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Research article

New results of positive doubly periodic solutions to telegraph equations

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Abstract: The paper is devoted to obtain new results of positive doubly periodic solutions to telegraph equations. One of the interesting features in our proof is that we give a new attempt to solve telegraph equation by using the theory of Hilbert's metric. Then we apply the eigenvalue theory to analyze the existence, multiplicity, nonexistence and asymptotic behavior of positive doubly periodic solutions. We also study a corresponding eigenvalue problem in a more general case.

Keywords: telegraph equation; nontrivial doubly periodic solution; Hilbert's metric; Eigenvalue theory; uniqueness; existence; multiplicity and asymptotic behavior

1. Introduction

Consider the following telegraph equation

$$u_{tt} - u_{xx} + cu_t + a(t,x)u = \omega(t,x)u^{\gamma} \tag{1.1}$$

with doubly periodic boundary conditions

$$u(t+2\pi, x) = u(t, x+2\pi) = u(t,x),\ (t,x) \in \mathbb{R}^2, \tag{1.2}$$

where $c > 0$ is a constant, $a,\ \omega \in C(\mathbb{R}^2, \mathbb{R}^+)$ is 2π-periodic in t and x and γ satisfies $0 < |\gamma| < 1$.

Telegraph equation is a typical partial differential equation, and there are important applications in the propagation of electromagnetic waves in an electrically conducting medium, the motion of a viscoelastic fluid under the Maxwell body theory, the motion of a string or membrane with external damping and the damped wave equation in a thermally conducting medium. For details and explanations, we refer the readers to Barbu [1] and Roussy-Pearcy [2].

Many authors have demonstrated increasing interest in the subject of telegraph equations with various boundary conditions by different methods: maximum principles and the method of upper and lower solutions, see Li [3], Mawhin-Ortega-Robles-Pérez [4–6] and Ortega-Robles-Pérez [7]; an integral equation approach, see Gilding-Kersner [8], and the fixed point theorem in a cone, see Li [9], and Wang and An [10, 11].

In this paper, we choose different strategy of proof which relies essentially on the theory of Hilbert's metric, the eigenvalue theory, the fixed point index in a cone and the theory of α-concave operator.

On the one hand, we notice that Hilbert introduced the Hilbert's metric in 1895 in an early paper [12] which is on the foundations of geometry. Birkhoff [13] made clear the usefulness of Hilbert's metric in algebra and analysis in 1957. In suitable metric spaces, Birkhoff proved that the Jentzch's theorem for integral operators with positive kernel and Perron-Frobenius theorem for non-negative matrices could both be verified by an application of the Banach contraction mapping theorem. Applications of Hilbert's metric to positive integral operators and ordinary differential equations are given by Bushell [14].

But many important kernels are not positive kernel, which plays an important role in Birkhoff [13]. For example, Green's functions of some boundary value problems are nonnegative, not positive; for instance, when $n = 3$, the Green's function $G_1(x, y)$ for the elliptic boundary value problem

$$\begin{cases} -\Delta u = 0 \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega \end{cases}$$

satisfies that

$$0 \le G_1(x, y) \le \frac{1}{4\pi r},$$

where

$$r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

For the case of one-dimension, it is well known that the Green's function $G_2(t, s)$ for the following boundary value problem

$$\begin{cases} -x'' = 0 \ \text{ in } (0, 1), \\ x(0) = x(1) = 0 \end{cases}$$

is that

$$G_2(t, s) = \begin{cases} s(1 - t), & 0 \le s \le t \le 1, \\ t(1 - s), & 0 \le t \le s \le 1, \end{cases} \tag{1.3}$$

It so follows from (1.3) that

$$0 \le G_2(t, s) \le \frac{1}{4}.$$

Therefore, the theory of Hilbert's metric has been somewhat neglected, perhaps because the Green's function is lack of the strictly positivity for many boundary value problems. In this paper, we give a new attempt to consider the existence and uniqueness of positive solution for problem (1.1) with (1.2) by using the theory of Hilbert's metric.

Let $\mathbb{Z}, \mathbb{R}$, and $\mathbb{R}^+$ denote the set of all integers, real numbers and nonnegative real numbers, respectively and let

$$\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z})$$

denote the the torus.

Throughout this paper, a doubly 2π-periodic function will be identified to be a function defined on $\mathbb{T}^2$. We also let

$$L^p(\mathbb{T}^2),\ C(\mathbb{T}^2),\ C^\alpha(\mathbb{T}^2),\ \mathcal{D}(\mathbb{T}^2) = C^\infty(\mathbb{T}^2), \ldots$$

respectively denote the spaces of doubly periodic functions with the indicated degree of regularity. And $\mathcal{D}'(\mathbb{T}^2)$ is the space of distributions on $\mathbb{T}^2$.

We call a function $u \in L^1(\mathbb{T}^2)$ being a doubly periodic solution to (1.1) with (1.2), if u satisfies (1.1) and (1.2) in the distribution sense, that is

$$\int_{\mathbb{T}^2} u(\phi_{tt} - \phi_{xx} + c\phi_t + a(t,x)\phi)dtdx = \int_{\mathbb{T}^2} \omega(t,x)u^\gamma(t,x)dtdx,\ \forall \phi \in \mathcal{D}(\mathbb{T}^2).$$

For convenience, we assume that a and ω satisfy

($\mathbf{H_1}$) $a \in C(\mathbb{T}^2),\ 0 \leq a(t,x) \leq \frac{c^2}{4}$ for $(t,x) \in \mathbb{R}^2$, and $\int_{\mathbb{T}^2} a(t,x)dtdx > 0$;

($\mathbf{H_2}$) $\omega \in C(\mathbb{T}^2),\ \omega(t,x) \geq 0$ for $(t,x) \in \mathbb{R}^2$, and $\int_{\mathbb{T}^2} \omega(t,x)dtdx > 0$.

Theorem 1.1. *If* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *hold and* $0 < |\gamma| < 1$*, then problem* (1.1) *with* (1.2) *admits a unique positive doubly periodic solution.*

It is not difficult to see that if $\gamma = -1$, then the approach to prove Theorem 1.1 is invalid. We hence need introduce some different techniques, such as contraction ratio $R(A)$ and projective diameter $D(A)$ for some positive operator A, to prove the existence and uniqueness of positive doubly periodic solution for the following problem

$$u_{tt} - u_{xx} + cu_t + a(t,x)u = \omega(t,x)u^{-1}. \tag{1.4}$$

Theorem 1.2. *If* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *hold, then problem* (1.4) *with* (1.2) *admits a unique positive doubly periodic solution.*

Remark 1.1. The method used in the proof of Theorem 1.1 and Theorem 1.2 is invalid when we consider the case $\gamma > 1$. Therefore we need to introduce a different technique to approach this case.

On the other hand, we notice that the study of asymptotic behavior of solutions for various partial differential equations is also a hot topic, see Aviles [15], Feng-Zhang [16], Gidas-Spruck [17], Zhang-Feng [18, 19] and the references therein. Recently, Feng [20] employs the fixed point theorem of cone expansion and compression of norm type to analyze the existence and asymptotic behavior of nontrivial radial convex solutions for a Monge-Ampère system. Comparing with the fixed point theorem used in [20], the eigenvalue theory is often a very effective approach to analyze the existence and asymptotic behavior of positive solutions for various boundary value problems.

Next, we employ the eigenvalue theory to analyze the existence and asymptotic behavior of positive doubly periodic solutions for

$$u_{tt} - u_{xx} + cu_t + a(t,x)u = \lambda\omega(t,x)u^\gamma \tag{1.5}$$

with (1.2), where $\lambda > 0$ is a parameter.

Theorem 1.3. *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then, for* $\gamma > 0$*, we have the following conclusions:*

(i) If $\gamma > 1$*, then for any* $\lambda > 0$*, problem* (1.5) *with* (1.2) *admits a positive doubly periodic solution* $u_\lambda \in C(\mathbb{T}^2)$ *with* $u_\lambda \not\equiv 0$*, and*

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = +\infty, \quad \lim_{\lambda\to+\infty} \|u_\lambda\| = 0. \tag{1.6}$$

(ii) If $0 < \gamma < 1$*, then for any* $\lambda > 0$*, problem* (1.5) *with* (1.2) *admits a positive doubly periodic solution* $u_\lambda \in C(\mathbb{T}^2)$ *with* $u_\lambda \not\equiv 0$*, and*

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0. \tag{1.7}$$

(iii) If $\gamma = 1$*, then there exits* $\lambda_* > 0$ *such that problem* (1.5) *with* (1.2) *admits no positive doubly periodic solution for* $0 < \lambda < \lambda_*$*.*

We also consider the more general equation

$$u_{tt} - u_{xx} + cu_t + a(t, x)u = \lambda\omega(t, x)f(u), \tag{1.8}$$

where $\lambda > 0$ is a parameter.

We first give new results of existence, nonexistence and asymptotic behavior of positive doubly periodic solutions for problem (1.8) with (1.2) in Theorems 3.1–3.3. We will use the eigenvalue theory again to prove these assertions.

Then we consider the multiplicity of positive doubly periodic solutions for problem (1.8) with (1.2) by employing the fixed point index in a cone, which is used in Hu and Wang [21] and Zhang [22].

We obtain the following theorem:

Theorem 1.4. *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. In addition, if* $f : [0, +\infty) \to [0, +\infty)$ *is continuous and satisfies the following conditions:*

$(\mathbf{H_3})$ $\lim_{u\to 0^+} \frac{f(u)}{u} = 0$*;*

$(\mathbf{H_4})$ $\lim_{u\to+\infty} \frac{f(u)}{u} = 0$*;*

$(\mathbf{H_5})$ $0 < \underline{\lim}_{u\to+\infty} f(u) \le +\infty$.

Then, for any given $\tau > 0$*, there exists* $\delta > 0$ *such that, for* $\lambda > \delta$*, problem* (1.8) *with* (1.2) *admits at least two positive doubly periodic solutions* $u_\lambda^{(1)}(t, x)$*,* $u_\lambda^{(2)}(t, x) \in C(\mathbb{T}^2)$ *and* $\max_{(t,x)\in\mathbb{T}^2} u_\lambda^{(1)}(t, x) > \tau$*.*

Next, we apply the theory of α-concave operator to analyze the uniqueness and continuity of positive doubly periodic solution on the parameter λ for problem (1.8) with (1.2).

More precisely, we have:

Theorem 1.5. *Suppose that* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $f(u) : [0, +\infty) \to [0, +\infty)$ *is a nondecreasing function with* $f(u) > 0$ *for* $u > 0$*, and satisfies* $f(\rho u) \ge \rho^\alpha f(u)$*, for any* $0 < \rho < 1$*, where* $0 \le \alpha < 1$*. Then, for any* $\lambda \in (0, \infty)$*, problem* (1.8) *with* (1.2) *admits a unique positive doubly periodic solution* $u_\lambda(t)$*. Furthermore, such a solution* $u_\lambda(t)$ *satisfies the following properties:*

(i) $u_\lambda(t)$ *is strong increasing in* λ*. That is,* $\lambda_1 > \lambda_2 > 0$ *implies* $u_{\lambda_1}(t) \gg u_{\lambda_2}(t)$ *for* $t \in J$*.*

(ii) $\lim_{\lambda\to 0^+} \|u_\lambda\| = 0, \quad \lim_{\lambda\to+\infty} \|u_\lambda\| = +\infty$.

(iii) $u_\lambda(t)$ *is continuous with respect to* λ*. That is,* $\lambda \to \lambda_0 > 0$ *implies* $\|u_\lambda - u_{\lambda_0}\| \to 0$.

Remark 1.2. We only need f to be monotonic, but not continuous in Theorem 1.5.

The rest of the paper is organized as follows. In Section 2, we shall recall some necessary definitions, lemmas and theorems about Hilbert's metric, which will be used to prove Theorem 1.1 and 1.2. Section 3 is devoted to proving the existence, nonexistence and asymptotic behavior of positive doubly periodic solutions by using the eigenvalue theory and the inequality technique. The multiplicity of positive doubly periodic solutions for (1.8) with (1.2) is discussed in Section 4 and we will prove Theorem 1.4 there. In Section 5, we will give the proof of Theorem 1.5.

2. Hilbert's metric and positive doubly periodic solution

In this section, we will give a new attempt to solve problem (1.1) with (1.2) by using Hilbert's metric. As Bushell [14] pointed out that Hilbert's original definition of the projective metric is the logarithm of the cross-ratio for certain points in the interior of a convex cone in $\mathbb{R}^n$. Let us begin with the definition of the Hilbert metric in a general setting.

Let E be a real Banach space with a closed solid cone K^0 (K^0 denotes the interior of K). Then for $u, v \in K^0$ we define

$$M(\frac{u}{v}) = \min\{\xi : u \leq \xi v\},$$

$$m(\frac{u}{v}) = \max\{\zeta : \zeta u \leq v\}.$$

It is shown in Bushell [14] that

$$0 < m(\frac{u}{v}) \leq M(\frac{u}{v}),\ \ m(\frac{u}{v})v \leq u \leq M(\frac{u}{v})v.$$

Definition 2.1. (Definition 2.2, Bushell [14]) If $A : K \to K$ we say that A is non-negative, and if $A : K^0 \to K^0$ we say that A is positive.

Definition 2.2. (Definition 2.2, Bushell [14]) Hilbert's projective metric $d(\cdot,\cdot)$ is defined in K^0 by

$$d(u, v) = \ln\left\{M(\frac{u}{v})\Big/m(\frac{u}{v})\right\}.$$

Lemma 2.1. *(Theorem 2.1, Bushell [14])* $\{K^0, d\}$ *is a pseudo-metric space and* $X = \{K^0 \cap B, d\}$ *is a metric space, where* B *is the unit sphere in* E*.*

Lemma 2.2. *(Lemma 2.2, Bushell [14]) If* $u, v \in K^0$*, then* $d(\xi u, \zeta v) = d(u, v)$ *for all* $\xi,\ \zeta > 0$.

Moreover, Bushell [24] proved that if the norm is monotone with respect to P (i.e. $0 \leq u \leq v \Rightarrow \|u\| \leq \|v\|$), then X is complete.

Definition 2.3. (Definition 1, Botter [23]) If $A(\xi u) = \xi^\gamma Au$ for all $u \in K^0$, $\xi > 0$, we say that A is positive homogeneous of degree γ in K^0.

Definition 2.4. (Definition 3, Botter [23]) If $A : E \to E$, then A is said to be monotone decreasing (increasing) if $u \leq v$ implies $Au \leq Av (Au \geq Av)$.

We first consider the linear equation

$$u_{tt} - u_{xx} + cu_t - \lambda u = h(t, x) \text{ in } \mathcal{D}'(\mathbb{T}^2), \tag{2.1}$$

where $\lambda \in \mathbb{R}$, and $h(t,x) \in L^1(\mathbb{T}^2)$.

Let $\mathfrak{L}_\lambda$ denote the differential operator

$$\mathfrak{L}_\lambda = u_{tt} - u_{xx} + cu_t - \lambda u,$$

acting on functions on $\mathbb{T}^2$. It follows from Lemma 1 in [9] that if $\lambda < 0$, then $\mathfrak{L}_\lambda$ admits the resolvent $\mathfrak{R}_\lambda$ defined by

$$\mathfrak{R}_\lambda : L^1(\mathbb{T}^2) \to C(\mathbb{T}^2), \quad h \mapsto u,$$

where u is the unique solution of (2.1). Moreover, if we restrict $\mathfrak{R}_\lambda$ on $L^p(\mathbb{T}^2)$ or $C(\mathbb{T}^2)$, then $\mathfrak{R}_\lambda$ is compact. Specifically, $\mathfrak{R}_\lambda : C(\mathbb{T}^2) \to C(\mathbb{T}^2)$ is completely continuous.

Letting $\mu = -\frac{c^2}{4}$, then the Green's function $G(t,x)$ of $\mathfrak{L}_\mu$ can be explicitly expressed, which was obtained in Lemma 5.2 of [7]. Thus, it follows from Lemma 5.1 in [7] that the unique solution of (2.1) can be defined by convolution product

$$u(t,x) = (R_\mu h)(t,x) = \int_{\mathbb{T}^2} G(t-s, x-y) h(s,y) ds dy. \tag{2.2}$$

By the definition of $G(t,x)$, one can get

$$\underline{G} \le G(t,s) \le \overline{G}, \tag{2.3}$$

where

$$\underline{G} := \operatorname{ess\,inf} G(t,x) = \frac{e^{\frac{-3c\pi}{2}}}{(1-e^{-c\pi})^2},$$

$$\overline{G} := \operatorname{ess\,sup} G(t,x) = \frac{1+e^{-c\pi}}{2(1-e^{-c\pi})^2}.$$

Letting $h \in L^1(\mathbb{T}^2)$ with $h(t,x) \ge 0$ for *a.e.* $(t,x) \in \mathbb{T}^2$, then it follows from (2.2) that

$$\underline{G} \|h\|_{L^1(\mathbb{T}^2)} \le (R_\mu h)(t,x) \le \overline{G} \|h\|_{L^1(\mathbb{T}^2)}. \tag{2.4}$$

Now take the Banach space to be $C(\mathbb{T}^2) := E$ with supremum norm.

Let $K \subset E$ by

$$K = \Big\{ u \in E : u(t,x) \ge 0, \ \forall (t,x) \in \mathbb{T}^2 \Big\}. \tag{2.5}$$

Then

$$K^0 = \Big\{ u \in E : u(t,x) > 0, \ \forall (t,x) \in \mathbb{T}^2 \Big\}.$$

Thus, E is an ordered Banach space with K.

For convenience, we denote the norm in Banach E by $\|\cdot\|$, and in $L^p(\mathbb{T}^2)$ by $\|\cdot\|_p$ hereafter.

Next, we consider (2.1) with (1.2) when $-\lambda$ is replaced by $a(t,x)$. In [9], the author obtained the following conclusions.

Lemma 2.3. *(Lemma 2, Li [9]) Suppose that $h(t,x) \in L^2(\mathbb{T}^2)$. Then (2.1) with (1.2) admits a unique solution $u := Ph$, and $P : L^1(\mathbb{T}^2) \to C(\mathbb{T}^2)$ is a linear bounded operator with the following properties.*

(i) $P : C(\mathbb{T}^2) \to C(\mathbb{T}^2)$ is a completely continuous operator.

(ii) If $h(t,x) > 0$ for a.e. $(t,x) \in \mathbb{T}^2$, then

$$\underline{G}\|h\|_1 \leq (Ph)(t,x) \leq \frac{\overline{G}}{\underline{G}\|a\|_1}\|h\|_1, \quad \forall (t,x) \in \mathbb{T}^2. \tag{2.6}$$

Let $T : K \to E$ be defined by

$$(Tu)(t,x) = P(\omega(t,x)u^\gamma(t,x)), \ \forall u \in E. \tag{2.7}$$

It follows from Lemma 2.3 that $T : K \to E$ is completely continuous, and the doubly periodic solution of (1.1) with (1.2) is equivalent to the fixed point of T.

Proof of Theorem 1.1. Let $u \in K$ and

$$h(t,x) = \omega(t,x)u^\gamma(t,x) \tag{2.8}$$

for $(t,x) \in \mathbb{R}^2$. Then $h \in E$ and $Tu = Ph$, and it follows from the proof of Lemma 2 in [9] that

$$(Ph)(t,x) \geq (R_\mu h)(t,x). \tag{2.9}$$

Set $u \in K^0$, and define

$$l = \min_{(t,x)\in\mathbb{T}^2} u(t,x), \quad L = \max_{(t,x)\in\mathbb{T}^2} u(t,x).$$

Then it follows from ($\mathbf{H_2}$), (2.2), (2.7) and (2.9) that

$$\begin{aligned}
(Tu)(t,x) &= (Ph)(t,x) \\
&\geq (R_\mu h)(t,x) \\
&= \int_{\mathbb{T}^2} G(t-s, x-y)\omega(s,y)u^\gamma(s,y)dsdy \\
&\geq \begin{cases} l^\gamma \underline{G} \int_{\mathbb{T}^2} \omega(s,y)dsdy > 0 \text{ for } 0 < \gamma < 1, \\ \\ L^\gamma \underline{G} \int_{\mathbb{T}^2} \omega(s,y)dsdy > 0 \text{ for } -1 < \gamma < 0, \end{cases}
\end{aligned}$$

which shows that $Tu \in K^0$. Further $T : K^0 \to K^0$.

Plainly T is monotone increasing (decreasing) and positive homogeneous of degree γ when $0 < \gamma < 1$ $(-1 < \gamma < 0)$. Next we analyze the existence and uniqueness of positive fixed point of T in K^0.

Case 1) $0 < \gamma < 1$

Since $0 < \gamma < 1$, one can prove

$$d(Tu, Tv) \leq \gamma d(u,v), \quad \forall u, v \in K^0. \tag{2.10}$$

In fact, it follows from

$$m(\frac{u}{v})v \leq u \leq M(\frac{u}{v})v$$

and T is monotone increasing and positive homogeneous of degree γ that

$$\left[m(\frac{u}{v})\right]^{\gamma} Tv \leq Tu \leq \left[M(\frac{u}{v})\right]^{\gamma} Tv,$$

which shows that

$$M\left(\frac{Tu}{Tv}\right) \leq \left[M(\frac{u}{v})\right]^{\gamma}, \quad m\left(\frac{Tu}{Tv}\right) \geq \left[m(\frac{u}{v})\right]^{\gamma}.$$

Thus

$$\begin{aligned} d(Tu, Tv) &= \ln\left\{\frac{M\left(\frac{Tu}{Tv}\right)}{m\left(\frac{Tu}{Tv}\right)}\right\} \\ &\leq \gamma \ln\left\{\frac{M\left(\frac{u}{v}\right)}{m\left(\frac{u}{v}\right)}\right\} \\ &= \gamma d(u, v). \end{aligned}$$

Consider the mapping $\hat{T} : E \to E$ defined by

$$\hat{T}u = \frac{Tu}{\|Tu\|}, \quad u \in E.$$

We hence get from Lemma 2.2 that

$$\begin{aligned} d(\hat{T}u, \hat{T}v) &= d(\tfrac{Tu}{\|Tu\|}, \tfrac{Tv}{\|Tv\|}) \\ &= d(Tu, Tv) \\ &\leq \gamma d(u, v). \end{aligned}$$

It is not difficult to see that the norm is monotone with respect to K, so E is complete. Thus $\hat{T}$ is a contraction in Hilbert's metric and so admits a unique fixed point in E; call it v^*.

Let

$$u^* = \|Tv^*\|^{\frac{1}{1-\gamma}} v^*.$$

Next we prove that T possesses a unique fixed point in K^0. In fact, obviously $u^* \in K^0$, and

$$Tu^* = \|Tv^*\|^{\frac{\gamma}{1-\gamma}} Tv^* = \|Tv^*\|^{\frac{\gamma}{1-\gamma}+1} \hat{T}v^* = u^*.$$

Moreover, suppose there exists another point $u_1 \in K^0$ such that $Tu_1 = x_1$. Then it follows from (2.10) that

$$d(u^*, u_1) = d(Tu^*, Tu_1) \leq \gamma d(u^*, u_1).$$

This shows that $d(u^*, u_1) = 0$. So we get $u^* = \varsigma u_1$ for $\varsigma > 0$, and hence

$$u^* = Tu^* = T(\varsigma u_1) = \varsigma^{\gamma} T(u_1) = \varsigma^{\gamma} u_1,$$

which shows $\varsigma = 1, \ \ u^* = u_1$.

Case 2) Let $-1 < \gamma < 0$. Then for $u, v \in K^0$, it follows from

$$m(\frac{u}{v})v \leq u \leq M(\frac{u}{v})v$$

and T is monotone decreasing and positive homogeneous of degree γ that

$$\left[M(\frac{u}{v})\right]^{\gamma} Tv \leq Tu \leq \left[m(\frac{u}{v})\right]^{\gamma} Tv,$$

which shows that

$$M\Big(\frac{Tu}{Tv}\Big) \le \Big[m(\frac{u}{v})\Big]^{\gamma},\ \ m\Big(\frac{Tu}{Tv}\Big) \ge \Big[M(\frac{u}{v})\Big]^{\gamma}.$$

Thus

$$\begin{aligned} d(Tu,Tv) &= \ln\left\{\frac{M\big(\frac{Tu}{Tv}\big)}{m\big(\frac{Tu}{Tv}\big)}\right\} \\ &\le \gamma \ln\left\{\frac{m\big(\frac{u}{v}\big)}{M\big(\frac{u}{v}\big)}\right\} \\ &\le (-\gamma)\ln\left\{\frac{M\big(\frac{u}{v}\big)}{m\big(\frac{u}{v}\big)}\right\} \\ &= |\gamma| d(u,v). \end{aligned}$$

Let

$$\hat{T}u = \frac{Tu}{||Tu||},\ \ u \in E.$$

Then one can prove $\hat{T} : E \to E$, and

$$d(\hat{T}u, \hat{T}v) \le |\gamma| d(u,v),\ \forall u,v \in E.$$

So $\hat{T}$ admits a unique fixed point v^* in E by using Banach's contraction mapping theorem. Let

$$u^* = ||Tv^*||^{\frac{1}{1-\gamma}} v^*.$$

One can prove that u^* is the unique fixed point of T in K^0. □

It is not difficult to see that if $\gamma = -1$, then the Banach's contraction mapping theorem is invalid in the proof of Theorem 1.1. We hence need introduce some different techniques to prove the existence and uniqueness of positive doubly periodic solution for problem (1.1) with (1.2).

Definition 2.5. (Definition 2, Botter [23]) The contraction ratio, $R(A)$, of A is defined by

$$R(A) = \inf\{\xi : d(Au, Av) \le \xi d(u,v) \text{ for all } u, v \in K^0\}.$$

Definition 2.6. (Definition 3.3, Bushell [14]) If A is positive, the projective diameter $D(A)$ of A is defined by

$$D(A) = \sup\{d(Au, Av) : u, v \in K^0\}.$$

Lemma 2.4. *(Corollary, Bushell [14]) Let A be a positive linear mapping. Then*

$$R(A) \le 1.$$

Lemma 2.5. *(Theorem 3.2, Bushell [14]) Let A be a positive linear mapping in X. Then*

$$R(A) = \tanh\frac{1}{4}D(A)$$

Lemma 2.6. *(Theorem 1, Botter [23]) Let A be a monotone decreasing mapping which is positive homogeneous of degree $-\alpha(\alpha > 0)$. Then the contraction ratio, $R(A)$, does not exceed α*

Proof of Theorem 1.2. Define $\mathbb{B} : K^0 \to K^0$ by

$$\mathbb{B} = \frac{1}{u(t,x)}.$$

Then one can verify that B is a monotone decreasing and positive homogeneous of degree -1. Lemma 2.6 tells us that

$$R(\mathbb{B}) \leq 1.$$

Let $T^* : K^0 \to K^0$ be defined by

$$(T^* u)(t,x) = P(\omega(t,x)u(t,x)).$$

Plainly T^* is a positive linear mapping of E into E. And then it follows from Lemma 2.5 that

$$R(T^*) \leq \tanh \frac{D(T)}{4}.$$

But T^* admits finite projective diameter, so we get $R(T^*) < 1$.

Define $T : K^0 \to K^0$ by

$$Tu = T^* \mathbb{B} u, \ u \in K^0.$$

As in the proof of Theorem 1.1 one can define $\hat{T} : E \to E$ defined by

$$\hat{T} u = \frac{Tu}{||Tu||}, \quad u \in E.$$

We hence get from Lemma 2.2 that

$$\begin{aligned} d(\hat{T}u, \hat{T}v) &= d(\tfrac{Tu}{||Tu||}, \tfrac{Tv}{||Tv||}) \\ &= d(Tu, Tv) \\ &\leq \gamma d(T^*\mathbb{B}u, T^*\mathbb{B}v) \\ &\leq R(T^*) d(\mathbb{B}u, \mathbb{B}v) \\ &\leq R(T^*) R(\mathbb{B}) d(u,v). \end{aligned}$$

But $R(T^*)R(\mathbb{B}) < 1$ and hence $\hat{T}$ is a contraction. Therefore it follows from E is complete that $\hat{T}$ possesses a unique point in E. From this we can construct (as in Theorem 1.1) an element u of K^0 such that

$$(Tu)(t,x) = u(t,x) = P(\omega(t,x)u^{-1}(t,x)).$$

This finishes the proof of Theorem 1.2 □

Remark 2.1. One of the contributions of this section is to give the application of Hilbert's projective metric to boundary value problems.

3. Existence and asymptotic behavior of positive doubly periodic solution

In this section, we apply the eigenvalue theory to study the existence and asymptotic behavior of positive continuous solutions to problem (1.5) with (1.2). So we first collect some known results of the eigenvalue theory, which will be used in the subsequent proofs.

Lemma 3.1. *(Corollary of Theorem 1, Guo [26]) Let $A : E \to E$ be completely continuous. Suppose that $A\theta = \theta$,*

$$\lim_{\|x\|\to 0} \frac{\|Ax\|}{\|x\|} = 0$$

and

$$\lim_{\|x\|\to +\infty} \frac{\|Ax\|}{\|x\|} = +\infty.$$

Then the following two conclusions hold:

i) Every $\mu \neq 0$ is an eigenvalue of A, i.e., there exists $x_\mu \in E,\ x_\mu \neq 0$ such that $Ax_\mu = \mu x_\mu$;

ii) $\lim_{\mu\to\infty} \|x_\mu\| = +\infty$.

Lemma 3.2. *(Corollary of Theorem 2, Guo [26]) Let $A : E \to E$ be completely continuous. Suppose that $A\theta = \theta$,*

$$\lim_{\|x\|\to 0} \frac{\|Ax\|}{\|x\|} = +\infty$$

and

$$\lim_{\|x\|\to +\infty} \frac{\|Ax\|}{\|x\|} = 0.$$

Then the following two conclusions hold:

i) Every $\mu \neq 0$ is an eigenvalue of A, i.e., there exists $x_\mu \in E,\ x_\mu \neq 0$ such that $Ax_\mu = \mu x_\mu$;

ii) $\lim_{\mu\to\infty} x_\mu = 0$.

Proof of Theorem 1.3. Here we only prove the conclusion (i) holds since the proof is similar when we verify (ii). Let

$$K^* = \{u \in K : u(t,x) \geq \sigma\|u\|,\ \forall (t,x) \in \mathbb{T}^2\}, \tag{3.1}$$

where

$$\sigma = \frac{\underline{G}^2\|a\|_1}{\overline{G}} = \frac{2e^{-3c\pi}\|a\|_1}{(1-e^{-c\pi})^2(1+e^{-c\pi})}.$$

Then, it is easy to verify that T is completely continuous from K^* to K^*.

Moreover, it follows from $(\mathbf{H_1})$ and the the definitions of $\underline{G}$ and $\overline{G}$ that $0 < \sigma < 1$.

Next, we prove that all the conditions of the Lemma 3.1 and Lemma 3.2 are satisfied.

On the one hand, for $u \in K^*$, it follows from (2.7) and (2.8) that

$$\|Tu\| = \|Ph\| \leq \frac{\overline{G}}{\underline{G}\|a\|_1}\|h\|_1 \leq \frac{\overline{G}}{\underline{G}\|a\|_1}\|\omega\|_1\|u\|^\gamma. \tag{3.2}$$

It hence follows from (3.2) that: for $\gamma > 1$, we get

$$\lim_{\|u\|\to 0} \frac{\|Tu\|}{\|u\|} = 0; \tag{3.3}$$

for $0 < \gamma < 1$, we have

$$\lim_{\|u\|\to 0} \frac{\|Tu\|}{\|u\|} = +\infty. \tag{3.4}$$

On the other hand, for any $u \in K^*$, we deduce from (2.7), (2.8) and (2.9) that

$$\|Tu\| = \|Ph\| \geq \underline{G}\|h\|_1 \geq \underline{G}\sigma^\gamma \|u\|^\gamma \|\omega\|_1. \tag{3.5}$$

Thus, for $\gamma > 1$, we get

$$\lim_{\|u\|\to +\infty} \frac{\|Tu\|}{\|u\|} = +\infty. \tag{3.6}$$

For $0 < \gamma < 1$, we get

$$\lim_{\|u\|\to +\infty} \frac{\|Tu\|}{\|u\|} = 0. \tag{3.7}$$

Then from (3.3) and (3.6) or (3.4) and (3.7), together with Lemma 3.1 and Lemma 3.2, we respectively get that: for any $\lambda^* > 0$, there exists $u_{\lambda^*} \in E$ with $u_{\lambda^*} \neq \theta$ such that $Tu_{\lambda^*} = \lambda^* u_{\lambda^*}$; and

$$\lim_{\lambda^*\to +\infty} \|u_{\lambda^*}\| = +\infty \quad \text{or} \quad \lim_{\lambda^*\to +\infty} u_{\lambda^*} = 0. \tag{3.8}$$

Moreover, we obtain from (3.5) that

$$\lambda^* \|u_{\lambda^*}\| = \|Tu_{\lambda^*}\| \geq \underline{G}\|h\|_1 \geq \underline{G}\sigma^\gamma \|u_{\lambda^*}\|^\gamma \|\omega\|_1,$$

which shows that

$$\|u_{\lambda^*}\| \leq (\lambda^*)^{\frac{1}{\gamma-1}} (\underline{G}\sigma^\gamma \|\omega\|_1)^{\frac{1}{1-\gamma}}. \tag{3.9}$$

But

$$\left|\lambda^* u_{\lambda^*}\right| \leq \frac{\overline{G}}{\underline{G}\|a\|_1} \|\omega\|_1 \|u_{\lambda^*}\|^\gamma.$$

It so follows from (3.9) that

$$\begin{aligned} \|u_{\lambda^*}\| &\leq (\lambda^*)^{-1} \tfrac{\overline{G}}{\underline{G}\|a\|_1} \|\omega\|_1 \|u_{\lambda^*}\|^\gamma \\ &\leq (\lambda^*)^{\frac{1}{\gamma-1}} \tfrac{\overline{G}}{\|a\|_1} \left(\|\omega\|_1 \underline{G}^{2\gamma-1} \sigma^{\gamma^2} \right)^{\frac{1}{1-\gamma}}. \end{aligned} \tag{3.10}$$

Thus, when $\gamma > 1$, we have

$$\lim_{\lambda^*\to 0^+} \|u_{\lambda^*}\| = 0.$$

Similarly, when $0 < \gamma < 1$, we get from (3.10) that

$$\lim_{\lambda^* \to +\infty} \|u_{\lambda^*}\| = 0.$$

Let $\lambda = \frac{1}{\lambda^*}$. Thus, we finish the proof of (i) and (ii) in Theorem 1.3.

Next, we give the proof of (iii) in Theorem 1.3. Assume u is a positive solution for problem (1.1) with (1.2). We will prove that this leads to a contradiction for $0 < \lambda < \lambda_*$, where

$$\lambda_* = \frac{\underline{G}\|a\|_1}{\overline{G}\|\omega\|_1\|u\|^\gamma}.$$

Since $(Tu)(t, x) = \frac{1}{\lambda}u(t, x)$ for $x \in E$, it follows from (3.2) that

$$\begin{aligned} \|u\| &\le \lambda \frac{\overline{G}}{\underline{G}\|a\|_1}\|\omega\|_1\|u\|^\gamma \\ &< \lambda_* \frac{\overline{G}}{\underline{G}\|a\|_1}\|\omega\|_1\|u\|^\gamma \\ &= \|u\|^\gamma = \|u\|. \end{aligned}$$

This is a contradiction, and our proof is finished. □

Remark 3.1. The approach to prove Theorem 1.3 can be applied to the more general problem (1.8) with (1.2).

Remark 3.2. Although we also use the eigenvalue theory to study problem (1.8) with (1.2), since the nonlinear term is in a general form, some new techniques are needed. For detail to see the proof of Theorems 3.1–3.3.

To consider the existence and asymptotic behavior of positive continuous solutions for eigenvalue problem (1.8) with (1.2), we need to introduce the following notations:

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u^\gamma}, \ \ f_\infty = \lim_{u \to +\infty} \frac{f(u)}{u^\gamma}, \ \ \mathbb{R}^+ = [0, \infty),$$

where $\gamma > 0$.

Theorem 3.3. *Suppose that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. If* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *is continuous and*

$$f_0 = 0, \ f_\infty = \infty,$$

then for $\gamma > 0$ *and* $\gamma \neq 1$, *we get the following conclusions:*

(i) If $\gamma > 1$, *then for any* $\lambda > 0$, *problem* (1.8) *with* (1.2) *admits a positive doubly periodic solution* u_λ *with* $u_\lambda \not\equiv 0$, *and*

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = +\infty, \tag{3.11}$$

$$\lim_{\lambda \to +\infty} \|u_\lambda\| = 0. \tag{3.12}$$

(ii) If $0 < \gamma < 1$, *then for any* $\lambda > 0$, *problem* (1.8) *with* (1.2) *admits a positive doubly periodic solution* u_λ *with* $u_\lambda \not\equiv 0$, *and*

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0. \tag{3.13}$$

Proof. Let $T^* : K \to E$ be defined by

$$(T^*u)(t,x) = P(\omega(t,x)f(u(t,x)),\ \forall u \in E. \tag{3.14}$$

It follows from Lemma 2.3 that $T^* : K \to E$ is completely continuous, and the doubly periodic solution of (1.8) with (1.2) is equivalent to the fixed point of λT^*.

Similarly, one can verify that T^* is completely continuous from K^* to K^*, where K^* is defined in (3.1).

Let $u \in K$ and $h^*(t,x) = \omega(t,x)f(u(t,x))$ for $(t,x) \in \mathbb{R}^2$. Then $h^* \in E$ and $T^*u = Ph^*$, and it follows from the proof of Lemma 2 in [9] that

$$(Ph^*)(t,x) \geq (R_\mu h^*)(t,x). \tag{3.15}$$

Considering $f_\infty = \infty$, there exists $r_1 > 0$ such that $f(u) \geq \varepsilon_1 u^\gamma$, for $u \geq r_1$, where $\varepsilon_1 > 0$ satisfies

$$\underline{G}\varepsilon_1 \|\omega\|_1 \sigma^\gamma \geq 1.$$

Let $\partial K^*_{r_1} = \{u \in K^* : \|u\| = r_1\}$. Then, for $x \in K^*_{r_1}$, we get from (3.14) and (3.15) that

$$\begin{aligned} \|T^*u\| &= \|Ph^*\| \\ &\geq \|R_\mu h^*\| \\ &\geq \underline{G}\|h^*\|_1 \\ &\geq \underline{G}\varepsilon_1(\sigma\|u\|)^\gamma \|\omega\|_1 \\ &\geq \|u\|^\gamma. \end{aligned} \tag{3.16}$$

We so have the following two conclusions:

1) $\gamma > 1$

Since $\gamma > 1$, we get

$$\lim_{\|u\| \to +\infty} \frac{\|T^*u\|}{\|u\|} = +\infty. \tag{3.17}$$

2) $0 < \gamma < 1$

Since $0 < \gamma < 1$, we obtain

$$\lim_{\|u\| \to +\infty} \frac{\|T^*u\|}{\|u\|} = 0. \tag{3.18}$$

Next, turning to $f^0 = 0$, there exists $r_2 : 0 < r_2 < r_1$ so that $f(u) \leq \varepsilon_2 u^\gamma$, for $0 \leq u \leq r_2$, where $\varepsilon_2 > 0$ satisfies

$$\frac{\overline{G}}{\underline{G}\|a\|_1}\|\omega\|_1 \varepsilon_2 \leq 1.$$

Thus, for $u \in \partial K^*_{r_2}$, we have from (3.14)

$$\begin{aligned} \|T^*u\| &= \|Ph^*\| \\ &\leq \tfrac{\overline{G}}{\underline{G}\|a\|_1}\|h^*\|_1 \\ &\leq \tfrac{\overline{G}}{\underline{G}\|a\|_1}\|\omega\|_1 \varepsilon_2 \|u\|^\gamma \\ &\leq \|u\|^\gamma. \end{aligned} \tag{3.19}$$

We hence obtain from (3.19) the following two conclusions:

1) $\gamma > 1$

Since $\gamma > 1$, we get

$$\lim_{\|u\|\to 0} \frac{\|T^* u\|}{\|u\|} = 0. \tag{3.20}$$

2) $0 < \gamma < 1$

Since $0 < \gamma < 1$, we get

$$\lim_{\|u\|\to 0} \frac{\|T^* u\|}{\|u\|} = +\infty. \tag{3.21}$$

Thus, observing $\gamma > 1$, (3.17), (3.20) and Lemma 3.1, we find that: for any $\bar{\lambda} > 0$, there exists $u_{\bar{\lambda}} \in E$ with $u_{\bar{\lambda}} \neq \theta$ such that $T^* u_{\bar{\lambda}} = \bar{\lambda} u_{\bar{\lambda}}$; and

$$\lim_{\bar{\lambda}\to +\infty} \|u_{\bar{\lambda}}\| = +\infty. \tag{3.22}$$

Moreover, it follows from (3.16) that

$$\bar{\lambda}\|u_{\bar{\lambda}}\| = \|T^* u_{\bar{\lambda}}\| \geq \|u_{\bar{\lambda}}\|^{\gamma},$$

which shows that

$$\|u_{\bar{\lambda}}\| \leq (\bar{\lambda})^{\frac{1}{\gamma-1}}. \tag{3.23}$$

This proves that

$$\lim_{\bar{\lambda}\to 0^+} \|u_{\bar{\lambda}}\| = 0$$

when $\gamma > 1$.

When considering $0 < \gamma < 1$, then from (3.18) and (3.21), together with Lemma 3.2, we obtain that: for any $\bar{\lambda} > 0$, there exists $u_{\bar{\lambda}} \in E$ with $u_{\bar{\lambda}} \neq \theta$ such that $T^* u_{\bar{\lambda}} = \bar{\lambda} u_{\bar{\lambda}}$; and $\lim\limits_{\bar{\lambda}\to\infty} u_{\bar{\lambda}} = 0$.

Let $\lambda = \frac{1}{\bar{\lambda}}$. Then the proof of Theorem 3.3 is completed. □

In Theorem 3.3, we consider the existence of positive solution for (1.8) with (1.2) in the case $\gamma > 1$ and $0 < \gamma < 1$. Next we discuss what happen in the case $\gamma = 1$?

In fact, we will obtain two nonexistence results when $\gamma = 1$.

Theorem 3.4. *Suppose that* ($\mathbf{H_1}$), ($\mathbf{H_2}$)*,* $\gamma = 1$ *and* $f_0 = 0$*. Then problem* (1.8) *with* (1.2) *possesses no positive doubly periodic solution for* $0 < \lambda < 1$.

Proof. Let u be a positive solutions to problem (1.8) with (1.2). We next show that this leads to a contradiction for $0 < \lambda < 1$. Since $(T^* u)(t, x) = \frac{1}{\lambda} u(t, x)$ for $(t, x) \in \mathbb{T}^2$, it follows from (3.19) that

$$\|u\| \leq \lambda \|u\|^{\gamma} < \|u\|^{\gamma} = \|u\|,$$

which is a contradiction, and our proof is completed. □

Theorem 3.5. *Suppose that* $(\mathbf{H_1})$, $(\mathbf{H_2})$, $\gamma = 1$ *and* $f_\infty = 0$. *Then problem* (1.8) *with* (1.2) *possesses no positive doubly periodic solution for* $\lambda > 1$.

Proof. Let u be a positive solutions to problem (1.8) with (1.2). We next show that this leads to a contradiction for $\lambda > 1$. Since $(T^*u)(t,x) = \frac{1}{\lambda}u(t,x)$ for $(t,x) \in \mathbb{T}^2$, it follows from (3.16) that

$$\|u\| \geq \lambda\|u\|^\gamma > \|u\|^\gamma = \|u\|.$$

This is a contradiction, and our proof is finished. □

4. Proof of Theorem 1.4

In this section, we consider the multiplicity of positive solutions for problem (1.8) with (1.2) by using a completely different method from that of Ortega-Robles-Pérez [7], Mawhin-Ortega-Robles-Pérez [4–6], Li [3], Gilding-Kersner [8], Li [9], and Wang and An [10, 11], namely the following lemma about fixed point index in a cone, which is used in Hu and Wang [21] and Zhang [22].

Lemma 4.1. *([25]) Let E be a real Banach space and K be a cone in E. For $r > 0$, define $K_r = \{x \in K : \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{x \in K : \|x\| = r\}$.*
(i) If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.
(ii) If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.

Let $T_\lambda : K \to E$ be defined by

$$(T_\lambda u)(t,x) = P(\lambda\omega(t,x)f(u(t,x))), \ \forall u \in E, \tag{4.1}$$

where K is defined in (2.5) and $E := C(\mathbb{T}^2)$.

It follows from Lemma 2.3 that $T_\lambda : K \to E$ is completely continuous, and the doubly periodic solution of (1.8) with (1.2) is equivalent to the fixed point of T_λ.

Proof of Theorem 1.4. Let $u \in K$ and

$$h_\lambda(t,x) = \lambda\omega(t,x)f(u(t,x)) \tag{4.2}$$

for $(t,x) \in \mathbb{R}^2$. Then $h_\lambda \in E$ and $T_\lambda u = Ph_\lambda$, and it follows from the proof of Lemma 2 in [9] that

$$(Ph_\lambda)(t,x) \geq (R_\mu h_\lambda)(t,x), \ \forall (t,x) \in C(\mathbb{T}^2). \tag{4.3}$$

For any given $\tau > 0$, it follows from $(\mathbf{H_5})$ that there exist $\eta > 0$ and $d > \tau$ such that

$$f(u) \geq \eta \quad \text{for } u \geq d. \tag{4.4}$$

Letting $\delta = \frac{d}{\underline{G}\eta\|\omega\|_1}$, then for $\lambda > \delta$, (4.1) and Lemma 2.3 imply that $T_\lambda : K \to K$ is completely continuous.

Considering $(\mathbf{H_3})$, there exists $0 < r < d$ such that $f(u) \leq \frac{\Lambda}{2}u$ for $0 \leq u \leq r$, where

$$\Lambda = \frac{\underline{G}\|a\|_1}{\overline{G}\lambda\|\omega\|_1}.$$

So, for $u \in \partial K_r = \{u \in K : \|u\| = r\}$, we have from (4.1) and (4.2) that

$$\|T_\lambda u\| = \|Ph_\lambda\| \le \frac{\overline{G}}{\underline{G}\|a\|_1}\|h\|_1 \le \frac{\lambda\overline{G}}{\underline{G}\|a\|_1}\|\omega\|_1\frac{\Lambda}{2}\|u\| < \|u\|. \tag{4.5}$$

Consequently, for $u \in \partial K_r$, we have $\|T_\lambda u\| < \|u\|$. Thus, by Lemma 4.1, we get

$$i(T_\lambda, K_r, K) = 1. \tag{4.6}$$

Now turning to ($\mathbf{H_4}$), there exists $\sigma > 0$, for $u > \sigma$, such that $f(u) \le \frac{\Lambda}{2}u$. Letting $\eta = \max_{0\le u\le\sigma} f(u)$, then

$$0 \le f(u) \le \frac{\Lambda}{2}u + \eta. \tag{4.7}$$

Choosing

$$R > \max\left\{d, \frac{2\lambda\overline{G}}{\underline{G}\|a\|_1}\|\omega\|_1\eta\right\}. \tag{4.8}$$

So for $x \in \partial K_R$, from (4.1) and (4.2) we have

$$\begin{aligned}\|T_\lambda u\| &= \|Ph_\lambda\| \\ &\le \tfrac{\overline{G}}{\underline{G}\|a\|_1}\|h_\lambda\|_1 \\ &\le \tfrac{\lambda\overline{G}}{\underline{G}\|a\|_1}\|\omega\|_1(\tfrac{\Lambda}{2}\|u\| + \eta) \\ &< \tfrac{\|u\|}{2} + \tfrac{R}{2} = \|u\|.\end{aligned} \tag{4.9}$$

Thus, it follows from Lemma 4.1 that

$$i(T_\lambda, K_R, K) = 1. \tag{4.10}$$

On the other hand, for $u \in \bar{K}_d^R = \{u \in K : \|u\| \le R, \min_{(t,x)\in\mathbb{T}^2} u(t,x) \ge d\}$, (4.1), (4.2), (4.7) and (4.8) yield that

$$\|T_\lambda u\| = \|Ph_\lambda\| \le \frac{\overline{G}}{\underline{G}\|a\|_1}\|h_\lambda\|_1 \le \frac{\lambda\overline{G}}{\underline{G}\|a\|_1}\|\omega\|_1(\frac{\Lambda}{2}\|u\| + \eta) < R.$$

Furthermore, for $u \in \bar{K}_d^R$, from (4.1), (4.2), (4.3), and ($\mathbf{H_5}$), we obtain

$$\begin{aligned}\min_{(t,x)\in C(\mathbb{T}^2)}(Tu)(t,x) = \min_{(t,x)\in C(\mathbb{T}^2)}(Ph_\lambda)(t,x) &\ge (R_\mu h_\lambda)(t,x) \\ &= \lambda \int_{\mathbb{T}^2} G(t-s, x-y)\omega(s,y)f(u(s,y))dsdy \\ &\ge \lambda\underline{G}\eta \int_{\mathbb{T}^2}\omega(s,y)dsdy \\ &> \delta\underline{G}\eta\|\omega\|_1 \\ &= d.\end{aligned}$$

Letting $u_0 \equiv \frac{d+R}{2}$ and $H(x,u) = (1-x)T_\lambda u + xu_0$, then $H : [0,1]\times\bar{K}_d^R \to K$ is completely continuous, and from the analysis above, we obtain for $(x,u) \in [0,1]\times\bar{K}_d^R$

$$H(x,u) \in K_d^R. \tag{4.11}$$

Therefore, for $x \in [0,1], u \in \partial K_d^R$, we have $H(x,u) \neq u$. Hence, by the normality property and the homotopy invariance property of the fixed point index, we obtain

$$i(T_\lambda, K_d^R, K) = i(u_0, K_d^R, K) = 1. \tag{4.12}$$

Consequently, by the solution property of the fixed point index, T_λ admits a fixed point $u_\lambda^{(1)}$ with $u_\lambda^{(1)} \in K_d^R$, and

$$\max_{(t,x)\in\mathbb{T}^2} u_\lambda^{(1)}(t,x) \geq \min_{(t,x)\in\mathbb{T}^2} u_\lambda^{(1)}(t,x) > d > \tau.$$

On the other hand, it follows from (4.6), (4.10) and (4.12) together with the additivity of the fixed point index that

$$\begin{aligned} & i(T_\lambda, K_R \backslash (\bar{K}_r \cup \bar{K}_d^R), K) \\ &= i(T_\lambda, K_R, K) - i(T_\lambda, K_d^R, K) - i(T_\lambda, K_r, K) \\ &= 1 - 1 - 1 = -1. \end{aligned} \tag{4.13}$$

By the solution property of the fixed point index, T_λ so admits a fixed point $u_\lambda^{(2)}$ and $u_\lambda^{(2)} \in K_R \backslash (\bar{K}_r \cup \bar{K}_d^R)$. It is not difficult to see that $u_\lambda^{(1)} \neq u_\lambda^{(2)}$. This gives the proof of Theorem 1.4. □

5. Proof of Theorem 1.5

In this section, we intend to analyze the uniqueness and continuity of positive solution on the parameter λ to problem (1.8) with (1.2). In order to prove Theorem 1.5, we need the following results and some definitions, which can be found in Guo-Lakshmikantham [27].

Let E be a real Banach space, K is a cone of E. Every cone $K \subset E$ induces an semi-order in E given by "$\leq$". That is, $x \leq y$ if and only if $y - x \in K$. If cone K is solid and $y - x \in K^\circ$, we write $x \ll y$.

Definition 5.1. Let K be a cone of a real Banach space E. K is a solid cone, if K° is not empty, where K° is the interior of K.

Definition 5.2. Let K be a solid cone of a real Banach space E. $A : K^\circ \to K^\circ$ is an operator. A is called an α-concave operator ($-\alpha$-convex operator), if

$$A(tx) \geq t^\alpha Ax \ (A(tx) \leq t^{-\alpha} Ax), \ \forall x \in K^\circ, \ 0 < t < 1,$$

where $0 \leq \alpha < 1$. The operator A is increasing (decreasing), if $x_1, x_2 \in K^\circ$ and $x_1 \leq x_2$ imply $Ax_1 \leq Ax_2$ ($Ax_1 \geq Ax_2$), and further, the operator A is strong increasing (decreasing), if $x_1, x_2 \in K^\circ$ and $x_1 < x_2$ imply $Ax_2 - Ax_1 \in K^\circ (Ax_1 - Ax_2 \in K^\circ)$. Let x_λ be a proper element of the enginvalve λ of A, that is $Ax_\lambda = \lambda x_\lambda$. x_λ is called strong increasing (decreasing), if $\lambda_1 > \lambda_2$ implies that $x_{\lambda_1} - x_{\lambda_2} \in K^\circ (x_{\lambda_2} - x_{\lambda_1} \in K^\circ)$, which is denoted by $x_{\lambda_1} \gg x_{\lambda_2} (x_{\lambda_2} \gg x_{\lambda_1})$.

Lemma 5.1. *Suppose that P is a normal cone of a real Banach space, $A : P^\circ \to P^\circ$ is an $\alpha-$concave increasing (or $-\alpha-$convex decreasing) operator. Then A has exactly one fixed point in P°.*

Proof of Theorem 1.5. Let $\Psi = \lambda A$, where $A : K \to E$ be defined by

$$(Au)(t,x) = P(\omega(t,x) f(u(t,x))), \ \forall u \in E, \tag{5.1}$$

K is defined in (2.5) and $E := C(\mathbb{T}^2)$. In fact, one can prove that the operator Ψ maps K into K. In view of $G(t,s) > 0$, $(\mathbf{H_2})$ and $f(u) > 0$ for $u > 0$, it is easy to see that $\Psi: K^0 \to K^0$. We prove that $\Psi: K^0 \to K^0$ is an α-concave increasing operator.

In fact, it follows from (5.1), Lemma 2.3 and $f(\rho u) \geq \rho^\alpha f(u)$ that

$$\begin{aligned}\Psi(\rho u) &= \lambda P(\omega(t,x) f(\rho u(t,x)))\\ &\geq \rho^\alpha \lambda P(\omega(t,x) f(u(t,x)))\\ &= \rho^\alpha \Psi(u), \quad \forall 0 < \rho < 1,\end{aligned}$$

where $0 \leq \alpha < 1$. Since $f(u)$ is nondecreasing, then

$$\begin{aligned}(\Psi u_*)(t,x) &= \lambda P(\omega(t,x) f(u_*(t,x)))\\ &\leq \lambda P(\omega(t,x) f(u_{**}(t,x)))\\ &= (\Psi u_{**})(t,x) \ \ for\ u_* \leq u_{**}, \quad u_*, u_{**} \in E.\end{aligned}$$

Lemma 5.1 yields that Ψ admits a unique fixed point $u_\lambda \in K^0$. This shows that problem (1.8) with (1.2) admits a unique positive solution u_λ.

Next, we prove that (i)-(iii) hold. Let $\gamma = \frac{1}{\lambda}$, and denote $\lambda A u_\lambda = u_\lambda$ by $A u_\gamma = \gamma u_\gamma$. Assume $0 < \gamma_1 < \gamma_2$. Then $u_{\gamma_1} \geq u_{\gamma_2}$. In fact, set

$$\bar{\eta} = \sup\Big\{\eta : \ u_{\gamma_1} \geq \eta u_{\gamma_2}\Big\}. \tag{5.2}$$

We prove $\bar{\eta} \geq 1$. If it is not, then $0 < \bar{\eta} < 1$, and further

$$\gamma_1 u_{\gamma_1} = A u_{\gamma_1} \geq A(\bar{\eta} u_{\gamma_2}) \geq \bar{\eta}^\alpha A u_{\gamma_2} = \bar{\eta}^\alpha \gamma_2 u_{\gamma_2}.$$

This shows

$$u_{\gamma_1} \geq \bar{\eta}^\alpha \frac{\gamma_2}{\gamma_1} u_{\gamma_2} \gg \bar{\eta}^\alpha u_{\gamma_2} \gg \bar{\eta} u_{\gamma_2},$$

which is a contradiction to (5.2).

It follows from the discussion above that

$$u_{\gamma_1} = \frac{1}{\gamma_1} A u_{\gamma_1} \geq \frac{1}{\gamma_1} A u_{\gamma_2} = \frac{\gamma_2}{\gamma_1} u_{\gamma_2} \gg u_{\gamma_2}. \tag{5.3}$$

This proves that $u_\gamma(t)$ is strong decreasing in γ. Namely $u_\lambda(t)$ is strong increasing in λ. This gives the proof of Theorem 1.5 (i).

Let $\gamma_2 = \gamma$ and fix γ_1 in (5.3), we get $u_{\gamma_1} \geq \frac{\gamma}{\gamma_1} u_\gamma$, for $\gamma > \gamma_1$. Further

$$\|u_\gamma\| \leq \frac{\gamma_1 N_1}{\gamma} \|u_{\gamma_1}\|, \tag{5.4}$$

where $N_1 > 0$ denotes a normal constant. Noting that $\gamma = \frac{1}{\lambda}$, we get $\lim\limits_{\lambda \to 0^+} \|u_\lambda(t)\| = 0$.

Similarly, set $\gamma_1 = \gamma$ and fix γ_2, again by (5.3) and the normality of K, we obtain $\lim\limits_{\lambda \to +\infty} \|u_\lambda(t)\| = +\infty$. This finishes the proof of (ii).

Finally, we prove the continuity of $u_\gamma(t)$. For given $\gamma_0 > 0$. From (i),

$$u_\gamma \ll u_{\gamma_0} \ \ for\ any\ \gamma > \gamma_0. \tag{5.5}$$

Set

$$l_\gamma = \sup\{\nu > 0 \ | \ u_\gamma \geq \nu u_{\gamma_0}, \ \gamma > \gamma_0\}.$$

It is clear to see that $0 < l_\gamma < 1$ and $u_\gamma \geq l_\gamma u_{\gamma_0}$. We so have

$$\gamma u_\gamma = Au_\gamma \geq A(l_\gamma u_{\gamma_0}) \geq l_\gamma^\alpha A u_{\gamma_0} = l_\gamma^\alpha \gamma_0 u_{\gamma_0},$$

and further

$$u_\gamma \geq \frac{\gamma_0}{\gamma} l_\gamma^\alpha u_{\gamma_0}.$$

It follows the definition of l_γ that

$$\frac{\gamma_0}{\gamma} l_\gamma^\alpha \leq l_\gamma \ \text{ or } \ l_\gamma \geq (\frac{\gamma_0}{\gamma})^{\frac{1}{1-\alpha}}.$$

Again from the definition of l_γ, we obtain

$$u_\gamma \geq (\frac{\gamma_0}{\gamma})^{\frac{1}{1-\alpha}} u_{\gamma_0} \ \text{ for any } \gamma > \gamma_0. \tag{5.6}$$

Noticing that K is a normal cone, it follows from (5.5) and (5.6) that

$$\|u_{\gamma_0} - u_\gamma\| \leq N_2[1 - (\frac{\gamma_0}{\gamma})^{\frac{1}{1-\alpha}}]\|u_{\gamma_0}\| \to 0, \ \ \gamma \to \gamma_0 + 0.$$

Similarly, we have

$$\|u_\gamma - u_{\gamma_0}\| \to 0, \ \ \gamma \to \gamma_0 - 0.$$

where $N_2 > 0$ denotes a normal constant. This shows that Theorem 1.5 (iii) holds. The proof of Theorem 1.5 is complete. □

Remark 5.1. The idea of the proof for Theorem 1.5 comes from Theorem 2.2.7 in Guo-Lakshmikantham [27], but there is almost no paper studying the uniqueness of positive doubly periodic solution for telegraph equations.

Remark 5.2. In Theorem 1.5, even though we do not suppose that A is continuous even completely continuous, we can prove that u_λ depends continuously on λ.

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Conflict of interest

The authors declare there is no conflicts of interest.

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