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Research article

Multiple solutions for the fourth-order Kirchhoff type problems in $\mathbb{R}^N$ involving concave-convex nonlinearities

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Abstract: In this paper, we study the multiplicity of solutions for the following fourth-order Kirchhoff type problem involving concave-convex nonlinearities and indefinite weight function

$$\Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + V(x)u = \lambda f(x)|u|^{q-2}u + |u|^{p-2}u,$$

where $u \in H^2(\mathbb{R}^N)(4 < N < 8), \lambda > 0, 1 < q < 2, 4 < p < 2_*(2_* = 2N/(N-4))$, $f(x)$ satisfy suitable conditions, and $f(x)$ may change sign in $\mathbb{R}^N$. Using Nehari manifold and fibering maps, the existense of multiple solutions is established. Moreover, the existence of sign-changing solution is obtained for $f(x) \equiv 0$. Our results generalize some recent results in the literature.

Keywords: fourth-order Kirchhoff type problems; multiple solutions; indefinite weight functions; Nehari manifold; fibering maps

1. Introduction and main results

In this paper we study the following fourth-order Kirchhoff type problem:

$$\Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + V(x)u = \lambda f(x)|u|^{q-2}u + |u|^{p-2}u, \tag{1.1}$$

where $u \in H^2(\mathbb{R}^N)(4 < N < 8)$, $1 < q < 2, 4 < p < 2_*(2_* = 2N/(N-4))$. The parameter $\lambda > 0$, a and b are positive constants, the potential function V and the weight function f satisfy the following conditions:

(V) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $V_0 := \inf_{\mathbb{R}^N} V(x) > 0$ and there exists a constant $l_0 > 0$ such that

$$\lim_{|y|\to\infty} meas(\{x \in \mathbb{R}^N \| x - y| \le l_0, V(x) \le M\}) = 0, \quad \forall M > 0,$$

where $meas(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$.
(F) $f \in C(\mathbb{R}^N) \cap L^{r_q}(\mathbb{R}^N)$, where $r_q = r/(r-q)$ for some $r \in (2, 2_*)$.
The general form of (1.1) can be written as

$$\Delta^2 u - \left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \Delta u + V(x)u = g(x, u), \tag{1.2}$$

where $u \in H^2(\mathbb{R}^N)$, a and b are positive constants, $V(x) : \mathbb{R}^N \to \mathbb{R}$ is a continuous potential. Different forms of the nonlinearity $g(x, u)$ will lead to different difficulties, such as the existence of critical sequence in sublinear case, the boundedness of critical sequence in superlinear case or the compactness in critical case. The nonlinearity $g(x, u)$ is superlinear, sublinear and critical growth, which has been widely studied by many scholars, see [1–3] and their references therein.

Let $V(x) = 0$, replace $\mathbb{R}^N$ by a smooth bounded domain $\Omega \subset \mathbb{R}^N$ and set $u = \nabla u = 0$ on $\partial\Omega$, the problem (1.2) is reduced to the following fourth-order Kirchhoff type problem:

$$\begin{cases} \Delta^2 u - \left(a + b \int_\Omega |\nabla u|^2 dx\right) \Delta u = g(x, u), & \text{in } \Omega, \\ u = \nabla u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

which is related to the following stationary analogue of the Kirchhoff type problem:

$$u_{tt} + \Delta^2 u - \left(a + b \int_\Omega |\nabla u|^2 dx\right) \Delta u = g(x, u), \quad \text{in } \Omega, \tag{1.4}$$

where Δ^2 is the biharmonic operator. In low dimensions, (1.4) is often used to describe the phenomenon of nonlinear vibration of beam or plate in physics and engineering (see [4,5]). Because of the existence of integral term $\int_\Omega |\nabla u|^2 dx$, this kind of problem is nonlocal, which indicates that Eq (1.4) is no longer pointwise identity. This phenomenon has caused some difficulties for mathematical research, so it has attracted the attention of a large number of scholars.

Here we focus on $g(x, u)$ with concave-convex nonlinearities. Semilinear elliptic equations with concave-convex nonlinearities in bounded domains are extensively researched. Ambrosetti et al. [6], for example, considered the following equation:

$$\begin{cases} -\Delta u = \lambda u^{q-1} + u^{p-1}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\lambda > 0$, $1 < q < 2 < p < 2^* = 2N/(N-2)$. They proved that there exists $\lambda_0 > 0$ such that (1.5) admits at least two positive solutions for all $\lambda \in (0, \lambda_0)$ and has one positive solution for $\lambda = \lambda_0$ and no positive solution for $\lambda > \lambda_0$. Actually, many scholars have also obtained this result in the unit ball $B^N(0; 1)$, see [7–9]. In addition, Chen, Kuo and Wu [10] investigated the following Kirchhoff type problem:

$$\begin{cases} -(a + b \int_\Omega |\nabla u|^2 dx)\Delta u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $a,\ b > 0$, $1 < q < 2 < p < 2^*$ and $f,\ g \in C(\bar{\Omega})$ are sign-changing weight functions. Using the Nehari manifold and fibering maps, the authors examined the existence of multiple positive solutions for three cases: $p > 4$, $p = 4$ and $p < 4$ when b and λ belong to specific intervals. For more results of problems involving concave-convex nonlinearities and sign-changing weights in bounded domain, the reader may see [11, 12] and the references therein. Furthermore, this kind of question in $\mathbb{R}^N$ also arouses the scholar's interest. Wu [13] has researched the following equation involving sign-changing weight functions:

$$\begin{cases} -\Delta u + u = a_\lambda(x)u^{q-1} + b_\mu(x)u^{p-1}, & \text{in } \mathbb{R}^N, \\ u > 0, & \text{in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

where $u \in H^1(\mathbb{R}^N)$, $1 < q < 2 < p < 2^*$, the parameters $\lambda, \mu > 0$. He assumed that $a_\lambda(x) = \lambda a_+(x) + a_-(x)$ is sign-changing and $b_\mu(x) = c(x) + \mu d(x)$, where $c(x)$ and $d(x)$ satisfy appropriate hypotheses, and obtained the multiplicity of positive solutions for the problem (1.7).

Inspired by the above work, the main aim of this paper is to study the Kirchhoff problem (1.1) in $\mathbb{R}^N$ involving conave-convex nonlinearities and sign-changing weight function. In addition, from the condition (F), we can see that f is allowed to be sign-changing. As far as we know, there are few articles to deal with the fourth-order Kirchhoff type problem (1.1). We are going to discuss the Nehari manifold and thoroughly check the relation between the Nehari manifold and the fibering maps; then using methods similar to those used in [14], we will prove the existence of two solutions by using Ekeland variational principle [15].

Set

$$\lambda_1 = \left(\frac{p-2}{(p-q)|f|_{r_q}}\right)\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} S_p^{\frac{p(2-q)}{2(p-2)}} S_r^{\frac{q}{2}} > 0$$

and $0 < \lambda_2 = \frac{q}{p-2}\lambda_1 < \lambda_1$, where $|f|_{r_q} = \left(\int_{\mathbb{R}^N} |f|^{r_q} dx\right)^{1/r_q}$ and S_p is described below. Now, we state the main result about the multiciplity of solution of (1.1) in $\mathbb{R}^N$.

Theorem 1.1. *Assume that* (V) *and* (F) *hold. If* $\lambda \in (0, \lambda_1)$*, then* (1.1) *admits at least two nontrivial solutions, one of which has negative energy. Furthermore, if* $\lambda \in (0, \lambda_2)$*, then* (1.1) *has at least one negative energy ground state solution and one positive energy solution.*

Theorem 1.2. *Assume* (V) *holds. Then* (1.1) *has a sign-changing solution for* $f(x) \equiv 0$.

In Eq (1.1), the unboundedness of the whole space $\mathbb{R}^N$ leads to no compactness, therefore we consider condition (V) to recover the compactness. The condition (V) was first mentioned by Bartsch and Wang in [11]. At the same time, we also have (V_1) and (V_2) conditions in the following remark to repair compactness. Therefore, using (V_1) or (V_2) instead of (V) can also get the same result. But the following two conditions are stronger than (V).

Remark 1.2. These conditions are usually used to restore compactness.
(V_1) $V(x) \in C(\mathbb{R}^N, \mathbb{R}), V_0 := \inf_{\mathbb{R}^N} V(x) > 0, V(x) \to +\infty$ as $|x| \to +\infty$ (see [16]).
(V_2) $V(x) \in C(\mathbb{R}^N, \mathbb{R}), V_0 := \inf_{\mathbb{R}^N} V(x) > 0$, for each $M > 0$, $meas(\{x \in \mathbb{R}^N | V(x) \leq M\}) < \infty$ (see [17]).

This paper is organized as follows. In Section 2, some notations and preliminaries are given, including lemmas that are required in proving the main theorem. In Section 3, we are concerned with the proof of Theorem 1.1. In Section 4, we are concerned with the proof of Theorem 1.2.

2. Preliminaries

Define our working space

$$E := \left\{ u \in H^2(\mathbb{R}^N) | \int_{\mathbb{R}^N} (|\Delta u|^2 + |\nabla u|^2 + V(x)u^2) dx < +\infty \right\}$$

with the inner product and norm

$$(u, v) = \int_{\mathbb{R}^N} (\Delta u \Delta v + a \nabla u \nabla v + V(x)uv) dx, \quad \|u\|^2 = (u, u).$$

where $H^2(\mathbb{R}^N)$ is the well known Sololev space.

Throughout this paper, under assumption (V), the embedding $E \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for $r \in [2, 2_*]$ and compact for $r \in [2, 2_*)$ [18]. We denote by S_r the best Sobolev constant for the embedding of E in $L^r(\mathbb{R}^N)$ with $r \in [2, 2_*)$. In particular,

$$|u|_r \leq S_r^{-1/2} \|u\| \quad \text{for all } u \in E \backslash \{0\},$$

where $L^r(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the standard norm $|u|_r = \left(\int_{\mathbb{R}^N} |u|^r dx \right)^{1/r}$ for $1 \leq r < \infty$.

The energy functional I_λ we consider that corresponds to (1.1) is given by, for each $u \in E$,

$$\begin{aligned} I_\lambda(u) = & \frac{1}{2} \int_{\mathbb{R}^N} (|\Delta u|^2 + a|\nabla u|^2 + V(x)u^2) dx + \frac{b}{4} (\int_{\mathbb{R}^N} |\nabla u|^2 dx)^2 - \frac{\lambda}{q} \int_{\mathbb{R}^N} f(x)|u|^q dx \\ & - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx. \end{aligned} \tag{2.1}$$

It is well known that the functional I_λ is of class C^1 in E and the solutions of (1.1) are the critical points of energy functional I_λ [19] and thus, by taking $(V)(F)$ and using a direct computation, we have

$$\begin{aligned} \langle I'_\lambda(u), v \rangle = & \int_{\mathbb{R}^N} (\Delta u \Delta v + a \nabla u \nabla v + V(x)uv) dx + b \int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla v dx \\ & - \lambda \int_{\mathbb{R}^N} f(x)|u|^{q-2} uv dx - \int_{\mathbb{R}^N} |u|^{p-2} uv dx, \end{aligned} \tag{2.2}$$

for any $u, v \in E$, and where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $H^2(\mathbb{R}^N)$. Moreover, it is clear that $lim_{t \to \infty} I_\lambda(tu) = -\infty$ and so I_λ is not bounded below on E. In order to obtain critical points, we consider the I_λ on the Nehari manifold

$$\mathcal{N}_\lambda = \{u \in E \backslash \{0\} | \langle I'_\lambda(u), u \rangle = 0\}.$$

Thus, $u \in \mathcal{N}_\lambda$ if and only if

$$\langle I'_\lambda(u), u \rangle = \|u\|^2 + b|\nabla u|_2^4 - \lambda \int_{\mathbb{R}^N} f(x)|u|^q dx - \int_{\mathbb{R}^N} |u|^p dx = 0.$$

Moreover, $\mathcal{N}_\lambda$ comprises all nontrivial solutions of problem (1.1). And we have the following lemma.

Lemma 2.1. *The energy functional I_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. For $u \in \mathcal{N}_\lambda$, by the Hölder and Sobolev inequalities,

$$
\begin{aligned}
I_\lambda(u) &= I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u), u\rangle \\
&= \frac{1}{4}\|u\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{4}\right)\int_{\mathbb{R}^N} f(x)|u|^q dx - \left(\frac{1}{p} - \frac{1}{4}\right)\int_{\mathbb{R}^N} |u|^p dx \\
&\geq \frac{1}{4}\|u\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{4}\right)|f|_{r_q} S_r^{-\frac{q}{2}}\|u\|^q.
\end{aligned}
$$

This ends the proof due to $1 < q < 2$. □

Distinctly, $\mathcal{N}_\lambda$ is a much smaller set than E and so it is simpler to discuss I_λ on $\mathcal{N}_\lambda$. The Nehari manifold $\mathcal{N}_\lambda$ is closely related to the character of functions of the form $\varphi_{\lambda,u} : t \to I_\lambda(tu)$ for $t > 0$. Such functions are known as fibering maps, which were studied by Brown and Wu in [20]. If $u \in E$, we have

$$
\begin{aligned}
\varphi_{\lambda,u}(t) &= I_\lambda(tu) = \frac{t^2}{2}\|u\|^2 + \frac{b}{4}t^4|\nabla u|_2^4 - \frac{\lambda}{q}t^q\int_{\mathbb{R}^N} f(x)|u|^q dx - \frac{t^p}{p}\int_{\mathbb{R}^N}|u|^p dx, \\
\varphi'_{\lambda,u}(t) &= t\|u\|^2 + bt^3|\nabla u|_2^4 - \lambda t^{q-1}\int_{\mathbb{R}^N} f(x)|u|^q dx - t^{p-1}\int_{\mathbb{R}^N}|u|^p dx, \\
\varphi''_{\lambda,u}(t) &= \|u\|^2 + 3bt^2|\nabla u|_2^4 - (q-1)\lambda t^{q-2}\int_{\mathbb{R}^N} f(x)|u|^q dx - (p-1)t^{p-2}\int_{\mathbb{R}^N}|u|^p dx.
\end{aligned}
$$

Evidently,

$$
t\varphi'_{\lambda,u}(t) = \langle I'_\lambda(tu), tu\rangle
$$

and so, for $u \in E\backslash\{0\}$ and $t > 0$, $\varphi'_{\lambda,u}(t) = 0$ if and only if $tu \in \mathcal{N}_\lambda$, that is, the critical points of $\varphi_{\lambda,u}$ correspond to points on the Nehari manifold, In particular, $u \in \mathcal{N}_\lambda$ if and only if $\varphi'_{\lambda,u} = 0$. Thus, it is inartificial to split $\mathcal{N}_\lambda$ into three parts [14]:

$$
\begin{aligned}
\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda | \varphi''_{\lambda,u}(1) > 0\}; \\
\mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda | \varphi''_{\lambda,u}(1) = 0\}; \\
\mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda | \varphi''_{\lambda,u}(1) < 0\}.
\end{aligned}
$$

It is easy to see that

$$
\varphi''_{\lambda,u}(1) = \|u\|^2 + 3b|\nabla u|_2^4 - (q-1)\lambda\int_{\mathbb{R}^N} f(x)|u|^q dx - (p-1)\int_{\mathbb{R}^N}|u|^p dx. \tag{2.3}
$$

Thus, for each $u \in \mathcal{N}_\lambda$, we have

$$
\begin{aligned}
\varphi''_{\lambda,u}(1) &= \varphi''_{\lambda,u}(1) - (p-1)\langle I'_\lambda(u), u\rangle \\
&= (2-p)\|u\|^2 + (4-p)b|\nabla u|_2^4 - (q-p)\lambda\int_{\mathbb{R}^N} f(x)|u|^q dx
\end{aligned} \tag{2.4}
$$

and

$$
\begin{aligned}
\varphi''_{\lambda,u}(1) &= \varphi''_{\lambda,u}(1) - (q-1)\langle I'_\lambda(u), u\rangle \\
&= (2-q)\|u\|^2 + (4-q)b|\nabla u|_2^4 - (p-q)\int_{\mathbb{R}^N}|u|^p dx.
\end{aligned} \tag{2.5}
$$

We now derive some basic properties of $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^0$ and $\mathcal{N}_\lambda^-$.

Lemma 2.2. *Assume that u is a local minimizer for I_λ on $\mathcal{N}_\lambda$ and $u \notin \mathcal{N}_\lambda^0$. Then $I'_\lambda(u) = 0$.*

Proof. The details of the proof can be referred to Brown and Zhang [12]. □

Lemma 2.3. *If $\lambda \in (0, \lambda_1)$, then $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof. Suppose the contrary. There exist $u \in \mathcal{N}_\lambda$ such that $\varphi''_{\lambda,u}(1) = 0$. From (2.5) and the Sobolev inequality, we have

$$\begin{aligned}(2-q)\|u\|^2 &\le (2-q)\|u\|^2 + (4-q)b|\nabla u|_2^4 \\ &= (p-q)\int_{\mathbb{R}^N} |u|^p dx \\ &\le (p-q)S_p^{-\frac{p}{2}}\|u\|^p\end{aligned}$$

and so

$$\|u\| \ge \left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{1}{p-2}}. \tag{2.6}$$

Similarly, using (2.4) and Hölder and Sobolev inequalities, we have

$$\begin{aligned}(p-2)\|u\|^2 &\le (p-2)\|u\|^2 + (p-4)b|\nabla u|_2^4 \\ &= (p-q)\lambda\int_{\mathbb{R}^N} f(x)|u|^q dx \\ &\le (p-q)\lambda|f|_{r_q} S_r^{-\frac{q}{2}}\|u\|^q\end{aligned}$$

which implies that

$$\|u\| \le \left(\frac{(p-q)\lambda|f|_{r_q}}{(p-2)S_r^{\frac{q}{2}}}\right)^{\frac{1}{2-q}}. \tag{2.7}$$

Combining (2.6) and (2.7) we deduce that

$$\lambda \ge \left(\frac{p-2}{(p-q)|f|_{r_q}}\right)\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} S_p^{\frac{p(2-q)}{2(p-2)}} S_r^{\frac{q}{2}} = \lambda_1,$$

which is a contradiction.This completes the proof. □

Lemma 2.4. *If $\lambda \in (0, \lambda_1)$, then the set $\mathcal{N}_\lambda^-$ is closed in E.*

Proof. Let $\{u_n\} \subset \mathcal{N}_\lambda^-$ such that $u_n \to u$ in E. In the following we show $u \in \mathcal{N}_\lambda^-$. In fact, by $\langle I'_\lambda(u_n), u_n\rangle = 0$ and

$$\langle I'_\lambda(u_n), u_n\rangle - \langle I'_\lambda(u), u\rangle = \langle I'_\lambda(u_n) - I'_\lambda(u), u\rangle + \langle I'_\lambda(u_n), u_n - u\rangle \to 0, \quad \text{as } n \to \infty,$$

we have $\langle I'_\lambda(u), u\rangle = 0$. So $u \in \mathcal{N}_\lambda$. For any $u \in \mathcal{N}_\lambda^-$, from (2.5) we have

$$\varphi''_{\lambda,u}(1) = (2-q)\|u\|^2 + (4-q)b|\nabla u|_2^4 - (p-q)\int_{\mathbb{R}^N} |u|^p dx < 0.$$

Then by Sobolev inequality, we have

$$\begin{aligned}(2-q)\|u\|^2 &< (2-q)\|u\|^2 + (4-q)b|\nabla u|_2^4\\ &< (p-q)\int_{\mathbb{R}^N}|u|^p dx\\ &\le (p-q)S_p^{-\frac{p}{2}}\|u\|^p,\end{aligned}$$

that is,

$$\|u\| > \left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{1}{p-2}} > 0.$$

Hence $\mathcal{N}_\lambda^-$ is bounded away from 0. Obviously, by (2.4), it follows that $\varphi''_{\lambda,u_n}(1) \to \varphi''_{\lambda,u}(1)$ as $n \to +\infty$. From $\varphi''_{\lambda,u_n}(1) < 0$, we have $\varphi''_{\lambda,u}(1) \le 0$. By Lemma 2.3, for $\lambda \in (0, \lambda_1)$, $\mathcal{N}_\lambda^0 = \emptyset$, then $\varphi''_{\lambda,u}(1) < 0$. Thus we deduce $u \in \mathcal{N}_\lambda^-$. This completes the proof. □

In order to obtain a better comprehension of the Nehari manifold and fibering maps, we consider the function $\psi_b : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\psi_b(t) = t^{2-q}\|u\|^2 + t^{4-q}b|\nabla u|_2^4 - t^{p-q}\int_{\mathbb{R}^N}|u|^p dx, \quad \text{for } t > 0.$$

Clearly $tu \in \mathcal{N}_\lambda$ if and only if $\psi_b(t) = \lambda\int_{\mathbb{R}^N} f(x)|u|^q dx$. Moreover,

$$\psi'_b(t) = (2-q)t^{1-q}\|u\|^2 + (4-q)t^{3-q}b|\nabla u|_2^4 - (p-q)t^{p-q-1}\int_{\mathbb{R}^N}|u|^p dx, \quad \text{for } t > 0,$$

and so it is easy to see that, if $tu \in \mathcal{N}_\lambda$, then $t^{q-1}\psi'_b(t) = \varphi''_{\lambda,u}(t)$. Hence, $tu \in \mathcal{N}_\lambda^+$ (or $tu \in \mathcal{N}_\lambda^-$) if and only if $\psi'_b(t) > 0$ (or $\psi'_b(t) < 0$). Furthermore, from $1 < q < 2$, $4 < p < 2_*$, $\psi'_b(t) = 0$ and $\psi_b(0) = 0$, we can deduce that there is a unique $t_{b,max} > 0$ such that $\psi_b(t)$ achieves its maximum at $t_{b,max}$, increasing for $t \in [0, t_{b,max})$ and decreasing for $t \in (t_{b,max}, +\infty)$ with $\lim_{t\to+\infty}\psi_b(t) = -\infty$.

The next lemma allows us to assume that $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are nonempty under the hypothesis.

Lemma 2.5. *Suppose that $\lambda \in (0, \lambda_1)$, $u \in E\backslash\{0\}$. Then*

(i) *if $\lambda\int_{\mathbb{R}^N} f(x)|u|^q dx \le 0$, then there is a unique $t^- > t_{b,max}$ such that $t^-u \in \mathcal{N}_\lambda^-$ and*

$$I_\lambda(t^-u) = \sup_{t\ge0} I_\lambda(tu).$$

(ii) *if $\lambda\int_{\mathbb{R}^N} f(x)|u|^q dx > 0$, then there are unique t^+ and t^- with $0 < t^+ < t_{b,max} < t^-$ such that $t^+u \in \mathcal{N}_\lambda^+$, $t^-u \in \mathcal{N}_\lambda^-$ and*

$$I_\lambda(t^+u) = \inf_{t_{b,max}\ge t\ge0} I_\lambda(tu), \quad I_\lambda(t^-u) = \sup_{t\ge t_{b,max}} I_\lambda(tu).$$

Proof. (i) if $\lambda\int_{\mathbb{R}^N} f(x)|u|^q dx \le 0$, noting that $\psi_b(t)$ achieves its maximum at $t_{b,max}$, increasing for $t \in [0, t_{b,max})$ and decreasing for $t \in (t_{b,max}, +\infty)$ with $\lim_{t\to+\infty}\psi_b(t) = -\infty$, then there is a unique $t^- > t_{b,max}$

such that $\psi_b(t^-) = \lambda \int_{\mathbb{R}^N} f(x)|u|^q dx$, that is $t^- u \in \mathcal{N}_\lambda$. Moreover by $\psi_b'(t) < 0$, we obtain that $t^- u \in \mathcal{N}_\lambda^-$. And by

$$\varphi_{\lambda,u}'(t) = \frac{dI_\lambda(tu)}{dt} = t^{q-1}\left(\psi_b(t) - \lambda \int_{\mathbb{R}^N} f(x)|u|^q dx\right),$$

we have $I_\lambda(t^- u) = \sup_{t\geq 0} I_\lambda(tu)$.
(ii) Since $b > 0$, $t > 0$, we have

$$\psi_b(t) > \psi_0(t) = t^{2-q}\|u\|^2 - t^{p-q}\int_{\mathbb{R}^N} |u|^p dx,$$

where $\psi_0(t) = \psi_b(t)|_{b=0}$. Clearly, $\psi_0(t)$ has a unique critical point at $t_{0,max} = t_{0,max}(u)$, where

$$t_{0,max} = \left(\frac{(2-q)\|u\|^2}{(p-q)\int_{\mathbb{R}^N} |u|^p dx}\right)^{\frac{1}{p-2}}.$$

Moreover, by Sobolev inequality, we obtain

$$\begin{aligned}
\psi_0(t_{0,max}) &= \left(\frac{(2-q)\|u\|^2}{(p-q)\int_{\mathbb{R}^N} |u|^p dx}\right)^{\frac{2-q}{p-2}} \|u\|^2 - \left(\frac{(2-q)\|u\|^2}{(p-q)\int_{\mathbb{R}^N} |u|^p dx}\right)^{\frac{p-q}{p-2}} \int_{\mathbb{R}^N} |u|^p dx \\
&= \|u\|^q \left(\frac{\|u\|^p}{\int_{\mathbb{R}^N} |u|^p dx}\right)^{\frac{2-q}{p-2}} \left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} \frac{p-2}{p-q} \\
&\geq \|u\|^q \left(\frac{\|u\|^p}{S_p^{-\frac{p}{2}}\|u\|^p}\right)^{\frac{2-q}{p-2}} \left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} \frac{p-2}{p-q} \\
&= \|u\|^q \left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{2-q}{p-2}} \frac{p-2}{p-q} > 0.
\end{aligned} \tag{2.8}$$

Thus, $\psi_b(t_{b,max}) > \psi_0(t_{0,max}) > 0$.

From $\lambda \in (0, \lambda_1)$, (2.8), Hölder and Sobolev inequalities we also have

$$\begin{aligned}
\lambda \int_{\mathbb{R}^N} f(x)|u|^q dx &\leq \lambda |f|_{r_q} S_r^{-\frac{q}{2}} \|u\|^q \\
&< \|u\|^q \left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{2-q}{p-2}} \frac{p-2}{p-q} \\
&\leq \psi_0(t_{0,max}) < \psi_b(t_{b,max}).
\end{aligned} \tag{2.9}$$

If $\lambda \int_{\mathbb{R}^N} f(x)|u|^q dx > 0$. Since (2.9), the equation $\psi_b(t) = \lambda \int_{R^N} f(x)|u|^q dx$ has exactly two solutions $0 < t^+ < t_{b,max} < t^-$ such that

$$\psi_b(t^+) = \lambda \int_{\mathbb{R}^N} f(x)|u|^q dx = \psi_b(t^-)$$

and

$$\psi_b'(t^+) > 0 > \psi_b'(t^-).$$

Thus, there exist exactly two multiples of u lying in $\mathcal{N}_\lambda$, that is, $t^+u \in \mathcal{N}_\lambda^+$ and $t^-u \in \mathcal{N}_\lambda^-$. Finally, by analyzing $\frac{dI_\lambda(tu)}{dt} = t^{q-1}\left(\psi_b(t) - \lambda \int_{\mathbb{R}^N} f(x)|u|^q dx\right)$, $I_\lambda(tu)$ is decreasing for $t \in (0, t^+)$ and increasing for $t \in (t^+, t_{b,max})$. Moreover, $I_\lambda(tu)$ is increasing for $t \in (t_{b,max}, t^-)$ and decreasing for $t \in (t_{b,max}, +\infty)$. therefore,

$$I_\lambda(t^+u) = \inf_{t_{b,max} \geq t \geq 0} I_\lambda(tu), \quad I_\lambda(t^-u) = \sup_{t \geq t_{b,max}} I_\lambda(tu),$$

□

3. Proof of Theorem 1.1

First, we remark that it follows from Lemma 2.3 that

$$\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$$

for all $\lambda \in (0, \lambda_1)$. Furthermore, by Lemma 2.5 it follows that $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are nonempty, and by Lemma 2.1 we may define

$$\alpha_\lambda = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u); \quad \alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} I_\lambda(u); \quad \alpha_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} I_\lambda(u).$$

Then we get the following result.

Lemma 3.1. *One has the following.*

(i) *If $\lambda \in (0, \lambda_1)$, then one has $\alpha_\lambda^+ < 0$.*

(ii) *If $\lambda \in (0, \lambda_2)$, then one has $\alpha_\lambda^- > d_0$ for some $d_0 > 0$.*

In particular, for each $\lambda \in (0, \lambda_2)$, one has $\alpha_\lambda^+ = \alpha_\lambda$.

Proof. (i) Let $u \in \mathcal{N}_\lambda^+$. By (2.4)

$$(p-2)\|u\|^2 + (p-4)b|\nabla u|_2^4 < (p-q)\lambda \int_{\mathbb{R}^N} f(x)|u|^q dx$$

and so

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{p}\langle I'_\lambda(u), u\rangle \\ &= \frac{p-2}{2p}\|u\|^2 + \frac{p-4}{4p} b|\nabla u|_2^4 - \frac{p-q}{pq}\lambda \int_{\mathbb{R}^N} f(x)|u|^q dx \\ &< \frac{p-2}{2p}\|u\|^2 + \frac{p-4}{4p} b|\nabla u|_2^4 - \frac{1}{pq}\left((p-2)\|u\|^2 + (p-4)b|\nabla u|_2^4\right) \\ &= \frac{(p-2)(q-2)}{2pq}\|u\|^2 + \frac{(p-4)(q-4)}{4pq} b|\nabla u|_2^4 < 0. \end{aligned}$$

Therefore, $\alpha_\lambda^+ < 0$.

(ii) Let $u \in \mathcal{N}_\lambda^-$. By Lemma 2.4, we have

$$\|u\| > \left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{1}{p-2}}.$$

Furthermore, by Hölder and Sobolev inequalities, we have

$$\begin{aligned}
I_\lambda(u) &= I_\lambda(u) - \frac{1}{4}\langle I'_\lambda(u), u\rangle \\
&\geq \frac{1}{4}\|u\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{4}\right)|f|_{r_q} S_r^{-\frac{q}{2}}\|u\|^q \\
&= \|u\|^q\Big(\frac{1}{4}\|u\|^{2-q} - \lambda\left(\frac{1}{q} - \frac{1}{4}\right)|f|_{r_q} S_r^{-\frac{q}{2}}\Big) \\
&> \left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{q}{p-2}}\left(\frac{1}{4}\left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{2-q}{p-2}} - \lambda\left(\frac{1}{q} - \frac{1}{4}\right)|f|_{r_q} S_r^{-\frac{q}{2}}\right) \\
&\geq \left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{q}{p-2}}\left(\frac{1}{4}\left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{2-q}{p-2}} - \lambda\left(\frac{p-q}{4q}\right)|f|_{r_q} S_r^{-\frac{q}{2}}\right) > 0.
\end{aligned}$$

Thus, if $\lambda \in (0, \lambda_2)$, then

$$I_\lambda(u) > d_0, \quad \forall u \in \mathcal{N}_\lambda^-,$$

for some positive constant d_0. This completes the proof. □

From Lemma 2.1 we can obtain the minimizing sequence of the $I_\lambda(u)$ on the Nehari manifold $\mathcal{N}_\lambda$. To gain a $(PS)_c$ sequence from the minimizing sequence of the $I_\lambda(u)$ on Nehari manifold $\mathcal{N}_\lambda$, we require the following three lemmas:

Lemma 3.2. *If $\lambda \in (0, \lambda_1)$, then for every $u \in \mathcal{N}_\lambda^+$, there exist $\epsilon > 0$ and a differentiable function $g^+ : B_\epsilon(0) \subset E \to \mathbb{R}^+ := (0, +\infty)$ such that*

$$g^+(0) = 1, \quad g^+(\omega)(u - \omega) \in \mathcal{N}_\lambda^+, \quad \forall \omega \in B_\epsilon(0)$$

and

$$\langle (g^+)'(0), v\rangle = \frac{2(u,v) + 4b\int_{\mathbb{R}^N}|\nabla u|^2dx\int_{\mathbb{R}^N}\nabla u\nabla v dx - q\lambda\int_{\mathbb{R}^N} f(x)|u|^{q-2}uvdx - p\int_{\mathbb{R}^N}|u|^{p-2}uvdx}{\varphi''_{\lambda,u}(1)} \tag{3.1}$$

for all $v \in E$. Moreover, if $0 < C_1 \leq \|u\| \leq C_2$, then there exists $C > 0$ such that

$$|\langle (g^+)'(0), v\rangle| \leq C\|v\|. \tag{3.2}$$

Proof. We define $F : \mathbb{R} \times E \to \mathbb{R}$ by

$$\begin{aligned}
F(t,\omega) &= \langle I'_\lambda(t(u-\omega)), (u-\omega)\rangle \\
&= t\|u-\omega\|^2 + t^3 b|\nabla(u-\omega)|_2^4 - \lambda t^{q-1}\int_{\mathbb{R}^N} f(x)|u-\omega|^q dx - t^{p-1}\int_{\mathbb{R}^N}|u-\omega|^p dx,
\end{aligned}$$

it is easy to see F is differentiable. Since $F(1,0) = \langle I'_\lambda(u), u\rangle = 0$ and $F_t(1,0) = \varphi''_{\lambda,u}(1) > 0$, we apply the implicit function theorem at point $(1,0)$ to get the existence of $\epsilon > 0$ and differentiable function $g^+ : B_\epsilon(0) \to \mathbb{R}^+$ such that $g^+(0) = 1$ and $F(g^+(\omega), \omega) = 0$ for $\forall \omega \in B_\epsilon(0)$. Thus,

$$g^+(\omega)(u - \omega) \in \mathcal{N}_\lambda, \quad \forall \omega \in B_\epsilon(0).$$

Next, we show $g^+(\omega)(u-\omega) \in \mathcal{N}_\lambda^+$, $\forall \omega \in B_\epsilon(0)$. By $u \in \mathcal{N}_\lambda^+$ and (2.3), we have

$$\|u\|^2 + 3b|\nabla u|_2^4 - (q-1)\lambda \int_{\mathbb{R}^N} f(x)|u|^q dx - (p-1)\int_{\mathbb{R}^N} |u|^p dx > 0.$$

Since $g^+(\omega)(u-\omega)$ is continuous with respect to ω, when ϵ is small enough, we know for $\omega \in B_\epsilon(0)$

$$\begin{aligned}&\|g^+(\omega)(u-\omega)\|^2 + 3b|\nabla(g^+(\omega)(u-\omega))|_2^4 \\ &- (q-1)\lambda \int_{\mathbb{R}^N} f(x)|g^+(\omega)(u-\omega)|^q dx - (p-1)\int_{\mathbb{R}^N} |g^+(\omega)(u-\omega)|^p dx > 0.\end{aligned}$$

Thus, $g^+(\omega)(u-\omega) \in \mathcal{N}_\lambda^+$, $\forall \omega \in B_\epsilon(0)$.

Also by the differentiability of the implicit function theorem, we know that

$$\langle (g^+)'(0), v\rangle = -\frac{\langle F_\omega(1,0), v\rangle}{F_t(1,0)}.$$

Note that

$$-\langle F_\omega(1,0), v\rangle = 2(u,v) + 4b\int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla v dx - q\lambda \int_{\mathbb{R}^N} f(x)|u|^{q-2}uv dx - p\int_{\mathbb{R}^N} |u|^{p-2}uv dx$$

and $F_t(1,0) = \varphi''_{\lambda,u}(1)$. So we prove (3.1).

Moreover, by (3.1), $0 < C_1 \le \|u\| \le C_2$ and Hölder inequality, we have

$$|\langle (g^+)'(0), v\rangle| \le \frac{\widetilde{C}\|v\|}{\varphi''_{\lambda,u}(1)}$$

for some $\widetilde{C} > 0$. Therefore, in order to prove (3.2), we only need to show that $|\varphi''_{\lambda,u}(1)| > d$ for some $d > 0$. We argue by contradiction. Assume that there exists a sequence $\{u_n\} \in \mathcal{N}_\lambda^+$, $C_1 \le \|u_n\| \le C_2$, we have $\varphi''_{\lambda,u_n}(1) = o_n(1)$, where $o_n(1) \to 0$ as $n \to +\infty$. Then for $C_1 \le \|u_n\| \le C_2$ by (2.5) and Sobolev inequality, we have

$$\begin{aligned}(2-q)\|u_n\|^2 &\le (2-q)\|u_n\|^2 + (4-q)b|\nabla u_n|_2^4 \\ &= (p-q)\int_{\mathbb{R}^N} |u_n|^p dx + o_n(1) \\ &\le (p-q)S_p^{-\frac{p}{2}}\|u_n\|^p + o_n(1)\end{aligned}$$

and so

$$\|u_n\| \ge \left(\frac{(2-q)S_p^{\frac{p}{2}}}{p-q}\right)^{\frac{1}{p-2}} + o_n(1). \tag{3.3}$$

Similarly, using (2.4), Hölder and Sobolev inequalities, we have

$$\begin{aligned}(p-2)\|u_n\|^2 &\le (p-2)\|u_n\|^2 + (p-4)b|\nabla u_n|_2^4 \\ &= (p-q)\lambda \int_{\mathbb{R}^N} f(x)|u_n|^q dx + o_n(1) \\ &\le (p-q)\lambda |f|_{r_q} S_r^{-\frac{q}{2}}\|u_n\|^q + o_n(1)\end{aligned}$$

which implies

$$\|u_n\| \le \left(\frac{(p-q)\lambda|f|_{r_q}}{(p-2)S_r^{\frac{q}{2}}}\right)^{\frac{1}{2-q}} + o_n(1). \tag{3.4}$$

Combining (3.3) and (3.4) as $n \to +\infty$, we deduce

$$\lambda \ge \left(\frac{p-2}{(p-q)|f|_{r_q}}\right)\left(\frac{2-q}{p-q}\right)^{\frac{2-q}{p-2}} S_p^{\frac{p(2-q)}{2(p-2)}} S_r^{\frac{q}{2}} = \lambda_1,$$

which is a contradiction. Thus if $0 < C_1 \le \|u\| \le C_2$, there exists $C > 0$ such that

$$|\langle (g^+)'(0), v\rangle| \le C\|v\|.$$

This completes the proof. □

Analogously, we establish the following lemma.

Lemma 3.3. *If $\lambda \in (0, \lambda_1)$, then for every $u \in \mathcal{N}_\lambda^-$, there exist $\epsilon > 0$ and a differentiable function $g^- : B_\epsilon(0) \subset E \to \mathbb{R}^+$ such that*

$$g^-(0) = 1, \quad g^-(\omega)(u-\omega) \in \mathcal{N}_\lambda^-, \quad \forall \omega \in B_\epsilon(0)$$

and

$$\langle (g^-)'(0), v\rangle = \frac{2(u,v) + 4b\int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla v dx - q\lambda \int_{\mathbb{R}^N} f(x)|u|^{q-2}uv dx - p\int_{\mathbb{R}^N} |u|^{p-2}uv dx}{\varphi''_{\lambda,u}(1)} \tag{3.5}$$

for all $v \in E$. Moreover, if $0 < C_1 \le \|u\| \le C_2$, then there exists $C > 0$ such that

$$|\langle (g^-)'(0), v\rangle| \le C\|v\|. \tag{3.6}$$

Lemma 3.4. *If $\lambda \in (0, \lambda_1)$, one has the following:*
(i) *there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda^+$ such that*

$$I_\lambda(u_n) = \alpha_\lambda^+ + o_n(1),$$
$$I'_\lambda(u_n) = o_n(1);$$

(ii) *there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ such that*

$$I_\lambda(u_n) = \alpha_\lambda^- + o_n(1),$$
$$I'_\lambda(u_n) = o_n(1).$$

Proof. (i) By Lemma 2.1 and the Ekeland variational principle on $\mathcal{N}_\lambda^+$, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda^+$ such that

$$\alpha_\lambda^+ \le I_\lambda(u_n) < \alpha_\lambda^+ + \frac{1}{n} \tag{3.7}$$

and

$$I_\lambda(u_n) \le I_\lambda(v) + \frac{1}{n}\|v - u_n\| \quad \text{for each } v \in \mathcal{N}_\lambda^+. \tag{3.8}$$

And we can show that there exists $C_1, C_2 > 0$ such that $0 < C_1 \leq \|u_n\| \leq C_2$. Indeed, if not, that is, $u_n \to 0$ in E, then $I_\lambda(u_n)$ would converge to zero, which contradict with $I_\lambda(u_n) \to \alpha_\lambda^+ < 0$. Moreover, by Lemma 2.1 we know that I_λ is coercive on $\mathcal{N}_\lambda^+$, $\{u_n\}$ is bounded in $\mathcal{N}_\lambda^+$.

Now, we show that

$$\|I_\lambda'(u_n)\| \to 0 \quad \text{as } n \to \infty.$$

Applying Lemma 3.2 with u_n to obtain the functions $g_n^+(\omega) : B_{\epsilon_n}(0) \to \mathbb{R}^+$ for some $\epsilon_n > 0$, such that

$$g_n^+(0) = 1, \quad g_n^+(\omega)(u_n - \omega) \in \mathcal{N}_\lambda^+, \quad \forall \omega \in B_{\epsilon_n}(0).$$

We choose $0 < \rho < \epsilon_n$. Let $u \in E\backslash\{0\}$ and $\omega_\rho = \rho u/\|u\|$. Since $g_n^+(\omega_\rho)(u_n - \omega_\rho) \in \mathcal{N}_\lambda^+$, we deduce from (3.8) that

$$\begin{aligned}
&\frac{1}{n}[|g_n^+(\omega_\rho) - 1|\|u_n\| + \rho g_n^+(\omega_\rho)] \\
\geq&\frac{1}{n}\|g_n^+(\omega_\rho)(u_n - \omega_\rho) - u_n\| \\
\geq&I_\lambda(u_n) - I_\lambda(g_n^+(\omega_\rho)(u_n - \omega_\rho)) \\
=&\frac{1}{2}\|u_n\|^2 + \frac{b}{4}|\nabla u_n|_2^4 - \frac{\lambda}{q}\int_{\mathbb{R}^N} f(x)|u_n|^q dx - \frac{1}{p}\int_{\mathbb{R}^N}|u_n|^p dx \\
&- \frac{1}{2}\left(g_n^+(\omega_\rho)\right)^2\|u_n - \omega_\rho\|^2 - \frac{b}{4}\left(g_n^+(\omega_\rho)\right)^4|\nabla(u_n - \omega_\rho)|_2^4 \\
&+ \frac{\lambda}{q}\left(g_n^+(\omega_\rho)\right)^q\int_{\mathbb{R}^N} f(x)|u_n - \omega_\rho|^q dx + \frac{1}{p}\left(g_n^+(\omega_\rho)\right)^p\int_{\mathbb{R}^N}|u_n - \omega_\rho|^p dx \\
=&-\frac{\left(g_n^+(\omega_\rho)\right)^2 - 1}{2}\|u_n - \omega_\rho\|^2 - \frac{1}{2}(\|u_n - \omega_\rho\|^2 - \|u_n\|^2) \\
&- b\frac{\left(g_n^+(\omega_\rho)\right)^4 - 1}{4}|\nabla(u_n - \omega_\rho)|_2^4 - \frac{b}{4}(|\nabla(u_n - \omega_\rho)|_2^4 - |\nabla u_n|_2^4) \\
&+ \lambda\frac{\left(g_n^+(\omega_\rho)\right)^q - 1}{q}\int_{\mathbb{R}^N} f(x)|u_n - \omega_\rho|^q dx + \frac{\lambda}{q}\left(\int_{\mathbb{R}^N} f(x)|u_n - \omega_\rho|^q dx - \int_{\mathbb{R}^N} f(x)|u_n|^q dx\right) \\
&+ \frac{\left(g_n^+(\omega_\rho)\right)^p - 1}{p}\int_{\mathbb{R}^N}|u_n - \omega_\rho|^p dx + \frac{1}{p}\left(\int_{\mathbb{R}^N}|u_n - \omega_\rho|^p dx - \int_{\mathbb{R}^N}|u_n|^p dx\right).
\end{aligned} \tag{3.9}$$

Note that

$$\lim_{\rho\to 0^+}\frac{g_n^+(\omega_\rho) - 1}{\rho} = \lim_{\rho\to 0^+}\frac{g_n^+(0 + \rho\frac{u}{\|u\|}) - g_n^+(0)}{\rho} = \langle (g_n^+)'(0), \frac{u}{\|u\|}\rangle.$$

If we divide the ends of (3.9) by ρ and let $\rho \to 0^+$, we have

$$
\begin{aligned}
&\frac{1}{n}\left[|\langle (g_n^+)'(0), \frac{u}{\|u\|}\rangle|\|u_n\| + 1\right] \\
\geq &- \langle (g_n^+)'(0), \frac{u}{\|u\|}\rangle\|u_n\|^2 - \int_{\mathbb{R}^N} \Delta u_n \Delta(-\frac{u}{\|u\|}) + a\nabla u_n \nabla(-\frac{u}{\|u\|}) + V(x)u_n(-\frac{u}{\|u\|})dx \\
&- b\langle (g_n^+)'(0), \frac{u}{\|u\|}\rangle|\nabla u_n|_2^4 - b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \int_{\mathbb{R}^N} \nabla u_n \nabla(-\frac{u}{\|u\|})dx \\
&+ \lambda\langle (g_n^+)'(0), \frac{u}{\|u\|}\rangle \int_{\mathbb{R}^N} f(x)|u_n|^q dx + \lambda \int_{\mathbb{R}^N} f(x)|u_n|^{q-2}u_n(-\frac{u}{\|u\|})dx \\
&+ \langle (g_n^+)'(0), \frac{u}{\|u\|}\rangle \int_{\mathbb{R}^N} |u_n|^p dx + \int_{\mathbb{R}^N} |u_n|^{p-2}u_n(-\frac{u}{\|u\|})dx \\
= &- \langle (g_n^+)'(0), \frac{u}{\|u\|}\rangle \left(\|u_n\|^2 + b|\nabla u_n|_2^4 - \lambda \int_{\mathbb{R}^N} f(x)|u_n|^q dx - \int_{\mathbb{R}^N} |u_n|^p dx\right) \\
&+ \frac{1}{\|u\|}\int_{\mathbb{R}^N} (\Delta u_n \Delta u + a\nabla u_n \nabla u + V(x)u_n u)dx + \frac{b}{\|u\|}\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \int_{\mathbb{R}^N} \nabla u_n \nabla u dx \\
&- \frac{\lambda}{\|u\|}\int_{\mathbb{R}^N} f(x)|u_n|^{q-2}u_n u dx - \frac{1}{\|u\|}\int_{\mathbb{R}^N} |u_n|^{p-2}u_n u dx \\
= &- \langle (g_n^+)'(0), \frac{u}{\|u\|}\rangle\langle I_\lambda'(u_n), u_n\rangle + \frac{1}{\|u\|}\langle I_\lambda'(u_n), u\rangle \\
= &\frac{1}{\|u\|}\langle I_\lambda'(u_n), u\rangle,
\end{aligned}
$$

that is,

$$\frac{1}{n}\left[|\langle (g_n^+)'(0), u\rangle|\|u_n\| + \|u\|\right] \geq \langle I_\lambda'(u_n), u\rangle.$$

By the boundedness of $\|u_n\|$ and Lemma 3.2, there exists $\hat{C} > 0$ such that

$$\frac{\hat{C}}{n} \geq \langle I_\lambda'(u_n), \frac{u}{\|u\|}\rangle.$$

Hence we have

$$\|I_\lambda'(u_n)\| = \sup_{u \in E\setminus\{0\}} \frac{\langle I_\lambda'(u_n), u\rangle}{\|u\|} \leq \frac{\hat{C}}{n},$$

that is, $I_\lambda'(u_n) = o(1)$ as $n \to +\infty$. This completes the proof of (i).
(ii) Similarly, by using Lemma 3.3, we can prove (ii). We will omit detailed proof here. □

Now, we establish the existence of a minimum for I_λ on $\mathcal{N}_\lambda^+$.

Theorem 3.5. *If $\lambda \in (0, \lambda_1)$, the functional I_λ has a minimizer u_0^+ in $\mathcal{N}_\lambda^+$ and it satisfies $I_\lambda(u_0^+) = \alpha_\lambda^+$.*

Proof. By Lemma 3.4, there exist a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda^+$ such that

$$I_\lambda(u_n) = \alpha_\lambda^+ + o_n(1), \quad \text{and} \quad I_\lambda'(u_n) = o_n(1).$$

Then by Lemma 2.1 and the compact embedding theorem, there exist a subsequence $\{u_n\}$ and $u_0^+ \in E$ such that

$$u_n \rightharpoonup u_0^+ \quad \text{in } E,$$
$$u_n \to u_0^+ \quad \text{in } L^r(\mathbb{R}^N) \text{ for } 2 \le r < 2_*.$$

Next we prove $u_n \to u_0^+$ in E. Note that

$$\begin{aligned}
&\langle I'_\lambda(u_n) - I'_\lambda(u_0^+), u_n - u_0^+\rangle = \langle I'_\lambda(u_n), u_n - u_0^+\rangle - \langle I'_\lambda(u_0^+), u_n - u_0^+\rangle \\
=&\int_{\mathbb{R}^N} \Delta u_n \Delta(u_n - u_0^+) + a\nabla u_n \nabla(u_n - u_0^+) + V(x)u_n(u_n - u_0^+)dx \\
&+ b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \int_{\mathbb{R}^N} \nabla u_n \nabla(u_n - u_0^+)dx \\
&- \lambda\int_{\mathbb{R}^N} f(x)|u_n|^{q-2}u_n(u_n - u_0^+)dx - \int_{\mathbb{R}^N} |u_n|^{p-2}u_n(u_n - u_0^+)dx \\
&- \int_{\mathbb{R}^N} \Delta u_0^+ \Delta(u_n - u_0^+) + a\nabla u_0^+ \nabla(u_n - u_0^+) + V(x)u_0^+(u_n - u_0^+)dx \\
&- b\int_{\mathbb{R}^N} |\nabla u_0^+|^2 dx \int_{\mathbb{R}^N} \nabla u_0^+ \nabla(u_n - u_0^+)dx \\
&+ \lambda\int_{\mathbb{R}^N} f(x)|u_0^+|^{q-2}u_0^+(u_n - u_0^+)dx + \int_{\mathbb{R}^N} |u_0^+|^{p-2}u_0^+(u_n - u_0^+)dx \\
=&\int_{\mathbb{R}^N} |\Delta(u_n - u_0^+)|^2 + a|\nabla(u_n - u_0^+)|^2 + V(x)|u_n - u_0^+|^2 dx + b\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \int_{\mathbb{R}^N} |\nabla(u_n - u_0^+)|^2 dx \\
&- b\left(\int_{\mathbb{R}^N} |\nabla u_0^+|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N} \nabla u_0^+ \nabla(u_n - u_0^+)dx \\
&- \lambda\int_{\mathbb{R}^N} f(x)(|u_n|^{q-2}u_n - |u_0^+|^{q-2}u_0^+)(u_n - u_0^+)dx - \int_{\mathbb{R}^N} (|u_n|^{p-2}u_n - |u_0^+|^{p-2}u_0^+)(u_n - u_0^+)dx \\
\ge&\|u_n - u_0^+\|^2 - b\left(\int_{\mathbb{R}^N} |\nabla u_0^+|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N} \nabla u_0^+ \nabla(u_n - u_0^+)dx \\
&- \lambda\int_{\mathbb{R}^N} f(x)(|u_n|^{q-2}u_n - |u_0^+|^{q-2}u_0^+)(u_n - u_0^+)dx - \int_{\mathbb{R}^N} (|u_n|^{p-2}u_n - |u_0^+|^{p-2}u_0^+)(u_n - u_0^+)dx,
\end{aligned}$$

then we can deduce that $\|u_n - u_0^+\| \to 0$ as $n \to \infty$. Indeed, from the boundedness of $\{u_n\}$ in E and the continuous embedding, $\{u_n\}$ is bounded in $L^r(\mathbb{R}^N)$, $r \in [2, 2_*]$. Using Hölder inequality we see that

$$\begin{aligned}
&|\lambda\int_{\mathbb{R}^N} f(x)(|u_n|^{q-2}u_n - |u_0^+|^{q-2}u_0^+)(u_n - u_0^+)dx| \\
\le&\lambda\left(\int_{\mathbb{R}^N} |f|^{r_q} dx\right)^{\frac{1}{r_q}}\left(\int_{\mathbb{R}^N} ||u_n|^{q-2}u_n - |u_0^+|^{q-2}u_0^+|^{\frac{r}{q}}|u_n - u_0^+|^{\frac{r}{q}} dx\right)^{\frac{q}{r}} \\
\le&C|f|_{r_q}(|u_n|_r^{q-1} + |u_0^+|_r^{q-1})|u_n - u_0^+|_r \to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

where C is a positive constant. Similarly, we obtain

$$|\int_{\mathbb{R}^N} (|u_n|^{p-2}u_n - |u_0^+|^{p-2}u_0^+)(u_n - u_0^+)dx| \to 0, \quad \text{as } n \to \infty.$$

From

$$b\left(\int_{\mathbb{R}^N} |\nabla u_0^+|^2 dx - \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^N} \nabla u_0^+ \nabla (u_n - u_0^+) dx \to 0, \quad \text{as } n \to \infty,$$

and

$$\langle I'_\lambda(u_n) - I'_\lambda(u_0^+), u_n - u_0^+ \rangle = \langle I'_\lambda(u_n), u_n - u_0^+ \rangle - \langle I'_\lambda(u_0^+), u_n - u_0^+ \rangle \to 0, \quad \text{as } n \to \infty,$$

we have $\|u_n - u_0^+\| \to 0$ as $n \to \infty$.

In addition, from the proof of Lemma 3.4 we know that there exists $C_1, C_2 > 0$ such that $0 < C_1 \leq \|u_n\| \leq C_2$, then $0 < C_1 \leq \|u_0^+\| \leq C_2$. Thus $u_0^+ \neq 0$.

Next we prove $u_0^+ \in \mathcal{N}_\lambda^+$. In fact, it follows from (2.4) that

$$\varphi''_{\lambda,u_n}(1) \to \varphi''_{\lambda,u_0^+}(1), \quad n \to \infty.$$

From $\varphi''_{\lambda,u_n}(1) > 0$, we have $\varphi''_{\lambda,u_0^+}(1) \geq 0$. By Lemma 2.3, we know $\varphi''_{\lambda,u_0^+}(1) > 0$. Thus we deduce

$$u_0^+ \in \mathcal{N}_\lambda^+, \quad I_\lambda(u_0^+) = \lim_{n\to\infty} I_\lambda(u_n) = \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u) = \alpha_\lambda^+.$$

This completes the proof. □

Next, we establish the existence of a minimum for I_λ on $\mathcal{N}_\lambda^-$.

Theorem 3.6. *If $\lambda \in (0, \lambda_1)$, the functional I_λ has a minimizer u_0^- in $\mathcal{N}_\lambda^-$ and it satisfies $I_\lambda(u_0^-) = \alpha_\lambda^-$.*

Proof. By Lemma 3.4, there exist a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ such that

$$I_\lambda(u_n) = \alpha_\lambda^- + o_n(1), \quad \text{and} \quad I'_\lambda(u_n) = o_n(1).$$

Then by Lemma 2.1 and the conpact embedding theorem, there exist a subsequence $\{u_n\}$ and $u_0^- \in E$ such that

$$u_n \rightharpoonup u_0^- \quad \text{in } E,$$
$$u_n \to u_0^- \quad \text{in } L^r(\mathbb{R}^N) \text{ for } 2 \leq r < 2_*.$$

In view of the proof of Lemma 3.4 we know that there exists $C_1, C_2 > 0$ such that $0 < C_1 \leq \|u_n\| \leq C_2$, then $0 < C_1 \leq \|u_0^-\| \leq C_2$. Thus $u_0^- \neq 0$. Moreover, in the same way as Theorem 3.5, we still have $u_n \to u_0^-$ in E. By Lemma 2.4 the set $\mathcal{N}_\lambda^-$ is closed in E, we know $u_0^- \in \mathcal{N}_\lambda^-$. Thus,

$$I_\lambda(u_0^-) = \lim_{n\to\infty} I_\lambda(u_n) = \inf_{u\in\mathcal{N}_\lambda^-} I_\lambda(u) = \alpha_\lambda^-.$$

This completes the proof. □

Now we can give the proof of the main result.

Proof of Theorem 1.1. From Theorems 3.5, 3.6 and Lemma 2.2, we know if $\lambda \in (0, \lambda_1)$, then Eq (1.1) has at least two solutions u_0^-, u_0^+ and $I_\lambda(u_0^+) < 0$. Since $u_0^+ \in \mathcal{N}_\lambda^+$, $u_0^- \in \mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, this implies that u_0^+ and u_0^- are different. In addition, if $\lambda \in (0, \lambda_2)$, by Lemma 3.1 we have $I_\lambda(u_0^+) < 0$ and $I_\lambda(u_0^-) > 0$, which implies $\alpha_\lambda = \alpha_\lambda^+ = I_\lambda(u_0^+)$. So u_0^+ is a ground state solution of Eq (1.1). It completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

In this section, we denote ${}^+u = \max\{u(x), 0\}$ and ${}^-u = \min\{u(x), 0\}$, then $u = {}^+u + {}^-u$. Define working space

$$\bar{E} = \{u \in E | \frac{\partial_i u}{\partial x_i} \in H \text{ for } i = 1, 2, \cdots, N\},$$

where $H = \{u \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty\}$. Moreover, the functional $I : \bar{E} \to \mathbb{R}$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\Delta u|^2 + a|\nabla u|^2 + V(x)u^2)dx + \frac{b}{4}(\int_{\mathbb{R}^N}|\nabla u|^2 dx)^2 - \frac{1}{p}\int_{\mathbb{R}^N}|u|^p dx.$$

In order to obtain a sign-changing solution of (1.1), we consider the minimization of the following manifold

$${}^{\pm}\mathcal{N} = \{u \in \bar{E}, {}^{\pm}u \neq 0 \text{ and } \langle I'(u), {}^+u\rangle = \langle I'(u), {}^-u\rangle = 0\}.$$

Define $\alpha = \inf_{u \in {}^{\pm}\mathcal{N}} I(u)$. Similar to Lemma 2.2, if there exists $u \in {}^{\pm}\mathcal{N}$ such that $I(u) = \alpha$, then u is a solution of (1.1).

Proof of Theorem 1.2. Without loss of generality, we can assume $b = 1$. Let $\{u_n\} \subset {}^{\pm}\mathcal{N}$ be a minimizing sequence of α. Going if necessary to a subsequence, one has

$$\frac{1}{4}\|u_n\|^2 + \frac{p-4}{4p}|u_n|_p^p = I(u_n) - \langle I'(u_n), u_n\rangle \leq 2\alpha,$$

that is, $\{u_n\}$ is a bounded sequence of $\bar{E}$. Then by the compact embedding theorem, there exist a subsequence $\{u_n\}$ and $u \in \bar{E}$ such that

$$u_n \rightharpoonup u, \quad {}^{\pm}u_n \rightharpoonup {}^{\pm}u \quad \text{in } E,$$

$${}^{\pm}u_n \to {}^{\pm}u \quad \text{in } L^r(\mathbb{R}^N) \text{ for } 2 \leq r < 2_*,$$

$$\nabla^{\pm}u_n \to \nabla^{\pm}u \quad \text{in } L^r(\mathbb{R}^N) \text{ for } 2 \leq r < 2_*,$$

as $n \to \infty$. We assert that there exists $C > 0$ such that $|{}^{\pm}u_n|_p \geq C$, which implies that ${}^{\pm}u \neq 0$. In fact, for any $u \in {}^{\pm}\mathcal{N}$, there exists $C > 0$ such that $|u|_p \geq C$. Suppose to the contrary that there exists a sequence $\{u_n\} \subset {}^{\pm}\mathcal{N}$ such that $|u_n|_p \to 0$ as $n \to \infty$. From $\langle I'(u_n), {}^+u_n\rangle = 0$, there holds

$$\|{}^+u_n\|^2 \leq \|{}^+u_n\|^2 + |\nabla u_n|_2^2|\nabla^+ u_n|_2^2 = |{}^+u_n|_p^p \leq C\|{}^+u_n\|^p.$$

Therefore, there exists $C > 0$ such that $\|{}^+u_n\| \geq C$. Moreover, it follows from $|{}^+u_n|_p \leq |u_n|_p \to 0$ and $\langle I'(u_n), {}^+u_n\rangle = 0$ that $\|{}^+u_n\| \to 0$ as $n \to \infty$, which contradicts $\|{}^+u_n\| \geq C > 0$. Therefore, there exists $C > 0$ such that $|u|_p \geq C$ for any $u \in {}^{\pm}\mathcal{N}$. Similar to the discussion of Lemma 2.5, there exists $0 < {}^-t \leq {}^+t$ such that ${}^-t^-u + {}^+t^+u \in {}^{\pm}\mathcal{N}$, which implies that

$$\|{}^+u\|^2 + {}^+t^2|\nabla^+u|_2^4 + {}^-t^2|\nabla^-u|_2^2|\nabla^+u|_2^2 = {}^+t^{p-2}|{}^+u|_p^p.$$

Since ${}^-t \leq {}^+t$, there holds

$${}^+t^{p-4}|{}^+u|_p^p \leq \frac{1}{{}^+t^2}\|{}^+u\|^2 + |\nabla u|_2^2|\nabla^+u|_2^2. \tag{4.1}$$

Moreover, it follows from $\{u_n\} \subset^{\pm} \mathcal{N}$ that

$$|^+u_n|_p^p = \|^+u_n\|^2 + |\nabla u_n|_2^2|\nabla^+ u_n|_2^2,$$

and by the weakly lower semicontinuity of norm, one has

$$|^+u|_p^p \geq \|^+u\|^2 + |\nabla u|_2^2|\nabla^+ u|_2^2. \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$(1 -^+ t^{p-4})|^+u|_p^p \geq (1 - \frac{1}{^+t^2})\|^+u\|^2,$$

which implies $^+t \leq 1$. Therefore, $0 <^- t \leq^+ t \leq 1$. By $^-t^-u +^+ t^+u \in^{\pm} \mathcal{N}$, one has

$$\begin{aligned}
\alpha \leq I(^-t^-u +^+ t^+u) &= I(^-t^-u +^+ t^+u) - \frac{1}{4}\langle I'(^-t^-u +^+ t^+u),^- t^-u +^+ t^+u\rangle \\
&= \frac{^+t^2}{4}\|^+u\|^2 + (\frac{1}{4} - \frac{1}{p})^+t^p|^+u|_p^p + \frac{^-t^2}{4}\|^-u\|^2 + (\frac{1}{4} - \frac{1}{p})^-t^p|^-u|_p^p \\
&\leq \frac{1}{4}\|^+u\|^2 + (\frac{1}{4} - \frac{1}{p})|^+u|_p^p + \frac{1}{4}\|^-u\|^2 + (\frac{1}{4} - \frac{1}{p})|^-u|_p^p \\
&= \frac{1}{4}\|u\|^2 + (\frac{1}{4} - \frac{1}{p})|u|_p^p \leq \lim_{n\to\infty}\inf[\frac{1}{4}\|u_n\|^2 + (\frac{1}{4} - \frac{1}{p})|u_n|_p^p] \\
&= \lim_{n\to\infty}\inf I(u_n) = \alpha,
\end{aligned}$$

which implies that $^+t =^- t = 1$, $u =^+ u +^- u \in^{\pm} \mathcal{N}$ and $I(u) = I(^+u +^- u) = \alpha$. Then, we conclude that $u =^+ u +^- u$ is a sign-changing solution of (1.1). It completes the proof of Theorem 1.2. □

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Conflict of interest

The authors declare there is no conflicts of interest.

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