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Research article

Zero-stability of waveform relaxation methods for ordinary differential equations

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Abstract: Zero-stability is the basic property of numerical methods of ordinary differential equations (ODEs). Little work on zero-stability is obtained for the waveform relaxation (WR) methods, although it is an important numerical method of ODEs. In this paper we present a definition of zero-stability of WR methods and prove that several classes of WR methods are zero-stable under the Lipschitz conditions. Also, some numerical examples are given to outline the effectiveness of the developed results.

Keywords: waveform relaxation methods; zero stability; ordinary differential equations; linear multi-step methods; Lipschitz conditions

1. Introduction

Waveform relaxation (WR) is an iterative method for systems of ordinary differential equations (ODEs). It is introduced for the first time by Lelarasmee et al. [1] for the time domain analysis of large-scale nonlinear dynamical systems. A lot of studies have been done on WR methods and most of them focus on convergence (see [2–9]). Zero-stability is the basic property of numerical methods of ODEs [10]. However, to our best knowledge, so far there exists little work on zero-stability of WR methods.

We first of all propose two examples to show the fact that tiny perturbation may lead to huge change of WR approximation solutions, at the same time, to confirm that it is necessary to study zero-stability of WR methods. For this, it is enough to consider a simple method, and suppose only the initial value has tiny perturbations.

Example 1. Consider the WR method

$$x_{n+2}^{(k+1)} - (1+\alpha)x_{n+1}^{(k+1)} + \alpha x_n^{(k+1)} = -(1-\alpha)h(x_n^{(k+1)} + x_n^{(k)}) \tag{1.1}$$

of solving numerically the initial value problem

$$\dot{x}(t) = -2x(t), t \in [0, T]; x(0) = x_0,$$

which is gotten by applying the two step method

$$x_{n+2} - (1+\alpha)x_{n+1} + \alpha x_n = -2(1-\alpha)hx_n$$

that is consistent for any α and zero-stable for $-1 < \alpha < 1$ to the iterative scheme

$$\dot{x}^{(k+1)}(t) = -x^{(k+1)}(t) - x^{(k)}(t).$$

Suppose that

$$N \text{ is a positive integer and } h = \frac{T}{N}, \tag{1.2}$$

$$x_n^{(0)} = x_0, n = 0, 1, \cdots, N, x_0^{(k)} = x_0, k = 0, 1, \cdots \tag{1.3}$$

and

$$x_1^{(k+1)} = x_0^{(k+1)} - h(x_0^{(k+1)} + x_0^{(k)}), k = 0, 1, \cdots. \tag{1.4}$$

We can get the approximate solution $\{\{x_n^{(k)}, n = 0, 1, \cdots, N\}, k = 0, 1, \cdots\}$ by using Eqs (1.1)–(1.4). Suppose that tiny perturbations denoted by δ lead to the initial values become

$$\tilde{x}_n^{(0)} = x_0 + \delta, n = 0, 1, \cdots, N, \tilde{x}_0^{(k)} = x_0 + \delta, k = 0, 1, \cdots \tag{1.5}$$

and the resulting perturbed solution of Eq (1.1) is denoted by $\{\tilde{x}_n^{(k)}\}$. Let $\delta = 1$ for simplicity and denote $\tilde{x}_n^{(k)} - x_n^{(k)}$ with $e_n^{(k)}$. Then we have

$$\begin{aligned} & e_{n+2}^{(k+1)} - (1+\alpha)e_{n+1}^{(k+1)} + \alpha e_n^{(k+1)} = -(1-\alpha)h(e_n^{(k+1)} + e_n^{(k)}) \\ & \qquad (n = 0, 1, \cdots, N-2, k = 0, 1, \cdots), \\ & e_0^{(k+1)} = 1, e_1^{(k+1)} = (1-2h)e_0^{(k+1)}(k = 0, 1, \cdots), e_n^{(0)} = 1(n = 0, 1, \cdots, N). \end{aligned} \tag{1.6}$$

By virtue of Eq (1.6) the following numerical results are obtained.

Table 1. Results obtained by (1.6) with $\alpha = 0.5$.

N	10	100	250	500	750	1000
$e_N^{(5)}$	0.10621	0.1319	0.13343	0.13388	0.13402	0.13409
$e_N^{(10)}$	0.10621	0.13238	0.13422	0.13478	0.13497	0.13506
$e_N^{(100)}$	0.10621	0.13238	0.13422	0.13478	0.13497	0.13506

The data in Tables 1 and 2 show that errors of the approximate solution of Eq (1.1) brought by perturbations of initial values are controllable for $\alpha = 0.5$, but they are unbounded for $\alpha = 2$ when h tends to zero.

In this numerical examples, taking $\delta = 1$ instead of tiny perturbations seem to be unreasonable. The main reason for this is that we are usually interested in the ratio $\frac{e_N^{(k)}}{\delta}$ instead of δ itself. In fact we have also performed some other experiments using smaller δ, in which the similar ratios are obtained for different δ.

Table 2. Results obtained by (1.6) with $\alpha = 2$.

N	10	100	250	500	1000
$e_N^{(5)}$	268.74	6.2863e+028	3.7892e+073	3.4922e+148	5.7695e+298
$e_N^{(10)}$	268.74	6.2867e+028	3.7897e+073	3.4927e+148	5.7704e+298
$e_N^{(100)}$	268.74	6.2867e+028	3.7897e+073	3.4927e+148	5.7704e+298

Example 2. Consider the one-step WR method

$$\begin{aligned} x_{n+1}^{(k+1)} &= x_n^{(k+1)} + h\left(500(x_n^{(k+1)})^{1.001} + 500(x_n^{(k)})^{1.001}\right), n = 0, 1, \cdots, \frac{2}{h}, \\ x_0^{(k+1)} &= 0;\ x_n^{(0)} = 0,\ \text{for all } n, \end{aligned} \tag{1.7}$$

where $2/h$ is an integer. Suppose that the initial value 0 becomes δ by the influence of perturbations and the solution $x_n^{(k)}$ becomes $\tilde{x}_n^{(k)}$ by the influence of δ, which satisfies

$$\begin{aligned} \tilde{x}_{n+1}^{(k+1)} &= \tilde{x}_n^{(k+1)} + h\left(500(\tilde{x}_n^{(k+1)})^{1.001} + 500(\tilde{x}_n^{(k)})^{1.001}\right), n = 0, 1, \cdots, \frac{2}{h}, \\ \tilde{x}_0^{(k+1)} &= \delta;\ \tilde{x}_n^{(0)} = \delta,\ \text{for all } n. \end{aligned} \tag{1.8}$$

Clearly the numerical solution generated by Eq (1.7) is the constant zero and Eq (1.8) can be regarded as the WR method of solving the following initial value problem

$$\dot{x}(t) = 1000(x(t))^{1.001}, t \in [0, 2]; x(0) = \delta. \tag{1.9}$$

It is easy to derive that $t_0 = \dfrac{1}{\delta^{0.001}}$ is the blow-up time of the solution of Eq (1.9) and $t_0 \in [0, 2]$ whenever $\delta > 10^{-301}$. Consequently, $\sup\limits_{0<t<2} x(t) = +\infty$ for any $\delta > 10^{-301}$. If the "2" in Eqs (1.8) and (1.9) is replaced by $t_0 - \varepsilon$, where $\varepsilon > 0$ is small enough, then Eq (1.8) converges to Eq (1.9), that is,

$$\lim_{h\to 0, k\to\infty} \sup_{0 \le n \le \frac{t_0-\varepsilon}{h}} |\tilde{x}_n^{(k)} - x(nh)| = 0,$$

see [4]. Thus

$$\lim_{h\to 0, k\to\infty} \sup_{0 \le n \le \frac{2}{h}} |\tilde{x}_n^{(k)}| = +\infty,\ \text{for any } \delta > 10^{-301},$$

that is, the change of the solution of Eq (1.8) brought by the tiny change of initial values is unbounded as h tends to zero.

The above two examples show that tiny perturbations can lead to huge change of WR approximation solutions, although the methods used in the examples are too simple to be practically useful.

A numerical method is said to be zero-stable if the change of its solution brought by the tiny perturbations is controllable when h tends to zero (see [10]). In this paper we will explore what conditions guarantee the zero-stability of WR methods.

2. Preliminary

Consider the initial problem

$$\dot{x}(t) = f(t, x(t)), t \in [0, T]; x(0) = x_0(x(t) \in \mathbb{R}^d).$$

Taking the splitting function $F(t, x, x) = f(t, x)$ we can construct the iterative scheme

$$\begin{aligned} &\dot{x}^{(k+1)}(t) = F(t, x^{(k+1)}(t), x^{(k)}(t)), t \in [0, T], k = 0, 1, \cdots ; \\ &x^{(0)}(t) \equiv x_0, x^{(k+1)}(0) = x_0. \end{aligned} \tag{2.1}$$

Using one-step methods with variable step-size to discrete Eq (2.1) one arrives at

$$x_{n+1}^{(k+1)} = x_n^{(k+1)} + h_n \Phi(t_n, h_n, x_{n+1}^{(k+1)}, x_n^{(k+1)}, x_{n+1}^{(k)}, x_n^{(k)}); x_0^{(k+1)} = x_0, \tag{2.2}$$

where $n = 0, 1, \cdots, N-1, k = 0, 1, \cdots, \sum_{n=0}^{N-1} h_n = T$ and $x_n^{(0)} = x_0$ for all n. Applying a class of multi-step methods with fixed step-size to Eq (2.1) we get

$$x_{n+1}^{(k+1)} = x_n^{(k+1)} + h\Psi(t_n, h, x_{n+1}^{(k+1)}, x_n^{(k+1)}, \cdots, x_{n-p}^{(k+1)}, x_{n+1}^{(k)}, x_n^{(k)}, \cdots, x_{n-p}^{(k)}), \tag{2.3}$$

where $n = p, p+1, \cdots, N-1, k = 0, 1, \cdots, h = T/N$, the initial values except $x_n^{(0)}(n = 0, 1, \cdots, N)$ and $x_0^{(k)}(k = 0, 1, \cdots)$ are generated by a suitable one-step method. Here we take the initial values as

$$x_n^{(k)} = c_n^{(k)}, n = 0, 1, \cdots, p, k = 0, 1, \cdots (c_n^{(k)} = x_0, \text{ if } n \cdot k = 0). \tag{2.4}$$

A classical example of Eq (2.3) is Admas-type linear multi-step methods:

$$x_{n+1}^{(k+1)} = x_n^{(k+1)} + h \sum_{j=0}^{p+1} \beta_j F(t_{n+1-j}, x_{n+1-j}^{(k+1)}, x_{n+1-j}^{(k)}), n = p, p+1, \cdots, N-1,$$

where $\beta_j, j = 0, 1, \cdots, p+1$ are constants. The linear multi-step methods use generally the fixed step-size and therefore we only consider the Eq (2.3) with the fixed step-size.

Let ${}_i x^{(k)}$ denote the component of $x^{(k)}$ satisfying

$$x^{(k)} = \left(\left({}_1 x^{(k)} \right)^{\mathrm{T}}, \left({}_2 x^{(k)} \right)^{\mathrm{T}}, \cdots, \left({}_m x^{(k)} \right)^{\mathrm{T}} \right)^{\mathrm{T}},$$

where ${}_i x^{(k)} \in \mathbb{R}^{d_i}$, $x^{(k)} \in \mathbb{R}^d$ and $d_1 + d_2 + \cdots + d_m = d$. The Eq (2.1) can be rewritten as

$${}_i\dot{x}^{(k+1)} = F_i(t, {}_1 x^{(k+1)}, {}_2 x^{(k+1)}, \cdots, {}_m x^{(k+1)}, {}_1 x^{(k)}, {}_2 x^{(k)}, \cdots, {}_m x^{(k)}),$$

with $i = 1, 2, \cdots, m$, that is, the large Eq (2.1) is divided into m subsystems. When taking

$$F_i(t, \cdots) = f_i(t, {}_1 x^{(k)}, \cdots, {}_{i-1} x^{(k)}, {}_i x^{(k+1)}, {}_{i+1} x^{(k)}, \cdots, {}_m x^{(k)}), i = 1, 2, \cdots, m$$

one arrives at Guass-Jacobi WR method used frequently in actual computation, which consists of m independent subsystems and is hence parallel in nature(see [2, 6, 9]). In Eq (2.2) the unique mesh

$0 = t_0 < t_1 < \cdots < t_N = T$ is applied to all subsystems. However the subsystems of Gauss-Jocobi WR methods are independent and may have distinct behaviors, and consequently it is better to use different meshes for the subsystems with different behaviors. Applying the mesh

$$0 = {}_it_0 < {}_it_1 < \cdots < {}_it_{N_i} = T,\ {}_ih_j = {}_it_{j+1} - {}_it_j,\ j = 0, 1, \cdots, N_i - 1$$

to ith subsystem yields the following multi-rate WR method

$$\begin{aligned}{}_ix_{n+1}^{(k+1)} = {}_ix_n^{(k+1)} + {}_ih_n\,\Phi_i\Big(&{}_it_n,\ {}_ih_n,\ {}_1y^{(k)}({}_it_{n+1}), \cdots, {}_{i-1}y^{(k)}({}_it_{n+1}),\ {}_ix_{n+1}^{(k+1)},\\ &{}_{i+1}y^{(k)}({}_it_{n+1}), \cdots, {}_my^{(k)}({}_it_{n+1}),\ {}_1y^{(k)}({}_it_n), \cdots, {}_{i-1}y^{(k)}({}_it_n),\\ &{}_ix_n^{(k+1)},\ {}_{i+1}y^{(k)}({}_it_n), \cdots, {}_my^{(k)}({}_it_n)\Big), n = 0, 1, \cdots, N_i - 1,\end{aligned} \tag{2.5}$$

where $i = 1, 2, \cdots, m, k = 0, 1, \cdots$. Here the initial values ${}_ix_0^{(k+1)} = {}_ix_n^{(0)} = {}_ix_0$ for all k and n, $\left({}_1x_0{}^{\mathrm{T}}, {}_2x_0{}^{\mathrm{T}}, \cdots, {}_mx_0{}^{\mathrm{T}}\right)^{\mathrm{T}} = x_0$ and ${}_jy^{(k)}(t)$ is the interpolation function satisfying ${}_jy^{(k)}({}_jt_n) = {}_jx_n^{(k)}$ for $n = 0, 1, \cdots, N_j$.

Because of the presence of errors in actual computing, it is necessary to consider the following perturbed systems of Eqs (2.2), (2.3) and (2.5) generated by the perturbations $\{\delta_n^{(k)}, n = 0, 1, \cdots, N, k = 0, 1, \cdots\}$

$$\begin{aligned}\tilde{x}_{n+1}^{(k+1)} &= \tilde{x}_n^{(k+1)} + h_n\Phi(t_n, h_n, \tilde{x}_{n+1}^{(k+1)}, \tilde{x}_n^{(k+1)}, \tilde{x}_{n+1}^{(k)}, \tilde{x}_n^{(k)}) + h_n\delta_{n+1}^{(k+1)},\\ &\qquad n = 0, 1, \cdots, N - 1;\\ \tilde{x}_0^{(k+1)} &= x_0 + \delta_0^{(k+1)}, \tilde{x}_n^{(0)} = x_0 + \delta_n^{(0)}(n = 0, 1, \cdots, N),\end{aligned} \tag{2.6}$$

$$\begin{aligned}\tilde{x}_{n+1}^{(k+1)} &= \tilde{x}_n^{(k+1)} + h\Psi\Big(t_n, h, \tilde{x}_{n+1}^{(k+1)}, \tilde{x}_n^{(k+1)}, \cdots, \tilde{x}_{n-p}^{(k+1)},\\ &\qquad \tilde{x}_{n+1}^{(k)}, \tilde{x}_n^{(k)}, \cdots, \tilde{x}_{n-p}^{(k)}\Big) + h\delta_{n+1}^{(k+1)}, n = p, p + 1, \cdots, N - 1;\\ \tilde{x}_n^{(k+1)} &= c_n^{(k+1)} + \delta_n^{(k+1)}, n = 0, 1, \cdots, p, \tilde{x}_n^{(0)} = x_0 + \delta_n^{(0)}(n = 0, 1, \cdots, N),\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}{}_i\tilde{x}_{n+1}^{(k+1)} = {}_i\tilde{x}_n^{(k+1)} + {}_ih_n\,\Phi_i\Big(&{}_it_n,\ {}_ih_n,\ {}_1\tilde{y}^{(k)}({}_it_{n+1}), \cdots, {}_{i-1}\tilde{y}^{(k)}({}_it_{n+1}),\ {}_i\tilde{x}_{n+1}^{(k+1)},\\ &{}_{i+1}\tilde{y}^{(k)}({}_it_{n+1}), \cdots, {}_m\tilde{y}^{(k)}({}_it_{n+1}),\ {}_1\tilde{y}^{(k)}({}_it_n), \cdots, {}_{i-1}\tilde{y}^{(k)}({}_it_n),\\ &{}_i\tilde{x}_n^{(k+1)},\ {}_{i+1}\tilde{y}^{(k)}({}_it_n), \cdots, {}_m\tilde{y}^{(k)}({}_it_n)\Big) + {}_ih_n \cdot {}_i\delta_{n+1}^{(k+1)},\\ &\qquad n = 0, 1, \cdots, N_i - 1;\\ {}_i\tilde{x}_0^{(k+1)} = {}_ix_0 + {}_i\delta_0^{(k+1)},\ &{}_i\tilde{x}_n^{(0)} = {}_ix_0 + {}_i\delta_n^{(0)}(n = 0, 1, \cdots, N_i),\end{aligned} \tag{2.8}$$

where $i = 1, 2, \cdots, m$ and ${}_j\tilde{y}^{(k)}(t)$ is the interpolation function satisfying ${}_j\tilde{y}^{(k)}({}_jt_n) = {}_j\tilde{x}_n^{(k)}$ for $n = 0, 1, \cdots, N_j, j = 0, 1, \cdots, m$.

Definition 2.1. (Page 32 of [10]) Let $\{\delta_n^{(k)}, n = 0, 1, \cdots, N, k = 0, 1, \cdots\}$ be any perturbation of Eq (2.2) or Eq (2.3) and Eq (2.4), and let $\{\tilde{x}_n^{(k)}, n = 0, 1, \cdots, N, k = 0, 1, \cdots\}$ be the solution of the resulting perturbed system Eq (2.6) or Eq (2.7). Then if there exist constants C and h_0 such that, for all $h \in (0, h_0](h = \max_n h_n$ for Eq (2.2))

$$\max_{0\le k, 0\le n\le N} \|\tilde{x}_n^{(k)} - x_n^{(k)}\| \le C\varepsilon,$$

whenever

$$\max_{0\le k, 0\le n\le N} \|\delta_n^{(k)}\| \le \varepsilon,$$

we say that the Eq (2.2) or Eqs (2.3) and Eq (2.4) is zero-stable.

Here and hereafter $\|\cdot\|$ denotes any norm defined in $\mathbb{R}^d$.

Definition 2.2. Let $\{{}_i\delta_n^{(k)}, i = 1, 2, \cdots, m, n = 0, 1, \cdots, N, k = 0, 1, \cdots\}$ be any perturbation of Eq (2.5), and let $\{{}_i\tilde{x}_n^{(k)}, i = 1, 2, \cdots, m, n = 0, 1, \cdots, N, k = 0, 1, \cdots\}$ be the solution of the resulting perturbed Eq (2.8). Then if there exist constants h_0 and C_k depended on k such that

$$\max_{\substack{1\le i\le m\\ 0\le l\le k}} \max_{0\le n\le N_i} \|{}_i\tilde{x}_n^{(l)} - {}_ix_n^{(l)}\| \le C_k\varepsilon$$

for all $h = \max_{1\le i\le m} \max_{0\le n\le N_i} {}_ih_n \in (0, h_0]$, whenever

$$\max_{\substack{1\le i\le m\\ 0\le l\le k}} \max_{0\le n\le N_i} \|{}_i\delta_n^{(l)}\| \le \varepsilon,$$

we say that the Eq (2.5) is weakly zero-stable.

The following several lemmas are useful for studying zero-stability of the WR methods mentioned above.

Lemma 2.3. *Let a, b and T be nonnegative constants, and N be a positive integer. Suppose that the sets $\{h_n > 0, n = 0, 1, \cdots, N-1\}$ and $\{u_n > 0, n = 0, 1, \cdots, N\}$ satisfy*

$$u_{n+1} \le (1 + ah_n)u_n + bh_n, n = 0, 1, \cdots, N-1. \tag{2.9}$$

Then

$$\max_{0\le n\le N} u_n \le \mathrm{e}^{aT}u_0 + \frac{b}{a}(\mathrm{e}^{aT} - 1)$$

provided that $\sum_{n=0}^{N-1} h_n = T$.

Proof. By using Eq (2.9) repeatedly we have

$$\begin{aligned} u_{n+1} \le &(1 + ah_n)(1 + ah_{n-1}) \cdots (1 + ah_0)u_0 \\ &+ (1 + ah_n)(1 + ah_{n-1}) \cdots (1 + ah_1)bh_0 \\ &+ (1 + ah_n)(1 + ah_{n-1}) \cdots (1 + ah_2)bh_1 \\ &+ \cdots + (1 + ah_n)bh_{n-1} + bh_n, \end{aligned}$$

where $n = 0, 1, \cdots, N-1$. Let $t_0 = 0, t_{n+1} = t_n + h_n, n = 0, 1, \cdots, N-1$, then $t_N = T$. Noting that $1 + x < \mathrm{e}^x$ for $x > 0$ we can derive easily from above inequality

$$\begin{aligned} u_{n+1} \le &\mathrm{e}^{a(t_{n+1}-t_0)}u_0 + \mathrm{e}^{a(t_{n+1}-t_1)}b(t_1 - t_0) + \mathrm{e}^{a(t_{n+1}-t_2)}b(t_2 - t_1) \\ &+ \cdots + \mathrm{e}^{a(t_{n+1}-t_n)}b(t_n - t_{n-1}) + \mathrm{e}^{a(t_{n+1}-t_{n+1})}b(t_{n+1} - t_n) \\ \le &\mathrm{e}^{a(T-t_0)}u_0 + \mathrm{e}^{aT}b\left(\mathrm{e}^{-at_1}(t_1 - t_0) + \cdots + \mathrm{e}^{-at_n}(t_n - t_{n-1})\right. \\ &\left.+ \cdots + \mathrm{e}^{-at_N}(t_N - t_{N-1})\right) \end{aligned}$$

for $n = 0, 1, \cdots, N-1$. This together with monotonicity of the function e^{-at} imply

$$u_{n+1} \le e^{a(T-t_0)}u_0 + e^{aT}b\int_{t_0}^{T} e^{-at}dt = e^{aT}u_0 + \frac{b}{a}\left(e^{aT}-1\right)$$

for all $n = 0, 1, \cdots, N-1$. □

Lemma 2.4. *Let a, b, c, d and T be nonnegative constants, N be a positive integer and $h_n, n = 0, 1, \cdots, N-1$, be positive real numbers.*

Suppose that the sequence of positive numbers $\{e_n^{(k)} > 0, n = 0, 1, \cdots N, k = 0, 1, \cdots\}$ satisfies

$$e_{n+1}^{(k+1)} \le (1 + ah_n)e_n^{(k+1)} + bh_n e_n^{(k)} + ch_n e_{n+1}^{(k)} + dh_n, n = 0, 1, \cdots N-1. \tag{2.10}$$

Then for any $0 < h = \max_{0\le n\le N-1} h_n < \dfrac{1}{2c}$

$$\max_{0\le k, 0\le n\le N} e_n^{(k)} \le e^{2(a+b+c)T} \max\left\{\max_{0\le k} e_0^{(k)}, \max_{0\le n\le N} e_n^{(0)}\right\} + \frac{d}{a+b+c}(e^{2(a+b+c)T}-1) \tag{2.11}$$

provided that $\sum_{n=0}^{N-1} h_n = T$.

Proof. By Eq (2.10) we have

$$\max_{0\le k} e_{n+1}^{(k+1)} \le (1 + ah_n)\max_{0\le k} e_n^{(k+1)} + bh_n \max_{0\le k} e_n^{(k)} + ch_n \max_{0\le k} e_{n+1}^{(k)} + dh_n \tag{2.12}$$

for $n = 0, 1, \cdots, N-1$. Let $\varepsilon = \max_{0\le n\le N} e_n^{(0)}$ and $u_n = \max\left\{\max_{0\le k} e_n^{(k)}, \varepsilon\right\}, n = 0, 1, \cdots, N$. Clearly $\max\left\{\max_{0\le k} e_n^{(k+1)}, \varepsilon\right\} = u_n$. Hence we can get by Eq (2.12)

$$\begin{aligned} u_{n+1} &\le (1 + ah_n)u_n + bh_n u_n + ch_n u_{n+1} + dh_n \\ &\le \frac{1 + ah_n + bh_n}{1 - ch_n}u_n + \frac{dh_n}{1 - ch_n}, n = 0, 1, \cdots, N-1. \end{aligned}$$

Suppose that $0 < h < \dfrac{1}{2c}$. Consequently, $\dfrac{1}{1 - ch_n} < 2$ and the above inequality thus yield

$$u_{n+1} \le (1 + 2(a + b + c)h_n)u_n + 2dh_n, n = 0, 1, \cdots, N-1.$$

This together with Lemma 2.3 yield

$$\max_{0\le n\le N} u_n \le e^{2(a+b+c)T}u_0 + \frac{d}{a+b+c}(e^{2(a+b+c)T}-1).$$

Thus Eq (2.11) holds true. The proof is complete. □

Lemma 2.5. *Let* $a_j, b_j (j = 0, 1, \cdots, p), c, d$ *and* T *be nonnegative constants. Let* N *be a positive integer and* $h = T/N$. *Suppose that the sequence of positive numbers* $\{e_n^{(k)} > 0, n = 0, 1, \cdots N, k = 0, 1, \cdots\}$ *satisfies*

$$e_{n+1}^{(k+1)} \leq e_n^{(k+1)} + h \sum_{j=0}^{p} \left(a_j e_{n-j}^{(k+1)} + b_j e_{n-j}^{(k)} \right) + che_{n+1}^{(k)} + dh, n = p, p+1, \cdots, N-1. \tag{2.13}$$

Then for any $0 < h < \dfrac{1}{2c}$

$$\max_{0 \leq k, p+1 \leq n \leq N} e_n^{(k)} \leq \mathrm{e}^{2(A+c)T} \varepsilon + \frac{d}{A+c} \left(\mathrm{e}^{2(A+c)T} - 1 \right),$$

where $\varepsilon = \max \left\{ \max\limits_{0 \leq n \leq N} e_n^{(0)}, \max\limits_{0 \leq k, 0 \leq n \leq p} e_n^{(k)} \right\}$, $A = \sum\limits_{j=0}^{p} (a_j + b_j)$.

Proof. Note that $e_n^{(k)} \leq \varepsilon$ for all $n \leq p$ and all $k \geq 0$. By using Eq (2.13) repeatedly we have

$$\begin{aligned} e_{p+1}^{(k+1)} \leq & (1 + ch + \cdots + (ch)^k)(1 + Ah)\varepsilon + (ch)^{k+1}\varepsilon \\ & + (1 + ch + \cdots + (ch)^k) dh, k = 0, 1, \cdots . \end{aligned} \tag{2.14}$$

Let $0 < h < \dfrac{1}{2c}$. With the notation $\alpha_h = \dfrac{1 + Ah}{1 - ch}$ and $\beta_h = \dfrac{dh}{1 - ch}$ we can derive from Eq (2.14)

$$e_{p+1}^{(k)} \leq \alpha_h \varepsilon + \beta_h, k = 0, 1, \cdots .$$

Noting that $\varepsilon \leq \alpha_h \varepsilon + \beta_h$ and therefore $e_n^{(k)} \leq \alpha_h \varepsilon + \beta_h$ for all $n \leq p + 1$ and all $k \geq 0$ we thus have by repeating the above process

$$e_{p+2}^{(k)} \leq \alpha_h (\alpha_h \varepsilon + \beta_h) + \beta_h = (\alpha_h)^2 \varepsilon + \beta_h (1 + \alpha_h), k = 0, 1, \cdots .$$

Consequently, we get

$$e_{p+m}^{(k)} \leq (\alpha_h)^m \varepsilon + \beta_h (1 + \alpha_h + (\alpha_h)^2 + \cdots + (\alpha_h)^{m-1}), k = 0, 1, \cdots$$

hold true for $m = 1, 2, \cdots, N - p$. Note that $\alpha_h \geq 1$. Thus

$$e_n^{(k)} \leq (\alpha_h)^N \varepsilon + \beta_h (1 + \alpha_h + (\alpha_h)^2 + \cdots + (\alpha_h)^{N-1})$$

for $k = 0, 1, \cdots, n = 0, 1, \cdots, N$. This together with $0 < ch < 1/2$ yield

$$\begin{aligned} e_n^{(k)} \leq & (\alpha_h)^N \varepsilon + \frac{(\alpha_h)^N - 1}{\alpha_h - 1} \beta_h \\ \leq & (1 + 2(A + c)h)^{T/h} \varepsilon + \frac{d}{A + c} \left((1 + 2(A + c)h)^{T/h} - 1 \right) \\ \leq & \mathrm{e}^{2(A+c)T} \varepsilon + \frac{d}{A + c} \left(\mathrm{e}^{2(A+c)T} - 1 \right) \end{aligned}$$

for $k = 0, 1, \cdots, n = 0, 1, \cdots, N$. The proof is complete. □

3. Main results

Theorem 3.1. *Suppose that there exist constants* L_1,L_2,L_3 *and* L_4 *such that*

$$||\Phi(t,h,x_1,x_2,x_3,x_4) - \Phi(t,h,y_1,y_2,y_3,y_4)|| \le \sum_{i=1}^{4} L_i||x_i - y_i|| \tag{3.1}$$

for any real numbers $t, h \in \mathbb{R}, x_i, y_i \in \mathbb{R}^d, i = 1,2,3,4$. *Then the WR Eq* (2.2) *is zero-stable.*

Proof. Let $\varepsilon_n^{(k)} = ||\tilde{x}_n^{(k)} - x_n^{(k)}||$ and for any $\varepsilon > 0$ the perturbations in Eq (2.6) $\{\delta_n^{(k)}, k = 0,1,\cdots,n = 0,1,\cdots,N\}$ satisfies $\max\limits_{0\le k,0\le n\le N} ||\delta_n^{(k)}|| < \varepsilon$. Then by virtue of Eqs (2.2) and (2.6) and the condition (3.1) we have

$$\varepsilon_{n+1}^{(k+1)} \le \varepsilon_n^{(k+1)} + h_n\left(L_1\varepsilon_{n+1}^{(k+1)} + L_2\varepsilon_n^{(k+1)} + L_3\varepsilon_{n+1}^{(k)} + L_4\varepsilon_n^{(k)}\right) + h_n\varepsilon \tag{3.2}$$

for $n = 0,1,\cdots,N-1, k = 0,1,\cdots$ and

$$\varepsilon_0^{(k+1)} \le \varepsilon, k = 0,1,\cdots, \varepsilon_n^{(0)} \le \varepsilon, n = 0,1,\cdots,N. \tag{3.3}$$

Let $h_n < \frac{1}{2L_1}$. Then $\frac{1}{1-h_nL_1} < 2$. We therefore derive from Eq (3.2)

$$\varepsilon_{n+1}^{(k+1)} \le (1 + 2h_n(L_1 + L_2))\varepsilon_n^{(k+1)} + 2h_nL_3\varepsilon_{n+1}^{(k)} + 2h_nL_4\varepsilon_n^{(k)} + 2h_n\varepsilon \tag{3.4}$$

for $n = 0,1,\cdots,N-1, k = 0,1,\cdots$. Consequently, by Eqa (3.3), (3.4) and Lemma 2.4 we have

$$\varepsilon_n^{(k)} \le e^{4(L_1+L_2+L_3+L_4)T}\varepsilon + \frac{\varepsilon}{L_1+L_2+L_3+L_4}\left(e^{4(L_1+L_2+L_3+L_4)T} - 1\right)$$

for any $k = 0,1,\cdots,n = 0,1,\cdots,N$ if $0 < h < \min\left\{\frac{1}{2L_1}, \frac{1}{4L_3}\right\}$. The proof is therefore complete. □

Theorem 3.2. *Suppose that there exist constants* $L_i, i = 1,2,\cdots,2p+4$ *such that*

$$||\Psi(t,h,x_1,x_2,\cdots,x_{2p+4}) - \Psi(t,h,y_1,y_2,\cdots,y_{2p+4})|| \le \sum_{i=1}^{2p+4} L_i||x_i - y_i|| \tag{3.5}$$

for any real numbers $t, h \in \mathbb{R}, x_i, y_i \in \mathbb{R}^d, i = 1,2,\cdots,2p+4$. *Then the WR Eq* (2.3) *and Eq* (2.4) *is zero-stable.*

Proof. Let $\varepsilon_n^{(k)}$ denote $||\tilde{x}_n^{(k)} - x_n^{(k)}||$ and the perturbations $\{\delta_n^{(k)}, k = 0,1,\cdots,n = 0,1,\cdots,N\}$ in Eq (2.7) satisfy $\max\limits_{0\le k,0\le n\le N} ||\delta_n^{(k)}|| < \varepsilon$, where ε is any positive number. Then by virtue of Eqs (2.3), (2.4), (2.7) and the condition (3.5) we have

$$\begin{aligned}\varepsilon_{n+1}^{(k+1)} \le& \varepsilon_n^{(k+1)} + h(L_1\varepsilon_{n+1}^{(k+1)} + L_2\varepsilon_n^{(k+1)} + \cdots + L_{p+2}\varepsilon_{n-p}^{(k+1)} \\ &+ L_{p+3}\varepsilon_{n+1}^{(k)} + L_{p+4}\varepsilon_n^{(k)} + \cdots + L_{2p+4}\varepsilon_{n-p}^{(k)}) + h\varepsilon, \\ &\qquad n = p, p+1,\cdots,N-1, \\ \varepsilon_n^{(k+1)} \le& \varepsilon, n = 0,1,\cdots,p, \varepsilon_n^{(0)} \le \varepsilon, n = 0,1,\cdots,N,\end{aligned} \tag{3.6}$$

where $k = 0, 1, \cdots$. Let $h < \dfrac{1}{2L_1}$. Consequently, $\dfrac{1}{1 - hL_1} < 2$. We therefore derive from Eq (3.6)

$$\begin{aligned}\varepsilon_{n+1}^{(k+1)} \leq &(1 + 2h(L_1 + L_2))\varepsilon_n^{(k+1)} + 2h(L_3\varepsilon_{n-1}^{(k+1)} + \cdots + L_{p+2}\varepsilon_{n-p}^{(k+1)} + L_{p+3}\varepsilon_{n+1}^{(k)} \\ &+ L_{p+4}\varepsilon_n^{(k)} + \cdots + L_{2p+4}\varepsilon_{n-p}^{(k)}) + 2h\varepsilon, n = p, p+1, \cdots, N-1, \\ \varepsilon_n^{(k+1)} \leq &\varepsilon, n = 0, 1, \cdots, p, \varepsilon_n^{(0)} \leq \varepsilon, n = 0, 1, \cdots, N\end{aligned} \tag{3.7}$$

for $n = 0, 1, \cdots, N-1, k = 0, 1, \cdots$. By Eq (3.7) and Lemma 2.5 we have

$$\varepsilon_n^{(k)} \leq \mathrm{e}^{4(L_1+L_2+\cdots+L_{2p+4})T}\varepsilon + \frac{\varepsilon}{L_1 + L_2 + \cdots + L_{2p+4}}\left(\mathrm{e}^{4(L_1+L_2+\cdots+L_{2p+4})T} - 1\right)$$

for any $k = 0, 1, \cdots, n = 0, 1, \cdots, N$ if $0 < h < \min\left\{\dfrac{1}{2L_1}, \dfrac{1}{4L_{p+3}}\right\}$. The proof is complete. □

Theorem 3.3. *For Eqs (2.5) and (2.8) let $\Phi = (\Phi_1^{\mathrm{T}}, \Phi_2^{\mathrm{T}}, \cdots, \Phi_m^{\mathrm{T}})^{\mathrm{T}}$ satisfy the Lipschitz condition: there exist constants L_1 and L_2 such that*

$$\|\Phi(t, h, x_1, x_2) - \Phi(t, h, y_1, y_2)\| \leq \sum_{i=1}^{2} L_i\|x_i - y_i\|, \textit{ for any } x_1, x_2, y_1, y_2 \in \mathbb{R}^d, \tag{3.8}$$

and the interpolation functions ${}_j\tilde{y}^{(k)}(t)$ and ${}_jy^{(k)}(t)$ satisfy that

$$\sup_{0\leq t\leq T} \|{}_j\tilde{y}^{(k)}(t) - {}_jy^{(k)}(t)\| \leq C \max_{0\leq n\leq N_j} \|{}_j\tilde{x}_n^{(k)} - {}_jx_n^{(k)}\|, j = 1, 2, \cdots, m, \tag{3.9}$$

where C is the positive constant. Then the multi-rate Eq (2.5) is weakly zero-stable.

Proof. Define $\|x\|_a = \max\limits_{1\leq i\leq m} \|{}_ix\|$, $\|x\|_b = \sum\limits_{i=1}^{m} \|{}_ix\|$ for $x = ({}_1x^{\mathrm{T}}, {}_2x^{\mathrm{T}}, \cdots, {}_mx^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^d$, where ${}_ix \in \mathbb{R}^{d_i}, i = 1, 2, \cdots, m, d_1 + d_2 + \cdots + d_m = d$. It is easy to prove that $\|\cdot\|_a$ and $\|\cdot\|_b$ are the norm. Consequently, there exist constants C_1 and C_2 such that

$$C_1\|x\|_a \leq \|x\| \leq C_2\|x\|_b \text{ for any } x \in \mathbb{R}^d. \tag{3.10}$$

Eqs (3.8) and (3.10) thus yield

$$\begin{aligned}C_1\|\Phi(t, h, x_1, x_2) - \Phi(t, h, y_1, y_2)\|_a &\leq \|\Phi(t, h, x_1, x_2) - \Phi(t, h, y_1, y_2)\| \\ \leq \sum_{j=1}^{2} L_j\|x_j - y_j\| \leq \sum_{j=1}^{2} L_jC_2\|x_j - y_j\|_b &= \sum_{j=1}^{2} L_jC_2 \sum_{i=1}^{m} \|{}_ix_j - {}_iy_j\|\end{aligned}$$

for any $x_1, x_2, y_1, y_2 \in \mathbb{R}^d$ and therefore

$$\|\Phi_i(t, h, x_1, x_2) - \Phi_i(t, h, y_1, y_2)\| \leq \sum_{j=1}^{2} L_j\frac{C_2}{C_1} \sum_{i=1}^{m} \|{}_ix_j - {}_iy_j\|. \tag{3.11}$$

With the notation ${}_ie_n^{(k)} = \|{}_i\tilde{x}_n^{(k)} - {}_ix_n^{(k)}\|$ we can derive from Eqs (2.5), (2.8), (3.9) and (3.11)

$$\begin{aligned}
{}_ie_{n+1}^{(k+1)} \leq {}_ie_n^{(k+1)} + {}_ih_n\frac{L_1C_2C}{C_1}\Bigg(&\max_{0\leq j\leq N}{}_1e_j^{(k)} + \cdots + \max_{0\leq j\leq N_i}{}_{i-1}e_j^{(k)} + {}_ie_{n+1}^{(k+1)} \\
&+ \max_{0\leq j\leq N_i}{}_{i+1}e_j^{(k)} + \cdots + \max_{0\leq j\leq N_i}{}_me_j^{(k)}\Bigg) + {}_ih_n\frac{L_2C_2C}{C_1}\Bigg(\max_{0\leq j\leq N_i}{}_1e_j^{(k)} \\
&+ \cdots + \max_{0\leq j\leq N_i}{}_{i-1}e_j^{(k)} + {}_ie_n^{(k+1)} + \max_{0\leq j\leq N_i}{}_{i+1}e_j^{(k)} + \cdots + \max_{0\leq j\leq N_i}{}_me_j^{(k)}\Bigg) \\
&+ {}_ih_n\|{}_i\delta_{n+1}^{(k+1)}\|.
\end{aligned} \tag{3.12}$$

Let $h = \max_{1\leq i\leq m}\max_{0\leq n\leq N_i}{}_ih_n < \frac{C_1}{2L_1C_2C}$. Consequently, ${}_ih_n\frac{L_1C_2C}{C_1} < \frac{1}{2} < 1 - {}_ih_n\frac{L_1C_2C}{C_1}$ which together with Eq (3.12) imply that

$$\begin{aligned}
{}_ie_{n+1}^{(k+1)} &\leq (1 + a\,{}_ih_n)\,{}_ie_n^{(k+1)} + \left(a\sum_{\substack{l=1\\ l\neq i}}^{m}\max_{0\leq j\leq N_i}{}_le_j^{(k)} + 2\|{}_i\delta_{n+1}^{(k+1)}\|\right){}_ih_n \\
&\leq (1 + a\,{}_ih_n)\,{}_ie_n^{(k+1)} + \left(a\sum_{i=1}^{m}\max_{0\leq j\leq N_i}{}_ie_j^{(k)} + 2\max_{1\leq i\leq m}\max_{0\leq j\leq N_i}\|{}_i\delta_j^{(k+1)}\|\right){}_ih_n
\end{aligned} \tag{3.13}$$

with $a = \frac{2(L_1 + L_2)C_2C}{C_1}$. By using Eq (3.13) with $k = 0$ and Lemma 2.3 we get

$${}_ie_n^{(1)} \leq e^{aT}\,{}_ie_0^{(1)} + \frac{1}{a}(e^{aT} - 1)\left(a\sum_{i=1}^{m}\max_{0\leq j\leq N_i}{}_ie_j^{(0)} + 2\max_{1\leq i\leq m}\max_{0\leq j\leq N_i}\|{}_i\delta_j^{(1)}\|\right).$$

Thus

$$\max_{1\leq i\leq m}\max_{0\leq n\leq N_i}{}_ie_n^{(1)} \leq a_1\max\left(\max_{1\leq i\leq m}{}_ie_0^{(1)}, \sum_{i=1}^{m}\max_{0\leq j\leq N_i}{}_ie_j^{(0)}, \max_{1\leq i\leq m}\max_{0\leq j\leq N_i}\|{}_i\delta_j^{(1)}\|\right)$$

with $a_1 = e^{aT} + \frac{1}{a}(e^{aT} - 1)(a + 2)$. Similarly, using the above inequality, Eq (3.13) with $k = 1$ and Lemma 2.3 we have

$$\max_{1\leq i\leq m}\max_{0\leq n\leq N_i}{}_ie_n^{(2)} \leq a_2\max\left(\max_{\substack{1\leq i\leq m\\ 1\leq l\leq 2}}{}_ie_0^{(l)}, \sum_{i=1}^{m}\max_{0\leq j\leq N_i}{}_ie_j^{(0)}, \max_{\substack{1\leq i\leq m\\ 1\leq l\leq 2}}\max_{0\leq j\leq N_i}\|{}_i\delta_j^{(l)}\|\right)$$

with $a_2 = e^{aT} + (e^{aT} - 1)(ma_1 + \frac{2}{a})$. Repeating the above process yields that for any k there exists the constant a_k such that

$$\begin{aligned}
&\max_{\substack{1\leq i\leq m\\ 0\leq l\leq k}}\max_{0\leq n\leq N_i}\|{}_i\tilde{x}_n^{(l)} - {}_ix_n^{(l)}\| \\
&\leq a_k\max\left(\max_{\substack{1\leq i\leq m\\ 0\leq l\leq k}}\|{}_i\delta_0^{(l)}\|, \sum_{i=1}^{m}\max_{0\leq n\leq N_i}\|{}_i\delta_n^{(0)}\|, \max_{\substack{1\leq i\leq m\\ 0\leq l\leq k}}\max_{0\leq n\leq N_i}\|{}_i\delta_n^{(l)}\|\right)
\end{aligned}$$

for any mesh $\{{}_it_n, n = 0, 1, \cdots, N, i = 1, 2, \cdots, m\}$. The proof is therefore complete. □

Remark 1. In the above discussions we assume that the perturbation at one step is $\delta_n^{(k)}h$, which depends on step-size h. This assumption is necessary for the proof of the theorems mentioned above. However it seems to be more reasonable to regard $\delta_n^{(k)}$, a small number, as the perturbation, if the step-size is very small. In this case the tiny perturbation at each step may bring out the huge change of the numerical solution. We will explore these in next section by some computer experiments.

Remark 2. Note that the convergence property of the WR methods is not be used in the proof of Theorem 3.1-3.3. Hence, a divergent WR method may be zero-stable. In other words, zero-stability does not imply convergence.

4. Numerical experiments

In this section some numerical experiments are performed to verify the theorems obtained in Section 3.

Consider the equation

$$x' = Ax = \begin{pmatrix} T_1 & I & 0 \\ I & T_2 & I \\ 0 & I & T_3 \end{pmatrix} x, t \in [0,1], x(0) = x_0, \tag{4.1}$$

where $x \in \mathbb{R}^{15}$, $T_i = \begin{pmatrix} -2^i & 1 & & \\ 1 & -2^i & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2^i \end{pmatrix} \in \mathbb{R}^{5\times5}$ and I is the identity matrix. The special case of such system has been examined by Burrage [11], which is obtained by discretizing the heat equation in two variables.

Let $A_1 = \begin{pmatrix} T_1 & & \\ & T_2 & \\ & & T_3 \end{pmatrix}$ and $A_2 = A - A_1$. First, choose the following WR methods of solving the Eq (4.1):

(i)Euler method with variable step-size

$$\begin{aligned} &x_{n+1}^{(k+1)} = x_n^{(k+1)} + h_n(A_1x_n^{(k+1)} + A_2x_n^{(k)}), n = 0, 1, \cdots, N-1, k = 0, 1, \cdots, \\ &x_0^{(k+1)} = x_0 \text{ for all } k, x_n^{(0)} = x_0 \text{ for all } n. \end{aligned}$$

(ii)Two step methods with fixed step-size

$$\begin{aligned} x_{n+1}^{(k+1)} &= x_n^{(k+1)} + \frac{h}{2}\left[3(A_1x_n^{(k+1)} + A_2x_n^{(k)}) - (A_1x_{n-1}^{(k+1)} + A_2x_{n-1}^{(k)})\right], \\ &\qquad\qquad n = 1, 2, \cdots, N-1, k = 0, 1, \cdots, \\ x_0^{(k+1)} &= x_0 \text{ for all } k, x_n^{(0)} = x_0 \text{ for all } n, \end{aligned}$$

where $x_1^{(k+1)}$ is obtained by a suitable one-step method. Here let

$$x_1^{(k+1)} = x_0 + hAx_0 + \frac{h^2}{2}A^2x_0.$$

(iii)Multi-rate Jacobi methods based on Euler method

$$\left.\begin{aligned} {}_ix_{n+1}^{(k+1)} = {}_ix_n^{(k+1)} + {}_ih_n\left(T_i \cdot {}_ix_{n+1}^{(k+1)} + {}_{i-1}x_{n+1}^{(k)} + {}_{i+1}x_{n+1}^{(k)}\right), \\ n = 0, 1, \cdots, N_i - 1, \\ {}_ix_0^{(k+1)} = {}_ix_n^{(0)} = {}_ix_0,\ {}_0x_n^{(k)} = {}_4x_n^{(k)} = (0, 0, 0, 0, 0)^{\mathrm{T}} \text{ for all } k \text{ and } n \end{aligned}\right\} i = 1, 2, 3.$$

Second, assume the initial values and the approximation solution at each calculation step have tiny perturbations as in Eqs (2.6)–(2.8) and let $\delta_n^{(k)} \equiv \Delta = \delta \cdot (1, 1, \cdots, 1)^{\mathrm{T}} \in \mathbb{R}^{15}$,${}_i\delta_n^{(k)} \equiv \Delta_p = \delta \cdot (1, 1, 1, 1, 1)^{\mathrm{T}} \in \mathbb{R}^5$, $\tilde{x}_n^{(k)}$ denote the resulting perturbed solution and $u_n^{(k)}$ denote $\tilde{x}_n^{(k)} - x_n^{(k)}$, which yields

$$\begin{aligned} &u_{n+1}^{(k+1)} = u_n^{(k+1)} + h_n(A_1u_n^{(k+1)} + A_2u_n^{(k)}) + h_n\Delta, n = 0, 1, \cdots, N-1, k = 0, 1, \cdots, \\ &u_0^{(k+1)} = \Delta \text{ for all } k, u_n^{(0)} = \Delta \text{ for all } n, \end{aligned} \tag{4.2}$$

$$\begin{aligned} u_{n+1}^{(k+1)} &= u_n^{(k+1)} + \frac{h}{2}\left[3(A_1u_n^{(k+1)} + A_2u_n^{(k)}) - (A_1u_{n-1}^{(k+1)} + A_2u_{n-1}^{(k)})\right] + \frac{h}{2}\Delta, \\ &\qquad n = 1, 2, \cdots, N-1, k = 0, 1, \cdots, \\ u_0^{(k+1)} &= \Delta, u_1^{(k+1)} = (I + hA + \frac{h^2}{2}A^2)\Delta, \text{ for all } k, u_n^{(0)} = \Delta \text{ for all } n, \end{aligned} \tag{4.3}$$

and

$$\left.\begin{aligned} {}_iu_{n+1}^{(k+1)} = {}_iu_n^{(k+1)} + {}_ih_n\left(T_i \cdot {}_iu_{n+1}^{(k+1)} + {}_{i-1}u_{n+1}^{(k)} + {}_{i+1}u_{n+1}^{(k)} + \Delta_p\right), \\ n = 0, 1, \cdots, N_i - 1, \\ {}_iu_0^{(k+1)} = {}_iu_n^{(0)} = \Delta_p,\ {}_0u_n^{(k)} = {}_4u_n^{(k)} = (0, 0, 0, 0, 0)^{\mathrm{T}}, \forall k, \forall n \end{aligned}\right\} i = 1, 2, 3. \tag{4.4}$$

Lastly, reach a reasonable conclusion by analysing data generated by Eqs (4.2)–(4.4).

We choose A, A_1, A_2 and $\delta_n^{(k)}$ as above to guarantee that $u_n^{(k)}$ increase with k and n that is the worst case of error propagation.

In our experiments the step-sizes h_n in Eq (4.2) and ${}_ih_n$ in Eq (4.4) are taken as

$$h_n = \min\{10h, \max\{0.1h, |\xi_n|h\}\}$$

and

$${}_ih_n = \min\{10^ih, \max\{10^{i-2}h, 10^{i-1}|\xi_n|h\}\},$$

where $h > 0$, $\{\xi_0, \xi_1, \cdots\}$ is a sequence of independent and identically distributed random variables satisfying $\xi_i \sim N(0, 1)$.

The data in Tables 3, 5 and 7 show clearly the errors resulted by the perturbations $h_n\Delta(h\Delta/2)$ in Eq (4.2) (Eq (4.3) are controllable as k and n tend to infinity which support the theorems developed in Section 3). However the data in Table 4 and 6 show that the errors increase with k and $1/h$ when the perturbations $h_n\Delta(h\Delta/2)$ are replaced by the constant perturbation $\Delta(\Delta/2)$.

Table 3. The computing results of Eq (4.2) with $\delta = 0.0001$ ($\varepsilon = \max_{0\le l\le k}\max_{0\le n\le N} u_n^{(l)}$).

h	0.1		0.001	
k	5	10	10	15
ε	3.1483e-04	3.1483e-04	3.1605e-04	3.1605e-04
h	0.00001			
k	10	15		
ε	3.1606e-04	3.1606e-04		

Table 4. The computing results of Eq (4.2) with $\delta = 0.0001$ ($\varepsilon = \max_{0\le l\le k}\max_{0\le n\le N} u_n^{(l)}$) for the case of replacing perturbation $h_n\Delta$ with the constant perturbation Δ.

h	0.1		0.001		0.00001	
k	5	10	10	15	10	15
ε	0.0018	0.0018	0.1824	0.1824	17.6723	17.6723
h	0.000001					
k	10	15				
ε	176.8079	176.8079				

Table 5. The computing results of Eq (4.3) with $\delta = 0.0001$ ($\varepsilon = \max_{0\le l\le k}\max_{0\le n\le N} u_n^{(l)}$).

h	0.1		0.001	
k	10	15	10	15
ε	3.1568e-04	3.1568e-04	3.1606e-04	3.1606e-04
h	0.00001			
k	10	15		
ε	3.1606e-04	3.1606e-04		

Table 6. The computing results of Eq (4.3) with $\delta = 0.0001$ ($\varepsilon = \max_{0\le l\le k}\max_{0\le n\le N} u_n^{(l)}$) for the case of replacing perturbation $h\Delta/2$ with $\Delta/2$.

h	0.1		0.001		0.00001	
k	10	15	10	15	10	15
ε	0.1782	0.1782	0.1435	0.1435	14.1724	14.1724
h	0.000001					
k	10	15				
ε	141.7252	141.7252				

Table 7. The computing results of Eq (4.4) with $\delta = 0.0001$ ($\varepsilon = \max_{0\le l\le k}\max_{0\le n\le N} u_n^{(l)}$).

h	0.1		0.001	
k	5	10	10	15
ε	3.1085e-04	3.1085e-04	3.1567e-04	3.1567e-04
h	0.00001			
k	10	15		
ε	3.1606e-04	3.1606e-04		

5. Further work

It is well known that a linear multi-step method with fixed step-size is convergent if, and only if, it is both consistent and zero-stable. In this paper we only present the sufficient conditions of zero-stability for some special methods. We will explore the relationship between convergence and zero-stability for WR methods based on general linear multi-step methods in the future. Moreover, we shall apply our methods to the numerical study of some inverse problems [12–14].

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Conflict of interest

The author declares that he has no conflicts of interest regarding the publication of this paper.

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