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Research article

Image restoration via Picard's and Mountain-pass Theorems

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Abstract: In this work, we present existence results for some problems which arise in image processing namely image restoration. Our essential tools are Picard's fixed point theorem for a strict contraction and Mountain-pass Theorem for critical point.

Keywords: image restoration; fixed point; Picard Theorem; Mountain-pass Theorem; image processing

1. Introduction and mathematical preliminaries

In image processing, some degradations often affect an image during its acquisition or transmission. These deteriorations can be related to the image acquisition modality or the noise coming from any signal transmission. Image restoration is an important step in image processing. It permits to improve the quality of the image. There are some purposes of image restoration such as removing the noise or the blur, and adding some informations to the image. The main idea is to estimate the original image from the degraded.

The image restoration problem is modeled in [1] as follows. Let $\Omega \subset \mathbb{R}^2$ be a bounded subset (the domain of the image u) where $u : \Omega \to \mathbb{R}$ is the original image describing a real scene, and f be the observed image of the same scene (that is the degraded version of u). Assume that the image u has been degraded by a noise η and a linear operator φ which is not necessarily invertible, even if it is invertible, its inverse is difficult to calculate numerically.

We seek to recover u from f such that

$$f = \varphi u + \eta. \tag{1.1}$$

The idea is to find an approximation of u by solving the problem

$$\inf_u \int_\Omega |f - \varphi u|^2 dx. \tag{1.2}$$

When the noise is neglected and if u is a minimum of (1.2), then u satisfies the equation

$$\varphi^* f = \varphi^* \varphi u \tag{1.3}$$

with φ^* which is the adjoint of φ. Since φ is not necessarily invertible, one can get u from (1.3).

The problem of image restoration is an ill posed problem, generally its resolution is based on numerical approaches; such as gradient descent which gives approximate solutions. The direct method in the calculus of variations [1, 2] is the most used when we study the problem of restoration from a theoretical point of view seeking to show that the problem admits at least a solution, and therefore one wonders if other minimization theorems such as Mountain-Pass Theorem can be applicable. Recently, authors have introduced fixed point theorems in image processing [3–6]. For detailed information on metric spaces and well-known fixed point theorems, see [7–9].

Besides, recent papers deal with image restoration using fixed point theory [10, 11] where authors propose new TVL1, TVL2 regularization model for image restoration. It is obvious to see the efficiency of fixed point theorems in image restoration using other regularization terms. However, the variational approach in the image restoration is the most classic. Indeed, the idea of Tikhonov and Arsenin in 1977 was to regularize the problem (1.2), that is to consider a related problem that admits a unique solution. The authors proposed to consider the problem

$$\inf_u \left(\int_\Omega |f - \varphi u|^2 dx \ + \ \alpha \int_\Omega |\nabla|^2 dx \right), \qquad \alpha > 0. \tag{1.4}$$

The functional

$$E_\alpha = \int_\Omega |f - \varphi u|^2 dx \ + \ \alpha \int_\Omega |\nabla|^2 dx, \qquad \alpha > 0 \tag{1.5}$$

is well-defined in the space $H^1(\Omega)$ and the Euler-Lagrange equation associated to the problem

$$\inf_u \left\{ E_\alpha, \ u \in H^1(\Omega) \right\} \tag{1.6}$$

is

$$\varphi^* \varphi u - \varphi^* f - \alpha \Delta u = 0 \tag{1.7}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial N} = 0 \quad \text{on} \quad \partial\Omega \tag{1.8}$$

where N is the outward normal to $\partial\Omega$, $\int_\Omega |f - \varphi u|^2 dx$ is the fidelity term to the data, $\alpha \int_\Omega |\nabla|^2 dx$ is a smoothing term and α is a positive weighting constant. It was proven in [1], that under suitable assumptions on φ, the problem (1.6) has a unique solution in the space $H^1(\Omega)$.

2. Main results

Our first main result is to prove existence and uniqueness of solution for the problem (1.6) in the case where the operator φ designates the identity operator. Our tool is a fixed point theorem. That is to rewrite the boundary value problem

$$\begin{cases} -\alpha\Delta u + u = f \quad \text{on } \Omega \\ \dfrac{\partial u}{\partial N} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.1}$$

as a fixed point problem, namely $u = Au$, where the operator A will be defined later; then applying fixed point theorems such as Banach's and Picard's contraction principles.

Let us consider the following boundary value problem:

$$\begin{cases} -\alpha\Delta u + u = f, \quad \text{on } \Omega \\ \dfrac{\partial u}{\partial N} = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $u \in H^1(\Omega)$ and $f \in L^2(\Omega)$. This problem can be written as follows:

$$\begin{cases} -\Delta u = \dfrac{1}{\alpha}(f - u) \quad \text{on } \Omega \\ \dfrac{\partial u}{\partial N} = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.3}$$

Our main result is stated as follows:

Theorem 2.1. *Let $f \in L^2(\Omega)$ and assume that*

$$0 < \frac{1}{\alpha}\|(-\Delta)^{-1}\|_{\mathcal{L}(L^2(\Omega))} < 1.$$

Then the problem (2.3) has a unique solution u_0 in $H^1(\Omega)$.

To proof Theorem 2.1, we need Picard's Theorem.

Theorem 2.2. *(Picard's Theorem)([12]) Let (X, d) be a complete metric space and let $T : X \to X$ be a strict contraction. Then T has a unique fixed point $u_0 = Tu_0 \in X$. Furthermore, for any $v \in X$, the sequence $(T^n(v))$ converges to u_0 as $n \to +\infty$ where $T^0(v) = v$ and*

$$T^{n+1}(v) = T(T^n(v)), \quad v \in X, \quad n \in \{0, 1, 2, \ldots\}.$$

Remark 2.3. Theorem 2.2 allows us to prove the existence and uniqueness of the solution and to give an approximation of this solution at the same time.

Our proof for Theorem 2.1 is based on a fixed point approach where we define a compact operator such that assumptions of Theorem 2.2 will be satisfied.

Proof. Let us consider the operator A defined from $L^2(\Omega)$ to $L^2(\Omega)$ such that:

$$Au = \frac{1}{\alpha}(-\Delta)^{-1}(f - u). \tag{2.4}$$

One can see that a fixed point of the operator A satisfies (2.3). Hence to solve (2.3) we search fixed points of the operator A. It is well-known [12] that the linear operator $(-\Delta)^{-1}$ is continuous from $L^2(\Omega)$ to H_0^1, it follows from the compact embedding $i : H_0^1 \to L^2(\Omega)$ that $i \circ (-\Delta)^{-1}$ is compact from $L^2(\Omega)$ to $L^2(\Omega)$. Then the linear operator $(-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is continuous. Now, we can prove that under assumptions of Theorem 2.1, the operator A is a strict contraction on $L^2(\Omega)$. Indeed, for every $u_1, u_2 \in L^2(\Omega)$, we obtain

$$\begin{aligned}
\|Au_1 - Au_2\|_{L^2} &= \frac{1}{\alpha}\|(-\Delta)^{-1}(f-u_1) - (-\Delta)^{-1}(f-u_2)\|_{L^2} \\
&= \frac{1}{\alpha}\|(-\Delta)^{-1}(f-u_1-f+u_2)\|_{L^2} \\
&= \frac{1}{\alpha}\|(-\Delta)^{-1}(u_2-u_1)\|_{L^2} \\
&\leq \frac{1}{\alpha}\|(-\Delta)^{-1}\|_{\mathcal{L}(L^2(\Omega))}\|u_1-u_2\|_{L^2} \\
&\leq \|(-\Delta)^{-1}\|_{\mathcal{L}(L^2(\Omega))}\|u_1-u_2\|_{L^2} \\
&\leq k\|u_1-u_2\|_{L^2},
\end{aligned}$$

where $k = \frac{1}{\alpha}\|(-\Delta)^{-1}\|_{\mathcal{L}(L^2(\Omega))}$.

As $k < 1$, then A is a strict contraction and the conclusion follows from Picard's Theorem which confirms that the operator A has a unique fixed point $u_0 \in L^2(\Omega) \supset H^1(\Omega)$. Hence, the problem (2.2) has a unique solution in $H^1(\Omega)$.

Remark 2.4. This theorem allows us to choose a good value of α which gives the right approximation of the restored image.

Unfortunately, the presence of the Laplacian does not lead to a good restoration of the degraded image, given that the Laplacian is a very strong smoothing operator and does not preserve the edges. So the obtained image does not contain noise but it is very blurry. For this reason, several authors have sought to find regularization terms which allow the contours to be preserved. In this direction, authors [1] proposed another term of regularization and the study the following energy:

$$J(u) = \frac{1}{2}\int_\Omega |f - \varphi u|^2 dx \ + \ \alpha \int_\Omega \phi(|\nabla u|)dx, \quad \alpha > 0, \quad u \in V \tag{2.5}$$

where $V = \{u \in L^2(\Omega),\ \nabla u \in L^1(\Omega)\}$ is a suitable space and $u : \Omega \subset \mathbb{R}^{\varkappa} \to \mathbb{R}$, ϕ is chosen so that the smoothing is the same in all directions in places where the gradient is low and at the same time controls the smoothing in the neighborhood of contours in order to preserve them. If u is a minimum point to $J(u)$, then it satisfies the Euler-Lagrange equation:

$$\varphi^*\varphi u - \varphi^* f - \lambda div\left(\frac{\phi'(\|\nabla u\|)}{\|\nabla u\|}.\nabla u\right) = 0. \tag{2.6}$$

This equation cannot be solved in the space V because it is not reflexive. Authors in [1] calculated the relaxed functional of (2.5) in the BV_w^* and then used direct method of the calculus of variations to prove that the relaxed problem has a unique solution and then deduced the existence of a unique solution for the original problem. Recall that BV_w^* denotes the space of functions of bounded variation endowed with weak topology.

In [13], authors studied the problem

$$-div(\psi(\nabla u)) = \lambda g(x)||u||^{p-2}u, \quad x \in \mathbb{R}^{\mathbb{N}}, \tag{2.7}$$

where $1 < p < N$, $N \geq 3$ and $g(x)$ is a positive and measurable function. Using Mountain Pass Theorem, it was proven that under suitable conditions, the problem (2.7) admits a weak solution in $W^{1,p}(\mathbb{R}^N)$.

Inspired by [1, 13], we seek to show that the problem

$$-div(\psi(\nabla u)) = F(u), \quad u \in W_0^{1,2}(\Omega)$$

has a weak solution under some assumptions on F and ψ, where Ω is an open bounded set $\Omega \subseteq \mathbb{R}^{\mathbb{N}}$.

Let us consider the Hilbert space $H_0^1(\Omega)$ endowed with the scalar product

$$< u, v >_{H_0^1} =< \nabla u, \nabla v >_{L^2(\Omega)} = \int_\Omega \nabla u . \nabla v \, dx$$

and the associated norm

$$||u||_{H_0^1} = \left(\int_\Omega |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}} = ||\nabla u||_{L^2}.$$

Now, let $J(u)$ be the functional defined by

$$J(u) = \frac{1}{2} \left(||u - u_0||_{H_0^1}^2 - ||u_0||_{H_0^1}^2 \right) - \alpha \int_\Omega \phi(|\nabla u|) dx, \quad u \in H_0^1(\Omega), \quad \alpha > 0 \tag{2.8}$$

where ϕ is strictly convex and nondecreasing on $\mathbb{R}^+$ into $\mathbb{R}^+$ with

$$\phi(0) = 0 \quad \text{and} \quad \phi'(0) = 0,$$

$$\text{there exists } c > 0, \ b \geq 0, \quad cs - b \leq \phi(s) \leq cs, \quad \forall s > 0 \tag{2.9}$$

and

$$\text{there exists } \psi \in H_0^1(\Omega) \ \text{ such that } ||\psi||_{H_0^1} < 1. \tag{2.10}$$

We also assume that:

$$\exists\, M > 0 \ \text{ such that } \frac{\phi'(s)}{s} \leq M, \quad \forall s > 0. \tag{2.11}$$

The functional $J(u)$ in (2.8) is well-defined and is class C^1 (see [12]). Using (2.8), we obtain the following:

$$J'(u) \cdot v = \int_\Omega \nabla(u - u_0) . \nabla v \, dx - \alpha \int_\Omega \frac{\phi'(|\nabla u|)}{|\nabla u|} \nabla u . \nabla v \, dx.$$

Then we have

$$J'(u) \cdot v =< u - u_0, v >_{H_0^1} - \alpha \int_\Omega \frac{\phi'(|\nabla u|)}{|\nabla u|} \nabla u . \nabla v \, dx.$$

Let z be such that $\nabla z = \dfrac{\phi'(|\nabla u|)}{|\nabla u|}\nabla u$, then

$$J'(u)\cdot v =< u-u_0, v >_{H_0^1} -\alpha\int_\Omega \nabla z\nabla u,$$

hence

$$J'(u)\cdot v =< u-u_0, v >_{H_0^1} -\alpha < z, v >_{H_0^1},$$

it follows that

$$J'(u) = (u-u_0) - \alpha z.$$

If u is such that $J'(u) = 0$ that is u is a critical point for J in other words to solve the equation $J'(u) = 0$, we seek for critical points for J. An important theorem in this area is a minimization theorem named Mountain Pass Theorem.

Theorem 2.5. *(Mountain Pass Theorem [14, 15]) Let H be a Banach space and* $J \in C^1(H, \mathbb{R})$ *satisfies the Palais-Smale condition. Assume* $J(0) = 0$ *and there exist positive numbers* ρ *and* δ *such that:*

1) $J(u) \geq \delta$ *if* $\|u\| = \rho$,

2) There exists $\vartheta \in H$ *such that* $\|\vartheta\| > \rho$ *and* $J(\vartheta) < \delta$.

Then μ *is a critical value of J with* $\mu \geq \delta$, *where*

$$\mu = \min_{A\in\Gamma}\max_{u\in A} J(u),$$

and

$$\Gamma = \{\gamma([0,1]) :\ \gamma \in C([0,1], H),\ \gamma(0) = 0,\ \gamma(1) = \vartheta\}.$$

Definition 2.6. (Palais-Smale condition) [15–17] Let $J \in C^1(H, \mathbb{R})$. If any sequence $(u_n) \subset H$ for which $(J(u_n))$ is bounded in $\mathbb{R}$ and $J'(u_n) \to 0$ as $n \to +\infty$ in H possesses a convergent subsequence, then we say that J satisfies Palais-Smale condition.

Our second main result is the following theorem:

Theorem 2.7. *Under assumptions* (2.9)–(2.11)*, the equation* $J'(u) = 0$ *admits at least a nontrivial solution.*

Proof. We will show that all conditions of Theorem 2.5 are satisfied.

1) It can be easily seen that $J(0) = 0$.

2) $J(u)$ satisfies the Palais-Smale condition. Indeed, let (u_n) be a sequence $H_0^1(\Omega)$ such that $J(u_n)$ is bounded and $J'(u_n) \to 0$ in $H^{-1}(\Omega)$ which is the dual space of $H_0^1(\Omega)$, we prove that (u_n) admits a convergent subsequence in $H_0^1(\Omega)$. Since $J(u_n)$ is bounded, there exists $K > 0$ such that

$$\begin{aligned} K &\geq J(u_n) \\ &\geq \frac{1}{2}\|u_n\|^2_{H_0^1} - \langle u_n,\, u_0\rangle - \alpha\int_\Omega c|\nabla u_n|dx \\ &\geq \frac{1}{2}\|u_n\|^2_{H_0^1} - \|u_n\|_{H_0^1}\|u_0\|_{H_0^1} - \alpha c\|\nabla u_n\|_{L^1} \end{aligned}$$

$$\geq \frac{1}{2}\|u_n\|^2_{H^1_0} - \|u_n\|_{H^1_0}\|u_0\|_{H^1_0} - \alpha\eta c\|\nabla u_n\|_{L^2}$$
$$\geq \frac{1}{2}\|u_n\|^2_{H^1_0} - \|u_n\|_{H^1_0}\|u_0\|_{H^1_0} - \alpha\eta c\|u_n\|_{H^1_0}.$$

As Ω is bounded, we use Cauchy-Schwartz inequality and the continuous embedding of $L^2(\Omega)$ in $L^1(\Omega)$. (u_n) is bounded in the reflexive space $H^1_0(\Omega)$. Then there exist a subsequence u_{n_k} and w such that $u_{n_k} \to w$ in $H^1_0(\Omega)$ as $n_k \to +\infty$. Using the compact embedding $H^1_0(\Omega)$ in $L^2(\Omega)$, we have $u_{n_k} \to w$ in $L^2(\Omega)$ as $n_k \to +\infty$. Now, we have to show that $\nabla u_{n_k} \to \nabla w$ in $L^2(\Omega)$ as $n_k \to +\infty$ to deduce that $u_{n_k} \to w$ in $H^1_0(\Omega)$ as $n_k \to +\infty$. From the definition of weak convergence, we have

$$u_{n_k} - w \to 0 \text{ in } H^1_0(\Omega)$$

and

$$\lim_{n_k\to+\infty} \langle J'(u_{n_k} - w), (u_{n_k} - w)\rangle \to 0.$$

On the other hand, by the following statements

$$0 = \lim_{n_k\to+\infty} J'(u_{n_k}) \cdot (u_{n_k} - w) = \lim_{n_k\to+\infty} \int_\Omega \nabla (u_{n_k} - u_0) \nabla(u_{n_k} - w)\, dx$$
$$- \alpha \lim_{n_k\to+\infty} \int_\Omega \frac{\phi'(|\nabla(u_{n_k})|)}{|\nabla(u_{n_k})|} \nabla(u_{n_k}).\nabla(u_{n_k} - w) dx,$$

this last result cannot be true unless

$$\nabla(u_{n_k} - w) \rightharpoonup 0 \text{ in } L^2(\Omega),$$

i.e., $\nabla(u_{n_k}) \rightharpoonup \nabla w$ in $L^2(\Omega)$. Indeed, if $\nabla(u_{n_k} - w) \in L^2(\Omega)$, then

$$\lim_{n_k\to+\infty} \int_\Omega \nabla (u_{n_k} - u_0) \nabla(u_{n_k} - w)\, dx = 0$$

when $\nabla(u_{n_k}) \rightharpoonup \nabla w$ in $L^2(\Omega)$. On the other hand, by (2.11) and the fact that $u_{n_k} \in H^1_0(\Omega)$, we have

$$\frac{\phi'(|\nabla(u_{n_k})|)}{|\nabla(u_{n_k})|} \nabla(u_{n_k}) \text{ in } L^2(\Omega).$$

Then,

$$\lim_{n_k\to+\infty} \int_\Omega \frac{\phi'(|\nabla(u_{n_k})|)}{|\nabla(u_{n_k})|} \nabla(u_{n_k}).\nabla(u_{n_k} - w) dx = 0$$

when $\nabla(u_{n_k}) \rightharpoonup \nabla w$ in $L^2(\Omega)$.

3) Let $\eta > 0$ be the constant of the continuous embedding of $L^2(\Omega)$ in $L^1(\Omega)$ and

$$\rho > 2\left(\|u_0\|_{H^1_0} + \alpha\eta c\right).$$

Then there exists $\delta > 0$ such that if $\|u\| = \rho$, then $J(u) > \delta$. In fact,

$$J(u) \geq \frac{1}{2}\|u\|^2_{H^1_0} - \|u_0\|_{H^1_0}\|u\|_{H^1_0} - \alpha \int_\Omega c|\nabla u|\, dx$$

$$
\begin{aligned}
&\geq \frac{1}{2}\|u\|_{H_0^1}^2 - \|u_0\|_{H_0^1}\|u\|_{H_0^1} - \alpha c\eta\|\nabla u\|_{L^2} \\
&\geq \frac{1}{2}\|u\|_{H_0^1}^2 - \|u_0\|_{H_0^1}\|u\|_{H_0^1} - \alpha c\eta\|u\|_{H_0^1} \\
&\geq \|u\|_{H_0^1}\left[\frac{1}{2}\|u\|_{H_0^1} - \|u_0\|_{H_0^1} - \alpha c\eta\right] \\
&\geq \rho\left(\frac{1}{2}\rho - \|u_0\|_{H_0^1} - \alpha c\eta\right) \\
&\geq \delta,
\end{aligned}
$$

with $\delta = \rho\left(\frac{1}{2}\rho - \|u_0\|_{H_0^1} - \alpha c\eta\right)$.

4) Now, we will show that $\exists\vartheta \in H_0^1(\Omega)$ such that $\|\vartheta\| > \rho$ and $J(\vartheta) < \delta$. Putting $\vartheta = \lambda\psi$ with $\lambda > 0$ and

$$\langle\psi,\ u_0\rangle_{H_0^1} + \alpha c\|\nabla\psi\|_{L^1} > \frac{1}{2}\lambda,$$

then we have

$$
\begin{aligned}
J(\vartheta) &\leq \frac{1}{2}\lambda^2\|\psi\|_{H_0^1}^2 - \lambda\langle\psi,\ u_0\rangle_{H_0^1} + \alpha\int_\Omega (-c|\nabla\lambda\psi| + b)\ dx \\
&\leq \frac{1}{2}\lambda^2\|\psi\|_{H_0^1}^2 - \lambda\langle\psi,\ u_0\rangle_{H_0^1} - \lambda\alpha c\|\nabla\psi\|_{L^1} + b\alpha|\Omega| \\
&\leq \frac{1}{2}\lambda^2\|\psi\|_{H_0^1}^2 - \frac{1}{2}\lambda^2 + b\alpha|\Omega| \\
&\leq \frac{1}{2}\lambda^2\left[\|\psi\|_{H_0^1} - 1 + \frac{\alpha b|\Omega|}{2\lambda^2}\right]
\end{aligned}
$$

and thus $J(\vartheta) \to -\infty$ as $\lambda \to +\infty$. Since $\delta > 0$, it is obvious that $J(\vartheta) < \delta$.

The functional J satisfies the hypotheses of Mountain Pass Theorem, so J has a critical point.

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Conflict of interest

The authors declare there is no conflicts of interest.

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