, $2T$, $4T$ states for $\rho = 1.26$ (see Figure 10(b)).



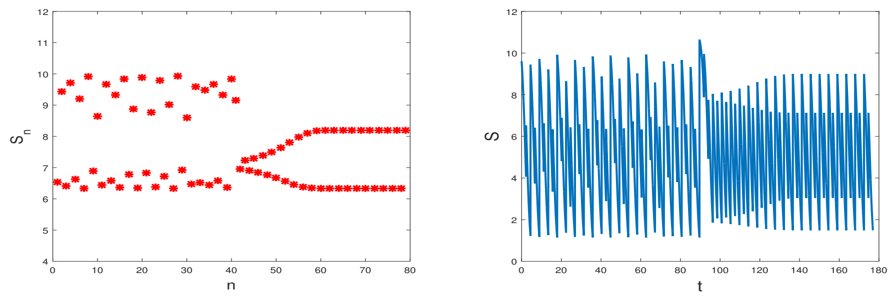
(a) (b)

Figure 9. For system with $\rho = 1.1$, $\Delta S(t) = \frac{\alpha}{S(t)}$, $\Delta S(t) = \frac{\alpha}{S(t)} + 0.8$, (a) fixed points, (b) eigenvalues of fixed points S_{02} and $\bar{S}_{02}$.



(a) (b)

Figure 10. Bifurcation diagrams with $\Delta S(t) = \frac{\alpha}{S(t)}$ and $\Delta S(t) = \frac{\alpha}{S(t)} + 0.8$, (a) $\rho = 1.1$, (b) $\rho = 1.26$.

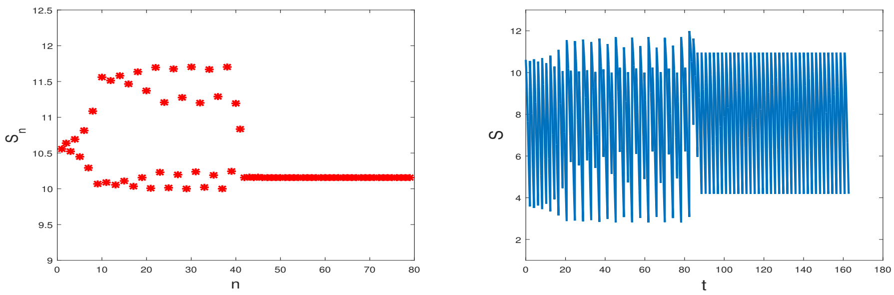


(a) (b)

Figure 11. Control measures $\Delta S(t) = \frac{\alpha}{S(t)} + 0.8$ are taken at $t = nT$ ($n \geq 40, T = 2.2590$) in system (4) with $\rho = 1.1$ and $\alpha = 10$, (a) time series of S_n, (b) time series of $S(t)$.

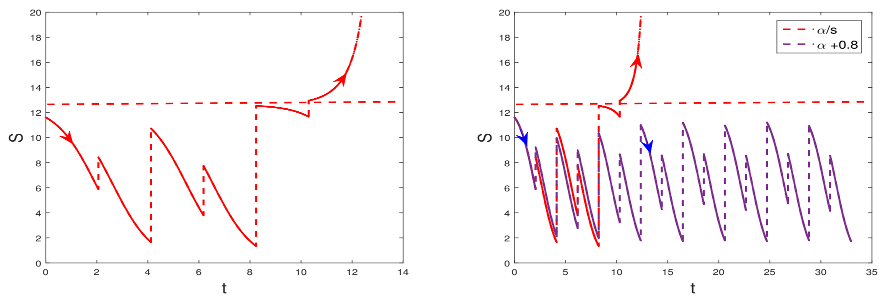
Figure 11 shows that control measures $\Delta S(t) = \frac{\alpha}{S(t)} + 0.8$ are taken at $t = nT$ ($n \geq 40, T = \frac{1}{1.1}\ln(1 + 10 * 1.1) = 2.2590$) in system (4) with $\rho = 1.1$ and $\alpha = 10$. A chaotic solution slowly evolves into a stable period-$2T$ solution.

For system (4) with $\rho = 1.26$ and $\alpha = 25$, suppose that control measures $\Delta S(t) = \frac{\alpha}{S(t)} + 0.8$ are taken at $t = nT$ $(n \geq 40, T = \frac{1}{1.26}\ln(1 + 10 * 1.26)) = 2.0715$. A chaotic solution of system (4) slowly evolves into a stable period-T solution (Figure 12).



(a) (b)

Figure 12. Control measures $\Delta S(t) = \frac{\alpha}{S(t)} + 0.8$ are taken at $t = nT$ $(n \geq 40, T = 2.0715)$ in system (4) with $\rho = 1.26$ and $\alpha = 25$, (a) time series of S_n, (b) time series of $S(t)$



(a) (b)

Figure 13. The solution from the initial value $S(0) = 11.61$ under (a) promotion strategy $\Delta S(t) = \frac{15}{S(t)}$, (b) control strategy $\Delta S(t) = \frac{15}{S(t)} + 0.8$.

In the bifurcation diagram Figure 6(d) of system (4) with $\rho = 1.26$, there is a blank area for $\alpha \in (11.42, 18.91)$. The reason for this phenomenon is that the promotion strategy $\Delta S(t) = \frac{\alpha}{S(t)}$ makes sales level $S(t)$ greater than the maximum $\frac{\rho}{\gamma}$. So the promotion strategy $\Delta S(t) = \frac{\alpha}{S(t)}$ is ineffective for $\alpha \in (11.42, 18.91)$. It's seen from Figure 10(b) that the control strategy $\Delta S(t) = \frac{\alpha}{S(t)} + \beta$ can not only control chaos, but also control sales level under the maximum value $\frac{\rho}{\gamma} = 12.6$. The solution from the initial value $S(0) = 11.61$ crosses the maximum line at $t = 5T$ under promotion strategy $\Delta S(t) = \frac{\alpha}{S(t)}$ for $\alpha = 15$ (see Figure 13(a)). Figures 10(b) and 13(b) show that the solution from the same initial value $S(0) = 11.61$ tends a stable period-$4T$ solution and is always less than $\frac{\rho}{\gamma}$ under the control strategy $\Delta S(t) = \frac{15}{S(t)} + 0.8$.

5. Conclusions

In this paper, single parameter sales promotion strategy is introduced into a nonlinear differential advertising model. Theoretical analysis and numerical results show that the system possesses complex dynamic behavior.

The coexistence of multi periodic solutions, which include unstable and stable period-nT (n = 1,2,...) solutions, occurs in the system for parameter α satisfies some conditions. Interestingly, both flip bifurcation and inverse flip bifurcation occur in nonlinear differential advertising model with sales promotion. System (4) enters into chaos from stable state through flip bifurcation and enters into stable state from chaos through inverse flip bifurcation. By suppressing flip bifurcation and promoting inverse flip bifurcation, an effective control strategy is proposed to eliminate chaos. The parameter α plays an important role in the complex dynamics of advertising model. According to theoretical analysis and actual situation, firms determine the value of parameters α to develop sales promotion strategy and avoid disorderly changes in sales volume.

The purpose of the proposed sales promotion strategy is to improve sales level and maximize firm's profit. The condition to make sales meet given target is obtained. Considering unit profit margin, constant advertising expenditure, and unit promotion cost, the profit function is constructed. The optimal sales promotion strategy is obtained and used to maximize firm's profit.

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Conflict of interest

The authors declare there is no conflict of interest.

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