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Research article

Abelian extensions of Lie triple systems with derivations

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Abstract: Let $\mathfrak{L}$ and A be Lie triple systems, and let θ_A be a representation of $\mathfrak{L}$ on A. We first construct the third-order cohomology classes by derivations of A and $\mathfrak{L}$, then obtain a Lie algebra G_{θ_A} with a representation Φ on $H^3(\mathfrak{L}, A)$, where θ_A is given by an abelian extension

$$0 \longrightarrow A \longrightarrow \tilde{\mathfrak{L}} \xrightarrow{\pi} \mathfrak{L} \longrightarrow 0 .$$

We study obstruction classes for extensibility of derivations of A and $\mathfrak{L}$ to those of $\tilde{\mathfrak{L}}$. An application of Φ is discussed.

Keywords: Lie triple system; cohomology; derivation; abelian extension

1. Introduction

Lie triple systems were introduced by E. Cartan in his studies on Riemannian geometry. Since then, it has been studied by many scholars. For example, Jacobson studied Lie triple systems by an algebraic method in [1, 2]. The cohomology theories have been given by Yamaguti in [3] and some problems about cohomology have been solved, see [4–6]. Note that Lie triple systems are closely related to Lie algebras, so it is natural to generalize some properties of Lie algebras to Lie triple systems.

Automorphisms are very important subjects in the research of algebras. In [7], the authors studied extension of a pair of automorphisms of Lie algebras, and they gave a necessary and sufficient condition for a pair of automorphisms to be extensible. Since derivations are infinitesimals of automorphisms, it is interesting to study extension of a pair of derivations. Recently, extension of a pair of derivations on Lie algebras, 3-Lie algebras, Leibniz algebras and associative algebras have been studied, refer to [8–11]. We attempt to consider the same problems on Lie triple systems. Inspired by [11], we define a Lie algebra G_{θ_A}, where θ_A is a representation of a Lie triple system $\mathfrak{L}$ on a Lie triple system A, using compatible derivations of $\mathfrak{L}$ and A. Then we show that a certain representation of G_{θ_A} can characterize the extensibility of above compatible derivations.

This paper is organized as follows. In Section 2, we recall some basic definitions and properties of Lie triple systems. Then we construct a Lie algebra G_{θ_A} and consider its representation on $H^3(\mathfrak{L}, A)$ in Section 3. In Section 4, we investigate the abelian extension and extensibility of a pair of derivations. We prove that a compatible pair (∂_a, ∂_l) is extensible if and only if $[\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(\partial_a,\partial_l)}] \in H^3(\mathfrak{L}, A)$ is trivial and (∂_a, ∂_l) is extensible in every reversible extension if and only if $\Phi(\partial_a, \partial_l) = 0$.

In this paper, all Lie triple systems $\mathfrak{L}$ are defined over a fixed but arbitrary field $\mathbb{F}$.

2. Cohomology of Lie triple systems

In this section, we first recall some basic definitions and properties of Lie triple systems, then we show that $H^3(\mathfrak{L}, A) = 0$ for an abelian extension implies split property.

Definition 2.1. [2] A Lie triple system is a vector space $\mathfrak{L}$ endowed with a trilinear operation $[\cdot,\cdot,\cdot] : \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L}$ satisfying

$$\begin{aligned}
&[a, a, b] = 0,\\
&[a, b, c] + [b, c, a] + [c, a, b] = 0,\\
&[a, b, [c, d, e]] = [[a, b, c], d, e] + [c, [a, b, d], e] + [c, d, [a, b, e]],
\end{aligned}$$

for all $a, b, c, d, e \in \mathfrak{L}$.

For a Lie triple system $\mathfrak{L}$, a linear map $\partial : \mathfrak{L} \longrightarrow \mathfrak{L}$ is called a derivation of $\mathfrak{L}$, if for all $a, b, c \in \mathfrak{L}$,

$$\partial([a, b, c]) = [\partial(a), b, c] + [a, \partial(b), c] + [a, b, \partial(c)].$$

Denote by $\mathrm{Der}(\mathfrak{L})$ the space of derivations of $\mathfrak{L}$.

Definition 2.2. [3] An $\mathfrak{L}$-module is a vector space V with a bilinear map

$$\begin{aligned}
\theta_V : \mathfrak{L} \times \mathfrak{L} &\longrightarrow \mathrm{End}(V)\\
(a, b) &\longrightarrow \theta_V(a, b)
\end{aligned}$$

such that the following conditions hold:

$$\begin{aligned}
&\theta_V(c, d)\theta_V(a, b) - \theta_V(b, d)\theta_V(a, c) - \theta_V(a, [b, c, d]) + D_V(b, c)\theta_V(a, d) = 0,\\
&\theta_V(c, d)D_V(a, b) - D_V(a, b)\theta_V(c, d) + \theta_V([a, b, c], d) + \theta_V(c, [a, b, d]) = 0,
\end{aligned}$$

where $D_V(a, b) = \theta_V(b, a) - \theta_V(a, b)$, for all $a, b, c, d \in \mathfrak{L}$. Also θ_V is called a representation of $\mathfrak{L}$ on V.

Definition 2.3. [3] Let θ_V be a representation of a Lie triple system $\mathfrak{L}$. An n-linear map $f : \underbrace{\mathfrak{L} \times \cdots \times \mathfrak{L}}_{n \text{ times}} \to V$ satisfying

$$f(x_1, x_2, ..., x_{n-3}, x, x, x_n) = 0$$

and

$$f(x_1, x_2, ..., x_{n-3}, x, y, z) + f(x_1, x_2, ..., x_{n-3}, y, z, x) + f(x_1, x_2, ..., x_{n-3}, z, x, y) = 0$$

is called an n-cochain of $\mathfrak{L}$ on V. Denote by $C^n(\mathfrak{L}, V)$ the set of all n-cochains, for $n \geq 0$.

Definition 2.4. [3] Let $\mathfrak{L}$ be a Lie triple system and θ_V a representation of $\mathfrak{L}$ on V. The coboundary operator $\delta : C^{2n-1}(\mathfrak{L}, V) \to C^{2n+1}(\mathfrak{L}, V)$ is given by

$$\begin{aligned}
&(\delta f)(x_1, x_2, ..., x_{2n+1}) \\
&= \theta_V(x_{2n}, x_{2n+1}) f(x_1, x_2, ..., x_{2n-1}) - \theta_V(x_{2n-1}, x_{2n+1}) f(x_1, x_2, ..., x_{2n}) \\
&\quad + \sum_{i=1}^{n} (-1)^{i+1} D_V(x_{2i-1}, x_{2i}) f(x_1 ..., x_{2i-2}, x_{2i+1}, ..., x_{2n+1}) \\
&\quad + \sum_{i=1}^{n} \sum_{j=2i+1}^{2n+1} (-1)^{i+n+1} f(x_1 ..., x_{2i-2}, x_{2i+1}, ..., x_j, [x_{2i-1}, x_{2i}, x_j], ..., x_{2n+1})
\end{aligned} \tag{2.1}$$

for any $x_1, x_2, ..., x_{2n+1} \in \mathfrak{L}$, $f \in C^{2n-1}(\mathfrak{L}, V)$.

Let $\mathfrak{L}$ be a Lie triple system and V an $\mathfrak{L}$-module. The set

$$Z^{2n+1}(\mathfrak{L}, V) = \{f \in C^{2n+1}(\mathfrak{L}, V) \mid \delta f = 0\}$$

is called the space of $(2n + 1)$-cocycles of $\mathfrak{L}$ on V.

The set

$$B^{2n+1}(\mathfrak{L}, V) = \{\delta f \mid f \in C^{2n-1}(\mathfrak{L}, V)\}$$

is called the space of $(2n + 1)$-coboundaries of $\mathfrak{L}$ on V.

The n-th cohomology group is

$$H^n(\mathfrak{L}, V) := Z^n(\mathfrak{L}, V)/B^n(\mathfrak{L}, V).$$

Note that, by Eq 2.1, for $f \in C^3(\mathfrak{L}, V)$ we have

$$\begin{aligned}
\delta f(x_1, x_2, x_3, x_4, x_5) = {} & \theta_V(x_4, x_5) f(x_1, x_2, x_3) - \theta_V(x_3, x_5) f(x_1, x_2, x_4) \\
& + D_V(x_1, x_2) f(x_3, x_4, x_5) - D_V(x_3, x_4) f(x_1, x_2, x_5) \\
& + f([x_1, x_2, x_3], x_4, x_5) + f(x_3, [x_1, x_2, x_4], x_5) \\
& + f(x_3, x_4, [x_1, x_2, x_5]) - f(x_1, x_2, [x_3, x_4, x_5]).
\end{aligned} \tag{2.2}$$

Definition 2.5. [6] Suppose that $\mathfrak{L}$ and A are Lie triple systems. If

$$0 \longrightarrow A \longrightarrow \tilde{\mathfrak{L}} \xrightarrow{\pi} \mathfrak{L} \longrightarrow 0$$

is an exact sequence of Lie triple systems, and $[\tilde{\mathfrak{L}}, A, A] = 0$ (which implies that $[A, \tilde{\mathfrak{L}}, A] = [A, A, \tilde{\mathfrak{L}}] = 0$), then we call $\tilde{\mathfrak{L}}$ an abelian extension of $\mathfrak{L}$ by A. A linear map $s : \mathfrak{L} \to \tilde{\mathfrak{L}}$ is called a section of $\tilde{\mathfrak{L}}$ if it satisfies $\pi \circ s = \mathrm{id}_{\mathfrak{L}}$. If there exists a section which is also a homomorphism between Lie triple systems, we say that the abelian extension is split.

The following properties have been proved in [6], which will be used in Sections 3 and 4.

Let $\tilde{\mathfrak{L}}$ be an abelian extension of $\mathfrak{L}$ by A. We construct a representation of $\mathfrak{L}$ on A. Fix any section $s : \mathfrak{L} \longrightarrow \tilde{\mathfrak{L}}$ of π and define $\theta_A : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathrm{End}(A)$ by

$$\theta_A(x, y)(v) = [v, s(x), s(y)]_{\tilde{\mathfrak{L}}}, \tag{2.3}$$

for all $x, y \in \mathfrak{L}$, $v \in A$. In particular, $D_A(x, y)(v) = (\theta_A(y, x) - \theta_A(x, y))(v) = [s(x), s(y), v]_{\tilde{\mathfrak{L}}}$. Note that θ_A is independent on the choice of s. Moreover, since

$$[s(x), s(y), s(z)]_{\tilde{\mathfrak{L}}} - s([x, y, z]_{\mathfrak{L}}) \in A,$$

for any $x, y, z \in \mathfrak{L}$, we have a map $\omega : \mathfrak{L} \times \mathfrak{L} \times \mathfrak{L} \longrightarrow A$ given by

$$\omega(x, y, z) = [s(x), s(y), s(z)]_{\tilde{\mathfrak{L}}} - s([x, y, z]_{\mathfrak{L}}) \in A, \tag{2.4}$$

for all $x, y, z \in \mathfrak{L}$.

Lemma 2.6. *Let $\tilde{\mathfrak{L}}$ be an abelian extension of a Lie triple system $\mathfrak{L}$ by A. Then*
(1) *θ_A given by Eq* 2.3 *is a representation of $\mathfrak{L}$ on A*;
(2) *ω given by Eq* 2.4 *is a* 3*-cocycle associated to θ_A.*

Corollary 2.7. *Let $\tilde{\mathfrak{L}}$ be an abelian extension of a Lie triple system $\mathfrak{L}$ by A. Keep the same notations as in Lemma* 2.6*. Then the cohomology class $[\omega]$ does not depend on the choice of s.*

Proposition 2.8. *If θ_A is a representation of $\mathfrak{L}$ on A and ω is a* 3*-cocycle i.e., $\delta\omega = 0$ in Eq* 2.2 *then $\mathfrak{L} \oplus A$ is a Lie triple system with the bracket given by*

$$[x + u, y + v, z + w]_{\mathfrak{L}\oplus A} = [x, y, z]_{\mathfrak{L}} + \omega(x, y, z) + D_A(x, y)(w) + \theta_A(y, z)(u) - \theta_A(x, z)(v),$$

where $x, y, z \in \mathfrak{L}$, $u, v, w \in A$.

Corollary 2.9. *Retain all the notations and assumptions in Proposition* 2.8*. Let $\pi : \mathfrak{L} \oplus A \longrightarrow \mathfrak{L}$ be the canonical projection. Then there is an abelian extension $\tilde{\mathfrak{L}} = \mathfrak{L} \oplus A$ of Lie triple systems $\mathfrak{L}$ by A.*

Based on the previous notations, we have the following

Proposition 2.10. *Let θ_A be a representation of a Lie triple system $\mathfrak{L}$ on a Lie triple system A. If $H^3(\mathfrak{L}, A) = 0$ then any abelian extension of $\mathfrak{L}$ by A is split.*

Proof. It suffices to show that there is a section s of π which is a homomorphism. Recall that θ_A given by Eq 2.3 is independent on the choice of s. Consider the 3-cocycle ω given by Lemma 2.6, since $H^3(\mathfrak{L}, A) = 0$, there exists an $\alpha \in C^1(\mathfrak{L}, A)$ such that $\omega = \delta\alpha$. For any $x, y, z \in \mathfrak{L}$, it holds that

$$\omega(x, y, z) = -\alpha([x, y, z]_{\mathfrak{L}}) + D_A(x, y)(\alpha(z)) + \theta_A(y, z)(\alpha(x)) - \theta_A(x, z)(\alpha(y)).$$

Define a linear map $s' : \mathfrak{L} \longrightarrow \tilde{\mathfrak{L}}$ by $s'(x) = s(x) - \alpha(x)$. Note that s' is also a section of π. Then for any $x, y, z \in \mathfrak{L}$, we have

$$\begin{aligned}
[s'(x), s'(y), s'(z)]_{\tilde{\mathfrak{L}}} &= [s(x) - \alpha(x), s(y) - \alpha(y), s(z) - \alpha(z)]_{\tilde{\mathfrak{L}}} \\
&= [s(x), s(y), s(z)]_{\tilde{\mathfrak{L}}} - [s(x), s(y), \alpha(z)] - [\alpha(x), s(y), s(z)] - [s(x), \alpha(y), s(z)] \\
&= [s(x), s(y), s(z)]_{\tilde{\mathfrak{L}}} - D_A(x, y)(\alpha(z)) + \theta_A(x, z)(\alpha(y)) - \theta_A(y, z)(\alpha(x)) \\
&= s([x, y, z]_{\mathfrak{L}}) + \omega(x, y, z) - D_A(x, y)(\alpha(z)) + \theta_A(x, z)(\alpha(y)) - \theta_A(y, z)(\alpha(x)) \\
&= s([x, y, z]_{\mathfrak{L}}) - \alpha([x, y, z]_{\mathfrak{L}}) \\
&= s'([x, y, z]_{\mathfrak{L}}).
\end{aligned}$$

Therefore, s' is a homomorphism. □

3. Cohomology classes and a Lie algebra

In this section, $\mathfrak{L}$ and A denote Lie triple systems. We choose derivations of $\mathfrak{L}$ and A, and use these derivations to construct third-order cohomology classes. Based on these preparations, we construct a Lie algebra G_{θ_A} and its representation on $H^3(\mathfrak{L}, A)$.

Given a representation θ_A of $\mathfrak{L}$ on A. Suppose $\omega \in C^3(\mathfrak{L}, A)$. For any pair $(\partial_a, \partial_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$, define a 3-cochain $\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)} \in C^3(\mathfrak{L}, A)$ as

$$\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)} = \partial_a\omega - \omega(\partial_l \otimes \mathrm{id} \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes \partial_l \otimes \mathrm{id}) - \omega(\mathrm{id} \otimes \mathrm{id} \otimes \partial_l), \tag{3.1}$$

or equivalently,

$$\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}(x, y, z) = \partial_a\omega(x, y, z) - \omega(\partial_l(x), y, z) - \omega(x, \partial_l(y), z) - \omega(x, y, \partial_l(z)),$$

for all $x, y, z \in \mathfrak{L}$.

Lemma 3.1. *Let θ_A be a representation of $\mathfrak{L}$ on A and $\omega \in C^3(\mathfrak{L}, A)$ with respect to the representation θ_A. Assume that a pair $(\partial_a, \partial_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$ satisfies that*

$$\partial_a\theta_A(x, y) - \theta_A(x, y)\partial_a = \theta_A(\partial_l(x), y) + \theta_A(x, \partial_l(y)), \tag{3.2}$$

for all $x, y \in \mathfrak{L}$. If ω is a 3-cocycle then $\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}$ given by Eq 3.1 is also a 3-cocycle.

Proof. We only need to prove that $\delta\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)} = 0$. Since ω is a 3-cocycle, $\delta\omega = 0$, by Eq 2.2 it follows that, for any $x_1, x_2, x_3, x_4, x_5 \in \mathfrak{L}$,

$$\begin{aligned} 0 = {} & \theta_A(x_4, x_5)\omega(x_1, x_2, x_3) - \theta_A(x_3, x_5)\omega(x_1, x_2, x_4) + D_A(x_1, x_2)\omega(x_3, x_4, x_5) \\ & - D_A(x_3, x_4)\omega(x_1, x_2, x_5) + \omega([x_1, x_2, x_3], x_4, x_5) + \omega(x_3, [x_1, x_2, x_4], x_5) \\ & + \omega(x_3, x_4, [x_1, x_2, x_5]) - \omega(x_1, x_2, [x_3, x_4, x_5]). \end{aligned} \tag{3.3}$$

Then for any $x_1, x_2, x_3, x_4, x_5 \in \mathfrak{L}$, we have

$$\begin{aligned} & (\delta\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)})(x_1, x_2, x_3, x_4, x_5) \\ = {} & \underbrace{\theta_A(x_4, x_5)\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}(x_1, x_2, x_3)}_{(1)} - \underbrace{\theta_A(x_3, x_5)\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}(x_1, x_2, x_4)}_{(2)} + \underbrace{D_A(x_1, x_2)\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}(x_3, x_4, x_5)}_{(3)} \\ & - \underbrace{D_A(x_3, x_4)\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}(x_1, x_2, x_5)}_{(4)} + \underbrace{\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}([x_1, x_2, x_3], x_4, x_5)}_{(5)} + \underbrace{\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}(x_3, [x_1, x_2, x_4], x_5)}_{(6)} \\ & + \underbrace{\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}(x_3, x_4, [x_1, x_2, x_5])}_{(7)} - \underbrace{\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}(x_1, x_2, [x_3, x_4, x_5])}_{(8)}. \end{aligned} \tag{3.4}$$

Since $(\partial_a, \partial_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$ and $\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}$ satisfies Eq 3.1, we have

$$\begin{aligned} (1) = {} & \theta_A(x_4, x_5)\partial_a\omega(x_1, x_2, x_3) - \theta_A(x_4, x_5)\omega(\partial_l(x_1), x_2, x_3) - \theta_A(x_4, x_5)\omega(x_1, \partial_l(x_2), x_3) \\ & - \theta_A(x_4, x_5)\omega(x_1, x_2, \partial_l(x_3)), \end{aligned} \tag{3.5}$$

$$(2) = -\theta_A(x_3,x_5)\partial_a\omega(x_1,x_2,x_4) + \theta_A(x_3,x_5)\omega(\partial_l(x_1),x_2,x_4) + \theta_A(x_3,x_5)\omega(x_1,\partial_l(x_2),x_4) + \theta_A(x_3,x_5)\omega(x_1,x_2,\partial_l(x_4)), \tag{3.6}$$

$$(3) = D_A(x_1,x_2)\partial_a\omega(x_3,x_4,x_5) - D_A(x_1,x_2)\omega(\partial_l(x_3),x_4,x_5) - D_A(x_1,x_2)\omega(x_3,\partial_l(x_4),x_5) - D_A(x_1,x_2)\omega(x_3,x_4,\partial_l(x_5)), \tag{3.7}$$

$$(4) = -D_A(x_3,x_4)\partial_a\omega(x_1,x_2,x_5) + D_A(x_3,x_4)\omega(\partial_l(x_1),x_2,x_5) + D_A(x_3,x_4)\omega(x_1,\partial_l(x_2),x_5) + D_A(x_3,x_4)\omega(x_1,x_2,\partial_l(x_5)), \tag{3.8}$$

$$(5) = \partial_a(\omega([x_1,x_2,x_3],x_4,x_5)) - \omega([\partial_l(x_1),x_2,x_3],x_4,x_5) - \omega([x_1,\partial_l(x_2),x_3],x_4,x_5) - \omega([x_1,x_2,\partial_l(x_3)],x_4,x_5) - \omega([x_1,x_2,x_3],\partial_l(x_4),x_5) - \omega([x_1,x_2,x_3],x_4,\partial_l(x_5)), \tag{3.9}$$

$$(6) = \partial_a(\omega(x_3,[x_1,x_2,x_4],x_5)) - \omega(\partial_l(x_3),[x_1,x_2,x_4],x_5) - \omega(x_3,[\partial_l(x_1),x_2,x_4],x_5) - \omega(x_3,[x_1,\partial_l(x_2),x_4],x_5) - \omega(x_3,[x_1,x_2,\partial_l(x_4)],x_5) - \omega(x_3,[x_1,x_2,x_4],\partial_l(x_5)), \tag{3.10}$$

$$(7) = \partial_a(\omega(x_3,x_4,[x_1,x_2,x_5])) - \omega(\partial_l(x_3),x_4,[x_1,x_2,x_5]) - \omega(x_3,\partial_l(x_4),[x_1,x_2,x_5]) - \omega(x_3,x_4,[\partial_l(x_1),x_2,x_5]) - \omega(x_3,x_4,[x_1,\partial_l(x_2),x_5]) - \omega(x_3,x_4,[x_1,x_2,\partial_l(x_5)]), \tag{3.11}$$

$$(8) = -\partial_a(\omega(x_1,x_2,[x_3,x_4,x_5])) + \omega(\partial_l(x_1),x_2,[x_3,x_4,x_5]) + \omega(x_1,\partial_l(x_2),[x_3,x_4,x_5]) + \omega(x_1,x_2,[\partial_l(x_3),x_4,x_5]) + \omega(x_1,x_2,[x_3,\partial_l(x_4),x_5]) + \omega(x_1,x_2,[x_3,x_4,\partial_l(x_5)]). \tag{3.12}$$

For Eqs 3.5–3.12, by suitable combination and with the aid of Eq 3.3, we get the following

$$\begin{aligned} &-\theta_A(x_4,x_5)\omega(\partial_l(x_1),x_2,x_3) + \theta_A(x_3,x_5)\omega(\partial_l(x_1),x_2,x_4) - \omega([\partial_l(x_1),x_2,x_3],x_4,x_5) \\ &+ D_A(x_3,x_4)\omega(\partial_l(x_1),x_2,x_5) - \omega(x_3,[\partial_l(x_1),x_2,x_4],x_5) - \omega(x_3,x_4,[\partial_l(x_1),x_2,x_5]) \\ &+ \omega(\partial_l(x_1),x_2,[x_3,x_4,x_5]) \\ =\ & D_A(\partial_l(x_1),x_2)\omega(x_3,x_4,x_5), \end{aligned} \tag{3.13}$$

$$\begin{aligned} &-\theta_A(x_4,x_5)\omega(x_1,\partial_l(x_2),x_3) + \theta_A(x_3,x_5)\omega(x_1,\partial_l(x_2),x_4) - \omega([x_1,\partial_l(x_2),x_3],x_4,x_5) \\ &+ D_A(x_3,x_4)\omega(x_1,\partial_l(x_2),x_5) - \omega(x_3,[x_1,\partial_l(x_2),x_4],x_5) - \omega(x_3,x_4,[x_1,\partial_l(x_2),x_5]) \\ &+ \omega(x_1,\partial_l(x_2),[x_3,x_4,x_5]) \\ =\ & D_A(x_1,\partial_l(x_2))\omega(x_3,x_4,x_5), \end{aligned} \tag{3.14}$$

$$\begin{aligned} &-\theta_A(x_4,x_5)\omega(x_1,x_2,\partial_l(x_3)) - D_A(x_1,x_2)\omega(\partial_l(x_3),x_4,x_5) - \omega([x_1,x_2,\partial_l(x_3)],x_4,x_5) \\ &- \omega(\partial_l(x_3),[x_1,x_2,x_4],x_5) - \omega(\partial_l(x_3),x_4,[x_1,x_2,x_5]) + \omega(x_1,x_2,[\partial_l(x_3),x_4,x_5]) \\ =\ & -\theta_A(\partial_l(x_3),x_5)\omega(x_1,x_2,x_4) - D_A(\partial_l(x_3),x_4)\omega(x_1,x_2,x_5), \end{aligned} \tag{3.15}$$

$$\begin{aligned}
&\theta_A(x_3,x_5)\omega(x_1,x_2,\partial_l(x_4)) - D_A(x_1,x_2)\omega(x_3,\partial_l(x_4),x_5) - \omega([x_1,x_2,x_3],\partial_l(x_4),x_5)\\
&- \omega(x_3,\partial_l(x_4),[x_1,x_2,x_5]) + \omega(x_1,x_2,[x_3,\partial_l(x_4),x_5]) - \omega(x_3,[x_1,x_2,\partial_l(x_4)],x_5)\\
=\ &\theta_A(\partial_l(x_4),x_5)\omega(x_1,x_2,x_3) - D_A(x_3,\partial_l(x_4))\omega(x_1,x_2,x_5),
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
&- D_A(x_1,x_2)\omega(x_3,x_4,\partial_l(x_5)) - \omega([x_1,x_2,x_3],x_4,\partial_l(x_5)) - \omega(x_3,[x_1,x_2,x_4],\partial_l(x_5))\\
&+ D_A(x_3,x_4)\omega(x_1,x_2,\partial_l(x_5)) - \omega(x_3,x_4,[x_1,x_2,\partial_l(x_5)]) + \omega(x_1,x_2,[x_3,x_4,\partial_l(x_5)])\\
=\ &\theta_A(x_4,\partial_l(x_5))\omega(x_1,x_2,x_3) - \theta_A(x_3,\partial_l(x_5))\omega(x_1,x_2,x_4).
\end{aligned} \tag{3.17}$$

Next, let us substituting Eqs 3.13–3.17 into Eq 3.4 having the following

$$\begin{aligned}
&(\delta \mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)})(x_1,x_2,x_3,x_4,x_5)\\
=\ &\theta_A(x_4,x_5)\partial_a\omega(x_1,x_2,x_3) - \theta_A(x_3,x_5)\partial_a\omega(x_1,x_2,x_4) + D_A(x_1,x_2)\partial_a\omega(x_3,x_4,x_5)\\
&- D_A(x_3,x_4)\partial_a\omega(x_1,x_2,x_5) + \partial_a\omega([x_1,x_2,x_3],x_4,x_5) + \partial_a\omega(x_3,[x_1,x_2,x_4],x_5)\\
&+ \partial_a\omega(x_3,x_4,[x_1,x_2,x_5]) - \partial_a\omega(x_1,x_2,[x_3,x_4,x_5]) + D_A(\partial_l(x_1),x_2)\omega(x_3,x_4,x_5)\\
&+ D_A(x_1,\partial_l(x_2))\omega(x_3,x_4,x_5) - \theta_A(\partial_l(x_3),x_5)\omega(x_1,x_2,x_4)\\
&- D_A(\partial_l(x_3),x_4)\omega(x_1,x_2,x_5) - D_A(x_3,\partial_l(x_4))\omega(x_1,x_2,x_5)\\
&+ \theta_A(\partial_l(x_4),x_5)\omega(x_1,x_2,x_3) + \theta_A(x_4,\partial_l(x_5))\omega(x_1,x_2,x_3)\\
&- \theta_A(x_3,\partial_l(x_5))\omega(x_1,x_2,x_4).
\end{aligned} \tag{3.18}$$

By Eq 3.3, we have

$$\begin{aligned}
&\partial_a\omega([x_1,x_2,x_3],x_4,x_5) + \partial_a\omega(x_3,[x_1,x_2,x_4],x_5) + \partial_a\omega(x_3,x_4,[x_1,x_2,x_5])\\
&- \partial_a\omega(x_1,x_2,[x_3,x_4,x_5])\\
=\ &- \partial_a\theta_A(x_4,x_5)\omega(x_1,x_2,x_3) + \partial_a\theta_A(x_3,x_5)\omega(x_1,x_2,x_4) - \partial_a D_A(x_1,x_2)\omega(x_3,x_4,x_5)\\
&+ \partial_a D_A(x_3,x_4)\omega(x_1,x_2,x_5).
\end{aligned} \tag{3.19}$$

Here, inserting Eq 3.19 into Eq 3.18 gives that

$$\begin{aligned}
&(\delta \mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)})(x_1,x_2,x_3,x_4,x_5)\\
=\ &\theta_A(x_4,x_5)\partial_a\omega(x_1,x_2,x_3) - \theta_A(x_3,x_5)\partial_a\omega(x_1,x_2,x_4) + D_A(x_1,x_2)\partial_a\omega(x_3,x_4,x_5)\\
&- D_A(x_3,x_4)\partial_a\omega(x_1,x_2,x_5) - \partial_a\theta_A(x_4,x_5)\omega(x_1,x_2,x_3) + \partial_a\theta_A(x_3,x_5)\omega(x_1,x_2,x_4)\\
&- \partial_a D_A(x_1,x_2)\omega(x_3,x_4,x_5) + \partial_a D_A(x_3,x_4)\omega(x_1,x_2,x_5) + D_A(\partial_l(x_1),x_2)\omega(x_3,x_4,x_5)\\
&+ D_A(x_1,\partial_l(x_2))\omega(x_3,x_4,x_5) - \theta_A(\partial_l(x_3),x_5)\omega(x_1,x_2,x_4) - D_A(\partial_l(x_3),x_4)\omega(x_1,x_2,x_5)\\
&+ \theta_A(\partial_l(x_4),x_5)\omega(x_1,x_2,x_3) - D_A(x_3,\partial_l(x_4))\omega(x_1,x_2,x_5) + \theta_A(x_4,\partial_l(x_5))\omega(x_1,x_2,x_3)\\
&- \theta_A(x_3,\partial_l(x_5))\omega(x_1,x_2,x_4)\\
=\ &(-\partial_a\theta_A(x_4,x_5) + \theta_A(x_4,x_5)\partial_a + \theta_A(\partial_l(x_4),x_5) + \theta_A(x_4,\partial_l(x_5)))\omega(x_1,x_2,x_3)\\
&+ (\partial_a\theta_A(x_3,x_5) - \theta_A(x_3,x_5)\partial_a - \theta_A(\partial_l(x_3),x_5) - \theta_A(x_3,\partial_l(x_5)))\omega(x_1,x_2,x_4)\\
&+ (-\partial_a D_A(x_1,x_2) + D_A(x_1,x_2)\partial_a + D_A(\partial_l(x_1),x_2) + D_A(x_1,\partial_l(x_2)))\omega(x_3,x_4,x_5)\\
&+ (\partial_a D_A(x_3,x_4) - D_A(x_3,x_4)\partial_a - D_A(\partial_l(x_3),x_4) - D_A(x_3,\partial_l(x_4)))\omega(x_1,x_2,x_5).
\end{aligned}$$

Since (∂_a, ∂_l) satisfies Eq 3.2, we only need to show that

$$\partial_a D_A(x,y) - D_A(x,y)\partial_a = D_A(\partial_l(x), y) + D_A(x, \partial_l(y)),$$

by $D_A(x,y) = \theta_A(y,x) - \theta_A(x,y)$, we have

$$\begin{aligned}\partial_a D_A(x,y) - D_A(x,y)\partial_a &= \partial_a(\theta_A(y,x) - \theta_A(x,y)) - (\theta_A(y,x) - \theta_A(x,y))\partial_a \\ &= \Big(\partial_a\theta_A(y,x) - \theta_A(y,x)\partial_a\Big) - \Big(\partial_a\theta_A(x,y) - \theta_A(x,y)\partial_a\Big) \\ &= \Big(\theta_A(\partial_l(y), x) + \theta_A(y, \partial_l(x))\Big) - \Big(\theta_A(\partial_l(x), y) + \theta_A(x, \partial_l(y))\Big) \\ &= \Big(\theta_A(\partial_l(y), x) - \theta_A(x, \partial_l(y))\Big) + \Big(\theta_A(y, \partial_l(x)) - \theta_A(\partial_l(x), y)\Big) \\ &= D_A(x, \partial_l(y)) + D_A(\partial_l(x), y).\end{aligned}$$

Then we have $\delta \mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)} = 0$ as required. □

Definition 3.2. Let θ_A be a representation of $\mathfrak{L}$ on A. A pair $(\partial_a, \partial_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$ is called compatible with respect to θ_A if Eq 3.2 holds.

Based on the previous works, we are ready to construct a Lie algebra and its representation on the third cohomology group. Set

$$G_{\theta_A} = \{(\partial_a, \partial_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L}) \mid (\partial_a, \partial_l) \text{ is compatible with respect to } \theta_A\}.$$

Lemma 3.3. *There is a linear map* $\Phi : G_{\theta_A} \longrightarrow \mathrm{End}(H^3(\mathfrak{L}, A))$ *given by*

$$\Phi(\partial_a, \partial_l)([\omega]) = [\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}], \; for\ any\ \omega \in Z^3(\mathfrak{L}, A), \tag{3.20}$$

where $\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}$ *is given by Eq* 3.1.

Proof. By Lemma 3.1 and (∂_a, ∂_l) is compatible with respect to θ_A it follows that $\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}$ is a 3-cocycle whenever ω is a 3-cocycle. Therefore, we only need to show that if $\delta\lambda$ is a 3-coboundary, then $\Phi(\partial_a, \partial_l)(\delta\lambda) = 0$, which means that Φ is well-defined.

$$\begin{aligned}&(\Phi(\partial_a, \partial_l)(\delta\lambda))(x,y,z) \\ =& \Big(\partial_a(\delta\lambda) - (\delta\lambda)(\partial_l \otimes \mathrm{id} \otimes \mathrm{id}) - (\delta\lambda)(\mathrm{id} \otimes \partial_l \otimes \mathrm{id}) - (\delta\lambda)(\mathrm{id} \otimes \mathrm{id} \otimes \partial_l)\Big)(x,y,z) \\ =& \partial_a\Big(- \lambda([x,y,z]) + \theta_A(y,z)(\lambda(x)) - \theta_A(x,z)(\lambda(y)) + D_A(x,y)(\lambda(z))\Big) \\ &- \Big(- \lambda([\partial_l(x),y,z]) + \theta_A(y,z)(\lambda(\partial_l(x))) - \theta_A(\partial_l(x),z)(\lambda(y)) + D_A(\partial_l(x),y)(\lambda(z))\Big) \\ &- \Big(- \lambda([x,\partial_l(y),z]) + \theta_A(\partial_l(y),z)(\lambda(x)) - \theta_A(x,z)(\lambda(\partial_l(y))) + D_A(x,\partial_l(y))(\lambda(z))\Big) \\ &- \Big(- \lambda([x,y,\partial_l(z)]) + \theta_A(y,\partial_l(z))(\lambda(x)) - \theta_A(x,\partial_l(z))(\lambda(y)) + D_A(x,y)(\lambda(\partial_l(z)))\Big).\end{aligned}$$

Since ∂_l is a derivation, we have

$$\lambda([\partial_l(x), y, z]) + \lambda([x, \partial_l(y), z]) + \lambda([x, y, \partial_l(z)]) = \lambda(\partial_l([x,y,z])).$$

Then

$$
\begin{aligned}
&(\Phi(\partial_a,\partial_l)(\delta\lambda))(x,y,z)\\
=&\,\partial_a\theta_A(y,z)(\lambda(x))-\partial_a\theta_A(x,z)(\lambda(y))+\partial_aD_A(x,y)(\lambda(z))-\theta_A(y,z)(\lambda(\partial_l(x)))\\
&+\theta_A(\partial_l(x),z)(\lambda(y))-D_A(\partial_l(x),y)(\lambda(z))-\theta_A(\partial_l(y),z)(\lambda(x))+\theta_A(x,z)(\lambda(\partial_l(y)))\\
&-D_A(x,\partial_l(y))(\lambda(z))-\theta_A(y,\partial_l(z))(\lambda(x))+\theta_A(x,\partial_l(z))(\lambda(y))-D_A(x,y)(\lambda(\partial_l(z)))\\
&-\partial_a(\lambda([x,y,z]))+\lambda(\partial_l([x,y,z]))\\
=&-\theta_A(y,z)(\lambda(\partial_l(x)))+\theta_A(x,z)(\lambda(\partial_l(y)))-D_A(x,y)(\lambda(\partial_l(z)))+\theta_A(y,z)\partial_a(\lambda(x))\\
&-\theta_A(x,z)\partial_a(\lambda(y))+D_A(x,y)\partial_a(\lambda(z))-\partial_a(\lambda([x,y,z]))+\lambda(\partial_l([x,y,z]))\\
=&\,\delta(\partial_a\circ\lambda-\lambda\circ\partial_l)(x,y,z)=0,
\end{aligned}
$$

it is obvious that Φ is a linear map. □

We will end this section with the following conclusion.

Theorem 3.4. *Let θ_A be a representation of $\mathfrak{L}$ on A. Then G_{θ_A} is a Lie subalgebra of* $\mathrm{Der}(A)\times\mathrm{Der}(\mathfrak{L})$ *and the map Φ given by Eq* 3.20 *is a Lie algebra homomorphism.*

Proof. Note that G_{θ_A} is a subalgebra of $\mathrm{Der}(A)\times\mathrm{Der}(\mathfrak{L})$. First, we prove that if $(\partial_{a_1},\partial_{l_1}),(\partial_{a_2},\partial_{l_2})\in G_{\theta_A}$, then the commutator $[(\partial_{a_1},\partial_{l_1}),(\partial_{a_2},\partial_{l_2})]\in G_{\theta_A}$. For all $x,y\in\mathfrak{L}$, we have

$$
\begin{aligned}
&(\partial_{a_1}\partial_{a_2}-\partial_{a_2}\partial_{a_1})\theta_A(x,y)-\theta_A(x,y)(\partial_{a_1}\partial_{a_2}-\partial_{a_2}\partial_{a_1})\\
=&\,\partial_{a_1}(\partial_{a_2}\theta_A(x,y))-\partial_{a_2}(\partial_{a_1}\theta_A(x,y))-\theta_A(x,y)\partial_{a_1}\partial_{a_2}+\theta_A(x,y)\partial_{a_2}\partial_{a_1}\\
=&\,\underbrace{\partial_{a_1}\Big(\theta_A(x,y)\partial_{a_2}+\theta_A(\partial_{l_2}(x),y)+\theta_A(x,\partial_{l_2}(y))\Big)}_{I_1}-\underbrace{\partial_{a_2}\Big(\theta_A(x,y)\partial_{a_1}+\theta_A(\partial_{l_1}(x),y)+\theta_A(x,\partial_{l_1}(y))\Big)}_{I_2}\\
&-\theta_A(x,y)\partial_{a_1}\partial_{a_2}+\theta_A(x,y)\partial_{a_2}\partial_{a_1}.
\end{aligned}
\tag{3.21}
$$

By Eq 3.2 it follows that

$$
\begin{aligned}
I_1=&\,\theta_A(x,y)\partial_{a_1}\partial_{a_2}+\theta_A(\partial_{l_1}(x),y)\partial_{a_2}+\theta_A(x,\partial_{l_1}(y))\partial_{a_2}+\theta_A(\partial_{l_2}(x),y)\partial_{a_1}+\theta_A(\partial_{l_1}\partial_{l_2}(x),y)\\
&+\theta_A(\partial_{l_2}(x),\partial_{l_1}(y))+\theta_A(x,\partial_{l_2}(y))\partial_{a_1}+\theta_A(\partial_{l_1}(x),\partial_{l_2}(y))+\theta_A(x,\partial_{l_1}\partial_{l_2}(y)),
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
I_2=&\,\theta_A(x,y)\partial_{a_2}\partial_{a_1}+\theta_A(\partial_{l_2}(x),y)\partial_{a_1}+\theta_A(x,\partial_{l_2}(y))\partial_{a_1}+\theta_A(\partial_{l_1}(x),y)\partial_{a_2}+\theta_A(\partial_{l_2}\partial_{l_1}(x),y)\\
&+\theta_A(\partial_{l_1}(x),\partial_{l_2}(y))+\theta_A(x,\partial_{l_1}(y))\partial_{a_2}+\theta_A(\partial_{l_2}(x),\partial_{l_1}(y))+\theta_A(x,\partial_{l_2}\partial_{l_1}(y)).
\end{aligned}
\tag{3.23}
$$

Then taking Eqs 3.22 and 3.23 into Eq 3.21 gives that

$$
(\partial_{a_1}\partial_{a_2}-\partial_{a_2}\partial_{a_1})\theta_A(x,y)-\theta_A(x,y)(\partial_{a_1}\partial_{a_2}-\partial_{a_2}\partial_{a_1})=\theta_A\Big((\partial_{l_1}\partial_{l_2}-\partial_{l_2}\partial_{l_1})(x),y\Big)+\theta_A\Big(x,(\partial_{l_1}\partial_{l_2}-\partial_{l_2}\partial_{l_1})(y)\Big),
$$

which implies that $[(\partial_{a_1},\partial_{l_1}),(\partial_{a_2},\partial_{l_2})]$ is compatible.

Next, for any $(\partial_{a_1},\partial_{l_1}),(\partial_{a_2},\partial_{l_2})\in G_{\theta_A}$, $[\omega]\in H^3(\mathfrak{L},A)$, we have

$$
[\Phi(\partial_{a_1},\partial_{l_1}),\Phi(\partial_{a_2},\partial_{l_2})]([\omega])=\Phi(\partial_{a_1},\partial_{l_1})\Phi(\partial_{a_2},\partial_{l_2})([\omega])-\Phi(\partial_{a_2},\partial_{l_2})\Phi(\partial_{a_1},\partial_{l_1})([\omega]).
\tag{3.24}
$$

By Eqs 3.1 and 3.20, we have

$$
\begin{aligned}
&\Phi(\partial_{a_1},\partial_{l_1})\Phi(\partial_{a_2},\partial_{l_2})([\omega])\\
&=\Phi(\partial_{a_1},\partial_{l_1})\Big(\partial_{a_2}\omega-\omega(\partial_{l_2}\otimes\mathrm{id}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\partial_{l_2}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\mathrm{id}\otimes\partial_{l_2})\Big)\\
&=\Big[\partial_{a_1}\Big(\partial_{a_2}\omega-\omega(\partial_{l_2}\otimes\mathrm{id}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\partial_{l_2}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\mathrm{id}\otimes\partial_{l_2})\Big)\\
&\quad-\Big(\partial_{a_2}\omega-\omega(\partial_{l_2}\otimes\mathrm{id}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\partial_{l_2}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\mathrm{id}\otimes\partial_{l_2})\Big)(\partial_{l_1}\otimes\mathrm{id}\otimes\mathrm{id})\\
&\quad-\Big(\partial_{a_2}\omega-\omega(\partial_{l_2}\otimes\mathrm{id}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\partial_{l_2}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\mathrm{id}\otimes\partial_{l_2})\Big)(\mathrm{id}\otimes\partial_{l_1}\otimes\mathrm{id})\\
&\quad-\Big(\partial_{a_2}\omega-\omega(\partial_{l_2}\otimes\mathrm{id}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\partial_{l_2}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\mathrm{id}\otimes\partial_{l_2})\Big)(\mathrm{id}\otimes\mathrm{id}\otimes\partial_{l_1})\Big].
\end{aligned}
$$

Similarly, one obtains

$$
\begin{aligned}
&\Phi(\partial_{a_2},\partial_{l_2})\Phi(\partial_{a_1},\partial_{l_1})([\omega])\\
&=\Phi(\partial_{a_2},\partial_{l_2})\Big(\partial_{a_1}\omega-\omega(\partial_{l_1}\otimes\mathrm{id}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\partial_{l_1}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\mathrm{id}\otimes\partial_{l_1})\Big)\\
&=\Big[\partial_{a_2}\Big(\partial_{a_1}\omega-\omega(\partial_{l_1}\otimes\mathrm{id}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\partial_{l_1}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\mathrm{id}\otimes\partial_{l_1})\Big)\\
&\quad-\Big(\partial_{a_1}\omega-\omega(\partial_{l_1}\otimes\mathrm{id}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\partial_{l_1}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\mathrm{id}\otimes\partial_{l_1})\Big)(\partial_{l_2}\otimes\mathrm{id}\otimes\mathrm{id})\\
&\quad-\Big(\partial_{a_1}\omega-\omega(\partial_{l_1}\otimes\mathrm{id}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\partial_{l_1}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\mathrm{id}\otimes\partial_{l_1})\Big)(\mathrm{id}\otimes\partial_{l_2}\otimes\mathrm{id})\\
&\quad-\Big(\partial_{a_1}\omega-\omega(\partial_{l_1}\otimes\mathrm{id}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\partial_{l_1}\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\mathrm{id}\otimes\partial_{l_1})\Big)(\mathrm{id}\otimes\mathrm{id}\otimes\partial_{l_2})\Big].
\end{aligned}
$$

Inserting the above two identities into Eq 3.24, we deduce

$$
\begin{aligned}
&[\Phi(\partial_{a_1},\partial_{l_1}),\Phi(\partial_{a_2},\partial_{l_2})]([\omega])\\
&=[(\partial_{a_1}\partial_{a_2}-\partial_{a_2}\partial_{a_1})\omega-\omega((\partial_{l_1}\partial_{l_2}-\partial_{l_2}\partial_{l_1})\otimes\mathrm{id}\otimes\mathrm{id})\\
&\quad-\omega(\mathrm{id}\otimes(\partial_{l_1}\partial_{l_2}-\partial_{l_2}\partial_{l_1})\otimes\mathrm{id})-\omega(\mathrm{id}\otimes\mathrm{id}\otimes(\partial_{l_1}\partial_{l_2}-\partial_{l_2}\partial_{l_1}))]\\
&=[\mathrm{Ob}^{\omega}_{(\partial_{a_1}\partial_{a_2}-\partial_{a_2}\partial_{a_1},\partial_{l_1}\partial_{l_2}-\partial_{l_2}\partial_{l_1})}]\\
&=\Phi(\partial_{a_1}\partial_{a_2}-\partial_{a_2}\partial_{a_1},\partial_{l_1}\partial_{l_2}-\partial_{l_2}\partial_{l_1})([\omega])\\
&=\Phi[(\partial_{a_1},\partial_{l_1}),(\partial_{a_2},\partial_{l_2})]([\omega])
\end{aligned}
$$

as desired. □

4. Abelian extensions and extensibility of derivations

In this section, we construct obstruction classes for extensibility of derivations by Lemma 3.1. Also, for G_{θ_A} we give a representation in terms of extensibility of derivations.

Lemma 4.1. *Keep notation as above. The cohomology class* $[\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}]\in H^3(\mathfrak{L},A)$ *does not depend on the choice of sections of* π. (*Hence we will denote* $[\mathrm{Ob}^{\omega}_{(\partial_a,\partial_l)}]$ *by* $[\mathrm{Ob}^{\mathfrak{L}}_{(\partial_a,\partial_l)}]$).

Proof. Suppose that s_1 and s_2 are sections of π and ω_1,ω_2 are defined by Eq 2.4, while $\mathrm{Ob}^{\omega_1}_{(\partial_a,\partial_l)}$, $\mathrm{Ob}^{\omega_2}_{(\partial_a,\partial_l)}$

are defined by Eq 3.1 with respect to ω_1, ω_2. Then

$$\begin{aligned}
&\mathrm{Ob}^{\omega_1}_{(\partial_a,\partial_l)}(x,y,z) - \mathrm{Ob}^{\omega_2}_{(\partial_a,\partial_l)}(x,y,z)\\
&= \partial_a(\omega_1(x,y,z)) - \omega_1(\partial_l(x),y,z) - \omega_1(x,\partial_l(y),z) - \omega_1(x,y,\partial_l(z))\\
&\quad - \partial_a(\omega_2(x,y,z)) + \omega_2(\partial_l(x),y,z) + \omega_2(x,\partial_l(y),z) + \omega_2(x,y,\partial_l(z))\\
&= \underbrace{\partial_a(\omega_1(x,y,z) - \omega_2(x,y,z))}_{I_1} - \underbrace{(\omega_1(\partial_l(x),y,z) - \omega_2(\partial_l(x),y,z))}_{I_2}\\
&\quad - \underbrace{(\omega_1(x,\partial_l(y),z) - \omega_2(x,\partial_l(y),z))}_{I_3} - \underbrace{(\omega_1(x,y,\partial_l(z)) - \omega_2(x,y,\partial_l(z)))}_{I_4}.
\end{aligned} \tag{4.1}$$

A map $\lambda : \mathfrak{L} \longrightarrow A$ is defined by $\lambda(x) = s_1(x) - s_2(x)$, for all $x \in \mathfrak{L}$. Recall the proof of Corollary 2.7, we have

$$\omega_1(x,y,z) - \omega_2(x,y,z) = -\lambda([x,y,z]) + \theta_A(y,z)(\lambda(x)) - \theta_A(x,z)(\lambda(y)) + D_A(x,y)(\lambda(z)),$$

for any $x,y,z \in \mathfrak{L}$. So

$$I_1 = \partial_a\Big(-\lambda([x,y,z]) + \theta_A(y,z)(\lambda(x))) - \theta_A(x,z)(\lambda(y)) + D_A(x,y)(\lambda(z))\Big).$$

Similarly, we have

$$I_2 = -\lambda([\partial_l(x),y,z]) + \theta_A(y,z)(\lambda(\partial_l(x))) - \theta_A(\partial_l(x),z)(\lambda(y)) + D_A(\partial_l(x),y)(\lambda(z)),$$

$$I_3 = -\lambda([x,\partial_l(y),z]) + \theta_A(\partial_l(y),z)(\lambda(x)) - \theta_A(x,z)(\lambda(\partial_l(y))) + D_A(x,\partial_l(y))(\lambda(z)),$$

and

$$I_4 = -\lambda([x,y,\partial_l(z)]) + \theta_A(y,\partial_l(z))(\lambda(x)) - \theta_A(x,\partial_l(z))(\lambda(y)) + D_A(x,y)(\lambda(\partial_l(z))).$$

By I_1, I_2, I_3, I_4 and Eq 4.1 have

$$\begin{aligned}
&\mathrm{Ob}^{\omega_1}_{(\partial_a,\partial_l)}(x,y,z) - \mathrm{Ob}^{\omega_2}_{(\partial_a,\partial_l)}(x,y,z)\\
&= \Big(\partial_a\theta_A(y,z) - \theta_A(\partial_l(y),z) - \theta_A(y,\partial_l(z))\Big)(\lambda(x)) - \Big(\partial_a\theta_A(x,z) - \theta_A(\partial_l(x),z) - \theta_A(x,\partial_l(z))\Big)(\lambda(y))\\
&\quad + \Big(\partial_a D_A(x,y) - D_A(\partial_l(x),y) - D_A(x,\partial_l(y))\Big)(\lambda(z)) - \theta_A(y,z)(\lambda(\partial_l(x))) + \theta_A(x,z)(\lambda(\partial_l(y)))\\
&\quad - D_A(x,y)(\lambda(\partial_l(z))) - \partial_a(\lambda([x,y,z])) + \lambda(\partial_l([x,y,z])).
\end{aligned}$$

Since (∂_a, ∂_l) satisfies Eq 3.2 it follows that

$$\begin{aligned}
&\mathrm{Ob}^{\omega_1}_{(\partial_a,\partial_l)}(x,y,z) - \mathrm{Ob}^{\omega_2}_{(\partial_a,\partial_l)}(x,y,z)\\
&= \theta_A(y,z)\partial_a(\lambda(x)) - \theta_A(x,z)\partial_a(\lambda(y)) + D_A(x,y)\partial_a(\lambda(z)) - \theta_A(y,z)(\lambda(\partial_l(x)))\\
&\quad + \theta_A(x,z)(\lambda(\partial_l(y))) - D_A(x,y)(\lambda(\partial_l(z))) - \partial_a(\lambda([x,y,z])) + \lambda(\partial_l([x,y,z]))\\
&= \delta(\partial_a \circ \lambda - \lambda \circ \partial_l)(x,y,z),
\end{aligned}$$

i.e., $[\mathrm{Ob}^{\omega_1}_{(\partial_a,\partial_l)}] = [\mathrm{Ob}^{\omega_2}_{(\partial_a,\partial_l)}] \in H^3(\mathfrak{L}, A)$, so we have done the proof. □

Next, we will define the extensibility of derivations, and we will give a necessary and sufficient condition that (∂_a, ∂_l) is extensible.

Definition 4.2. Let $\tilde{\mathfrak{L}}$ be an abelian extension of a Lie triple system $\mathfrak{L}$ by A. A pair $(\partial_a, \partial_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$ is called extensible if there is a derivation $\partial_{\tilde{l}} \in \mathrm{Der}(\tilde{\mathfrak{L}})$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota"] \arrow[d, "\partial_a"] & \tilde{\mathfrak{L}} \arrow[r, "\pi"] \arrow[d, "\partial_{\tilde{l}}"] & \mathfrak{L} \arrow[r] \arrow[d, "\partial_l"] & 0 \\
0 \arrow[r] & A \arrow[r, "\iota"] & \tilde{\mathfrak{L}} \arrow[r, "\pi"] & \mathfrak{L} \arrow[r] & 0
\end{tikzcd} \tag{4.2}$$

is commutative, where $\iota : A \longrightarrow \tilde{\mathfrak{L}}$ is the inclusion map.

Proposition 4.3. *Assume that $\tilde{\mathfrak{L}}$ is an abelian extension of a Lie triple system $\mathfrak{L}$ by A. If $(\partial_a, \partial_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$ is extensible, then (∂_a, ∂_l) is compatible with respect to θ_A.*

Proof. Since (∂_a, ∂_l) is extensible, there is a derivation $\partial_{\tilde{l}} \in \mathrm{Der}(\tilde{\mathfrak{L}})$ such that the diagram (4.2) is commutative, i.e., $\iota \circ \partial_a = \partial_{\tilde{l}} \circ \iota$, hence there is a map $\mu : \mathfrak{L} \longrightarrow A$ given by

$$\mu(x) = \partial_{\tilde{l}}(s(x)) - s(\partial_l(x)). \tag{4.3}$$

Since $\iota \circ \partial_a = \partial_{\tilde{l}} \circ \iota$, (equivalent, $\partial_{\tilde{l}}|_A = \partial_a$) and $\partial_{\tilde{l}} \in \mathrm{Der}(\tilde{\mathfrak{L}})$, we obtain

$$\begin{aligned}
\partial_a(\theta_A(x,y)(v)) - \theta_A(x,y)(\partial_a(v)) &= \partial_a([v, s(x), s(y)]_{\tilde{\mathfrak{L}}}) - [\partial_a(v), s(x), s(y)]_{\tilde{\mathfrak{L}}} \\
&= \partial_{\tilde{l}}([v, s(x), s(y)]_{\tilde{\mathfrak{L}}}) - [\partial_a(v), s(x), s(y)]_{\tilde{\mathfrak{L}}} \\
&= [\partial_{\tilde{l}}(v), s(x), s(y)]_{\tilde{\mathfrak{L}}} + [v, \partial_{\tilde{l}}(s(x)), s(y)]_{\tilde{\mathfrak{L}}} \\
&\quad + [v, s(x), \partial_{\tilde{l}}(s(y))]_{\tilde{\mathfrak{L}}} - [\partial_a(v), s(x), s(y)]_{\tilde{\mathfrak{L}}} \\
&= [\partial_{\tilde{l}}(v), s(x), s(y)]_{\tilde{\mathfrak{L}}} + [v, s(\partial_l(x)), s(y)]_{\tilde{\mathfrak{L}}} \\
&\quad + [v, \mu(x), s(y)]_{\tilde{\mathfrak{L}}} + [v, s(x), s(\partial_l(y))]_{\tilde{\mathfrak{L}}} \\
&\quad + [v, s(x), \mu(y)]_{\tilde{\mathfrak{L}}} - [\partial_a(v), s(x), s(y)]_{\tilde{\mathfrak{L}}} \\
&= [v, s(\partial_l(x)), s(y)]_{\tilde{\mathfrak{L}}} + [v, s(x), s(\partial_l(y))]_{\tilde{\mathfrak{L}}} \\
&= \theta_A(\partial_l(x), y)(v) + \theta_A(x, \partial_l(y))(v),
\end{aligned}$$

hence (∂_a, ∂_l) is compatible. ◻

Theorem 4.4. *Let $\tilde{\mathfrak{L}}$ be an abelian extension of a Lie triple system $\mathfrak{L}$ by A and $(\partial_a, \partial_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$ compatible with respect to θ_A. Then (∂_a, ∂_l) is extensible if and only if $[\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(\partial_a,\partial_l)}] \in H^3(\mathfrak{L}, A)$ is trivial.*

Proof. ($\Rightarrow$) Fix any linear section s of π. Since (∂_a, ∂_l) is extensible, there exists a derivation $\partial_{\tilde{l}} \in \mathrm{Der}(\tilde{\mathfrak{L}})$ such that the associated diagram (4.2) is commutative. By $\pi \circ \partial_{\tilde{l}} = \partial_l \circ \pi$, we have $\partial_{\tilde{l}}(s(x)) - s(\partial_l(x)) \in A$, for $x \in \mathfrak{L}$, then there exists a map $\mu : \mathfrak{L} \longrightarrow A$ given by Eq 4.3. It sufficient to show that

$$\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(\partial_a,\partial_l)}(x,y,z) = (\delta\mu)(x,y,z), \tag{4.4}$$

for all $x, y, z \in \mathfrak{L}$. Next, we compute both hand sides of the following identity

$$\begin{aligned}
&\partial_{\tilde{l}}\Big([s(x_1) + v_1, s(x_2) + v_2, s(x_3) + v_3]_{\tilde{\mathfrak{L}}}\Big) \\
&= [\partial_{\tilde{l}}(s(x_1) + v_1), s(x_2) + v_2, s(x_3) + v_3]_{\tilde{\mathfrak{L}}} + [s(x_1) + v_1, \partial_{\tilde{l}}(s(x_2) + v_2), s(x_3) + v_3]_{\tilde{\mathfrak{L}}} \\
&\quad + [s(x_1) + v_1, s(x_2) + v_2, \partial_{\tilde{l}}(s(x_3) + v_3)]_{\tilde{\mathfrak{L}}},
\end{aligned} \tag{4.5}$$

for any $x_1, x_2, x_3 \in \mathfrak{L}$, $v_1, v_2, v_3 \in A$.

At first, since $\tilde{\mathfrak{L}}$ is an abelian extension of $\mathfrak{L}$ by A, we have $[\tilde{\mathfrak{L}}, A, A] = 0$ and

$$\begin{aligned}
&[s(x_1) + v_1, s(x_2) + v_2, s(x_3) + v_3]_{\tilde{\mathfrak{L}}} \\
&= [s(x_1), s(x_2), s(x_3)]_{\tilde{\mathfrak{L}}} + [s(x_1), s(x_2), v_3]_{\tilde{\mathfrak{L}}} + [s(x_1), v_2, s(x_3)]_{\tilde{\mathfrak{L}}} + [v_1, s(x_2), s(x_3)]_{\tilde{\mathfrak{L}}} \\
&= [s(x_1), s(x_2), s(x_3)]_{\tilde{\mathfrak{L}}} + \theta_A(x_2, x_3)(v_1) - \theta_A(x_1, x_3)(v_2) + D_A(x_1, x_2)(v_3),
\end{aligned}$$

by Eq 2.4 the left-hand side of Eq 4.5 is

$$\begin{aligned}
\text{LHS of Eq 4.5} &= \partial_{\tilde{l}}\Big([s(x_1), s(x_2), s(x_3)]_{\tilde{\mathfrak{L}}} + \theta_A(x_2, x_3)(v_1) - \theta_A(x_1, x_3)(v_2) + D_A(x_1, x_2)(v_3)\Big) \\
&= \partial_{\tilde{l}}\Big(s([x_1, x_2, x_3]_{\mathfrak{L}}) + \omega(x_1, x_2, x_3) + \theta_A(x_2, x_3)(v_1) - \theta_A(x_1, x_3)(v_2) + D_A(x_1, x_2)(v_3)\Big).
\end{aligned}$$

Since the diagram (4.2) is commutative, we have

$$\begin{aligned}
\text{LHS of Eq 4.5} &= s(\partial_l([x_1, x_2, x_3]_{\mathfrak{L}})) + \mu([x_1, x_2, x_3]_{\mathfrak{L}}) + \partial_a(\omega(x_1, x_2, x_3)) \\
&\quad + \partial_a(\theta_A(x_2, x_3)(v_1)) - \partial_a(\theta_A(x_1, x_3)(v_2)) + \partial_a(D_A(x_1, x_2)(v_3)) \\
&= s([\partial_l(x_1), x_2, x_3]_{\mathfrak{L}}) + s([x_1, \partial_l(x_2), x_3]_{\mathfrak{L}}) + s([x_1, x_2, \partial_l(x_3)]_{\mathfrak{L}}) \\
&\quad + \mu([x_1, x_2, x_3]_{\mathfrak{L}}) + \partial_a(\omega(x_1, x_2, x_3)) + \partial_a(\theta_A(x_2, x_3)(v_1)) \\
&\quad - \partial_a(\theta_A(x_1, x_3)(v_2)) + \partial_a(D_A(x_1, x_2)(v_3)).
\end{aligned} \tag{4.6}$$

Now we compute the right-hand side of Eq 4.5. Note that, since $\partial_{\tilde{l}}|_A = \partial_a$, it holds that

$$\begin{aligned}
\partial_{\tilde{l}}(s(x_i) + v_i) &= \partial_{\tilde{l}}(s(x_i)) + \partial_a(v_i) \\
&= \partial_{\tilde{l}}(s(x_i)) - s(\partial_l(x_i)) + s(\partial_l(x_i)) + \partial_a(v_i) \\
&= s(\partial_l(x_i)) + \mu(x_i) + \partial_a(v_i) \in s(\mathfrak{L}) \oplus A.
\end{aligned}$$

By this the right-hand side of Eq 4.5 is

$$\begin{aligned}
\text{RHS of Eq 4.5} &= [s(\partial_l(x_1)) + \mu(x_1) + \partial_a(v_1), s(x_2) + v_2, s(x_3) + v_3]_{\tilde{\mathfrak{L}}} \\
&\quad + [s(x_1) + v_1, s(\partial_l(x_2)) + \mu(x_2) + \partial_a(v_2), s(x_3) + v_3]_{\tilde{\mathfrak{L}}} \\
&\quad + [s(x_1) + v_1, s(x_2) + v_2, s(\partial_l(x_3)) + \mu(x_3) + \partial_a(v_3)]_{\tilde{\mathfrak{L}}} \\
&= [s(\partial_l(x_1)), s(x_2), s(x_3)]_{\tilde{\mathfrak{L}}} + [s(\partial_l(x_1)), s(x_2), v_3]_{\tilde{\mathfrak{L}}} \\
&\quad + [s(\partial_l(x_1)), v_2, s(x_3)]_{\tilde{\mathfrak{L}}} + [\mu(x_1), s(x_2), s(x_3)]_{\tilde{\mathfrak{L}}} \\
&\quad + [s(x_1), \mu(x_2), s(x_3)]_{\tilde{\mathfrak{L}}} + [\partial_a(v_1), s(x_2), s(x_3)]_{\tilde{\mathfrak{L}}} \\
&\quad + [s(x_1), s(\partial_l(x_2)), v_3]_{\tilde{\mathfrak{L}}} + [v_1, s(\partial_l(x_2)), s(x_3)]_{\tilde{\mathfrak{L}}} \\
&\quad + [s(x_1), \partial_a(v_2), s(x_3)]_{\tilde{\mathfrak{L}}} + [s(x_1), s(x_2), s(\partial_l(x_3))]_{\tilde{\mathfrak{L}}} \\
&\quad + [s(x_1), s(x_2), \mu(x_3)]_{\tilde{\mathfrak{L}}} + [s(x_1), s(x_2), \partial_a(v_3)]_{\tilde{\mathfrak{L}}} \\
&\quad + [s(x_1), v_2, s(\partial_l(x_3))]_{\tilde{\mathfrak{L}}} + [v_1, s(x_2), s(\partial_l(x_3))]_{\tilde{\mathfrak{L}}} \\
&\quad + [s(x_1), s(\partial_l(x_2)), s(x_3)]_{\tilde{\mathfrak{L}}}.
\end{aligned} \tag{4.7}$$

By Eqs 4.6 and 4.7, we have

$$s([\partial_l(x_1), x_2, x_3]_{\mathfrak{L}}) + s([x_1, \partial_l(x_2), x_3]_{\mathfrak{L}}) + s([x_1, x_2, \partial_l(x_3)]_{\mathfrak{L}}) + \mu([x_1, x_2, x_3]_{\mathfrak{L}})$$

$$
\begin{aligned}
&+ \partial_a(\omega(x_1,x_2,x_3)) + \partial_a(\theta_A(x_2,x_3)(v_1)) - \partial_a(\theta_A(x_1,x_3)(v_2)) + \partial_a(D_A(x_1,x_2)(v_3))\\
= &[s(\partial_l(x_1)), s(x_2), s(x_3)]_{\tilde{\mathfrak{L}}} + D_A(\partial_l(x_1),x_2)(v_3) - \theta_A(\partial_l(x_1),x_3)(v_2) + \theta_A(x_2,x_3)(\mu(x_1))\\
&+ \theta_A(x_2,x_3)(\partial_a(v_1)) + [s(x_1), s(\partial_l(x_2)), s(x_3)]_{\tilde{\mathfrak{L}}} + D_A(x_1,\partial_l(x_2))(v_3) - \theta_A(x_1,x_3)(\mu(x_2))\\
&- \theta_A(x_1,x_3)(\partial_a(v_2)) + \theta_A(\partial_l(x_2),x_3)(v_1) + [s(x_1), s(x_2), s(\partial_l(x_3))]_{\tilde{\mathfrak{L}}} + D_A(x_1,x_2)(\mu(x_3))\\
&+ D_A(x_1,x_2)(\partial_a(v_3)) - \theta_A(x_1,\partial_l(x_3))(v_2) + \theta_A(x_2,\partial_l(x_3))(v_1).
\end{aligned}
$$

Then

$$
\begin{aligned}
0 = &- \omega(\partial_l(x_1),x_2,x_3) - \omega(x_1,\partial_l(x_2),x_3) - \omega(x_1,x_2,\partial_l(x_3)) + \partial_a(\omega(x_1,x_2,x_3))\\
&- D_A(x_1,x_2)(\mu(x_3)) - \theta_A(x_2,x_3)(\mu(x_1)) + \theta_A(x_1,x_3)(\mu(x_2)) + \mu([x_1,x_2,x_3]_{\mathfrak{L}})\\
&+ \Big(\partial_a D_A(x_1,x_2) - D_A(x_1,x_2)\partial_a - D_A(\partial_l(x_1),x_2) - D_A(x_1,\partial_l(x_2))\Big)(v_3)\\
&+ \Big(\partial_a \theta_A(x_2,x_3) - \theta_A(x_2,x_3)\partial_a - \theta_A(\partial_l(x_2),x_3) - \theta_A(x_2,\partial_l(x_3))\Big)(v_1)\\
&- \Big(\partial_a \theta_A(x_1,x_3) - \theta_A(x_1,x_3)\partial_a - \theta_A(\partial_l(x_1),x_3) - \theta_A(x_1,\partial_l(x_3))\Big)(v_2).
\end{aligned}
$$

Since (∂_a, ∂_l) is compatible and by the proof of Lemma 3.1,

$$
\begin{aligned}
&\Big(\partial_a D_A(x_1,x_2) - D_A(x_1,x_2)\partial_a - D_A(\partial_l(x_1),x_2) - D_A(x_1,\partial_l(x_2))\Big)(v_3) = 0,\\
&\Big(\partial_a \theta_A(x_2,x_3) - \theta_A(x_2,x_3)\partial_a - \theta_A(\partial_l(x_2),x_3) - \theta_A(x_2,\partial_l(x_3))\Big)(v_1) = 0,\\
&\Big(\partial_a \theta_A(x_1,x_3) - \theta_A(x_1,x_3)\partial_a - \theta_A(\partial_l(x_1),x_3) - \theta_A(x_1,\partial_l(x_3))\Big)(v_2) = 0.
\end{aligned}
\tag{4.8}
$$

Thus we have

$$
\begin{aligned}
&\partial_a(\omega(x_1,x_2,x_3)) - \omega(\partial_l(x_1),x_2,x_3) - \omega(x_1,\partial_l(x_2),x_3) - \omega(x_1,x_2,\partial_l(x_3))\\
&- D_A(x_1,x_2)(\mu(x_3)) - \theta_A(x_2,x_3)(\mu(x_1)) + \theta_A(x_1,x_3)(\mu(x_2)) + \mu([x_1,x_2,x_3]_{\mathfrak{L}}) = 0,
\end{aligned}
\tag{4.9}
$$

which is exactly Eq 4.4 due to Eq 3.1 and the definition of 3-cohomology group. So $[\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(\partial_a,\partial_l)}] = 0$ as required.

($\Leftarrow$) Assume that $[\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(\partial_a,\partial_l)}]$ is trivial. Then there is a map $\mu : \mathfrak{L} \longrightarrow A$ such that $\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(\partial_a,\partial_l)} = \delta\mu$. For any $s(x) + v \in \tilde{\mathfrak{L}}$, define $\partial_{\tilde{l}} : \tilde{\mathfrak{L}} \longrightarrow \tilde{\mathfrak{L}}$ by

$$
\partial_{\tilde{l}}(s(x) + v) = s(\partial_l(x)) + \mu(x) + \partial_a(v),
$$

then the associated diagram in (4.2) is commutative: for any $x \in \mathfrak{L}$, $v \in A$,

$$
\begin{aligned}
(\pi \circ \partial_{\tilde{l}})(s(x) + v) &= \pi(s(\partial_l(x)) + \mu(x) + \partial_a(v)) = \partial_l(x) = (\partial_l \circ \pi)(s(x) + v);\\
\partial_{\tilde{l}} \circ \iota(v) &= \partial_{\tilde{l}}(v) = \partial_a(v) = \iota \circ \partial_a(v).
\end{aligned}
$$

Moreover, since (∂_a, ∂_l) satisfies Eq 3.2, by Eqs 4.8 and 4.9, it follows that Eq 4.5 holds by Eqs 4.6 and 4.7, that is, $\partial_{\tilde{l}} \in \mathrm{Der}(\tilde{\mathfrak{L}})$. □

Hence, for a compatible pair $(\partial_a, \partial_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$, the cohomology class $[\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(\partial_a,\partial_l)}]$ can be regarded as an obstruction class to extensibility.

By Proposition 4.3 and Theorem 4.4 we have the following result.

Corollary 4.5. *Let $\tilde{\mathfrak{L}}$ be an abelian extension of a Lie triple system $\mathfrak{L}$ by A. If $H^3(\mathfrak{L}, A) = 0$, then any pair $(\partial_a, \partial_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$ is compatible if and only if it is extensible.*

We know that the condition $H^3(\mathfrak{L}, A) = 0$ is in general not equivalent to split property of extensions. However, we still have the following result.

Corollary 4.6. *Let $\tilde{\mathfrak{L}}$ be a split abelian extension of a Lie triple system $\mathfrak{L}$ by A. Then any pair $(\partial_a, \partial_l) \in \mathrm{Der}(A) \times \mathrm{Der}(\mathfrak{L})$ is compatible if and only if it is extensible.*

Proof. ($\Leftarrow$) It holds due to Proposition 4.3.

($\Rightarrow$) Since the extension is split there exists a section s', which is a homomorphism. Let $\theta_{s'}$, (resp. $\omega_{s'}$) be defined by Eq 2.3 (resp. Eq 2.4) with respect to s'. Then we get $\omega_{s'} = 0$. By Eq 3.1, we have $\mathrm{Ob}^{\omega_{s'}}_{(\partial_a,\partial_l)} = 0$. In view of Lemmas 3.1 and 4.1, one has $[\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(\partial_a,\partial_l)}] = [\mathrm{Ob}^{\omega_{s'}}_{(\partial_a,\partial_l)}] = 0$. Then by Theorem 4.4, we deduce that (∂_a, ∂_l) is extensible as required. □

Let us end this section with the relation between the representation Φ of G_{θ_A} and extensibility of derivations.

Definition 4.7. Let θ_A be a representation of $\mathfrak{L}$ on A and $\tilde{\mathfrak{L}}$ an abelian extension of Lie triple systems $\mathfrak{L}$ by A. So there exists a section s of π such that the representation θ_s defined by Eq 2.3 is the same as θ_A, then the extension is called reversible with respect to θ_A.

Example 4.8. Let θ_A be a representation of $\mathfrak{L}$ on A. By Proposition 2.8 we have a Lie triple system $\tilde{\mathfrak{L}} := \mathfrak{L} \oplus A$ with the bracket given by

$$[x + u, y + v, z + w]_{\tilde{\mathfrak{L}}} = [x, y, z]_{\mathfrak{L}} + \theta_A(y, z)(u) - \theta_A(x, z)(v) + D_A(x, y)(w),$$

where $x, y, z \in \mathfrak{L}$, $u, v, w \in A$. By Corollary 2.9, we have an abelian extension

$$0 \longrightarrow A \longrightarrow \tilde{\mathfrak{L}} \xrightarrow{p} \mathfrak{L} \longrightarrow 0\,,$$

and p is the canonical projection. Choose a section s of p given by $s(x) = x$, $x \in \mathfrak{L}$. Then $s : \mathfrak{L} \longrightarrow \tilde{\mathfrak{L}}$ is a homomorphism, and hence the representation θ_s given by Eq 2.3 is the same as θ_A which means that the extension

$$0 \longrightarrow A \longrightarrow \tilde{\mathfrak{L}} \xrightarrow{p} \mathfrak{L} \longrightarrow 0$$

is reversible with respect to θ_A.

Theorem 4.9. *Let θ_A be a representation of $\mathfrak{L}$ on A. The pair $(\partial_a, \partial_l) \in G_{\theta_A}$ is extensible in every reversible extension if and only if $\Phi(\partial_a, \partial_l) = 0$.*

Proof. ($\Rightarrow$) For any $[\varphi] \in H^3(\mathfrak{L}, A)$, by Corollary 2.9, there is an abelian extension

$$0 \longrightarrow A \longrightarrow \tilde{\mathfrak{L}} \xrightarrow{p} \mathfrak{L} \longrightarrow 0\,,$$

where p is the canonical projection and the bracket on $\tilde{\mathfrak{L}} := \mathfrak{L} \oplus A$ is given by

$$[x + u, y + v, z + w]_{\tilde{\mathfrak{L}}} = [x, y, z]_{\mathfrak{L}} + \varphi(x, y, z) + \theta_A(y, z)(u) - \theta_A(x, z)(v) + D_A(x, y)(w),$$

for any $x, y, z \in \mathfrak{L}$, $u, v, w \in A$. Choose a section s of p defined by $s(x) = x$, for any $x \in \mathfrak{L}$. Then the representation θ_s is given by Eq 2.3. Let $H^n_{\theta_s}(\mathfrak{L}, A)$ denote the cohomology group with respect to θ_s. We first show that θ_s is the same as θ_A for $x, y \in \mathfrak{L}$, $v \in A$,

$$\theta_s(x, y)(v) = [v, s(x), s(y)]_{\tilde{\mathfrak{L}}} = [v, x, y]_{\tilde{\mathfrak{L}}} = \theta_A(x, y)(v).$$

Hence we obtain that the cohomology group $H^n_{\theta_s}(\mathfrak{L}, A)$ associated to θ_s is the same as $H^n(\mathfrak{L}, A)$. Since we define ω_s by Eq 2.4 (resp. φ) is a 3-cocycle in $H^n_{\theta_s}(\mathfrak{L}, A)$ (resp. in $H^n(\mathfrak{L}, A)$), we have $[\omega_s] = [\varphi]$. Then

$$\begin{aligned}\Phi(\partial_a, \partial_l)([\varphi]) &= \Phi(\partial_a, \partial_l)([\omega_s]) \quad \text{(by Lemma 3.3)}\\ &= [\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(\partial_a, \partial_l)}]\\ &= 0. \quad \text{(by Theorem 4.4)}\end{aligned}$$

($\Leftarrow$) Suppose $\Phi(\partial_a, \partial_l) = 0$. For any reversible abelian extension

$$0 \longrightarrow A \longrightarrow \tilde{\mathfrak{L}} \xrightarrow{p} \mathfrak{L} \longrightarrow 0,$$

there exists a section s of π such that the representation θ_s is the same as θ_A. Therefore, ω_s defined by Eq 2.4 is a 3-cocycle in $H^3(\mathfrak{L}, A)$. Then we have

$$[\mathrm{Ob}^{\tilde{\mathfrak{L}}}_{(\partial_a, \partial_l)}] = \Phi(\partial_a, \partial_l)([\omega_s]) = 0.$$

By Theorem 4.4, (∂_a, ∂_l) is extensible. This completes the proof. □

The following corollary is straightforward.

Corollary 4.10. *Let θ_A be a representation of $\mathfrak{L}$ on A. Then any pair $(\partial_a, \partial_l) \in G_{\theta_A}$ is extensible in every reversible extension if and only if $\Phi \equiv 0$.*

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Conflict of interest

The authors declare there is no conflicts of interest.

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