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Research article

Efficient time second-order SCQ formula combined with a mixed element method for a nonlinear time fractional wave model

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Abstract: In this article, a kind of nonlinear wave model with the Caputo fractional derivative is solved by an efficient algorithm, which is formulated by combining a time second-order shifted convolution quadrature (SCQ) formula in time and a mixed element method in space. The stability of numerical scheme is derived, and an optimal error result for unknown functions which include an original function and two auxiliary functions are proven. Further, the numerical tests are conducted to confirm the theoretical results.

Keywords: nonlinear time fractional wave model; second-order SCQ formula; mixed finite element method; stability; optimal error estimate

1. Introduction

A large number of fractional partial differential equation (FPDE) models have been found in many fields of science and engineering, such as fractional wave model [1–5], fractional diffusion model [6–10], fractional FitzHugh-Nagumo monodomain model [11], fractional water wave model [12, 13], fractional Maxwell model [14,15], fractional Allen-Cahn model [16], fractional constitutive model [17] and fractional Fokker-Planck model [18]. With the continuous developments of scholars' research on FPDE models, the important fractional wave equations studied by theoretical or numerical methods [1–5] have received a lot of attention. However, as scholars know that due to the existing of the fractional derivative, their exact solutions for fractional wave equations are hard to be found by some theoretical methods. So, numerical solutions of fractional wave models are studied by designing efficient numerical algorithms, such as finite element method [2, 19–22], meshless method [3], finite difference method [23–33], spectral method [34, 35] and collocation method [36, 37]. In this article, we focus on the following wave model with a nonlinear term and a high-order Caputo time fractional derivative

$$
\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}+\frac{\partial^\beta u}{\partial t^\beta}-\frac{\partial^3 u}{\partial x^2\partial t}+f(u)=d(x,t),(x,t)\in\Omega\times J,\\
&u(a,t)=u(b,t)=0,t\in\overline{J},\\
&u(x,0)=0,\frac{\partial u}{\partial t}(x,0)=0,x\in\overline{\Omega},
\end{aligned}
\tag{1.1}
$$

where $\Omega=(a,b)$ is an open space domain and $J=(0,T]$ with $0<T<\infty$ is a time interval. The source term $d(x,t)$ is a known smooth function, the nonlinear term satisfies $f(u)\in C^2(R)$ with $f(0)=0$, and the Caputo fractional derivative is defined by

$$
\frac{\partial^\beta u(x,t)}{\partial t^\beta}=\frac{1}{\Gamma(2-\beta)}\int_0^t\frac{\frac{\partial^2 u(x,s)}{\partial s^2}ds}{(t-s)^{\beta-1}},\ 1<\beta<2. \tag{1.2}
$$

The fractional wave model (1.1), which describe many physical phenomena including nerve conduction and wave propagation, can be degenerated into the pseudo-hyperbolic equation for $\beta=1$ and the hyperbolic wave equation for $\beta=2$, respectively. In [38], Wang et al. developed a mixed element method with an $L2$-1_σ formula for solving the fractional wave model (1.1) with the time Caputo fractional derivative, which was proposed by improving the H^1-Galerkin mixed element method [39–44]. The improved mixed element method can approach three unknown functions simultaneously. However, in [38], the optimal theory error result for the auxiliary variable v depends on the parameter $\frac{\beta-1}{2}$, from which the optimal estimate result of $\|v(t_n)-v_h^n\|$ cannot abtained by choosing any fractional parameter $\beta\in(1,2)$.

In this article, we develop a fully discrete mixed finite element scheme, where the mixed element method is used to approximate the space direction and the generalized BDF2-θ [45] that is a shifted convolution quadrature (SCQ) method [46] is applied to the approxiamtion of the time direction at any time $t_{n-\theta}$. Based on the formulated fully discrete mixed element method with a second-order SCQ formula, we prove the stability and derive optimal error estimates for three unknown functions. More importantly, with a comparison to the theory error results in [38], we can obtain the optimal error result in L^2-norm for the auxiliary variable v at time t_n by choosing the shifted parameter $\theta=0$. Finally, we implement two numerical examples to verify our optimal theory results.

The rest of the article is outlined as follows: In Section 2, the fully discrete scheme based on the combination between a mixed element method and an SCQ formula (generalized BDF2-θ) is derived; In Section 3, the stability is proven by using useful lemmas; The optimal error estimates for the fully discrete scheme are derived in Section 4. Two experiments, in Section 5, are conducted to further confirm our theoretical results. Finally, in the last section we give the conclusions and advancements.

Throughout the article, we denote by C a positive generic constant which is free of time and space meshes, and may be changed at different occurrences.

2. Fully discrete scheme

By setting the parameter $\alpha = \beta - 1$ and an auxiliary variable $v = \frac{\partial u}{\partial t}$ as shown in [38], we get

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\frac{\partial v(x,s)}{\partial s} ds}{(t-s)^\alpha} = \frac{\partial^\alpha v(x,t)}{\partial t^\alpha}, 0 < \alpha < 1. \tag{2.1}$$

Further, by introducing the other auxiliary variable $\sigma = \frac{\partial v}{\partial x}$, we can rewrite the model (1.1) as the following coupled system with the low order space-time derivatives

$$\begin{aligned}
& v = \frac{\partial u}{\partial t}, \\
& \sigma = \frac{\partial v}{\partial x}, \\
& \frac{\partial v}{\partial t} + \frac{\partial^\alpha v}{\partial t^\alpha} - \frac{\partial \sigma}{\partial x} + f(u) = d(x,t), (x,t) \in \Omega \times J, \\
& u(a,t) = u(b,t) = v(a,t) = v(b,t) = 0, t \in \bar{J}, \\
& u(x,0) = v(x,0) = 0, x \in \bar{\Omega}.
\end{aligned} \tag{2.2}$$

For the fully discrete scheme, we first divide time interval $[0,T]$ by the nodes $t_n = n\tau$ $(n = 0,1,2,...,N)$ with the time step length size $\tau = T/N$, where t_n satisfy $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$, N is a positive integer. Setting $\phi^n = \phi(\cdot, t_n)$, the generalized BDF2-$\theta$ (See [45]) for the Caputo fractional differential operator with $\alpha \in (0,1]$ at time $t_{n-\theta}$ is

$$\begin{aligned}
\frac{\partial^\alpha \phi^{n-\theta}}{\partial t^\alpha} &= \tau^{-\alpha} \sum_{j=0}^{n} \omega_j^{(\alpha)} \phi^{n-j} + O(\tau^2) \\
&\doteq \Psi_\tau^{\alpha,n} \phi + O(\tau^2).
\end{aligned} \tag{2.3}$$

The convolution weights $\{\omega_j^{(\alpha)}\}_{j=0}^{\infty}$ are the coefficients of the following generating function with the relation $\omega^{(\alpha)}(\xi) = \sum_{j=0}^{\infty} \omega_j^{(\alpha)} \xi^j$,

$$\omega^{(\alpha)}(\xi) = \Big(\frac{3\alpha - 2\theta}{2\alpha} - \frac{2\alpha - 2\theta}{\alpha}\xi + \frac{\alpha - 2\theta}{2\alpha}\xi^2\Big)^\alpha, 0 \le \theta \le \min\{\alpha, \frac{1}{2}\}. \tag{2.4}$$

For the convenience of application in calculation, we provide the relationship among these convolution weights $\{\omega_j^{(\alpha)}\}_{j=0}^{\infty}$.

Lemma 2.1. *(See [45]) The convolution weights $\omega_k^{(\alpha)}$ for the generalized BDF2-θ can be arrived at by the recursive formula*

$$\begin{cases}
\omega_0^{(\alpha)} = \Big(\dfrac{3\alpha - 2\theta}{2\alpha}\Big)^\alpha, \ \omega_1^{(\alpha)} = 2(\theta - \alpha)\Big(\dfrac{2\alpha}{3\alpha - 2\theta}\Big)^{1-\alpha}, \\
\omega_k^{(\alpha)} = \dfrac{2\alpha}{k(3\alpha - 2\theta)}\Big[2(\alpha - \theta)\Big(\dfrac{k-1}{\alpha} - 1\Big)\omega_{k-1}^{(\alpha)} + (\alpha - 2\theta)\Big(1 - \dfrac{k-2}{2\alpha}\Big)\omega_{k-2}^{(\alpha)}\Big], \quad k \ge 2.
\end{cases}$$

For the term $\frac{\partial v(t_{n-\theta})}{\partial x}$, we have the following formula

$$\frac{\partial \phi(t_{n-\theta})}{\partial x} = (1-\theta)\frac{\partial \phi^n}{\partial x} + \theta\frac{\partial \phi^{n-1}}{\partial x} + O(\tau^2) \doteq \frac{\partial \phi^{n-\theta}}{\partial x} + O(\tau^2). \tag{2.5}$$

Now, by combining Eqs (2.2), (2.3) with (2.5), we have

$$\begin{aligned}
&(a)\ \Psi_\tau^{1,n}u = v^{n-\theta} + R_1^{n-\theta},\\
&(b)\ \sigma^{n-\theta} = \frac{\partial v^{n-\theta}}{\partial x} + R_2^{n-\theta},\\
&(c)\ \Psi_\tau^{1,n}v + \Psi_\tau^{\alpha,n}v - \frac{\partial \sigma^{n-\theta}}{\partial x} + f(u^{n-\theta}) = d(x,t_{n-\theta}) + R_3^{n-\theta},
\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}
R_1^{n-\theta} &= \Psi_\tau^{1,n}u - \frac{\partial u(t_{n-\theta})}{\partial t} + v(t_{n-\theta}) - v^{n-\theta} = O(\tau^2),\\
R_2^{n-\theta} &= \sigma^{n-\theta} - \sigma(t_{n-\theta}) + \frac{\partial v(t_{n-\theta})}{\partial x} - \frac{\partial v^{n-\theta}}{\partial x} = O(\tau^2),\\
R_3^{n-\theta} &= \Psi_\tau^{1,n}v - \frac{\partial v(t_{n-\theta})}{\partial t} + \Psi_\tau^{\alpha,n}v - \frac{\partial^\alpha v(t_{n-\theta})}{\partial t^\alpha} + \frac{\partial \sigma(t_{n-\theta})}{\partial x} - \frac{\partial \sigma^{n-\theta}}{\partial x} + f(u(t_{n-\theta})) - f(u^{n-\theta}) = O(\tau^2).
\end{aligned}$$

Based on Eq (2.6), the mixed weak formulation is to find $(u^n, v^n, \sigma^n) \in L^2 \times H_0^1 \times H^1$, such that

$$\begin{aligned}
&(a)\ (\Psi_\tau^{1,n}u, w) = (v^{n-\theta}, w) + (R_1^{n-\theta}, w),\\
&(b)\ \Big(\sigma^{n-\theta}, \frac{\partial \psi}{\partial x}\Big) = \Big(\frac{\partial v^{n-\theta}}{\partial x}, \frac{\partial \psi}{\partial x}\Big) + \Big(R_2^{n-\theta}, \frac{\partial \psi}{\partial x}\Big),\\
&(c)\ (\Psi_\tau^{1,n}\sigma, \chi) + (\tau^{-\alpha}\sum_{j=0}^{n}\omega_j^{(\alpha)}\sigma^{n-j}, \chi) + \Big(\frac{\partial \sigma^{n-\theta}}{\partial x}, \frac{\partial \chi}{\partial x}\Big)\\
&= -(g(u^{n-\theta})I_0^{n-\theta}\sigma, \chi) - \Big(d(x,t_{n-\theta}), \frac{\partial \chi}{\partial x}\Big) + (R_3^{n-\theta}, \frac{\partial \chi}{\partial x}) + (R_4^{n-\theta}, \chi),
\end{aligned} \tag{2.7}$$

where $g(u^{n-\theta}) = f'(u^{n-\theta}), I_0^{n-\theta}\sigma = \tau\Big(\frac{1}{2}\sigma^0 + \sum_{k=1}^{n-2}\sigma^k + (1-\frac{\theta}{2})\sigma^{n-1} + \frac{1}{2}(1-\theta)\sigma^n\Big), R_4^{n-\theta} = g(u^{n-\theta})(\int_0^{t_{n-\theta}}\sigma dt - I_0^{n-\theta}\sigma) = O(\tau^2)$.

Setting $(\bar{u}^n, \bar{v}^n, \bar{\sigma}^n) \in L^2 \times H_0^1 \times H^1$ be the time approximate solutions of (u^n, v^n, σ^n), we have

$$\begin{aligned}
&(a)\ (\Psi_\tau^{1,n}\bar{u}, w) = (\bar{v}^{n-\theta}, w),\\
&(b)\ \Big(\bar{\sigma}^{n-\theta}, \frac{\partial \psi}{\partial x}\Big) = \Big(\frac{\partial \bar{v}^{n-\theta}}{\partial x}, \frac{\partial \psi}{\partial x}\Big),\\
&(c)\ (\Psi_\tau^{1,n}\bar{\sigma}, \chi) + (\tau^{-\alpha}\sum_{j=0}^{n}\omega_j^{(\alpha)}\bar{\sigma}^{n-j}, \chi) + \Big(\frac{\partial \bar{\sigma}^{n-\theta}}{\partial x}, \frac{\partial \chi}{\partial x}\Big)\\
&= -(g(\bar{u}^{n-\theta})I_0^{n-\theta}\bar{\sigma}, \chi) - \Big(d(x,t_{n-\theta}), \frac{\partial \chi}{\partial x}\Big).
\end{aligned} \tag{2.8}$$

For formulating the fully discrete mixed element scheme, we provide the following mixed finite element spaces

$$\begin{aligned}
&L_h = \{u_h|u_h \in \mathbb{P}^m \text{ on each element}, m \in \mathbb{N}\},\\
&V_h = \{v_h|v_h \in \mathbb{P}^k \text{ on each element}, v_h(a) = v_h(b) = 0, \frac{\partial v_h}{\partial x} \in L^2, k \in \mathbb{Z}^+\},\\
&H_h = \{\sigma_h|\sigma_h \in \mathbb{P}^r \text{ on each element}, \frac{\partial \sigma_h}{\partial x} \in L^2, r \in \mathbb{Z}^+\},
\end{aligned}$$

where $\mathbb{P}^s$ the set of polynomials of x with the degree of $s \in \mathbb{N}$. Based on Eq (2.7), we obtain the mixed element scheme. That is to find $(u_h^n, v_h^n, \sigma_h^n) \in L_h \times V_h \times H_h \subset L^2 \times H_0^1 \times H^1$, such that

$$\begin{aligned}
&(a)\ (\Psi_\tau^{1,n} u_h, w_h) = (v_h^{n-\theta}, w_h), \quad \forall w_h \in L_h,\\
&(b)\ \Big(\sigma_h^{n-\theta}, \frac{\partial \psi_h}{\partial x}\Big) = \Big(\frac{\partial v_h^{n-\theta}}{\partial x}, \frac{\partial \psi_h}{\partial x}\Big), \quad \forall \psi_h \in V_h,\\
&(c)\ (\Psi_\tau^{1,n} \sigma_h, \chi_h) + (\tau^{-\alpha} \sum_{j=0}^{n} \omega_j^{(\alpha)} \sigma_h^{n-j}, \chi_h) + \Big(\frac{\partial \sigma_h^{n-\theta}}{\partial x}, \frac{\partial \chi_h}{\partial x}\Big)\\
&= -(g(u_h^{n-\theta}) I_0^{n-\theta} \sigma_h, \chi_h) - \Big(d(x, t_{n-\theta}), \frac{\partial \chi_h}{\partial x}\Big), \quad \forall \chi_h \in H_h.
\end{aligned} \tag{2.9}$$

Remark 2.2. 1) *For implementing the computation based on the system (2.9), we need to consider the following case for $n = 1$. For this case, we only need to take the semi-discrete approximation of the nonlinear term*

$$g(\bar{u}^{1-\theta}) I_0^{1-\theta} \bar{\sigma} = g(\bar{u}^0) I_0^1 \bar{\sigma} = g(\bar{u}^0) \tau \bar{\sigma}^0,$$

and the fully discrete approximation

$$g(\bar{u}_h^{1-\theta}) I_0^{1-\theta} \bar{\sigma}_h = g(\bar{u}_h^{0}) I_0^1 \bar{\sigma}_h = g(\bar{u}_h^{0}) \tau \bar{\sigma}_h^0.$$

2) *Now, we illustrate how to derive the Eq (2.7)(c). We multiply Eq (2.6)(c) by $-\frac{\partial \chi}{\partial x}$, and then make the inner product on the space domain $\bar{\Omega} = [a, b]$. Taking the first term as an example, we deduce it in detail. By the integration by part, we obtain for $v \in H_0^1(\Omega)$*

$$\Big(\Psi_\tau^{1,n} v, -\frac{\partial \chi}{\partial x}\Big) = \Big(\Psi_\tau^{1,n} \frac{\partial v}{\partial x}, \chi\Big) + [\chi \Psi_\tau^{1,n} v]|_a^b = (\Psi_\tau^{1,n} \sigma, \chi),$$

which also shows that χ only needs to belong to $H^1(\Omega)$. For this problem, readers can also see other references [39–41].

Remark 2.3. 1) *In Ref [45], one can see that the generalized BDF2-θ is given by*

$$\begin{aligned}
\frac{\partial^\alpha \phi^{n-\theta}}{\partial t^\alpha} &= \tau^{-\alpha} \sum_{j=0}^{n} \omega_j^{(\alpha)} \phi^{n-j} + \tau^{-\alpha} \sum_{j=1}^{k} w_{n,j}^{(\alpha)} \phi^j + O(\tau^2)\\
&\doteq \Psi_\tau^{\alpha,n} \phi + \mathcal{S}_{\tau,k}^{\alpha,n} \phi + O(\tau^2),
\end{aligned} \tag{2.10}$$

where $\Psi_\tau^{\alpha,n}\phi$ *and* $\mathcal{S}_{\tau,k}^{\alpha,n}$ *are called the convolution part and the starting part, respectively. If we only consider the model with a sufficiently smooth exact solution, the starting part will disappear. For this problem, readers can see the detailed illustrations in [45]. Here, we just study the case without the starting part.*
2) *Readers can know easily from many references that the following relationship between the Caputo fractional derivative and the Riemann-Liouville derivative holds*

$$ {}_0^{RL}D_t^\alpha \phi(t) = {}_0^C D_t^\alpha \phi(t) + \sum_{j=0}^{n-1} \frac{\phi^{(j)}(0)}{\Gamma(1+j-\alpha)} t^{j-\alpha}, n-1 \le \alpha < n, \tag{2.11}$$

which imply that if initial values $\phi^{(j)}(0) = 0$*, the equality* ${}_0^{RL}D_t^\alpha \phi(t) = {}_0^C D_t^\alpha \phi(t)$ *holds. In this article, the Caputo fractional derivative* ${}_0^C D_t^\alpha \phi(t)$ *is written as* $\frac{\partial^\alpha \phi(t)}{\partial t^\alpha}$.

3. Lemmas and stability

Now we need to introduce some useful lemmas for the next analysis.

Lemma 3.1. *(See [45]) For series* $\{\phi^m\}$ $m \ge 2$*, we have*

$$\Psi_\tau^{1,m}(\phi, \phi^m) \ge \frac{1}{4\tau}(\mathbb{H}_m(\phi) - \mathbb{H}_{m-1}(\phi)), \tag{3.1}$$

with

$$\mathbb{H}_m(\phi) = (3-2\theta)\|\phi^m\|^2 - (1-2\theta)\|\phi^{m-1}\|^2 + 2\|\phi^m - \phi^{m-1}\|^2. \tag{3.2}$$

In addition, we have

$$\mathbb{H}_m(\phi) \ge \|\phi^m\|^2, \tag{3.3}$$

where $\theta \in [0, \frac{1}{2}]$.

Proof. Here, we just need to take $\theta = \frac{\alpha}{2}$, and then follow the derived process as in [47] to get the result.

Lemma 3.2. *(See [45]) Let* $\omega_k^{(\alpha)}$ *be the coefficients of generating function* $\omega^{(\alpha)}(\xi)$ *and the parameter* θ *satisfies* $0 \le \theta \le \min\{\alpha, \frac{1}{2}\}$*, where* $\alpha \in (0,1)$*. Then we have for any vector* $(\phi^0, \phi^1, \cdots, \phi^n) \in \mathbb{R}^{n+1}$

$$\sum_{m=0}^{n}\sum_{k=0}^{m}(\omega_{m-k}^{(\alpha)}\phi^k, \phi^m) \ge 0, \forall n \ge 1. \tag{3.4}$$

Lemma 3.3. *(See [45]) With the shifted parameter* $\theta \le \frac{1}{2}$ *and* $\phi^0 = 0$*, we have for any vector* $(\phi^1, \phi^2, \cdots, \phi^n) \in \mathbb{R}^n$

$$\sum_{m=1}^{n}(\phi^{m-\theta}, \phi^m) \ge 0, \forall n \ge 1, \tag{3.5}$$

where $v^{m-\theta} \doteq (1-\theta)v^m + \theta v^{m-1}$.

Without loss of generality, we will analyze the stability of the numerical scheme Eq (2.9) for the case of the source term $d(x,t) = 0$.

Theorem 3.4. *For the fully discrete system (2.9), the following stability holds*

$$\|u_h^n\|^2 + \|\sigma_h^n\|^2 + \|v_h^{n-\theta}\|^2 + \Big\|\frac{\partial v_h^{n-\theta}}{\partial x}\Big\|^2 \le C(\|u_h^0\|^2 + \|\sigma_h^0\|^2), \tag{3.6}$$

where C is a positive constant independent on mesh parameters τ and h .

Proof. In Eq (2.9)(a), we take $w_h = u_h^n$, use Eq (3.1) and Cauchy-Schwarz inequality as well as Young inequality to get for $n \ge 2$

$$\frac{1}{4\tau}(\mathbb{H}_n(u_h) - \mathbb{H}_{n-1}(u_h)) \le (v_h^{n-\theta}, u_h^n) \le \frac{1-\theta}{2}\|v_h^n\|^2 + \frac{\theta}{2}\|v_h^{n-1}\|^2 + \frac{1}{2}\|u_h^n\|^2. \tag{3.7}$$

Sum Eq (3.7) for $j = 2$ to n, and use Eqs (3.2) and (3.3) to arrive at

$$\begin{aligned} \|u_h^n\|^2 \le& \mathbb{H}_n(u_h) \\ \le& \tau\sum_{j=2}^{n}\Big((2-\theta)\|v_h^j\|^2 + 2\theta\|v_h^{j-1}\|^2 + 2\|u_h^j\|^2\Big) + \mathbb{H}_1(u_h) \\ \le& \tau\sum_{j=2}^{n}\Big((2-\theta)\|v_h^j\|^2 + 2\theta\|v_h^{j-1}\|^2 + 2\|u_h^j\|^2\Big) + C(\|u_h^1\|^2 + \|u_h^0\|^2). \end{aligned} \tag{3.8}$$

Letting $\psi_h = v_h^{n-\theta}$ in Eq (2.9)(b), using Cauchy-Schwarz inequality as well as Young inequality, noting that $v_h^{n-\theta} \in V_h \subset H_0^1$ and making use of Poincaré inequality, we have

$$\|v_h^{n-\theta}\|^2 \le C\Big\|\frac{\partial v_h^{n-\theta}}{\partial x}\Big\|^2 \le C\|\sigma_h^{n-\theta}\|^2. \tag{3.9}$$

In Eq (2.9)(c), we set $\chi_h = \sigma_h^n$, replace n with k and sum for $k = 2$ to n to get

$$\begin{aligned} &\sum_{k=2}^{n}(\Psi_\tau^{1,k}\sigma_h, \sigma_h^k) + \tau^{-\alpha}\sum_{k=2}^{n}\Big(\sum_{j=0}^{k}\omega_{k-j}^{(\alpha)}\sigma_h^j, \sigma_h^k\Big) + \sum_{k=2}^{n}\Big(\frac{\partial\sigma_h^{k-\theta}}{\partial x}, \frac{\partial\sigma_h^k}{\partial x}\Big) \\ =& \sum_{k=2}^{n}(-\tau g(u_h^{k-\theta})\Big(\frac{1}{2}\sigma_h^0 + \sum_{j=1}^{k-2}\sigma_h^j + (1-\frac{\theta}{2})\sigma_h^{k-1} + \frac{1}{2}(1-\theta)\sigma_h^k\Big), \sigma_h^k), \end{aligned} \tag{3.10}$$

Use Hölder inequality as well as Young inequality to arrive at

$$\begin{aligned} &\frac{1}{4\tau}\sum_{k=2}^{n}(\mathbb{H}_k(\sigma_h) - \mathbb{H}_{k-1}(\sigma_h)) + \tau^{-\alpha}\sum_{k=2}^{n}\Big(\sum_{j=0}^{k}\omega_{k-j}^{(\alpha)}\sigma_h^j, \sigma_h^k\Big) + \sum_{k=2}^{n}\Big(\frac{\partial\sigma_h^{k-\theta}}{\partial x}, \frac{\partial\sigma_h^k}{\partial x}\Big) \\ \le& \tau\sum_{k=2}^{n}\Big(\|g(u_h^{k-\theta})\|_\infty\Big\|\frac{1}{2}\sigma_h^0 + \sum_{j=1}^{k-2}\sigma_h^j + (1-\frac{\theta}{2})\sigma_h^{k-1} + \frac{1}{2}(1-\theta)\sigma_h^k\Big\|\|\sigma_h^k\|\Big) \\ \le& C\tau\sum_{k=2}^{n}\Big(\|\sigma_h^k\|\sum_{j=0}^{k}\|\sigma_h^j\|\Big) \\ \le& C\sum_{k=2}^{n}\Big(\tau\sum_{j=0}^{k}\|\sigma_h^j\|^2 + \frac{\tau(k+1)}{2}\|\sigma_h^k\|^2\Big). \end{aligned} \tag{3.11}$$

Multiplying Eq (3.11) by 4τ and using Young inequality as well as triangle inequality, we have

$$\begin{aligned}\mathbb{H}_n(\sigma_h) + 4\tau^{1-\alpha}\sum_{k=2}^{n}\Big(\sum_{j=0}^{k}\omega_{k-j}^{(\alpha)}\sigma_h^j, \sigma_h^k\Big) + 4\tau\sum_{k=2}^{n}\Big(\frac{\partial\sigma_h^{k-\theta}}{\partial x}, \frac{\partial\sigma_h^k}{\partial x}\Big) \le& C\tau\sum_{k=0}^{n}\|\sigma_h^k\|^2 + \mathbb{H}_1(\sigma_h)\\ \le& C\tau\sum_{k=0}^{n}\|\sigma_h^k\|^2 + C(\|\sigma_h^1\|^2 + \|\sigma_h^0\|^2).\end{aligned} \tag{3.12}$$

Now we only need to estimate the case for $n = 1$. Similar to the processes of Eqs (3.8) and (3.12), we easily derive

$$\|u_h^1\|^2 \le C\tau(\|v_h^1\|^2 + \tau\|v_h^0\|^2 + \|u_h^1\|^2) + \|u_h^0\|^2, \tag{3.13}$$

and

$$\|\sigma_h^1\|^2 + \tau^{1-\alpha}\sum_{j=0}^{1}(\omega_{1-j}^{(\alpha)}\sigma_h^j, \sigma_h^1) + \tau\Big(\frac{\partial\sigma_h^{1-\theta}}{\partial x}, \frac{\partial\sigma_h^1}{\partial x}\Big) \le C\|\sigma_h^0\|^2. \tag{3.14}$$

Combining Eqs (3.8), (3.12), (3.13) with (3.14), we have

$$\|u_h^n\|^2 \le C(\|u_h^0\|^2 + \tau\sum_{j=0}^{n}\|v_h^j\|^2 + \tau\sum_{j=1}^{n}\|u_h^j\|^2), \tag{3.15}$$

$$\|\sigma_h^n\|^2 + \tau^{1-\alpha}\sum_{k=0}^{n}(\sum_{j=0}^{k}\omega_{k-j}^{(\alpha)}\sigma_h^j, \sigma_h^k) + \tau\sum_{k=1}^{n}\Big(\frac{\partial\sigma_h^{k-\theta}}{\partial x}, \frac{\partial\sigma_h^k}{\partial x}\Big) \le C(\|\sigma_h^0\|^2 + \tau\sum_{k=0}^{n}\|\sigma_h^k\|^2). \tag{3.16}$$

Combining Lemmas 3.2–3.3, Eqs (3.9), (3.15) with (3.16), we have

$$\begin{aligned}&\|u_h^n\|^2 + \|\sigma_h^n\|^2 + \|v_h^{n-\theta}\|^2 + \|\frac{\partial v_h^{n-\theta}}{\partial x}\|^2\\ \le& C(\|u_h^0\|^2 + \|\sigma_h^0\|^2 + \tau\sum_{k=0}^{n}(\|u_h^k\|^2 + \|v_h^k\|^2 + \|\sigma_h^k\|^2)).\end{aligned} \tag{3.17}$$

Use Gronwall lemma to finish the proof.

4. Optimal error analysis

In this section, we obtain an error estimate for the numerical scheme Eq (2.9). To facilitate the analysis, we first introduce three projection operators with the corresponding estimate inequalities.

Lemma 4.1. *Define an L^2-projection operator $\Lambda_h : L^2(\Omega) \to L_h$ by*

$$(\bar{u} - \Lambda_h\bar{u}, \omega_h) = 0, \quad \forall\omega_h \in L_h, \tag{4.1}$$

with an estimate inequality

$$\|\bar{u} - \Lambda_h\bar{u}\| + \|\bar{u}_t - \Lambda_h\bar{u}_t\| \le Ch^{m+1}(\|\bar{u}\|_{m+1} + \|\bar{u}_t\|_{m+1}), \quad \forall\bar{u} \in H^{m+1}(\Omega). \tag{4.2}$$

Lemma 4.2. *(See [41]). Define an elliptic projection operator $\Upsilon_h : H_0^1(\Omega) \to V_h$, such that*

$$\Big(\frac{\partial \bar{v}}{\partial x} - \Upsilon_h \frac{\partial \bar{v}}{\partial x}, \frac{\partial \phi_h}{\partial x}\Big) = 0, \quad \forall \phi_h \in V_h, \tag{4.3}$$

with an estimate inequality

$$\|\bar{v} - \Upsilon_h \bar{v}\| + h\|\bar{v} - \Upsilon_h \bar{v}\|_1 \le Ch^{k+1}\|\bar{v}\|_{k+1}, \quad \forall \bar{v} \in H_0^1(\Omega) \cap H^{k+1}(\Omega). \tag{4.4}$$

Lemma 4.3. *(See [41]) Define a Rize projection operator $\Pi_h : H^1(\Omega) \to H_h$ by*

$$\mathcal{A}(\bar{\sigma} - \Pi_h \bar{\sigma}, \chi_h) = 0, \quad \forall \chi_h \in H_h, \tag{4.5}$$

where $\mathcal{A}(\bar{\sigma}, \phi) \doteq \left(\frac{\partial \bar{\sigma}}{\partial x}, \frac{\partial \phi}{\partial x}\right) + \lambda(\bar{\sigma}, \phi)$ and $\mathcal{A}(\phi, \phi) \ge \mu_0 \|\phi\|_1^2, \mu_0 > 0$ is a constant. Further, the estimate inequality holds

$$\|\bar{\sigma} - \Pi_h \bar{\sigma}\| + \|\bar{\sigma}_t - \Pi_h \bar{\sigma}_t\| + h\|\bar{\sigma} - \Pi_h \bar{\sigma}\|_1 \le Ch^{r+1}(\|\bar{\sigma}\|_{r+1} + \|\bar{\sigma}_t\|_{r+1}), \quad \forall \bar{\sigma} \in H^{r+1}(\Omega). \tag{4.6}$$

Theorem 4.4. *With $\Lambda_h \bar{u}(0) = \bar{u}_h^0$, $\Upsilon_h \bar{v}(0) = \bar{v}_h^0$ and $\Pi_h \bar{\sigma}(0) = \bar{\sigma}_h^0$, there exists a positive constant C independent of (h, τ) such that*

$$\|u(t_n) - u_h^n\| + \|\sigma(t_n) - \sigma_h^n\| + \|v(t_{n-\theta}) - v_h^{n-\theta}\| \le C(h^{\min\{m+1, r+1, k+1\}} + \tau^2). \tag{4.7}$$

Proof. For convenience, we write errors as

$$\begin{aligned}
\bar{u}(t_n) - u_h^n &= (\bar{u}(t_n) - \Lambda_h \bar{u}^n) + (\Lambda_h \bar{u}^n - u_h^n) = \rho^n + \vartheta^n, \\
\bar{v}(t_n) - v_h^n &= (\bar{v}(t_n) - \Upsilon_h \bar{v}^n) + (\Upsilon_h \bar{v}^n - v_h^n) = \zeta^n + \xi^n, \\
\bar{\sigma}(t_n) - \sigma_h^n &= (\bar{\sigma}(t_n) - \Pi_h \bar{\sigma}^n) + (\Pi_h \bar{\sigma}^n - \sigma_h^n) = \eta^n + \delta^n,
\end{aligned}$$

Subtract Eq (2.9)(a) from Eq (2.8)(a), set $\omega_h = \vartheta^n$, apply the projection Eq (4.1) and use Cauchy-Schwarz inequality and Young inequality to obtain

$$\begin{aligned}
(\Psi_\tau^{1,n}\vartheta, \vartheta^n) =& -(\Psi_\tau^{1,n}\rho, \vartheta^n) + (\zeta^{n-\theta} + \xi^{n-\theta}, \vartheta^n) \\
&\le \frac{1}{2}(\|\Psi_\tau^{1,n}\rho\|^2 + \|\xi^{n-\theta}\|^2 + \|\zeta^{n-\theta}\|^2) + \frac{3}{2}\|\vartheta^n\|^2.
\end{aligned} \tag{4.8}$$

Replace n by m, sum from $m = 2$ to n, and use Lemmas 3.1 to have

$$\begin{aligned}
\|\vartheta^n\|^2 \le& \mathbb{H}_n(\vartheta) \\
\le& C\tau \sum_{m=2}^{n}(\|\Psi_\tau^{1,n}\rho\|^2 + \|\xi^{n-\theta}\|^2 + \|\zeta^{n-\theta}\|^2) + C\tau \sum_{m=2}^{n}\|\vartheta^m\|^2 + C(\|\vartheta^1\|^2 + \|\vartheta^0\|^2).
\end{aligned} \tag{4.9}$$

Subtract Eq (2.9)(b) from Eq (2.8)(b), take $\psi_h = \xi^{n-\theta}$ and apply projection Eq (4.3) to get

$$\Big(\delta^{n-\theta}, \frac{\partial \xi^{n-\theta}}{\partial x}\Big) = -\Big(\eta^{n-\theta}, \frac{\partial \xi^{n-\theta}}{\partial x}\Big) + \Big\|\frac{\partial \xi^{n-\theta}}{\partial x}\Big\|^2. \tag{4.10}$$

Use Cauchy-Schwarz inequality, Young inequality and Poincaré inequality to arrive at

$$\Big\|\frac{\partial \xi^{n-\theta}}{\partial x}\Big\|^2 + \|\xi^{n-\theta}\|^2 \le C(\|\eta^{n-\theta}\|^2 + \|\delta^n\|^2 + \|\delta^{n-1}\|^2). \tag{4.11}$$

Subtract Eq (2.9)(c) from Eq (2.8)(c), choose $\chi_h = \delta^n$, apply projection Eq (4.5) and use Hölder inequality as well as Young inequality to get

$$
\begin{aligned}
&(\Psi_\tau^{1,n}\delta,\delta^n)+(\Psi_\tau^{\alpha,n}\delta,\delta^n)+\Big(\frac{\partial\delta^{n-\theta}}{\partial x},\frac{\partial\delta^n}{\partial x}\Big)\\
=&-(\Psi_\tau^{1,n}\eta,\delta^n)-(\Psi_\tau^{\alpha,n}\eta,\delta^n)+\lambda(\eta^{n-\theta},\delta^n)\\
&-\Big(g(\bar{u}^{n-\theta})I_0^{n-\theta}\bar{\sigma}-g(u_h^{n-\theta})I_0^{n-\theta}\sigma_h,\delta^n\Big)\\
=&-(\Psi_\tau^{1,n}\eta,\delta^n)-(\Psi_\tau^{\alpha,n}\eta,\delta^n)+\lambda(\eta^{n-\theta},\delta^n)\\
&-\Big(g(\bar{u}^{n-\theta})I_0^{n-\theta}\bar{\sigma}-g(u_h^{n-\theta})I_0^{n-\theta}\bar{\sigma}+g(u_h^{n-\theta})I_0^{n-\theta}\bar{\sigma}\ -g(u_h^{n-\theta})I_0^{n-\theta}\sigma_h,\delta^n\Big)\\
=&-(\Psi_\tau^{1,n}\eta,\delta^n)-(\Psi_\tau^{\alpha,n}\eta,\delta^n)+\lambda(\eta^{n-\theta},\delta^n)\\
&-\Big((g(\bar{u}^{n-\theta})-g(u_h^{n-\theta}))I_0^{n-\theta}\bar{\sigma},\delta^n\Big)-\Big(g(u_h^{n-\theta})I_0^{n-\theta}(\bar{\sigma}-\sigma_h),\delta^n\Big)\\
\le&\frac{1}{2}\|\Psi_\tau^{1,n}\eta\|^2+\frac{1}{2}\|\Psi_\tau^{\alpha,n}\eta\|^2+\frac{\lambda(1-\theta)}{2}\|\eta^n\|^2+\frac{\lambda\theta}{2}\|\eta^{n-1}\|^2+\Big(\frac{3}{2}+\frac{\lambda}{2}\Big)\|\delta^n\|^2\\
&+\Big(\|(g(\bar{u}^{n-\theta})-g(u_h^{n-\theta})\|\|I_0^{n-\theta}\bar{\sigma}\|_\infty+\|g(u_h^{n-\theta})\|_\infty\|I_0^{n-\theta}(\bar{\sigma}-\sigma_h)\|\Big)\|\delta^n\|\\
\le&\frac{1}{2}\|\Psi_\tau^{1,n}\eta\|^2+\frac{1}{2}\|\Psi_\tau^{\alpha,n}\eta\|^2+\frac{\lambda(1-\theta)}{2}\|\eta^n\|^2+\frac{\lambda\theta}{2}\|\eta^{n-1}\|^2+\Big(\frac{3}{2}+\frac{\lambda}{2}\Big)\|\delta^n\|^2\\
&+\Big(C\|g'(\varsigma)\|_\infty(\|\rho^{n-\theta}+\vartheta^{n-\theta}\|+C\|\tau(\frac{1}{2}(\eta^0+\delta^0)+\sum_{j=1}^{n-1}(\eta^j+\delta^j)\\
&+\Big(1-\frac{\theta}{2}\Big)(\eta^{n-1}+\delta^{n-1})+\frac{1}{2}(1-\theta)(\eta^n+\delta^n))\|\Big)\|\delta^n\|\\
\le&\frac{1}{2}\|\Psi_\tau^{1,n}\eta\|^2+\frac{1}{2}\|\Psi_\tau^{\alpha,n}\eta\|^2+C(\|\eta^n\|^2+\|\eta^{n-1}\|^2+\|\rho^n\|^2+\|\rho^{n-1}\|^2)\\
&+C(\|\delta^n\|^2+\|\vartheta^n\|^2+\|\vartheta^{n-1}\|^2)+C\tau\sum_{k=0}^{n}(\|\eta^k\|^2+\|\delta^k\|^2).
\end{aligned}
\tag{4.12}
$$

Replace n by m and sum for $m = 2$ to n to arrive at

$$
\begin{aligned}
&\sum_{m=2}^{n}(\Psi_\tau^{1,m}\delta,\delta^m)+\tau^{-\alpha}\sum_{m=2}^{n}\Big(\sum_{j=0}^{m}\omega_{m-j}^{(\alpha)}\delta^j,\delta^m\Big)+\sum_{m=2}^{n}\Big(\frac{\partial\delta^{m-\theta}}{\partial x},\frac{\partial\delta^m}{\partial x}\Big)\\
\le&C\sum_{m=2}^{n}(\|\Psi_\tau^{\alpha,m}\eta\|^2+\|\Psi_\tau^{1,m}\eta\|^2+\|\delta^m\|^2)+C\sum_{m=1}^{n}(\|\rho^m\|^2+\|\eta^m\|^2+\|\vartheta^m\|^2)+C\tau\sum_{m=1}^{n}\sum_{k=0}^{m}(\|\eta^k\|^2+\|\delta^k\|^2).
\end{aligned}
\tag{4.13}
$$

Noting that the similar method in [45, 48], we have

$$
\|\Psi_\tau^{\alpha,n}\eta\|\le\Big\|\Pi_h(\frac{\partial^\alpha\bar{\sigma}^{n-\theta}}{\partial t^\alpha})-\frac{\partial^\alpha\bar{\sigma}^{n-\theta}}{\partial t^\alpha}\Big\|\le Ch^{r+1},\tag{4.14}
$$

and

$$
\|\Psi_\tau^{1,n}\eta\|\le Ch^{r+1}.\tag{4.15}
$$

Multiply Eq (4.13) by 4τ, and combine Lemmas 4.1 and 4.3 with Eqs (4.14) and (4.15) to arrive at

$$\begin{aligned}&\|\delta^n\|^2+\tau^{1-\alpha}\sum_{m=2}^{n}\Big(\sum_{j=0}^{m}\omega_{m-j}^{(\alpha)}\delta^j,\delta^m\Big)+\tau\sum_{m=2}^{n}\Big(\frac{\partial\delta^{m-\theta}}{\partial x},\frac{\partial\delta^m}{\partial x}\Big)\\ \le&C(h^{2k+2}+h^{2r+2})+C\tau\sum_{m=0}^{n}\|\delta^m\|^2+C\tau\sum_{m=1}^{n}\|\vartheta^m\|^2+\|\delta^1\|^2+\|\delta^0\|^2.\end{aligned}\tag{4.16}$$

Combining Eqs (4.16), (4.9) with (4.11), we have the estimate

$$\begin{aligned}&\|\delta^n\|^2+\|\vartheta^n\|^2+\|\xi^{n-\theta}\|^2+\Big\|\frac{\partial\xi^{n-\theta}}{\partial x}\Big\|^2+\tau^{1-\alpha}\sum_{m=2}^{n}\Big(\sum_{j=0}^{m}\omega_{m-j}^{(\alpha)}\delta^j,\delta^m\Big)+\tau\sum_{m=2}^{n}\Big(\frac{\partial\delta^{m-\theta}}{\partial x},\frac{\partial\delta^m}{\partial x}\Big)\\ \le&C(h^{2k+2}+h^{2r+2}+\tau\sum_{m=2}^{n}(\|\Psi_\tau^{1,n}\rho\|^2+\|\zeta^{n-\theta}\|^2)+C\tau\sum_{m=0}^{n}\|\delta^m\|^2+C\tau\sum_{m=1}^{n}\|\vartheta^m\|^2\\ &+C(\|\vartheta^1\|^2+\|\delta^1\|^2+\|\vartheta^0\|^2+\|\delta^0\|^2).\end{aligned}\tag{4.17}$$

For the case $n=1$, we use a similar derivation to Eq (4.17) to get

$$\begin{aligned}&\|\delta^1\|^2+\|\vartheta^1\|^2+\|\xi^{1-\theta}\|^2+\Big\|\frac{\partial\xi^{1-\theta}}{\partial x}\Big\|^2+\tau^{1-\alpha}\Big(\sum_{j=0}^{1}\omega_{1-j}^{(\alpha)}\delta^j,\delta^m\Big)+\tau\Big(\frac{\partial\delta^{1-\theta}}{\partial x},\frac{\partial\delta^1}{\partial x}\Big)\\ \le&C(h^{2k+2}+h^{2r+2}+h^{2m+2}+\|\vartheta^0\|^2+\|\delta^0\|^2).\end{aligned}\tag{4.18}$$

Combining Eq (4.17) with (4.18) and using Gronwall inequality, we have

$$\begin{aligned}&\|\delta^n\|^2+\|\vartheta^n\|^2+\|\xi^{n-\theta}\|^2+\Big\|\frac{\partial\xi^{n-\theta}}{\partial x}\Big\|^2+\tau^{1-\alpha}\sum_{m=0}^{n}\Big(\sum_{j=0}^{m}\omega_{m-j}^{(\alpha)}\delta^j,\delta^m\Big)+\tau\sum_{m=1}^{n}\Big(\frac{\partial\delta^{m-\theta}}{\partial x},\frac{\partial\delta^m}{\partial x}\Big)\\ \le&C(h^{2k+2}+h^{2r+2}+h^{2m+2}).\end{aligned}\tag{4.19}$$

Apply Lemmas 3.2–3.3 and combine Eqs (4.2), (4.4), (4.6) with triangle inequality to arrive at

$$\|\bar u(t_n)-u_h^n\|+\|\bar\sigma(t_n)-\sigma_h^n\|+\|\bar v(t_{n-\theta})-v_h^{n-\theta}\|\le Ch^{\min\{m+1,r+1,k+1\}}.\tag{4.20}$$

Combining Eq (4.20) with triangle inequality, we have

$$\begin{aligned}&\|u(t_n)-u_h^n\|+\|\sigma(t_n)-\sigma_h^n\|+\|v(t_{n-\theta})-v_h^{n-\theta}\|\\ \le&\|u(t_n)-\bar u(t_n)\|+\|\bar u(t_n)-u_h^n\|+\|\sigma(t_n)-\bar\sigma(t_n)\|\\ &+\|\bar\sigma(t_n)-\sigma_h^n\|+\|v(t_{n-\theta})-\bar v(t_{n-\theta})\|+\|\bar v(t_{n-\theta})-v_h^{n-\theta}\|\\ \le&\|\bar u(t_n)-u_h^n\|+\|\bar\sigma(t_n)-\sigma_h^n\|+\|\bar v(t_{n-\theta})-v_h^{n-\theta}\|+C\tau^2,\end{aligned}\tag{4.21}$$

which implies that we finish the proof.

5. Numerical tests

In this section, we will consider two numerical examples based on the linear element to validate our optimal theory results. In numerical experiments, we need to use the recursive formula provided in Lemma 2.1.

Table 1. The errors and convergence rates in time with $h = \frac{1}{1000}$.

β	θ	τ	$\|\|u - u_h\|\|$	Rate	$\|\|v - v_h\|\|$	Rate	$\|\|\sigma - \sigma_h\|\|$	Rate
		1/10	1.3016E-02	-	5.8621E-03	-	1.8795E-02	-
	0.1	1/14	6.7031E-03	1.9722	3.1046E-03	1.8891	9.8770E-03	1.9122
		1/18	4.0738E-03	1.9816	1.9155E-03	1.9214	6.0722E-03	1.9357
		1/10	1.0864E-02	-	5.1423E-03	-	1.6540E-02	-
1.3	0.2	1/14	5.5735E-03	1.9837	2.6895E-03	1.9263	8.5751E-03	1.9524
		1/18	3.3800E-03	1.9901	1.6452E-03	1.9558	5.2236E-03	1.9723
		1/10	8.7318E-03	-	4.2714E-03	-	1.3791E-02	-
	0.3	1/14	4.4618E-03	1.9954	2.1909E-03	1.9842	7.0052E-03	2.0131
		1/18	2.6999E-03	1.9989	1.3219E-03	2.0104	4.2068E-03	2.0292
		1/10	1.3694E-02	-	7.6728E-03	-	2.5054E-02	-
	0.1	1/14	7.0334E-03	1.9802	4.0506E-03	1.8986	1.3074E-02	1.9329
		1/18	4.2656E-03	1.9899	2.4853E-03	1.9438	7.9743E-03	1.9674
		1/10	1.1421E-02	-	6.9048E-03	-	2.2668E-02	-
1.5	0.2	1/14	5.8412E-03	1.9927	3.6073E-03	1.9296	1.1693E-02	1.9674
		1/18	3.5340E-03	1.9995	2.1967E-03	1.9738	7.0731E-03	2.0001
		1/10	9.1253E-03	-	5.9755E-03	-	1.9778E-02	-
	0.3	1/14	4.6449E-03	2.0070	3.0749E-03	1.9746	1.0033E-02	2.0170
		1/18	2.8024E-03	2.0105	1.8517E-03	2.0180	5.9965E-03	2.0480
		1/10	1.4835E-02	-	1.0860E-02	-	3.6502E-02	-
	0.1	1/14	7.6140E-03	1.9822	5.8365E-03	1.8454	1.9333E-02	1.8888
		1/18	4.6125E-03	1.9944	3.6181E-03	1.9027	1.1890E-02	1.9345
		1/10	1.2430E-02	-	1.0006E-02	-	3.3861E-02	-
1.7	0.2	1/14	6.3535E-03	1.9946	5.3398E-03	1.8665	1.7790E-02	1.9128
		1/18	3.8395E-03	2.0042	3.2932E-03	1.9232	1.0879E-02	1.9572
		1/10	9.9846E-03	-	8.9741E-03	-	3.0669E-02	-
	0.3	1/14	5.0785E-03	2.0092	4.7432E-03	1.8951	1.5940E-02	1.9450
		1/18	3.0599E-03	2.0159	2.9049E-03	1.9510	9.6719E-03	1.9879

5.1. Example 1

In this test, we calculate the convergence rate in time and space. By taking the space domain $\bar{\Omega} = [0, 1]$ and the time domain $[0, 1]$, the nonlinear term $f(u) = u^2$ and the source term

$$d(x, t) = (6t + \frac{6}{\Gamma(4-\beta)} t^{3-\beta} + 3\pi^2 t^2) \sin(\pi x) + (t^3 \sin(\pi x))^2,$$

with given initial and boundary conditions in Eq (1.1), we can validate easily that the exact solution is $u = t^3 \sin \pi x$.

In Table 1, with fixed space step length size $h = \frac{1}{1000}$ and changed time step length parameters $\tau = \frac{1}{10}, \frac{1}{14}, \frac{1}{18}$, we arrive at the approximating time second-order convergence rate in L^2-norm for three functions based on different parameters $\beta = 1.3, 1.5, 1.7$ and $\theta = 0.1, 0.2, 0.3$. Similarly, by choosing the same parameters β and θ as the ones in Table 1 with space-time step length parameters $\tau = \frac{1}{2000}$ and

$h = \frac{1}{10}, \frac{1}{30}, \frac{1}{50}$, we calculate the approximating a priori error results with the second-order convergence rate in Table 2. The data computed in Tables 1 and 2 show the optimal convergence results are achieved by using our method.

Table 2. The errors and convergence rates in space with $\tau = \frac{1}{2000}$.

β	θ	h	$\|\|u - u_h\|\|$	Rate	$\|\|v - v_h\|\|$	Rate	$\|\|\sigma - \sigma_h\|\|$	Rate
		1/10	1.6537E-02	-	4.9369E-02	-	3.2169E-01	-
	0.1	1/30	1.9382E-03	1.9514	5.7725E-03	1.9536	3.6219E-02	1.9880
		1/50	7.0494E-04	1.9800	2.0990E-03	1.9804	1.3053E-02	1.9978
		1/10	1.6537E-02	-	4.9369E-02	-	3.2169E-01	-
1.3	0.2	1/30	1.9383E-03	1.9514	5.7725E-03	1.9536	3.6219E-02	1.9880
		1/50	7.0499E-04	1.9799	2.0990E-03	1.9804	1.3053E-02	1.9978
		1/10	1.6537E-02	-	4.9369E-02	-	3.2169E-01	-
	0.3	1/30	1.9383E-03	1.9514	5.7725E-03	1.9536	3.6219E-02	1.9880
		1/50	7.0505E-04	1.9798	2.0990E-03	1.9804	1.3053E-02	1.9978
		1/10	1.6445E-02	-	4.9220E-02	-	3.0753E-01	-
	0.1	1/30	1.9279E-03	1.9512	5.7568E-03	1.9533	3.4563E-02	1.9896
		1/50	7.0124E-04	1.9799	2.0934E-03	1.9803	1.2451E-02	1.9986
		1/10	1.6445E-02	-	4.9220E-02	-	3.0753E-01	-
1.5	0.2	1/30	1.9280E-03	1.9511	5.7568E-03	1.9533	3.4563E-02	1.9896
		1/50	7.0130E-04	1.9797	2.0934E-03	1.9803	1.2451E-02	1.9986
		1/10	1.6445E-02	-	4.9220E-02	-	3.0753E-01	-
	0.3	1/30	1.9281E-03	1.9511	5.7568E-03	1.9533	3.4563E-02	1.9896
		1/50	7.0136E-04	1.9796	2.0934E-03	1.9803	1.2451E-02	1.9986
		1/10	1.6339E-02	-	4.9077E-02	-	2.9293E-01	-
	0.1	1/30	1.9161E-03	1.9509	5.7420E-03	1.9530	3.2857E-02	1.9914
		1/50	6.9697E-04	1.9797	2.0883E-03	1.9801	1.1831E-02	1.9995
		1/10	1.6339E-02	-	4.9077E-02	-	2.9293E-01	-
1.7	0.2	1/30	1.9161E-03	1.9508	5.7420E-03	1.9530	3.2857E-02	1.9914
		1/50	6.9704E-04	1.9796	2.0883E-03	1.9801	1.1831E-02	1.9995
		1/10	1.6339E-02	-	4.9077E-02	-	2.9293E-01	-
	0.3	1/30	1.9162E-03	1.9508	5.7420E-03	1.9530	3.2857E-02	1.9914
		1/50	6.9710E-04	1.9795	2.0883E-03	1.9801	1.1831E-02	1.9995

5.2. Example 2

In this numerical example, we consider the same space-time domain and the nonlinear term as shown in the first example, and choose the exact solution $u = t^3 \frac{\sin(3\pi x)}{x+1}$ with the corresponding source term. By choosing the space parameter $h = 1/30$, time step length size $\tau = 1/200$, fractional parameter $\beta = 1.7$ and shifted parameter $\theta = 0.3$, we obtain the comparison in Figures 1–3 between the figures of numerical solutions and the figures of exact solutions at $t = 0.25, 0.5, 0.75, 1$, from which one can visually see the approximation effect.

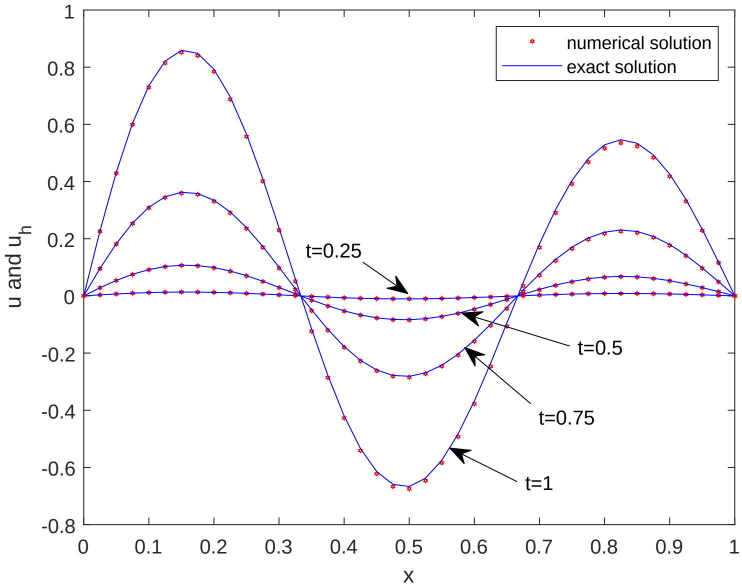

Figure 1. Comparison between u and u_h at different time t.

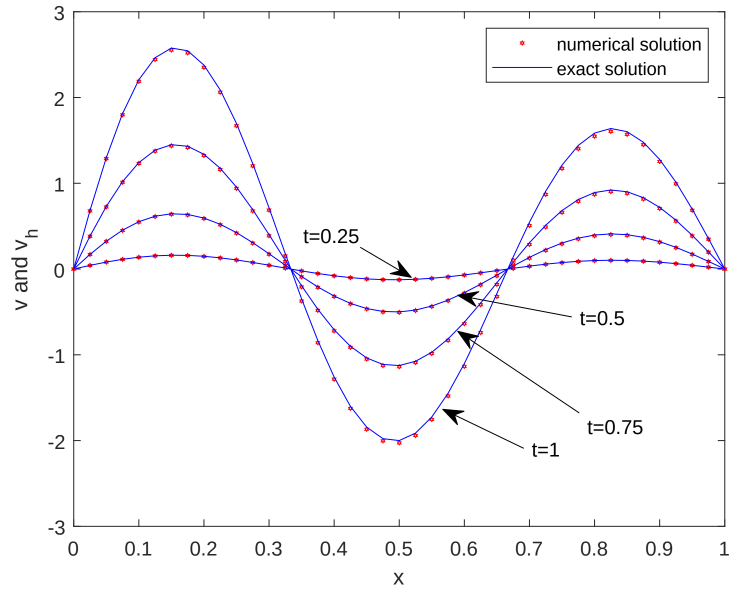

Figure 2. Comparison between v and v_h at different time t.

Remark 5.1. 1) *From these two examples, readers can see that the linear basis functions for three finite element spaces are used. In this article, the presented time second-order fully discrete mixed finite element scheme is derived by combining Pani's space H^1-mixed element method with a time second-order SCQ formula, so it also does not need to meet the LBB condition. Further, the degrees with k, m and r of three polynomial basis functions can be freely selected.*

2) *By introducing two auxiliary variables, the original problem is transformed into a low order coupled system in space-time directions. In this case, many efficient numerical approximation schemes in time for solving this system can be constructed. From the computational point of view, there are some small differences among them. However, the related technical difficulty of theoretical analysis by using different approximation technique is even a big difference, which will bring many challenges to researchers. For example, in this article, the positive definite property is used for analyzing the stability and error estimate, which differs from the iterative technique shown in [38].*

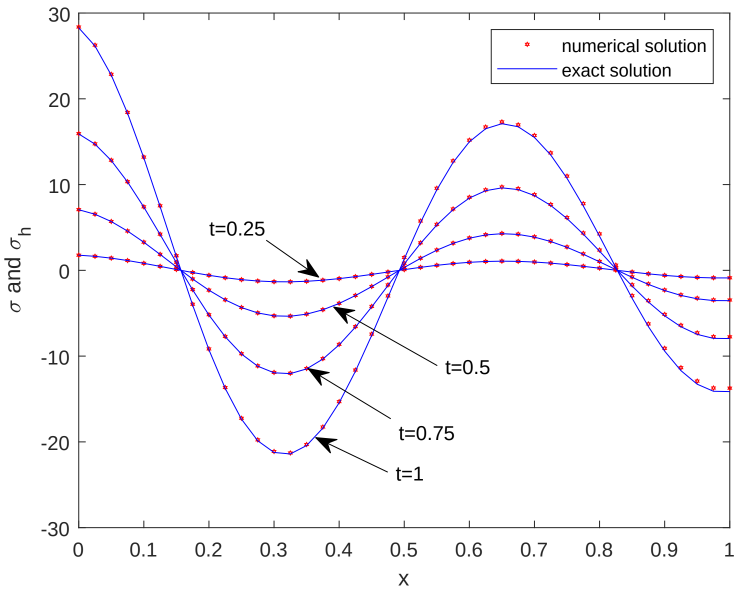

Figure 3. Comparison between σ and σ_h at different time t.

6. Conclusions and advancements

From the data computed by our fully discrete SCQ mixed element method, one can see clearly that the convergence orders for both space and time are optimal, which is in agreement with our theory result. With a comparison to the standard Galerkin finite element method for directly solving the studied fractional wave model, the advantage of this method is that three unknown functions can be approximated simultaneously. However, the computing time problem is its limitations, which urges us to further study the fast computing technology based on this method.

In the future, we will extend this method to solve multidimensional fractional wave models and multi-term time fractional wave equations [49], and consider other SCQ formulas [46, 50] and their numerical theories.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. Y. Luchko, Fractional wave equation and damped waves, *J. Math. Phys.*, **54** (2013), 031505. https://doi.org/10.1063/1.4794076
2. L. Feng, F. Liu, I. Turner, Finite difference/finite element method for a novel 2D multi-term time-fractional mixed sub-diffusion and diffusion-wave equation on convex domains, *Commun. Nonlinear Sci. Numer. Simulat.*, **70** (2019), 354–371. https://doi.org/10.1016/j.cnsns.2018.10.016
3. M. Dehghan, M. Abbaszadeh, A. Mohebbi, Analysis of a meshless method for the time fractional diffusion-wave equation, *Numer. Algor.*, **73** (2016), 445–476. https://doi.org/10.1007/s11075-016-0103-1
4. W. McLean, K. Mustapha, A second-order accurate numerical method for a fractional wave equation, *Numer. Math.*, **105** (2007), 481–510. https://doi.org/10.1007/s00211-006-0045-y
5. X. C. Zheng, H. Wang, Wellposedness and regularity of a nonlinear variable-order fractional wave equation, *Appl. Math. Lett.*, **95** (2019), 29–35. https://doi.org/10.1016/j.aml.2019.03.015
6. L. Feng, I. Turner, P. Perré, K. Burrage, An investigation of nonlinear time-fractional anomalous diffusion models for simulating transport processes in heterogeneous binary media, *Commun. Nonlinear Sci. Numer. Simulat.*, **92** (2020), 105454. https://doi.org/10.1016/j.cnsns.2020.105454
7. X. M. Gu, S. L. Wu, A parallel-in-time iterative algorithm for Volterra partial integro-differential problems with weakly singular kernel, *J. Comput. Phys.*, **417** (2020), 109576. https://doi.org/10.1016/j.jcp.2020.109576
8. K. Hosseini, M. Ilie, M. Mirzazadeh, D. Baleanu, An analytic study on the approximate solution of a nonlinear time-fractional Cauchy reaction-diffusion equation with the Mittag-Leffler law, *Math. Meth. Appl. Sci.*, **44** (2021) 6247–6258. https://doi.org/10.1002/mma.7059
9. R. H. Feng, Y. Liu, Y. Hou, H. Li, Z. C. Fang, Mixed element algorithm based on a second-order time approximation scheme for a two-dimensional nonlinear time fractional coupled sub-diffusion model, *Eng. Comput.*, 2022. https://doi.org/10.1007/s00366-020-01032-9
10. Z. C. Fang, R. X. Du, H. Li, Y. Liu, A two-grid mixed finite volume element method for nonlinear time fractional reaction-diffusion equations, *AIMS Math.*, **7** (2022), 1941–1970. 10.3934/math.2022112
11. W. P. Bu, Y. F. Tang, Y. C. Wu, J. Y. Yang, Crank-Nicolson ADI Galerkin finite element method for two-dimensional fractional FitzHugh-Nagumo monodomain model, *Appl. Math. Comput.*, **257** (2015), 355–364. https://doi.org/10.1016/j.amc.2014.09.034

12. Y. Liu, Z. D. Yu, H. Li, F. W. Liu, J. F. Wang, Time two-mesh algorithm combined with finite element method for time fractional water wave model, *Int. J. Heat Mass Transf.*, **120** (2018), 1132–1145. https://doi.org/10.1016/j.ijheatmasstransfer.2017.12.118

13. K. Hosseini, M. Ilie, M. Mirzazadeh, A. Yusuf, T. A. Sulaiman, D. Baleanu, S. Salahshour, An effective computational method to deal with a time-fractional nonlinear water wave equation in the Caputo sense, *Math. Comput. Simulat.*, **187** (2021), 248–260. https://doi.org/10.1016/j.matcom.2021.02.021

14. J. C. Li, Y. Q. Huang, Y. P. Lin, Developing finite element methods for Maxwell's equations in a Cole-Cole dispersive medium, *SIAM J. Sci. Comput.*, **33** (2011), 3153–3174. https://doi.org/10.1137/110827624

15. E. Y. Fan, J. F. Wang, Y. Liu, H. Li, Z. C. Fang, Numerical simulations based on shifted second-order difference finite element algorithms for the time fractional Maxwell's system, *Eng. Comput.*, 2020. https://doi.org/10.1007/s00366-020-01147-z

16. H. L. Liao, T. Tang, T. Zhou, An energy stable and maximum bound preserving scheme with variable time steps for time fractional Allen-Cahn equation, *SIAM J. Sci. Comput.*, **43** (2021), A3503–A3526. https://doi.org/10.1137/20M1384105

17. L. Liu, L. Feng, Q. Xu, L. Zheng, F. Liu, Flow and heat transfer of generalized Maxwell fluid over a moving plate with distributed order time fractional constitutive models, *Int. Commun. Heat Mass Transf.*, **116** (2020), 104679. https://doi.org/10.1016/j.icheatmasstransfer.2020.104679

18. M. Zheng, F. Liu, I. Turner, V. Anh, A novel high order space-time spectral method for the time fractional Fokker-Planck equation, *SIAM J. Sci. Comput.*, **37** (2015), A701–A724. https://doi.org/10.1137/140980545

19. Y. Cao, B. L. Yin, Y. Liu, H. Li, Crank-Nicolson WSGI difference scheme with finite element method for multi-dimensional time fractional wave problem, *Comput. Appl. Math.*, **37** (2018), 5126–5145. https://doi.org/10.1007/s40314-018-0626-2

20. B. L. Yin, Y. Liu, H. Li, F. H. Zeng, A class of efficient time-stepping methods for multi-term time-fractional reaction-diffusion-wave equations, *Appl. Numer. Math.*, **165** (2021), 56–82. https://doi.org/10.1016/j.apnum.2021.02.007

21. J. Ren, X. Long, S. Mao, J. Zhang, Superconvergence of finite element approximations for the fractional diffusion-wave equation, *J. Sci. Comput.*, **72** (2017), 917–935. https://doi.org/10.1007/s10915-017-0385-z

22. L. Li, D. Xu, M. Luo, Alternating direction implicit Galerkin finite element method for the two-dimensional fractional diffusion-wave equation, *J. Comput. Phys.*, **255** (2013), 471–485. https://doi.org/10.1016/j.jcp.2013.08.031

23. H. Y. Jian, T. Z. Huang, X. M. Gu, X. L. Zhao, Y. L. Zhao, Fast second-order implicit difference schemes for time distributed-order and Riesz space fractional diffusion-wave equations, *Comput. Math. Appl.*, **94** (2021), 136–154. https://doi.org/10.1016/j.camwa.2021.05.003

24. J. Y. Shen, X. M. Gu, Two finite difference methods based on an H2N2 interpolation for two-dimensional time fractional mixed diffusion and diffusion-wave equations, *Discrete Cont. Dyn. Syst. B*, **27** (2022), 1179-1207. 10.3934/dcdsb.2021086

25. M. H. Chen, W. H. Deng, A second-order accurate numerical method for the space-time tempered fractional diffusion-wave equation, *Appl. Math. Lett.*, **68** (2017), 87–93. https://doi.org/10.1016/j.aml.2016.12.010
26. H. F. Ding, A high-order numerical algorithm for two-dimensional time-space tempered fractional diffusion-wave equation, *Appl. Numer. Math.*, **135** (2019), 30–46. https://doi.org/10.1016/j.apnum.2018.08.005
27. F. H. Zeng, Second-order stable finite difference schemes for the time-fractional diffusion-wave equation, *J. Sci. Comput.*, **65** (2015), 411–430. https://doi.org/10.1007/s10915-014-9966-2
28. H. Sun, Z. Z. Sun, G. H. Gao, Some temporal second order difference schemes for fractional wave equations, *Numer. Meth. Part. Differ. Equations*, **32** (2016), 970–1001. https://doi.org/10.1002/num.22038
29. J. Huang, Y. Tang, L. Vázquez, J. Yang, Two finite difference schemes for time fractional diffusion-wave equation, *Numer. Algor.*, **64** (2013), 707–720. https://doi.org/10.1007/s11075-012-9689-0
30. Z. B. Wang, S. K. Vong, A high-order ADI scheme for the two-dimensional time fractional diffusion-wave equation, *Int. J. Comput. Math.*, **92** (2015), 970–979. https://doi.org/10.1080/00207160.2014.915960
31. F. Liu, M. M. Meerschaert, R. McGough, P. Zhuang, Q. Liu, Numerical methods for solving the multi-term time-fractional wave-diffusion equation, *Fract. Calc. Appl. Anal.*, **16** (2013), 9–25. https://doi.org/10.2478/s13540-013-0002-2
32. A. Chen, C. P. Li, Numerical solution of fractional diffusion-wave equation, *Numer. Funct. Anal. Opt.*, **37** (2016), 19–39.
33. Z. D. Luo, H. Wang, A highly efficient reduced-order extrapolated finite difference algorithm for time-space tempered fractional diffusion-wave equation, *Appl. Math. Lett.*, **102** (2020), 106090. https://doi.org/10.1016/j.aml.2019.106090
34. H. Zhang, X. Y. Jiang, Unconditionally convergent numerical method for the two-dimensional nonlinear time fractional diffusion-wave equation, *Appl. Numer. Math.*, **146** (2019), 1–12. https://doi.org/10.1016/j.apnum.2019.06.019
35. Y. Q. Liu, H. G. Sun, X. L. Yin, L. B. Feng, Fully discrete spectral method for solving a novel multi-term time-fractional mixed diffusion and diffusion-wave equation, *Zeitschrift für angewandte Mathematik und Physik*, **71** (2020), 1–19. https://doi.org/10.1007/s00033-019-1244-6
36. Y. Yang, Y. Chen, Y. Huang, H. Wei, Spectral collocation method for the time-fractional diffusion-wave equation and convergence analysis, *Comput. Math. Appl.* **73** (2017), 1218–1232. https://doi.org/10.1016/j.camwa.2016.08.017
37. G. Fairweather, X. H. Yang, D. Xu, H. X. Zhang, An ADI Crank-Nicolson orthogonal spline collocation method for the two-dimensional fractional diffusion-wave equation, *J. Sci. Comput.*, **65** (2015), 1217–1239. https://doi.org/10.1007/s10915-015-0003-x
38. J. F. Wang, B. L. Yin, Y. Liu, H. Li, Z. C. Fang, Mixed finite element algorithm for a nonlinear time fractional wave model, *Math. Comput. Simulat.*, **188** (2021), 60–76. https://doi.org/10.1016/j.matcom.2021.03.038

39. Y. Liu, H. Li, H^1-Galerkin Mixed finite element methods for pseudo-hyperbolic equations, *Appl. Math. Comput.*, **212** (2009) 446–457. https://doi.org/10.1016/j.amc.2009.02.039

40. L. Guo, H. Z. Chen, H^1-Galerkin Mixed finite element method for the regularized long wave equation, *Computing*, **77** (2006), 205–221. https://doi.org/10.1007/s00607-005-0158-7

41. A. K. Pani, An H^1-Galerkin mixed finite element method for parabolic partial differential equations, *SIAM J. Numer. Anal.*, **35** (1998), 712–727. https://doi.org/10.1137/S0036142995280808

42. D. Y. Shi, J. J. Wang, F. Yan, Unconditional superconvergence analysis of an H^1-Galerkin mixed finite element method for nonlinear Sobolev equations, *Numer. Meth. Part. Differ. Equations*, **34** (2018), 145–166. https://doi.org/10.1002/num.22189

43. Z. G. Shi, Y. M. Zhao, F. Liu, Y. F. Tang, F. L. Wang, Y. H. Shi, High accuracy analysis of an H^1-Galerkin mixed finite element method for two-dimensional time fractional diffusion equations, *Comput. Math. Appl.*, **74** (2017), 1903–1914. https://doi.org/10.1016/j.camwa.2017.06.057

44. C. Wen, Y. Liu, B. L. Yin, H. Li, J. F. Wang, Fast second-order time two-mesh mixed finite element method for a nonlinear distributed-order sub-diffusion model, *Numer. Algor.*, **88** (2021), 523–553. https://doi.org/10.1007/s11075-020-01048-8

45. B. L. Yin, Y. Liu, H. Li, A class of shifted high-order numerical methods for the fractional mobile/immobile transport equations, *Appl. Math. Comput.*, **368** (2020), 124799. https://doi.org/10.1016/j.amc.2019.124799

46. Y. Liu, B. Yin, H. Li, Z. Zhang, The unified theory of shifted convolution quadrature for fractional calculus, *J. Sci. Comput.*, **89** (2021), 2. https://doi.org/10.1007/s10915-021-01630-9

47. G. Gao, H. Sun, Z. Sun, Stability and convergence of finite difference schemes for a class of time-fractional sub-diffusion equations based on certain superconvergence, *J. Comput. Phys.*, **280** (2015), 510–528. https://doi.org/10.1016/j.jcp.2014.09.033

48. Y. Liu, Y. W. Du, H. Li, F. W. Liu, Y. J. Wang, Some second-order θ schemes combined with finite element method for nonlinear fractional Cable equation, *Numer. Algor.*, **80** (2019), 533–555. https://doi.org/10.1007/s11075-018-0496-0

49. L. F. Wu, Y. Y. Pan, X. Z. Yang, An efficient alternating segment parallel finite difference method for multi-term time fractional diffusion-wave equation, *Comput. Appl. Math.*, **40** (2021), 67. https://doi.org/10.1007/s40314-021-01455-0

50. B. L. Yin, Y. Liu, H. Li, Z. M. Zhang, Efficient shifted fractional trapezoidal rule for sub-diffusion problems with nonsmooth solutions on uniform meshes, *BIT Numer. Math.*, 2021. https://doi.org/10.1007/s10543-021-00890-z