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*Research article*

## On a study of the representation of solutions of a $\Psi$ -Caputo fractional differential equations with a single delay

Mustafa Aydin<sup>1,2,\*</sup>, Nazim I. Mahmudov<sup>1</sup>, Hüseyin Aktuğlu<sup>1</sup>, Erdem Baytunç<sup>1</sup> and Mehmet S. Atamert<sup>1</sup>

<sup>1</sup> Department of Mathematics, Eastern Mediterranean University, Famagusta 99628 T.R. North Cyprus, Mersin 10, Turkey

<sup>2</sup> Department of Mathematics, Istanbul Technical University, Sariyer, Istanbul 34469, Turkey

\* **Correspondence:** Email: [mustafaaydin@itu.edu.tr](mailto:mustafaaydin@itu.edu.tr).

**Abstract:** We give a representation of solutions to linear nonhomogeneous  $\Psi$ -fractional delayed differential equations with noncommutative matrices. We newly define  $\Psi$ -delay perturbation of Mittag-Leffler type matrix function with two parameters and apply the method of variation of constants to obtain the representation of the solutions. We investigate the existence and uniqueness of solutions for a class of  $\Psi$ -fractional delayed semilinear differential equations by using Banach Fixed Point Theorem. Further, we establish the Ulam-Hyers stability result for the analyzed problem. Finally, we provide some examples to illustrate the applicability of our results.

**Keywords:** Mittag-Leffler type matrix function; fractional linear and nonlinear systems;  $\Psi$ -Caputo derivative representation of solutions; delayed perturbation

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### 1. Introduction

The origins of fractional calculus may be traced back to the late seventeenth century, when Newton's and Leibniz's work provided a foundation for the development of traditional calculus. Leibniz created the notation  $\frac{d^n}{dt^n}g(t)$  to symbolize the derivative of the function  $g$  of order  $n$ . When he conveyed this to de l'Hôpital, the second one enquired what happens if  $n = \frac{1}{2}$ . This is at the turning point for fractional calculus. The Riemann–Liouville fractional integral and derivative are the most classical fractional calculus operators [1, 2]. Caputo contributed significantly by proposing a new concept of fractional derivatives that is more suited to specific physical circumstances [3, 4]. Moreover, numerous other fractional operator families have been introduced and researched up to this point, out of which Prabhakar, Hadamard, Hilfer, Grünwald–Letnikov, Marchaud, and Erdélyi–Kober are just a few to mention [1, 2, 5, 6]. Owing to the vast number of definitions related to fractional

operator, it is necessary to define so certain generalized fractional operators that the traditional ones are special instances. This allows researchers to operate on a broad scale and demonstrate results that may subsequently be applied to specific circumstances by applied researchers [7]. For example, several generic classes of fractional operators were presented in [8].

The widespread usage of fractional differential equations (FDEs) in engineering, economics, physics, and other research fields inspired us to work on this topic [1, 3, 5, 9–11]. There are numerous ways for solving FDEs analytically or numerically in the literature. One of the most difficult issues in the field of fractional calculus is to create some adequate methods for obtaining analytic solutions for specific kinds of FDEs numerically. Some academics have been occupied with adding fractional extensions related to well-known integral transforms like the Fourier and the Laplace transforms, in the last few years [12–17].

Over the last several decades, mathematical descriptions based on FDE linked to non-integral order derivatives have shown a highly valuable stuff for describing many phenomena such as viscoelasticity, anomalous diffusion, control and stability theory, etc. Delay (or retardation) is recognized to occur in chemical processes and many areas. The rate of evolution of these processes is generally dependent on prior history, which is a distinguishing property of the corresponding mathematical models. Differential equations are used to represent these issues and are referred to as delayed differential equations. The underlying qualitative theory about Eq (1.1), particularly in the linear case, is well understood. Time-delay of FDE, which include both delays and non-integer derivatives, allow single one of delayed differential equations. This method is valuable in technical applications for building extremely realistic simulations of specific processes and systems having memory. One can use in analysis and discussion of diverse time-delayed systems, as well as the stabilization and control of these systems via state feedback. A linear system's solution is well-known  $v'(t) = Av(t)$ ,  $t \in \mathbb{R}^+$  has the form  $v(t) = e^{At}v(0)$ , where  $e^{At}$  is known as fundamental matrix in the literature. We note that finding a fundamental matrix associated with a delayed linear system becomes more complicated.

$$\begin{aligned}v'(t) &= Av(t) + Bv(t-h), \quad t \geq 0, \quad h > 0 \\v(t) &= \eta(t), \quad -h \leq t \leq 0\end{aligned}$$

where  $A, B \in \mathbb{M}_{n \times n}(\mathbb{R})$ . Under the statement of permutation(commutation) of matrices  $A$  and  $B$ , the authors in [18] offered an excellent solution for the system which is both delayed and homogeneous and linear by defining the exponential delay matrix  $e_h^{Bt}$ . In the work of [19], researchers discussed the above problem with the fractional version when  $A = \Theta$ . Mahmudov in references [20, 21] handled the fractional delay differential equations with the classical Caputo and Riemann-Liouville derivative and noncommutative coefficient matrices. Almeida in the study of [22] investigated Caputo type fractional derivative with reference to further function and remarkable results relevant to this derivative.

Motivated by the pioneer works of [18–20, 22, 23], we consider the following nonhomogeneous linear  $\Psi$ -Caputo fractional delay differential system

$$\begin{aligned}{}_C D_{\Psi(x)}^\alpha v(x) &= Av(x) + Bv(x-h) + f(x), \quad 0 < x \leq T, \quad h > 0, \\v(x) &= \eta(x), \quad -h \leq x \leq 0.\end{aligned}\tag{1.1}$$

where  ${}_{-h^+}^C D_{\Psi(x)}^\alpha$  is  $\Psi$ -Caputo fractional derivative,  $\Psi(x) : \mathbb{R} \rightarrow \mathbb{R}$  is increasing and  $\Psi'(x) \neq 0$  for every  $x \in [-h, T]$ ,  $A$  and  $B$  are constant coefficient square matrices which do not have to be permutable,  $f \in C([0, T], \mathbb{R}^n)$ , and  $\eta(x) \in C^1([-h, 0], \mathbb{R}^n)$ . After obtaining a solution of the former system (1.1), we can extend it to the following system (1.2). Here is that the nonlinear  $\Psi$ -Caputo fractional delay differential system is

$$\begin{aligned} {}_{-h^+}^C D_{\Psi(x)}^\alpha v(x) &= Av(x) + Bv(x-h) + f(x, v(x)), \quad 0 < x \leq T, \quad h > 0, \\ v(x) &= \eta(x), \quad -h \leq x \leq 0. \end{aligned} \quad (1.2)$$

Each of details is as given in the system (1.1). Note that, by choosing  $\Psi(x) = x$  and  $\Psi(x) = \ln x$  we observe the above differential equations reduces to Caputo fractional linear delay differential equations (see [20]) and Hadamard fractional linear delay differential equations, respectively. It should be stressed out that the  $\Psi$ -fractional derivative is defined with respect to another function and unifies several definitions of fractional derivatives available in the literature. Thus the  $\Psi$ -fractional derivative covers a wide class of fractional derivatives and provides a platform to obtain a particular one by fixing the function  $\Psi$ . The function space and the physical meaning were recently provided in [24, 25].

Before finishing the introduction, we remind some notations which are valid in the rest of the paper. Let  $a, b \in \mathbb{R}$  with  $a < b$  which is the set of all real numbers. Then  $\mathbb{R}^n$  is the well-known Euclidean space whose dimension is  $n \in \{1, 2, 3, \dots\}$ . Also, let

$$C([a, b], \mathbb{R}^n) = \{\mu : [a, b] \rightarrow \mathbb{R}^n : \mu \text{ is continuous}\}$$

with the maximum norm  $\|\cdot\|_\infty$ , which is

$$\|\mu\|_\infty = \max_{t \in [a, b]} \|\mu(t)\|,$$

where  $\|\cdot\|$  is an arbitrary norm on  $\mathbb{R}^n$ . Let  $AC[a, b]$  be the space of functions which are absolutely continuous on  $[a, b]$ . For  $n \in \{1, 2, 3, \dots\}$  we denote by  $AC^n[a, b]$  the space of complex-valued functions  $f(x)$  which have continuous derivatives up to order  $n - 1$  on  $[a, b]$  such that  $f^{(n-1)}(x) \in AC[a, b]$ .

## 2. Preliminaries

**Definition 2.1.** Let a function  $f$  and an increasing function  $\Psi$  on  $[a, b]$  be integrable and continuously differentiable, respectively and let  $\Psi'(t) \neq 0$   $t \in [a, b]$ .  $\Psi$ -Riemann-Liouville (RL) fractional integrals of  $f$  of order  $\alpha > 0$  are given by [1]

$$\begin{aligned} {}_{a^+}^{RL} I_{\Psi(t)}^\alpha f(t) &:= \frac{1}{\Gamma(\alpha)} \int_a^t \Psi'(s) (\Psi(t) - \Psi(s))^{\alpha-1} f(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (\Psi(t) - \Psi(s))^{\alpha-1} f(s) d\Psi(s), \end{aligned}$$

and  $\Psi$ -RL fractional derivatives of  $f$  of order  $\alpha > 0$  are given by

$${}_{a^+}^{RL} D_{\Psi(t)}^\alpha f(t) := \left( \frac{1}{\Psi'(t)} \frac{d}{dt} \right)^\alpha {}_{a^+} I_{\Psi(t)}^{n-\alpha} f(t)$$

$$\begin{aligned}
&= \frac{1}{\Gamma(n-\alpha)} \left( \frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n \int_a^t \Psi'(s) (\Psi(t) - \Psi(s))^{n-\alpha-1} f(s) ds \\
&= \frac{1}{\Gamma(n-\alpha)} \left( \frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n \int_a^t (\Psi(t) - \Psi(s))^{n-\alpha-1} f(s) d\Psi(s),
\end{aligned}$$

where  $n = [\alpha] + 1$ .

**Definition 2.2.** Let  $\alpha \in \mathbb{R}^+$  and  $n \in \mathbb{N}$  [1]. If  $f, \Psi \in AC^n([a, b], \mathbb{R})$  with  $\Psi$  is increasing and  $\Psi'(x) \neq 0$  for every  $x \in [a, b]$ , then the left  $\Psi$ -Caputo fractional derivative of  $f$  of order  $\alpha$  is defined as

$${}^C D_{\Psi(t)}^\alpha f(t) := {}^C I_{\Psi(t)}^{n-\alpha} \left( \frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n f(t) \quad (2.1)$$

where  $n = [\alpha] + 1$ . Shortly, we use the abbreviation symbol as

$$f_{\Psi}^{[n]}(t) := \left( \frac{1}{\Psi'(t)} \frac{d}{dt} \right)^n f(t)$$

Clearly,

$${}^C D_{\Psi(t)}^\alpha c = 0$$

where  $c$  is a constant number.

**Theorem 2.3.** Let  $f \in AC^n([a, b], \mathbb{R})$  and  $\alpha \in \mathbb{R}^+$  [1], then

$$\left( {}^C D_{\Psi(t)}^\alpha f \right) (t) = {}^{RL} D_{\Psi(t)}^\alpha \left[ f(t) - \sum_{k=0}^{n-1} \frac{(\Psi(t) - \Psi(a))^k f_{\Psi}^{[k]}(a)}{k!} \right]$$

**Lemma 2.4.** Let  $\Re(\alpha) \geq 0$  and  $\Re(\beta) > 0$  [1], then we have

$${}^C D_{\Psi(t)}^\alpha (\Psi(x) - \Psi(a))^{\beta-1} (t) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} (\Psi(t) - \Psi(a))^{\beta-\alpha-1}.$$

**Definition 2.5.** Two parameters Mittag-Leffler type matrix function  $\Upsilon_{\alpha,\beta}(A, t) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  is defined by

$$\Upsilon_{\alpha,\beta}(t) := t^{\beta-1} E_{\alpha,\beta}(At^\alpha) := t^{\beta-1} \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, t \in \mathbb{R}$$

**Definition 2.6.** Delayed two parameters Mittag-Leffler type matrix  $E_{h,\alpha,\beta}^B : \mathbb{R} \rightarrow \mathbb{R}^n$  is given by

$$E_{h,\alpha,\beta}^B(t) = \begin{cases} \Theta, & -\infty < t \leq -h \\ I \frac{(h+t)^{\beta-1}}{\Gamma(\beta)}, & -h < t \leq 0 \\ \sum_{j=0}^l \frac{B^j (t-(j-1)h)^{j\alpha+\beta-1}}{\Gamma(j\alpha+\beta)}, & (l-1)h < t \leq lh \end{cases}$$

where  $l \in \mathbb{N}^+$  and  $\Theta$  and  $I$  are the zero and identity matrices.

**Definition 2.7.** The delayed perturbation of Mittag-Leffler type matrix function  $X_{h,\alpha,\beta}^{A,B}(\cdot) : [0, \infty) \rightarrow \mathbb{R}^n$  generated by  $A, B$  is defined by [20]

$$X_{h,\alpha,\beta}^{A,B}(t) := \begin{cases} \Theta, & -h \leq t < 0, \\ I, & t = 0, \\ \sum_{k=0}^{\infty} \sum_{j=0}^p Q_{k+1}(jh) \frac{(t-jh)^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}, & ph < t \leq (p+1)h, \end{cases}$$

where  $Q_{k+1}(jh) = AQ_k(jh) + BQ_k(jh-h)$ ,  $Q_0(s) = Q_k(-h) = \Theta$ ,  $Q_1(0) = I$  for  $k = 0, 1, 2, \dots$  and  $s = 0, h, 2h, \dots$ .  $\Theta$  and  $I$  are the zero and identity matrices.

**Definition 2.8.** If  $\forall \epsilon > 0$  and for any solution  $v \in C([0, T], \mathbb{R}^n)$  of inequality

$$\left\| \left( {}^C D_{\Psi(t)}^\alpha v \right) (t) - Av(t) - Bv(t-h) - f(t, v(t)) \right\| \leq \epsilon \quad (2.2)$$

then there exists a solution  $\mu \in C([0, T], \mathbb{R}^n)$  of (1.2), and a  $u_h \in \mathbb{R}^+$  such that

$$\|v(t) - \mu(t)\| \leq u_h \cdot \epsilon \quad (2.3)$$

$\forall t \in [0, T]$ , then (1.2) is called Ulam-Hyers stable.

**Remark 2.9.** A function  $\mu \in C^1([0, T], \mathbb{R}^n)$  is a solution of the inequality (2.2) if and only if there exist at least  $h \in C([0, T], \mathbb{R}^n)$  satisfying

- $\|h(t)\| \leq \epsilon$  ( $\epsilon > 0$ ),
- $\left( {}^C D_{\Psi(t)}^\alpha \mu \right) (t) = A\mu(t) + B\mu(t-h) + f(t, \mu(t)) + h(t)$ .

### 3. Main contributions

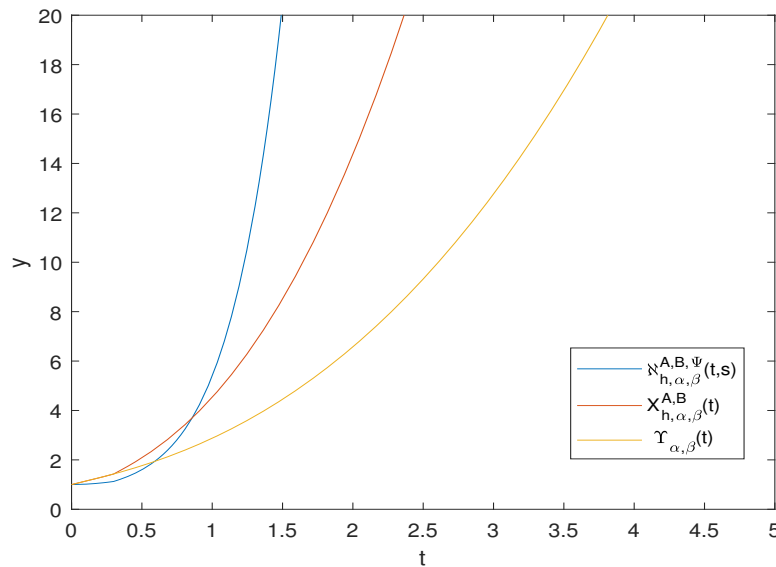
**Definition 3.1.**  $\Psi$ -delay perturbation of Mittag-Leffler type matrix function with two parameters  $\mathfrak{N}_{h,\alpha,\beta}^{A,B,\Psi} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^n$  is given as

$$\mathfrak{N}_{h,\alpha,\beta}^{A,B,\Psi}(t, s) = \begin{cases} \Theta, & t-s \in [-h, 0) \\ I, & t = s \\ \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} Q_{i+1}(jh) \frac{[\Psi(t)-\Psi(s+jh)]^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}, & t-s \in ((p-1)h, ph] \end{cases} \quad (3.1)$$

where  $\Psi(t) : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing function such that  $\Psi(0) = 0$  and  $\Psi'(t) \neq 0$  for every  $t \in [-h, T]$ ,  $\Theta$  and  $I$  represent the zero matrix and identity matrix, respectively. From [20], the matrices  $Q_k(s)$  are defined for  $s = kh$  with  $k = 0, 1, 2, \dots$  as

$$Q_0(s) = \Theta, \quad Q_1(0) = I, \quad Q_k(-h) = \Theta, \quad Q_{k+1}(s) = AQ_k(s) + BQ_k(s-h).$$

**Remark 3.2.** From Eq (3.1) choosing  $\Psi(t) = t$ , the  $\Psi$ -delay perturbation of Mittag-Leffler type matrix with two parameters reduces to the traditional one which is introduced and investigated in [20].



**Figure 1.** Comparison of functions  $\mathfrak{N}_{h, \alpha, \beta}^{A, B, \Psi}(t, s)$ ,  $X_{h, \alpha, \beta}^{A, B}(t)$ , and  $\Upsilon_{\alpha, \beta}(t)$  for  $\alpha = 0.9$ ,  $\beta = 1$ ,  $h = 0.3$ ,  $A = 1$ ,  $B = 1$ ,  $s = 0$ ,  $\Psi(t) = t^2$ .

**Lemma 3.3.**  $\mathfrak{N}_{h, \alpha, \beta}^{A, B, \Psi}(t, s)$  is jointly continuous in  $0 < s < t < \infty$ .

*Proof.* Without loss of generality we consider the case  $s = 0$ . For  $(p - 1)h < t_\eta < ph$ ,  $p = 1, 2, \dots$ . Then

$$\begin{aligned} \lim_{t \rightarrow t_\eta} \mathfrak{N}_{h, \alpha, \beta}^{A, B, \Psi}(t, 0) &= \lim_{t \rightarrow t_\eta} \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} Q_{i+1}(jh) \frac{[\Psi(t) - \Psi(jh)]^{i\alpha + \beta - 1}}{\Gamma(i\alpha + \beta)} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} Q_{i+1}(jh) \lim_{t \rightarrow t_\eta} \frac{[\Psi(t) - \Psi(jh)]^{i\alpha + \beta - 1}}{\Gamma(i\alpha + \beta)} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} Q_{i+1}(jh) \frac{[\Psi(t_\eta) - \Psi(jh)]^{i\alpha + \beta - 1}}{\Gamma(i\alpha + \beta)} \\ &= \mathfrak{N}_{h, \alpha, \beta}^{A, B, \Psi}(t_\eta, 0). \end{aligned}$$

For  $t_\eta = ph$ ,  $p = 1, 2, \dots$ . Then

$$\begin{aligned} \lim_{t \rightarrow t_\eta^-} \mathfrak{N}_{h, \alpha, \beta}^{A, B, \Psi}(t, 0) &= \lim_{t \rightarrow ph^-} \mathfrak{N}_{h, \alpha, \beta}^{A, B, \Psi}(t, 0) \\ &= \lim_{t \rightarrow ph^-} \left( \sum_{i=0}^{\infty} Q_{i+1}(0) \frac{[\Psi(t)]^{i\alpha + \beta - 1}}{\Gamma(i\alpha + \beta)} \right. \\ &\quad + \sum_{i=0}^{\infty} Q_{i+1}(h) \frac{[\Psi(t) - \Psi(h)]^{i\alpha + \beta - 1}}{\Gamma(i\alpha + \beta)} + \\ &\quad \left. \dots + \sum_{i=0}^{\infty} Q_{i+1}((p-1)h) \frac{[\Psi(t) - \Psi((p-1)h)]^{i\alpha + \beta - 1}}{\Gamma(i\alpha + \beta)} \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty} Q_{i+1}(0) \frac{[\Psi(ph)]^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} \\
&+ \sum_{i=0}^{\infty} Q_{i+1}(h) \frac{[\Psi(ph) - \Psi(h)]^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} + \\
&\cdots + \sum_{i=0}^{\infty} Q_{i+1}((p-1)h) \frac{[\Psi(ph) - \Psi((p-1)h)]^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} \\
&= \mathfrak{N}_{h,\alpha,\beta}^{A,B,\Psi}(ph, 0) = \mathfrak{N}_{h,\alpha,\beta}^{A,B,\Psi}(t_\eta, 0).
\end{aligned}$$

$$\begin{aligned}
\lim_{t \rightarrow t_\eta^+} \mathfrak{N}_{h,\alpha,\beta}^{A,B,\Psi}(t, 0) &= \lim_{t \rightarrow ph^+} \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} Q_{i+1}(jh) \frac{[\Psi(t) - \Psi(jh)]^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} Q_{i+1}(jh) \lim_{t \rightarrow ph^+} \frac{[\Psi(t) - \Psi(jh)]^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} \\
&+ \lim_{t \rightarrow ph^+} \sum_{i=0}^{\infty} Q_{i+1}(ph) \frac{[\Psi(t) - \Psi(ph)]^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} Q_{i+1}(jh) \frac{[\Psi(ph) - \Psi(jh)]^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} \\
&= \lim_{t \rightarrow ph^-} \mathfrak{N}_{h,\alpha,\beta}^{A,B,\Psi}(t, 0) = \lim_{t \rightarrow t_\eta^-} \mathfrak{N}_{h,\alpha,\beta}^{A,B,\Psi}(t, 0).
\end{aligned}$$

In brief,  $\mathfrak{N}_{h,\alpha,\beta}^{A,B,\Psi}(t, s)$  is continuous with respect to  $t \in (0, \infty)$ .  $\square$

Now, we give an explicit solution of homogenous part of (1.1), which is  $f = 0$ .

**Lemma 3.4.**  $\mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s)$  is a solution of

$${}_{-h^+}^C D_{\Psi(t)}^\alpha \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s) = A \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s) + B \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s+h). \quad (3.2)$$

*Proof.* We apply the mathematical induction method to prove that  $\mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s)$  satisfy the differential Eq (3.2) for all  $t-s \in ((p-1)h, ph]$ . For  $p=1$ ,  $0 < t-s \leq h$ , we have that

$$\mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s) = \sum_{i=0}^{\infty} Q_{i+1}(0) \frac{[\Psi(t) - \Psi(s)]^{i\alpha}}{\Gamma(i\alpha+1)}.$$

From the definition, we know that  $Q_{i+1}(0) = A^i$ . Therefore,

$$\mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s) = \sum_{i=0}^{\infty} A^i \frac{[\Psi(t) - \Psi(s)]^{i\alpha}}{\Gamma(i\alpha+1)} = E_\alpha(A[\Psi(t) - \Psi(s)]^\alpha)$$

and  $\mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s+h) = \Theta$ . By using these equalities, we get

$${}_{-h^+}^C D_{\Psi(t)}^\alpha \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s) = {}_{-h^+}^C D_{\Psi(t)}^\alpha \left( 1 + \sum_{k=1}^{\infty} A^k \frac{[\Psi(t) - \Psi(s)]^{k\alpha}}{\Gamma(k\alpha+1)} \right)$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{A^k}{\Gamma((k-1)\alpha + 1)} [\Psi(t) - \Psi(s)]^{\alpha(k-1)} \\
&= A \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha + 1)} [\Psi(t) - \Psi(s)]^{\alpha k} \\
&= A \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s) + B \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s + h).
\end{aligned}$$

Assume that the following relation

$$\mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s) = \sum_{k=0}^{\infty} \sum_{m=0}^{n-1} Q_{k+1}(mh) \frac{[\Psi(t) - \Psi(s + mh)]^{k\alpha}}{\Gamma(k\alpha + 1)}$$

holds for  $p - 1 = n - 1$ . Now, let  $p - 1 = n$ , we obtain

$$\begin{aligned}
{}_{-h^+}D_{\Psi(t)}^{\alpha} \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s) &= \sum_{k=0}^{\infty} \sum_{l=0}^n Q_{k+1}(lh) {}_{-h^+}D_{\Psi}^{\alpha} \frac{[\Psi(t) - \Psi(s + lh)]^{k\alpha}}{\Gamma(k\alpha + 1)} \\
&= \sum_{k=0}^{\infty} \sum_{l=0}^n Q_{k+1}(lh) \frac{[\Psi(t) - \Psi(s + lh)]^{(k-1)\alpha}}{\Gamma((k-1)\alpha + 1)} \\
&= \sum_{k=1}^{\infty} \sum_{l=0}^n [AQ_k(lh) + BQ_k((l-1)h)] \frac{[\Psi(t) - \Psi(s + lh)]^{(k-1)\alpha}}{\Gamma((k-1)\alpha + 1)} \\
&= \sum_{k=1}^{\infty} \sum_{l=0}^n AQ_k(lh) \frac{[\Psi(t) - \Psi(s + lh)]^{(k-1)\alpha}}{\Gamma((k-1)\alpha + 1)} \\
&\quad + \sum_{k=1}^{\infty} \sum_{l=1}^n BQ_k((l-1)h) \frac{[\Psi(t) - \Psi(s + lh)]^{(k-1)\alpha}}{\Gamma((k-1)\alpha + 1)} \\
&= \sum_{k=0}^{\infty} \sum_{l=0}^n AQ_{k+1}(lh) \frac{[\Psi(t) - \Psi(s + lh)]^{k\alpha}}{\Gamma(k\alpha + 1)} \\
&\quad + \sum_{k=0}^{\infty} \sum_{l=0}^{n-1} BQ_{k+1}(lh) \frac{[\Psi(t) - \Psi(s + lh + h)]^{k\alpha}}{\Gamma(k\alpha + 1)} \\
&= A \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s) + B \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, s + h).
\end{aligned}$$

Then, the proof is completed.  $\square$

**Lemma 3.5.** Let  $t - s \in ((p-1)h, ph]$ . We get the following equalities

$$\begin{aligned}
(a) \int_{s+mh}^t [\Psi(t) - \Psi(r)]^{-\alpha} [\Psi(r) - \Psi(s + mh)]^{(i+1)\alpha+1} d\Psi(r) &= [\Psi(t) - \Psi(s + mh)]^{i\alpha} B(1 - \alpha, i\alpha + \alpha) \\
(b) \int_s^t [\Psi(t) - \Psi(r)]^{-\alpha} \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(r, s) d\Psi(r) &= \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} Q_{i+1}(jh) [\Psi(t) - \Psi(s + jh)]^{i\alpha} \frac{\Gamma(1-\alpha)}{\Gamma(i\alpha+1)}
\end{aligned}$$

where  $B(., .)$  is the well-known beta function.

*Proof.* We apply simple substitution as  $z = \Psi(r) - \Psi(s + mh)$ , we get  $dz = d\Psi(r)$ . Thus, we have

$$\int_{s+mh}^t [\Psi(t) - \Psi(r)]^{-\alpha} [\Psi(r) - \Psi(s + mh)]^{(i+1)\alpha+1} d\Psi(r)$$



$$\begin{aligned}
&= \int_0^{\Psi(t)-\Psi(s+mh)} [\Psi(t) - \Psi(s + mh) - z]^{-\alpha} z^{(i+1)\alpha-1} dz \\
&= \int_0^{\Psi(t)-\Psi(s+mh)} [\Psi(t) - \Psi(s + mh)]^{-\alpha} \left[ 1 - \frac{z}{\Psi(t) - \Psi(s + mh)} \right]^{-\alpha} z^{(i+1)\alpha-1} dz.
\end{aligned}$$

Again, if we apply one more substitution as  $y = \frac{z}{\Psi(t)-\Psi(s+mh)}$ , then  $dy = \frac{dz}{\Psi(t)-\Psi(s+mh)}$ . Then we get,

$$\begin{aligned}
&\int_{s+mh}^t [\Psi(t) - \Psi(r)]^{-\alpha} [\Psi(r) - \Psi(s + mh)]^{(i+1)\alpha+1} d\Psi(r) \\
&= [\Psi(t) - \Psi(s + mh)]^{i\alpha} \int_0^1 (1 - y)^{-\alpha} y^{(i+1)\alpha-1} dy \\
&= [\Psi(t) - \Psi(s + mh)]^{i\alpha} B(1 - \alpha, i\alpha + \alpha). \tag{3.3}
\end{aligned}$$

By using the definition of  $\mathfrak{N}_{h,\alpha,\beta}^{A,B,\Psi}(t, s)$ , we have

$$\begin{aligned}
&\int_s^t [\Psi(t) - \Psi(r)]^{-\alpha} \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(r, s) d\Psi(r) \\
&= \int_s^t [\Psi(t) - \Psi(r)]^{-\alpha} \sum_{m=0}^{\infty} \sum_{j=0}^{p-1} \mathcal{Q}_{m+1}(jh) \frac{[\Psi(r) - \Psi(s + jh)]^{(m+1)\alpha+1}}{\Gamma((m+1)\alpha)} d\Psi(r) \\
&= \sum_{m=0}^{\infty} \frac{1}{\Gamma((m+1)\alpha)} \sum_{j=0}^{p-1} \mathcal{Q}_{m+1}(jh) \int_s^t [\Psi(t) - \Psi(r)]^{-\alpha} [\Psi(r) - \Psi(s + jh)]^{(m+1)\alpha+1} d\Psi(r) \\
&= \sum_{m=0}^{\infty} \frac{1}{\Gamma((m+1)\alpha)} \mathcal{Q}_{m+1}(0) \int_s^t [\Psi(t) - \Psi(r)]^{-\alpha} [\Psi(r) - \Psi(s)]^{(m+1)\alpha+1} d\Psi(r) \\
&\quad + \sum_{m=0}^{\infty} \frac{1}{\Gamma((m+1)\alpha)} \mathcal{Q}_{m+1}(h) \int_{s+h}^t [\Psi(t) - \Psi(r)]^{-\alpha} [\Psi(r) - \Psi(s + h)]^{(m+1)\alpha+1} d\Psi(r) \\
&\quad \vdots \\
&\quad + \sum_{m=0}^{\infty} \frac{1}{\Gamma((m+1)\alpha)} \mathcal{Q}_{m+1}((p-1)h) \\
&\quad \times \int_{s+(p-1)h}^t [\Psi(t) - \Psi(r)]^{-\alpha} [\Psi(r) - \Psi(s + (p-1)h)]^{(m+1)\alpha+1} d\Psi(r). \tag{3.4}
\end{aligned}$$

So, from Eqs (3.3) and (3.4), we get

$$\begin{aligned}
&\int_s^t [\Psi(t) - \Psi(r)]^{-\alpha} \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(r, s) \Psi(r) \\
&= \sum_{i=0}^{\infty} \frac{1}{\Gamma((i+1)\alpha)} \mathcal{Q}_{i+1}(0) [\Psi(t) - \Psi(s)]^{i\alpha} B(1 - \alpha, i\alpha + \alpha) \\
&\quad + \sum_{i=0}^{\infty} \frac{1}{\Gamma((i+1)\alpha)} \mathcal{Q}_{i+1}(h) [\Psi(t) - \Psi(s + h)]^{i\alpha} B(1 - \alpha, i\alpha + \alpha)
\end{aligned}$$

$$\begin{aligned}
& \vdots \\
& + \sum_{i=0}^{\infty} \frac{1}{\Gamma((i+1)\alpha)} Q_{i+1}((p-1)h) [\Psi(t) - \Psi(s + (p-1)h)]^{i\alpha} B(1-\alpha, i\alpha + \alpha) \\
& = \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} \frac{1}{\Gamma((i+1)\alpha)} Q_{i+1}(jh) [\Psi(t) - \Psi(s + jh)]^{i\alpha} B(1-\alpha, i\alpha + \alpha) \\
& = \sum_{k=0}^{\infty} \sum_{m=0}^{p-1} \frac{1}{\Gamma((k+1)\alpha)} Q_{k+1}(mh) [\Psi(t) - \Psi(s + mh)]^{k\alpha} \frac{\Gamma(1-\alpha)\Gamma(k\alpha + \alpha)}{\Gamma(k\alpha + 1)} \\
& = \sum_{k=0}^{\infty} \sum_{j=0}^{p-1} Q_{k+1}(mh) [\Psi(t) - \Psi(s + mh)]^{k\alpha} \frac{\Gamma(1-\alpha)}{\Gamma(k\alpha + 1)}. \tag{3.5}
\end{aligned}$$

□

**Theorem 3.6.** *If we consider the problem in Eq (1.1) with zero initial condition, which is  $v(t) = 0$ ,  $t \in [-h, 0]$ , then the solution  $v(t)$  has a form*

$$v(t) = \int_{-h}^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s) f(s) d\Psi(s), \quad t \geq 0.$$

*Proof.* By variation of constants method, if  $v(t)$  is any solution of nonhomogeneous system, then the form of  $v(t)$  must be satisfy the following form

$$v(t) = \int_{-h}^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s) c(s) d\Psi(s), \quad 0 \leq t \tag{3.6}$$

where  $v(0) = 0$  and  $c : [-h, t] \rightarrow \mathbb{R}^n$ , is a differentiable function which is not known. Applying the  $\Psi$ -Caputo fractional derivative in Eq (3.6), we get:

- 1) For  $p = 1$ , we have  $0 \leq t \leq h$ . It is clear that  $t - h < 0$ , by the zero initial condition we have  $v(t - h) = 0$ . So, according to Eq (1.1) we get

$$\begin{aligned}
{}_{-h^+}^C D_{\Psi(t)}^\alpha v(t) &= Av(t) + Bv(t-h) + f(t) \\
&= Av(t) + f(t) \\
&= A \int_{-h}^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s) c(s) d\Psi(s) + f(t).
\end{aligned}$$

On the other hand, by the assumption of zero initial condition we have  $v(-h) = 0$ . So, we get

$$\begin{aligned}
({}_{-h^+}^C D_{\Psi(t)}^\alpha v)(t) &= {}_{-h^+}^{RL} D_{\Psi(t)}^\alpha (v(t) - v(-h)) \\
&= {}_{-h^+}^{RL} D_{\Psi(t)}^\alpha (v(t)) \\
&= {}_{-h^+}^{RL} D_{\Psi(t)}^\alpha \left( \int_{-h}^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s) c(s) d\Psi(s) \right).
\end{aligned}$$

Now, according to Definition 2.1, we have

$$({}_{-h^+}^C D_{\Psi(t)}^\alpha v)(t)$$

$$\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)} \frac{1}{\Psi'(t)} \frac{d}{dt} \int_{-h}^t [\Psi(t) - \Psi(r)]^{-\alpha} \left( \int_{-h}^r \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(r,s) c(s) d\Psi(s) \right) d\Psi(r) \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{1}{\Psi'(t)} \sum_{i=0}^{\infty} A^i \frac{d}{dt} \int_{-h}^t c(s) \int_s^t [\Psi(t) - \Psi(r)]^{-\alpha} \left( \frac{[\Psi(r) - \Psi(s)]^{i\alpha + \alpha - 1}}{\Gamma(i\alpha + \alpha)} d\Psi(r) \right) d\Psi(s)
\end{aligned}$$

If we use Lemma 3.5 (a) to inner integral, we get

$$\begin{aligned}
& \left( {}_{-h}^C D_{\Psi(t)}^{\alpha} \nu \right) (t) \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{1}{\Psi'(t)} \sum_{i=0}^{\infty} A^i \frac{d}{dt} \int_{-h}^t c(s) \frac{(\Psi(t) - \Psi(s))^{i\alpha}}{\Gamma(i\alpha + \alpha)} B(1-\alpha, i\alpha + \alpha) d\Psi(s) \\
&= \frac{1}{\Psi'(t)} \sum_{i=0}^{\infty} A^i \frac{d}{dt} \int_{-h}^t c(s) \frac{(\Psi(t) - \Psi(s))^{i\alpha}}{\Gamma(i\alpha + 1)} d\Psi(s) \\
&= \frac{1}{\Psi'(t)} \frac{d}{dt} \int_{-h}^t c(s) d\Psi(s) + \frac{1}{\Psi'(t)} \sum_{i=1}^{\infty} A^i \frac{d}{dt} \int_{-h}^t c(s) \frac{(\Psi(t) - \Psi(s))^{i\alpha}}{\Gamma(i\alpha + 1)} d\Psi(s) \\
&= \frac{1}{\Psi'(t)} c(t) \Psi'(t) + \sum_{i=1}^{\infty} A^i \int_{-h}^t c(s) \frac{(\Psi(t) - \Psi(s))^{i\alpha - 1}}{\Gamma(i\alpha)} d\Psi(s) \\
&= c(t) + \int_{-h}^t A \sum_{i=0}^{\infty} A^i \frac{(\Psi(t) - \Psi(s))^{i\alpha + \alpha - 1}}{\Gamma(i\alpha + \alpha)} c(s) d\Psi(s) \\
&= c(t) + A \int_{-h}^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s) c(s) d\Psi(s).
\end{aligned}$$

Hence, we obtain the desired result  $c(t) = f(t)$ .

2) For  $p = n + 1$  according to Eq (1.1), we get, by remembering  $Q_k(-h) = \Theta$  from Definition 3.1,

$$\begin{aligned}
& \left( {}_{-h^+}^C D_{\Psi(t)}^{\alpha} \nu \right) (t) \\
&= A \int_{-h}^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s) c(s) d\Psi(s) + B \int_{-h}^{t-h} \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s+h) c(s) d\Psi(s) + f(t) \\
&= A \int_{-h}^t \sum_{k=0}^{\infty} \sum_{m=0}^n Q_{k+1}(mh) \frac{[\Psi(t) - \Psi(s+mh)]^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} c(s) d\Psi(s) + f(t) \\
&\quad + B \int_{-h}^{t-h} \sum_{k=0}^{\infty} \sum_{m=0}^{n-1} Q_{k+1}(mh) \frac{[\Psi(t) - \Psi(s+h+mh)]^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} c(s) d\Psi(s) \\
&= A \sum_{k=0}^{\infty} \sum_{m=0}^n Q_{k+1}(mh) \int_{-h}^{t-mh} \frac{[\Psi(t) - \Psi(s+mh)]^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} c(s) d\Psi(s) + f(t) \\
&\quad + B \sum_{k=0}^{\infty} \sum_{m=1}^n Q_{k+1}((m-1)h) \int_{-h}^{t-mh} \frac{[\Psi(t) - \Psi(s+mh)]^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} c(s) d\Psi(s) \\
&= \sum_{k=0}^{\infty} \sum_{m=0}^n Q_{k+2}(mh) \int_{-h}^{t-mh} \frac{[\Psi(t) - \Psi(s+mh)]^{k\alpha + \alpha - 1}}{\Gamma(k\alpha + \alpha)} c(s) d\Psi(s) + f(t).
\end{aligned}$$

However, by the assumption of zero initial condition we have  $\nu(-h) = 0$  and according to Theorem 2.3, we get

$$\begin{aligned} & \left({}_{-h^+}^C D_{\Psi(t)}^\alpha \nu\right)(t) \\ &= {}_{-h^+}^{RL} D_{\Psi(t)}^\alpha \left( \int_{-h}^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s) c(s) d\Psi(s) \right) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{1}{\Psi'(t)} \frac{d}{dt} \int_{-h}^t [\Psi(t) - \Psi(r)]^{-\alpha} \left( \int_{-h}^r \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(r,s) c(s) d\Psi(s) \right) d\Psi(r) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{1}{\Psi'(t)} \frac{d}{dt} \int_{-h}^t c(s) \int_s^t [\Psi(t) - \Psi(r)]^{-\alpha} \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(r,s) d\Psi(r) d\Psi(s). \end{aligned}$$

Now, if we apply Lemma 3.5 (b) on the above equality we get,

$$\begin{aligned} & \left({}_{-h^+}^C D_{\Psi(t)}^\alpha \nu\right)(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{1}{\Psi'(t)} \frac{d}{dt} \int_{-h}^t c(s) \sum_{k=0}^{\infty} \sum_{m=0}^n Q_{k+1}(mh) [\Psi(t) - \Psi(s+mh)]^{k\alpha} \frac{\Gamma(1-\alpha)}{\Gamma(i\alpha+1)} d\Psi(s) \\ &= \frac{1}{\Psi'(t)} \sum_{i=0}^{\infty} \sum_{j=0}^n Q_{i+1}(jh) \frac{d}{dt} \int_{-h}^{t-jh} c(s) \frac{[\Psi(t) - \Psi(s+jh)]^{i\alpha}}{\Gamma(i\alpha+1)} d\Psi(s) \\ &= \frac{1}{\Psi'(t)} \sum_{j=0}^n Q_1(jh) \frac{d}{dt} \int_{-h}^{t-jh} c(s) d\Psi(s) \\ &\quad + \frac{1}{\Psi'(t)} \sum_{i=1}^{\infty} \sum_{j=0}^n Q_{i+1}(jh) \frac{d}{dt} \int_{-h}^{t-jh} c(s) \frac{[\Psi(t) - \Psi(s+jh)]^{i\alpha}}{\Gamma(i\alpha+1)} d\Psi(s) \\ &= c(t) + \sum_{k=0}^{\infty} \sum_{m=0}^n Q_{k+2}(jh) \int_{-h}^{t-mh} c(s) \frac{[\Psi(t) - \Psi(s+mh)]^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)} d\Psi(s). \end{aligned}$$

So,  $c(t) = f(t)$ .

□

**Theorem 3.7.** If  $f = 0$ , then a solution  $\nu \in C(J, \mathbb{R}^n)$  of (1.1) can be expressed by

$$\nu(t) = \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t,-h)\eta(-h) + \int_{-h}^0 \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s) \left[ \left({}_{-h}^C D_{\Psi(t)}^\alpha \eta\right)(s) - A\eta(s) \right] d\Psi(s).$$

where  $J = ((p-1)h, ph]$  and  $p \in [0, l] \cap \mathbb{N}$ .

*Proof.* We will use the variation of constants method to prove this theorem again. In a similar way, the solution  $\nu(t)$  should search in the following form

$$\nu(t) = \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t,-h)c + \int_{-h}^0 \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s)g(s)d\Psi(s)$$

where  $c$  is a constant which is not known and  $g(t)$  is a continuously differentiable function which is not known. Note that,  $\nu(t)$  satisfies the initial condition  $\nu(t) = \eta(t)$  when  $t \in [-h, 0]$ , i.e.,

$$\nu(t) = \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t,-h)\eta(-h) + \int_{-h}^0 \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s)g(s)d\Psi(s) := \eta(t), \quad t \in [-h, 0].$$

Let  $t = -h$ , then we get

$$\mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(-h, s) = \begin{cases} \Theta, & s \in (-h, 0] \\ I, & s = -h \end{cases} \quad (3.7)$$

Therefore,  $c = \eta(-h)$ .

Now, let  $t \in [-h, 0]$ . We have that

$$\mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s) = \Theta$$

when  $s \in (t, 0]$  and

$$\mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s) = \sum_{i=0}^{\infty} A^i \frac{(\Psi(t) - \Psi(s))^{i\alpha + \alpha - 1}}{\Gamma(i\alpha + \alpha)}$$

when  $s \in [-h, t]$ . Hence on the interval  $t \in [-h, 0]$ , we derive that

$$\begin{aligned} \eta(t) &= \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, -h)\eta(-h) + \int_{-h}^0 \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s)g(s)d\Psi(s) \\ &= \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, -h)\eta(-h) + \int_{-h}^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s)g(s)d\Psi(s) + \int_t^0 \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s)g(s)d\Psi(s) \\ &= \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, -h)\eta(-h) + \int_{-h}^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s)g(s)d\Psi(s). \end{aligned}$$

If we take  $\Psi$ -Caputo fractional derivative on both sides for the above equality and employ Lemma 3.4 and Theorem 3.6, we get

$$\begin{aligned} & \left( {}_{-h^+}^C D_{\Psi(t)}^\alpha \eta \right) (t) \\ &= {}_{-h^+}^C D_{\Psi(t)}^\alpha \left( \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, -h) \right) \eta(-h) + {}_{-h^+}^C D_{\Psi(t)}^\alpha \left( \int_{-h}^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s)g(s)d\Psi(s) \right) \\ &= A \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, -h)\eta(-h) + A \int_{-h}^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s)g(s)d\Psi(s) + g(t) \\ &= A\eta(t) + g(t). \end{aligned}$$

Therefore,

$$g(t) = \left( {}_{-h^+}^C D_{\Psi(t)}^\alpha \eta \right) (t) - A\eta(t)$$

which is the desired result.  $\square$

By combining Theorem 3.6 and Theorem 3.7, we get the below upshot.

**Corollary 3.8.** *A solution  $v$  of system (1.1) is given by*

$$\begin{aligned} v(t) &= \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, -h)\eta(-h) + \int_{-h}^0 \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s) \left[ \left( {}_{-h^+}^C D_{\Psi(t)}^\alpha \eta \right) (s) - A\eta(s) \right] d\Psi(s) \\ &\quad + \int_0^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s)f(s)d\Psi(s) \end{aligned}$$

which belongs to  $C([-h, T], \mathbb{R}^n)$ .

**Remark 3.9.** By taking  $\Psi(t) = t$ , the above corollary corresponds to that of [20].

**Lemma 3.10.** If  $t \in [0, T]$ ,  $T = lh$  where  $l \in \mathbb{N}$  and  $h \in \mathbb{R}^+$ , then the next inequality is hold

$$\int_0^t \|\mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s)\| d\Psi(s) \leq [\Psi(T) - \Psi(0)] \mathfrak{N}_{h,\alpha,\alpha}^{\|A\|,\|B\|,\Psi}(T,0). \quad (3.8)$$

*Proof.* Suppose that all said conditions hold.

$$\begin{aligned} \int_0^t \|\mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s)\| d\Psi(s) &\leq \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} \|\mathcal{Q}_{i+1}(jh)\| \int_0^t \frac{[\Psi(t) - \Psi(s+jh)]^{i\alpha+\alpha-1}}{\Gamma(i\alpha+\alpha)} d\Psi(s) \\ &\leq \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} \|\mathcal{Q}_{i+1}(jh)\| \int_0^t \frac{[\Psi(t) - \Psi(jh)]^{i\alpha+\alpha-1}}{\Gamma(i\alpha+\alpha)} d\Psi(s) \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} \|\mathcal{Q}_{i+1}(jh)\| \frac{[\Psi(t) - \Psi(jh)]^{i\alpha+\alpha-1} [\Psi(t) - \Psi(0)]}{\Gamma(i\alpha+\alpha)} \\ &\leq [\Psi(T) - \Psi(0)] \mathfrak{N}_{h,\alpha,\alpha}^{\|A\|,\|B\|,\Psi}(T,0). \end{aligned}$$

□

Let:

$B_1 : f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a continuous function.

$B_2 : \exists L_f > 0$  such that  $\|f(t, \mu) - f(t, \nu)\| \leq L_f \|\mu - \nu\|$  for all  $t \in [0, T]$  and  $\mu, \nu \in \mathbb{R}^n$ .

$B_3 : L_f \mathfrak{N}_{h,\alpha,\alpha}^{\|A\|,\|B\|,\Psi}(T,0) [\Psi(T) - \Psi(0)] < 1$ .

Here is uniqueness and existence result of a solution of system (1.2).

**Theorem 3.11.** Assume that the conditions  $B_1 - B_3$  are hold. Then the system (1.2) has a unique solution in  $C([-h, T], \mathbb{R}^n)$ .

*Proof.* Let  $\mathcal{F}$  be an operator defined on  $C([-h, T], \mathbb{R}^n)$ :

$$\begin{aligned} \mathcal{F}\mu(t) &:= \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t,-h)\eta(-h) + \int_{-h}^0 \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s) \left[ \left( {}^C_{-h^+} D_{\Psi(t)}^\alpha \eta \right) (s) - A\eta(s) \right] d\Psi(s) \\ &\quad + \int_0^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s) f(s, \mu(s)) d\Psi(s). \end{aligned}$$

It is clear that the operator  $\mathcal{F}$  maps  $C([-h, T], \mathbb{R}^n)$  into itself, since  $\mathfrak{N}_{h,\alpha,\beta}^{A,B,\Psi}(t,s)$  is continuous with respect to  $t$ . Suppose that  $\mu$  and  $\nu$  are continuous on  $[-h, T]$ . Consider

$$\begin{aligned} \|\mathcal{F}\mu(t) - \mathcal{F}\nu(t)\| &\leq \int_0^t \|\mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s)\| \| [f(s, \mu(s)) - f(s, \nu(s))] \| d\Psi(s) \\ &\leq L_f \int_0^t \|\mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t,s)\| \|\mu(s) - \nu(s)\| d\Psi(s) \\ &\leq L_f [\Psi(T) - \Psi(0)] \mathfrak{N}_{h,\alpha,\alpha}^{\|A\|,\|B\|,\Psi}(T,0) \|\mu - \nu\|_\infty. \end{aligned}$$

So,  $\mathcal{F}$  is a contraction. In the light of Banach fixed point theorem,  $\mathcal{F}$  has a fixed point that it is unique on  $[-h, T]$ . In other words, there exists  $\mu \in C([-h, T], \mathbb{R}^n)$  that  $\mu = \mathcal{F}\mu$ . □

The following theorem provides the stability of system (1.2) in the sense of Ulam-Hyers.

**Theorem 3.12.** *The system given in system (1.2) is stable provided that all the statements of Theorem 3.11 are hold.*

*Proof.* Suppose that  $v \in C([0, T], \mathbb{R}^n)$  satisfies inequality (2.2), that is,

$$\left\| \left( {}^C D_{\Psi(t)}^\alpha v \right) (t) - Av(t) - Bv(t-h) - f(t, v(t)) \right\| \leq \epsilon \quad (3.9)$$

and  $\mu \in C([0, T], \mathbb{R}^n)$  is a unique solution of system (1.2), so that

$$\left( {}^C D_{\Psi(t)}^\alpha \mu \right) (t) = A\mu(t) + B\mu(t-h) + f(t, \mu(t))$$

$\forall t \in (0, T)$  and  $\alpha \in (0, 1)$ ;  $v(t) = \mu(t), t \in [-h, 0]$ . By Remark 2.9 and Eq (3.9), there exist so  $h \in C([0, T], \mathbb{R}^n)$  that  $h$  satisfies the inequality  $\|h(t)\| \leq \epsilon$  and the equation

$$\left( {}^C D_{\Psi(t)}^\alpha v \right) (t) = Av(t) + Bv(t-h) + f(t, v(t)) + h(t). \quad (3.10)$$

By using above equality, we get the solution  $v(t)$ :

$$\begin{aligned} v(t) &= \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, -h)\eta(-h) + \int_{-h}^0 \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s) \left[ \left( {}^C D_{\Psi(t)}^\alpha \eta \right) (s) - A\eta(s) \right] d\Psi(s) \\ &\quad + \int_0^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s) [f(s, v(s)) + h(s)] d\Psi(s) \\ &= \mathfrak{N}_{h,\alpha,1}^{A,B,\Psi}(t, -h)\eta(-h) + \int_{-h}^0 \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s) \left[ \left( {}^C D_{\Psi(t)}^\alpha \eta \right) (s) - A\eta(s) \right] d\Psi(s) \\ &\quad + \int_0^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s) f(s, v(s)) d\Psi(s) + \int_0^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s) h(s) d\Psi(s) \\ &= \mathcal{F}v(t) + \int_0^t \mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s) h(s) d\Psi(s). \end{aligned}$$

Taking the norm on the both sides of the above equality, we get

$$\begin{aligned} \|\mathcal{F}v(t) - v(t)\| &\leq \int_0^t \|\mathfrak{N}_{h,\alpha,\alpha}^{A,B,\Psi}(t, s)\| \|h(s)\| d\Psi(s) \\ &\leq \epsilon [\Psi(T) - \Psi(0)] \mathfrak{N}_{h,\alpha,\alpha}^{\|A\|, \|B\|, \Psi}(T, 0). \end{aligned} \quad (3.11)$$

One can easily infer  $\mu = \mathcal{F}\mu$  from the end of the statement and proof of Theorem 3.11. So,

$$\begin{aligned} \|\mu(t) - v(t)\| &\leq \|\mu(t) - \mathcal{F}v(t)\| + \|\mathcal{F}v(t) - v(t)\| \\ &= \|\mathcal{F}\mu(t) - \mathcal{F}v(t)\| + \|\mathcal{F}v(t) - v(t)\| \\ &\leq L_f [\Psi(T) - \Psi(0)] \mathfrak{N}_{h,\alpha,\alpha}^{\|A\|, \|B\|, \Psi}(T, 0) \|\mu - v\|_\infty \\ &\quad + \epsilon [\Psi(T) - \Psi(0)] \mathfrak{N}_{h,\alpha,\alpha}^{\|A\|, \|B\|, \Psi}(T, 0). \end{aligned}$$

Hence,

$$\left( 1 - L_f [\Psi(T) - \Psi(0)] \mathfrak{N}_{h,\alpha,\alpha}^{\|A\|, \|B\|, \Psi}(T, 0) \right) \|\mu - v\|_\infty \leq [\Psi(T) - \Psi(0)] \mathfrak{N}_{h,\alpha,\alpha}^{\|A\|, \|B\|, \Psi}(T, 0) \epsilon$$

then we get

$$\|\mu - \nu\|_\infty \leq u_h \cdot \epsilon$$

where

$$u_h = \frac{[\Psi(T) - \Psi(0)] \mathfrak{N}_{h,\alpha,\alpha}^{\|A\|,\|B\|,\Psi}(T, 0)}{1 - L_f [\Psi(T) - \Psi(0)] \mathfrak{N}_{h,\alpha,\alpha}^{\|A\|,\|B\|,\Psi}(T, 0)} > 0.$$

This completes the proof.  $\square$

#### 4. Illustrated examples

Here are examples to illustrate theoretical results.

**Example 4.1.** We now consider the following nonhomogeneous and nonlinear  $x^2$ -Caputo fractional delay differential system

$$\begin{aligned} {}_{-(0.2)^+}^C D_{x^2}^{\frac{1}{2}} \nu(x) &= \frac{1}{3} \cdot \nu(x) + 0 \cdot \nu(x - 0.2) + \frac{e^x}{25(1 + e^x)} \sin(\nu(x)), \quad 0 < x \leq 0.8, \\ \nu(x) &= 2(x^2 - 1), \quad -0.2 \leq x \leq 0. \end{aligned} \quad (4.1)$$

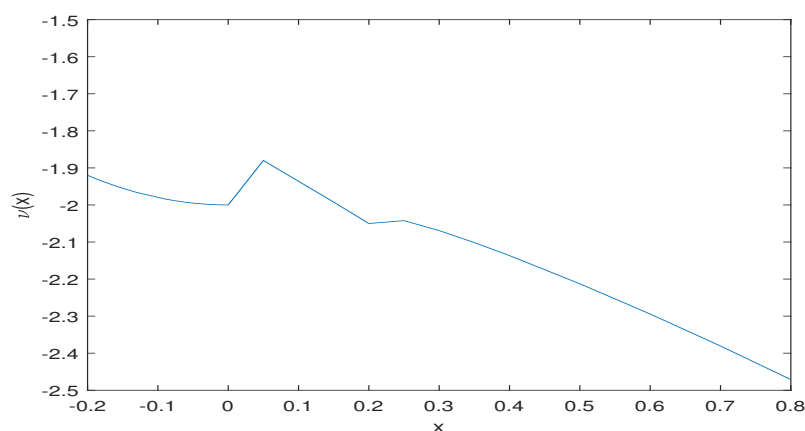
With the aid of Corollary 3.8, a solution of the above system (4.1) is given by

$$\begin{aligned} \nu(x) &= -1.82 \mathfrak{N}_{0.2, \frac{1}{2}, 1}^{\frac{1}{3}, 0, x^2}(x, -0.2) + 2 \int_{-1}^0 \mathfrak{N}_{0.2, \frac{1}{2}, \frac{1}{2}}^{\frac{1}{3}, 0, x^2}(x, s) \left[ \frac{4\sqrt{s^2 - 1}}{\sqrt{\pi}} - \frac{2(s^2 - 1)}{3} \right] s ds \\ &\quad + 2 \int_0^x \mathfrak{N}_{0.2, \frac{1}{2}, \frac{1}{2}}^{\frac{1}{3}, 0, x^2}(x, s) \frac{e^s}{25(1 + e^s)} \sin(\nu(s)) s ds, \end{aligned}$$

where  $\mathfrak{N}_{0.2, \frac{1}{2}, \beta}^{\frac{1}{3}, 0, x^2}(x, s) = (x^2 - s^2)^{\beta-1} E_{\frac{1}{2}, \beta} \left( \frac{1}{3} (x^2 - s^2)^{\frac{1}{2}} \right)$ . The graph of the solution  $\nu(x)$  can be found in Figure 2. One can easily see that  $f$  is continuous as well as being the Lipschitz function with the Lipschitz constant  $L_f = 0.04$  and

$$L_f \mathfrak{N}_{0.2, \frac{1}{2}, \frac{1}{2}}^{\frac{1}{3}, 0, x^2}(0.8, 0) [0.8^2 - 0^2] = 0.0269 < 1.$$

All of conditions  $B_{1-3}$  of Theorems 3.11 and 3.12 are satisfied, so system (4.1) is of an unique solution in addition to being Ulam-Hyers stable.



**Figure 2.** Graph of solution  $\nu(x)$  to system (4.1).



**Example 4.2.** We consider the following nonhomogeneous and nonlinear  $\sqrt{x}$ -Caputo fractional delay differential system

$$\begin{aligned} {}_{-(0.1)^+}^C D_{\sqrt{x}}^{0.8} v(x) &= Av(x) + Bv(x - 0.1) + \frac{\arctan v(x)}{2}, \quad 0 < x \leq 0.2, \\ v(x) &= x^3 + \sin x, \quad -0.1 \leq x \leq 0. \end{aligned} \quad (4.2)$$

where

$$A = \begin{bmatrix} 0.44 & 0.26 \\ 0.01 & 0.34 \end{bmatrix}, B = \begin{bmatrix} 0.3 & 0.14 \\ 0.1 & 0.5 \end{bmatrix}.$$

With the well-known maximum absolute row sum of the matrix  $\|\cdot\|_\infty$ , one can easily see  $\|A\|_\infty = 0.7$  and  $\|B\|_\infty = 0.6$ . Since arctan function is continuous, then  $f(x, v(x)) = \arctan v(x)/2$  is continuous. With a simple calculation

$$\left\| \frac{\arctan v(x)}{2} - \frac{\arctan \mu(x)}{2} \right\|_\infty \leq \frac{1}{2} \|v(x) - \mu(x)\|_\infty$$

which provides that  $f(x, v(x)) = \arctan v(x)$  satisfies Lipschitz condition with  $L_f = 0.5$ . We also have

$$L_f \mathfrak{N}_{0.1, 0.8, 0.8}^{0.7, 0.6, \sqrt{x}}(0.2, 0) [\sqrt{0.2} - \sqrt{0}] = 0.3778 < 1.$$

According to Theorem 3.12, system (4.2) is Ulam-Hyers stable because  $B_1$ ,  $B_2$ , and  $B_3$  are satisfied.

## 5. Conclusions

In this paper,  $\Psi$ -delay perturbation of Mittag-Leffler type matrix function with two parameters are defined and by using this definition, an explicit solution of nonhomogeneous linear  $\Psi$ -Caputo fractional delay differential system for noncommutative matrices are derived. Moreover, applying Banach Fixed Point theorem, the uniqueness and existence result of the solutions of system is given. Ulam-Hyers approach is used to provide the stability of the system.

The next further work can be devoted to study exponential stability, finite time stability, Lyapunov type stability and also controllability of the  $\Psi$ -Caputo fractional order time-delay differential linear nonhomogeneous systems. The above mentioned system also can be extended by adding multi-delayed terms, i.e.,  $\Psi$ -Caputo type fractional multi-delayed differential equations and it can be reconsidered from the similar aspects. Moreover, asymptotic stability, Ulam-Hyers stability, and approximate controllability results for multi-term fractional functional evolution equations can be investigated.

## Conflict of interest

The authors declare there is no conflicts of interest.

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