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Research article

On the global existence and extinction behavior for a polytropic filtration equation with variable coefficients

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Abstract: This article is devoted to the global existence and extinction behavior of the weak solution to an inhomogeneous polytropic filtration equation. Based on the integral norm estimate approach, the conditions on the global existence and the occurrence of the extinction singularity of the weak solution are given. Moreover, we also prove the non-extinction result under some appropriate assumptions by using the weak upper and lower solutions method.

Keywords: polytropic filtration equation; global existence; extinction behavior; non-extinction phenomenon; variable coefficients

1. Introduction

Here we are interested in the global existence, extinction and non-extinction phenomena of the weak solutions of the following polytropic filtration equation with variable coefficients

$$\begin{cases} |x|^{-s}u_t - \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) = \lambda |x|^{-\alpha}u^q, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, $u_0(x)$ is a non-negative non-trivial function and $u_0^m \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega)$, m, p, q, s, α and λ are positive parameters and satisfy

$$\tfrac{N-2}{N+2} < m \leq 1,\, 1 < p < 2,\, \tfrac{1-m}{2} < q \leq 1 \text{ and } \alpha < s. \tag{1.2}$$

Inhomogeneous parabolic problems like (1.1) are applied to describe many real natural phenomena (see [1, 2] and the references therein). Numerous literatures are devoted to deal with the qualitative properties of the solutions to various inhomogeneous parabolic problems. For example, one can refer to [3–10] for the researches on the well-posedness, comparison theorem, regularity, global existence and blow-up, interface blow-up phenomenon, and so on. The main purpose of the present article is to study the global existence and the conditions on the occurrence of the extinction behavior for solutions to problem (1.1). Problem (1.1) might not have classical solution due to the occurrence of the degeneration and singularity. Based on this reason, we work with the weak solution of problem (1.1) in the following sense.

Definition 1.1. *Let $T > 0$ and $\Omega_T = \Omega \times (0, T)$. A function $u(x,t) \in C\left([0,T]; L^1(\Omega)\right)$ with $\nabla u^m \in L^p(\Omega_T)$, $|x|^{-s} u_t \in L^2(\Omega_T)$, and $|x|^{-\alpha} u^q \in L^2(\Omega_T)$ is said to be a weak sub-solution of problem (1.1) if it fulfills the following assumptions*

- *For any nonnegative test function*

$$\phi \in \left\{\Phi : \Phi \in L^2(\Omega_T), \Phi \in C\left([0,T]; L^2(\Omega)\right), \nabla\Phi \in L^p(\Omega_T), \Phi_t \in L^2(\Omega_T), \Phi|_{\partial\Omega} = 0\right\},$$

  it holds that

$$\iint\limits_{\Omega_T} |x|^{-s} u_t \phi \mathrm{d}x\mathrm{d}t + \iint\limits_{\Omega_T} |\nabla u^m|^{p-2} \nabla u^m \cdot \nabla\phi \mathrm{d}x\mathrm{d}t \le \lambda \iint\limits_{\Omega_T} |x|^{-\alpha} u^q \phi \mathrm{d}x\mathrm{d}t. \tag{1.3}$$

- *$u(x,t) \le 0$ for $(x,t) \in \partial\Omega \times (0,T)$.*
- *$u(x,0) \le u_0(x)$ for $x \in \Omega$.*

The definition of the weak super-solution can be given by changing "≤" into "≥" in the above inequalities. Moreover, we call that $u(x,t)$ is a weak solution of problem (1.1) if $u(x,t)$ is a weak sub-solution as well as a weak super-solution of problem (1.1).

Next, let us review some related results on the extinction behaviors of the solutions to parabolic problem of the form

$$\begin{cases} |x|^{-s} u_t - \mathrm{div}(|\nabla u^m|^{p-2} \nabla u^m) = f(u,x,t), & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.4}$$

Under the assumptions $s = 0$, $f \equiv 0$ and $m = 1$, the author of [11] claimed that the solution of (1.4) possesses the extinction property if and only if $p \in (1,2)$. The authors of [12–15] studied problem (1.4) with $s = 0$ and $f = \lambda u^q$. Under the condition $m(p-1) \in (0,1)$, they proved that if $q > m(p-1)$, the extinction phenomenon of the solution to problem (1.4) will occur for appropriate small initial data $u_0(x)$, while if $q < m(p-1)$, the solution to problem (1.4) does not possess the extinction property. In the critical case $q = m(p-1)$, they concluded that whether the extinction phenomenon of the solution occurs or not depends strongly on the size of the positive parameter λ. Crespo and Alonso [16] investigated problem (1.4) with $s = 0$, $m = 1$, $p \in (1,2)$ and $f = \lambda |x|^{-p} u^{p-1}$. Based on Hardy inequality and comparison principle, they found the conditions on the occurrence of the extinction and non-extinction phenomena. To be more specific, they showed that if $\lambda > p^{-p}(N-p)^p$, then the solution does not possess the extinction behavior, while if λ fulfills the hypothesis (i) $0 < \lambda < p^{-p}(N-p)^p$ for $\frac{2N}{N+2} < p < 2$ or (ii) $0 < \lambda < [2N - p(N+1)]\, p^{p-1}(N-p)^p (p-N)^{-p} (p-2)^{-p}$ for $1 < p < \frac{2N}{N+2}$, then the solution will vanish in finite time. Recently, the authors of [17] considered problem (1.4) with $m = 1$ and $f = u^q$. Based on integral norm estimate approach and Hardy-Littlewood-Sobolev inequality, they showed that the solution of problem (1.4) is global providing that the parameters s, p and q fulfill the conditions $\frac{2N}{N+2} < p < 2$, $0 < q \le 1$ and $0 \le s < \frac{Nq}{q+1}$. Moreover, under some suitable assumptions, it is clarified that $q = p - 1$ plays a decisive role in dividing the situation between the extinction and non-extinction phenomena. Liu et al. [18] generalized the results in [17] to the more general case $m \in (0, 1]$.

Inspired by the above mentioned literatures, we consider the global existence and the extinction property of the solution to problem (1.1). We will focus our attention on the roles that the variable coefficients $|x|^{-s}$ and $|x|^{-\alpha}$ play. Our main results state as follows.

Theorem 1.1. *Suppose that* $\frac{N-2}{N+2} < m \le 1$, $1 < p < 2$, $0 < \alpha < s < N$ *and* $\frac{1-m}{2} < q \le 1$. *Then for any non-negative bounded initial data* $u_0(x)$*, the solution* $u(x,t)$ *of problem (1.1) satisfies that* $u^m(x,t)$ *is global in* $W^{1,p}$ *norm.*

Theorem 1.2. *Suppose that* $\frac{N-2}{N+2} < m \le 1$, $1 < p < 2$, $\max\left\{m(p-1), \frac{1-m}{2}\right\} < q \le 1$ *and*

$$0 < \alpha < s < \min\left\{N - \frac{(m+1)(N-p)}{mp}, \frac{Np[1-m(p-1)]}{2N[1-m(p-1)] + mp(p-1)}\right\}.$$

Then the solution $u(x,t)$ *of problem (1.1) will vanish in a finite time provided that the initial data* $u_0(x)$ *is suitably small.*

Theorem 1.3. *Suppose that* $\frac{N-2}{N+2} < m \le 1$, $1 < p < 2$, $0 < \alpha < s < N$, $\frac{1-m}{2} < q \le m(p-1) < 1$ *and* $O \notin \Omega$. *Then for any non-negative bounded initial data* $u_0(x)$*, problem (1.1) admits a non-extinction solution provided that the parametric* λ *is suitably large.*

Before leaving this section, let us introduce some notations and fundamental facts (see [17, 18]). Let Ω be a bounded domain in $\mathbb{R}^N$. $\|\cdot\|_r$ denotes the norm in the space $L^r(\Omega)$, and $\|\cdot\|_{W^{1,r}(\Omega)}$ denotes the norm in the space $W^{1,r}(\Omega)$. In other words, for any $\rho \in L^r(\Omega)$,

$$\|\rho\|_r = \begin{cases} \left(\int\limits_\Omega |\rho(x)|^r \,\mathrm{d}x\right)^{\frac{1}{r}}, & \text{if } 1 \le r < +\infty, \\ \operatorname*{ess\,sup}\limits_{x\in\Omega} |\rho(x)|, & \text{if } r = +\infty, \end{cases}$$

and for any $\rho \in W^{1,r}(\Omega)$,

$$\|\rho\|_{W^{1,r}(\Omega)} = \sqrt[r]{\|\rho\|_r^r + \|\nabla\rho\|_r^r}.$$

If $\rho \in W_0^{1,r}(\Omega)$, then Poincaré's inequality implies that $\|\nabla\rho\|_r$ is equivalent to $\|\rho\|_{W^{1,r}(\Omega)}$ in this case.

We denote $B(0,R)$ be a ball in $\mathbb{R}^N$ centered at origin with radius R. For any bounded domain $\Omega \subset \mathbb{R}^N$, there must be a constant $R = \sup\limits_{x\in\Omega} \sqrt{x_1^2 + \cdots + x_N^2}$ such that $\Omega \subseteq B(0,R)$. Furthermore, for any given number $\theta \in (0, N)$, one can verify that

$$0 < \int\limits_\Omega |x|^{-\theta} \,\mathrm{d}x \le \int\limits_{B(0,R)} |x|^{-\theta} \mathrm{d}x = \frac{\omega_N}{N-\theta} R^{N-\theta} < +\infty,$$

where

$$\omega_N = \frac{N\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}$$

denotes the surface area of the unit sphere $\partial B(0,1)$ and Γ is the usual Gamma function.

2. Proofs of the main results

Based on the integral norm estimate approach, we will discuss the conditions on the global existence and the occurrence of the extinction singularity of the weak solution. We will prove the non-extinction result under some appropriate assumptions by using the weak upper and lower solutions method. The proofs of Theorems 1.1, 1.2 and 1.3 will be given in this section.

Proof of Theorem 1.1. According to the different values of q, we shall divide the proof into two cases.

Case 1. $\frac{1-m}{2} < q < 1$. Multiplying the first equation of (1.1) by u^m and then integrating the result identity by parts, one has

$$\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x + \int_\Omega |\nabla u^m|^p\,\mathrm{d}x = \lambda\int_\Omega |x|^{-\alpha}u^{m+q}\mathrm{d}x. \tag{2.5}$$

By Hölder's inequality, it holds that

$$\int_\Omega |x|^{-\alpha}u^{m+q}\mathrm{d}x \le \underbrace{\left(\int_\Omega |x|^{\frac{s(m+q)-\alpha(m+1)}{1-q}}\mathrm{d}x\right)^{\frac{1-q}{m+1}}}_{\kappa_1}\left(\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x\right)^{\frac{m+q}{m+1}}, \tag{2.6}$$

and

$$\kappa_1 \le \begin{cases} R^{\frac{s(m+q)-\alpha(m+1)}{m+1}}|\Omega|^{\frac{1-q}{m+1}}, & \text{if } 0<\alpha\le\frac{s(m+q)}{m+1},\\ R^{\frac{s(m+q)-\alpha(m+1)+N(1-q)}{m+1}}\left(\frac{\omega_N(1-q)}{s(m+q)-\alpha(m+1)+N(1-q)}\right)^{\frac{1-q}{m+1}}, & \text{if } \frac{s(m+q)}{m+1}<\alpha<s.\end{cases}$$

Combining (2.5) with (2.6) leads us to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x \le \lambda\kappa_1(m+1)\left(\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x\right)^{\frac{m+q}{m+1}}. \tag{2.7}$$

Integrating both sides of (2.7) with respect to the time variable from 0 to t, it holds that

$$\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x \le \left[\lambda\kappa_1(1-q)t + \left(\int_\Omega |x|^{-s}u_0^{m+1}\mathrm{d}x\right)^{\frac{1-q}{m+1}}\right]^{\frac{m+1}{1-q}}. \tag{2.8}$$

Multiplying the first equation of (1.1) by $(u^m)_t$ and then integrating the result identity by parts, using Cauchy's inequality with ε, we find that

$$\begin{aligned}&\frac{4m}{(m+1)^2}\int_\Omega |x|^{-s}\left[\left(u^{\frac{m+1}{2}}\right)_t\right]^2 \mathrm{d}x+\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u^m|^p\,\mathrm{d}x\\ &\le \frac{2\lambda m\varepsilon}{m+1}\int_\Omega |x|^{-s}\left[\left(u^{\frac{m+1}{2}}\right)_t\right]^2 \mathrm{d}x+\frac{\lambda m}{2\varepsilon(m+1)}\int_\Omega |x|^{s-2\alpha}u^{m+2q-1}\mathrm{d}x.\end{aligned} \tag{2.9}$$

Taking $\varepsilon\in\left(0,\frac{2}{\lambda(m+1)}\right)$, then it follows from (2.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u^m|^p\,\mathrm{d}x\le \frac{\lambda mp}{2\varepsilon(m+1)}\int_\Omega |x|^{s-2\alpha}u^{m+2q-1}\mathrm{d}x. \tag{2.10}$$

With the help of Hölder's inequality and (2.8), one has

$$\begin{aligned}\int_\Omega |x|^{s-2\alpha}u^{m+2q-1}\mathrm{d}x&\le\left(\int_\Omega |x|^{\frac{s(m+q)-\alpha(m+1)}{1-q}}\mathrm{d}x\right)^{\frac{2(1-q)}{m+1}}\left(\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x\right)^{\frac{m+2q-1}{m+1}}\\ &\le\kappa_1^2\left[\lambda\kappa_1(1-q)t+\left(\int_\Omega |x|^{-s}u_0^{m+1}\mathrm{d}x\right)^{\frac{1-q}{m+1}}\right]^{\frac{m+2q-1}{1-q}}.\end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.10) and then integrating in the time variable on $(0,t)$, it holds that

$$\begin{aligned}\int_\Omega |\nabla u^m|^p\,\mathrm{d}x\le{}&\frac{mp\kappa_1}{2\varepsilon(m+1)(m+q)}\left[\lambda\kappa_1(1-q)t+\left(\int_\Omega |x|^{-s}u_0^{m+1}\mathrm{d}x\right)^{\frac{1-q}{m+1}}\right]^{\frac{m+q}{1-q}}\\ &+\int_\Omega \left|\nabla u_0^m\right|^p\mathrm{d}x-\frac{mp\kappa_1}{2\varepsilon(m+1)(m+q)}\left(\int_\Omega |x|^{-s}u_0^{m+1}\mathrm{d}x\right)^{\frac{m+q}{m+1}},\end{aligned}$$

which means that $u^m(x,t)$ is bounded in $W^{1,p}$ norm in the case $\frac{1-m}{2}<q<1$.

Case 2. $q=1$. Multiplying the first equation of (1.1) by u^m and then integrating the result identity by parts, one has

$$\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x+\int_\Omega |\nabla u^m|^p\,\mathrm{d}x=\lambda\int_\Omega |x|^{-\alpha}u^{m+1}\mathrm{d}x\le\lambda R^{s-\alpha}\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x,$$

which tells us that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x\le\lambda(m+1)R^{s-\alpha}\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x. \tag{2.12}$$

Integrating both sides of (2.12) with respect to the time variable from 0 to t, it holds that

$$\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x \le e^{\lambda(m+1)R^{s-\alpha}t}\int_\Omega |x|^{-s}u_0^{m+1}\mathrm{d}x. \tag{2.13}$$

Multiplying the first equation of (1.1) by $(u^m)_t$ and then integrating the result identity by parts, using Cauchy's inequality with ε, we obtain that

$$\begin{aligned}
&\frac{4m}{(m+1)^2}\int_\Omega |x|^{-s}\left[\left(u^{\frac{m+1}{2}}\right)_t\right]^2\mathrm{d}x + \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u^m|^p\,\mathrm{d}x\\
&\le \frac{2\lambda m\varepsilon}{m+1}\int_\Omega |x|^{-\alpha}\left[\left(u^{\frac{m+1}{2}}\right)_t\right]^2\mathrm{d}x + \frac{\lambda m}{2\varepsilon(m+1)}\int_\Omega |x|^{-\alpha}u^{m+1}\mathrm{d}x\\
&\le \frac{2\lambda\varepsilon mR^{s-\alpha}}{m+1}\int_\Omega |x|^{-s}\left[\left(u^{\frac{m+1}{2}}\right)_t\right]^2\mathrm{d}x + \frac{\lambda mR^{s-\alpha}}{2\varepsilon(m+1)}\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x.
\end{aligned} \tag{2.14}$$

Choosing $\varepsilon \in \left(0, \frac{2}{\lambda(m+1)R^{s-\alpha}}\right)$, then (2.13) and (2.14) imply that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla u^m|^p\,\mathrm{d}x &\le \frac{\lambda mpR^{s-\alpha}}{2\varepsilon(m+1)}\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x\\
&\le \frac{\lambda mpR^{s-\alpha}}{2\varepsilon(m+1)}e^{\lambda(m+1)R^{s-\alpha}t}\int_\Omega |x|^{-s}u_0^{m+1}\mathrm{d}x.
\end{aligned}$$

Integrating both sides of the above inequality from 0 to t yields that

$$\int_\Omega |\nabla u^m|^p\,\mathrm{d}x \le \int_\Omega \left|\nabla u_0^m\right|^p\mathrm{d}x + \frac{mp}{2\varepsilon(m+1)^2}\left(e^{\lambda(m+1)R^{s-\alpha}t}-1\right)\left(\int_\Omega |x|^{-s}u_0^{m+1}\mathrm{d}x\right),$$

which tells us that $u^m(x,t)$ is also bounded in $W^{1,p}$ norm in the case $q=1$. The proof of Theorem 1.1 is complete. □

In what follows, we will show that the solution $u(x,t)$ of problem (1.1) vanishes in finite time for $q \in \left(\max\left\{\frac{1-m}{2}, m(p-1)\right\}, 1\right]$ provided that the initial data is suitably small.

Proof of Theorem 1.2. Depending on whether the value of q is equal to one, we shall divide the proof into two parts. We first concern with the extinction property for $q \in \left(\max\left\{\frac{1-m}{2}, m(p-1)\right\}, 1\right)$. If $p \in \left(\frac{N+Nm}{Nm+m+1}, 2\right)$, then by (2.5) and (2.6), it holds that

$$\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x + \int_\Omega |\nabla u^m|^p\,\mathrm{d}x \le \lambda\kappa_1\left(\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x\right)^{\frac{m+q}{m+1}}. \tag{2.15}$$

Recalling the following sobolev inequality

$$\left(\int_\Omega u^{m\cdot\frac{Np}{N-p}}\mathrm{d}x\right)^{\frac{N-p}{Np}} \le \kappa_2\left(\int_\Omega |\nabla u^m|^p\,\mathrm{d}x\right)^{\frac{1}{p}},$$

where κ_2 is the optimal embedding constant, then from (2.15), one arrives at

$$\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x + \kappa_2^{-p}\left(\int_\Omega u^{m\cdot\frac{Np}{N-p}}\mathrm{d}x\right)^{\frac{N-p}{N}} \le \lambda\kappa_1\left(\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x\right)^{\frac{m+q}{m+1}}. \tag{2.16}$$

Under the assumptions $\frac{N+Nm}{Nm+m+1} < p < 2$ and

$$\alpha < s < \min\left\{N - \frac{(m+1)(N-p)}{mp}, \frac{Np\left[1-m(p-1)\right]}{2N+m(p-1)(p-2N)}\right\} < N,$$

Hölder's inequality can be used to get

$$\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x \le \underbrace{\left(\int_\Omega |x|^{-s\frac{Npm}{Npm-(m+1)(N-p)}}\mathrm{d}x\right)^{\frac{Npm-(m+1)(N-p)}{Npm}}}_{\kappa_3}\left(\int_\Omega u^{m\cdot\frac{Np}{N-p}}\mathrm{d}x\right)^{\frac{(N-p)(m+1)}{Npm}}, \tag{2.17}$$

where

$$\begin{aligned}\kappa_3 &\le \left(\int_{B(0,R)} |x|^{-s\frac{Npm}{Npm-(m+1)(N-p)}}\mathrm{d}x\right)^{\frac{Npm-(m+1)(N-p)}{Npm}}\\ &= \left(\frac{\omega_N[Npm-(m+1)(N-p)]}{N\left[mp(N-s)-(m+1)(N-p)\right]}R^{\frac{N[mp(N-s)-(m+1)(N-p)]}{Npm-(m+1)(N-p)}}\right)^{\frac{Npm-(m+1)(N-p)}{Npm}}\\ &< \infty.\end{aligned}$$

Let

$$y_1(t) = \int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x.$$

By virtue of (2.16) and (2.17), one observes

$$\frac{1}{m+1}\frac{\mathrm{d}y_1}{\mathrm{d}t} + \kappa_2^{-p}\kappa_3^{-\frac{mp}{m+1}}y_1^{\frac{mp}{m+1}} \le \lambda\kappa_1 y_1^{\frac{m+q}{m+1}},$$

namely,

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} \le (m+1)\,y_1^{\frac{mp}{m+1}}\left(\lambda\kappa_1 y_1^{\frac{q-m(p-1)}{m+1}} - \kappa_2^{-p}\kappa_3^{-\frac{mp}{m+1}}\right). \tag{2.18}$$

If $u_0(x)$ is sufficiently small such that

$$\kappa_4 = \lambda\kappa_1\left[y_1(0)\right]^{\frac{q-m(p-1)}{m+1}} - \kappa_2^{-p}\kappa_3^{-\frac{mp}{m+1}} < 0.$$

Then, from (2.18), it follows that

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} \le \kappa_4(m+1)y_1^{\frac{mp}{m+1}}.$$

Integrating the above inequality from 0 to t leads to

$$y_1 \leq \left\{[y_1(0)]^{\frac{1-m(p-1)}{m+1}} + [1-m(p-1)]\kappa_4 t\right\}^{\frac{m+1}{1-m(p-1)}}.$$

Therefore, there exists a finite time

$$T_0 = [m(p-1)-1]^{-1}\kappa_4^{-1}[y_1(0)]^{\frac{1-m(p-1)}{m+1}}$$

such that

$$\lim_{t\to T_0^-} y_1(t) = \lim_{t\to T_0^-}\int_\Omega |x|^{-s}u^{m+1}\mathrm{d}x = 0,$$

which implies that $u(x,t)$ will vanish in finite time T_0.

If $p \in \left(1, \frac{N+Nm}{Nm+m+1}\right]$. For this subcase, we denote $\ell = \frac{2N+2Nm}{p} - m - 2 - 2Nm$. It is easily seen that $\ell \geq m$. Multiplying the first equation of (1.1) by $u^{mp+\ell}$ and then integrating the result identity by parts, one gets

$$\lambda\int_\Omega |x|^{-\alpha}u^{mp+\ell+q}\mathrm{d}x = \frac{1}{mp+\ell+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s}u^{mp+\ell+1}\mathrm{d}x + \left(\ell m^{p-1}+pm^p\right)\left(\frac{2mp-m+\ell}{p}\right)^{-p}\int_\Omega \left|\nabla u^{\frac{2mp-m+\ell}{p}}\right|^p\mathrm{d}x. \tag{2.19}$$

Since

$$\int_\Omega u^{m+\ell+2}\mathrm{d}x = \int_\Omega u^{\frac{Np}{N-p}\cdot\frac{2mp-m+\ell}{p}}\mathrm{d}x \leq \kappa_5\left(\int_\Omega \left|\nabla u^{\frac{2mp-m+\ell}{p}}\right|^p\right)^{\frac{N}{N-P}},$$

where κ_5 is the optimal embedding constant, it holds that

$$\begin{aligned}\int_\Omega |x|^{-s}u^{mp+\ell+1}\mathrm{d}x &\leq \underbrace{\left(\int_\Omega |x|^{-s\frac{mp+\ell+2}{1-m(p-1)}}\mathrm{d}x\right)^{\frac{1-m(p-1)}{m+\ell+2}}}_{\kappa_6}\left(\int_\Omega u^{m+\ell+2}\mathrm{d}x\right)^{\frac{mp+\ell+1}{m+\ell+2}} \\ &\leq \kappa_6\kappa_5^{\frac{mp+\ell+1}{m+\ell+2}}\left(\int_\Omega \left|\nabla u^{\frac{2mp-m+\ell}{p}}\right|^p\mathrm{d}x\right)^{\frac{N}{N-p}\cdot\frac{mp+\ell+1}{m+\ell+2}},\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\kappa_6 &\leq \left(\int_{B(0,R)} |x|^{-s\frac{mp+\ell+2}{1-m(p-1)}}\mathrm{d}x\right)^{\frac{1-m(p-1)}{m+\ell+2}} \\ &= \left(\frac{\omega_N(1-m(p-1))}{N(1-m(p-1))-s(mp+\ell+2)}R^{\frac{N(1-m(p-1))-s(mp+\ell+2)}{1-m(p-1)}}\right)^{\frac{1-m(p-1)}{m+\ell+2}} \\ &< +\infty.\end{aligned}$$

On the other hand, Hölder's inequality tells us

$$\int_\Omega |x|^{-\alpha} u^{mp+\ell+q} \mathrm{d}x \le \underbrace{\left(\int_\Omega |x|^{\frac{s(mp+\ell+q)-\alpha(mp+\ell+1)}{1-q}} \mathrm{d}x \right)^{\frac{1-q}{mp+\ell+1}}}_{\kappa_7} \left(\int_\Omega |x|^{-s} u^{mp+\ell+1} \right)^{\frac{mp+\ell+q}{mp+\ell+1}}, \tag{2.21}$$

where

$$\kappa_7 \le \begin{cases} R^{\frac{s(mp+\ell+q)-\alpha(mp+\ell+1)}{mp+\ell+1}} |\Omega|^{\frac{1-q}{mp+\ell+1}}, & \text{if } 0 < \alpha \le \frac{s(mp+\ell+q)}{mp+\ell+1}, \\ \left(R^{\frac{s(mp+\ell+q)-\alpha(mp+\ell+1)+N(1-q)}{1-q}} \frac{\omega_N (1-q)}{s(mp+\ell+q)-\alpha(mp+\ell+1)+N(1-q)} \right)^{\frac{1-q}{mp+\ell+1}}, & \text{if } \frac{s(mp+\ell+q)}{mp+\ell+1} < \alpha < s. \end{cases}$$

Noticing that

$$\frac{N-p}{N} = \frac{2mp+\ell-m}{m+\ell+2},$$

combining (2.19) with (2.20) and (2.21), one obtains

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |x|^{-s} u^{mp+\ell+1} \mathrm{d}x \\ &\le \left(\int_\Omega |x|^{-s} u^{mp+\ell+1} \mathrm{d}x \right)^{\frac{2mp+\ell-m}{mp+\ell+1}} \left[\lambda \kappa_7 \left(\int_\Omega |x|^{-s} u^{mp+\ell+1} \right)^{\frac{q-m(p-1)}{mp+\ell+1}} - \kappa_8 \right], \end{aligned} \tag{2.22}$$

where

$$\kappa_8 = (mp+\ell+1) \left(\ell m^{p-1} + p m^p \right) \kappa_5^{\frac{m-\ell-2mp}{m+\ell+2}} \kappa_6^{\frac{m-\ell-2mp}{mp+\ell+1}} \left(\frac{\ell - m + 2pm}{p} \right)^{-p}.$$

Letting

$$y_2(t) = \int_\Omega |x|^{-s} u^{mp+\ell+1} \mathrm{d}x,$$

and choosing $u_0(x)$ so small that

$$\kappa_9 = \lambda \kappa_7 [y_2(0)]^{\frac{q-m(p-1)}{mp+\ell+1}} - \kappa_8 < 0,$$

then by (2.22), it holds that

$$\frac{\mathrm{d}y_2}{\mathrm{d}t} \le \kappa_9 y_2^{\frac{2mp+\ell-m}{mp+\ell+1}}.$$

Integrating both sides of the above inequality with respect to the time variable on $(0, t)$, on arrives at

$$y_2 \le \left\{ [y_2(0)]^{\frac{1-m(p-1)}{mp+\ell+1}} + [1 - m(p-1)] \kappa_9 t \right\}^{\frac{mp+\ell+1}{1-m(p-1)}},$$

which suggests that there exists a finite time

$$T_1 = [m(p-1) - 1]^{-1} \kappa_9^{-1} [y_2(0)]^{\frac{1-m(p-1)}{mp+\ell+1}}$$

such that

$$\lim_{t\to T_1^-} y_2(t) = \lim_{t\to T_1^-} \int_\Omega |x|^{-s} u^{mp+\ell+1} \mathrm{d}x = 0,$$

in other words, $u(x,t)$ will vanish in finite time T_1.

Now, we focus our attention on the proof of the extinction result for $q = 1$. We are also going to divide the proof into two subcases. If $p \in \left(\frac{N+Nm}{Nm+m+1}, 2\right)$, then from (2.5), it follows that

$$\frac{1}{m+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s} u^{m+1}\mathrm{d}x + \int_\Omega |\nabla u^m|^p \,\mathrm{d}x = \lambda \int_\Omega |x|^{-\alpha} u^{m+1}\mathrm{d}x \le \lambda R^{s-\alpha}\int_\Omega |x|^{-s} u^{m+1}\mathrm{d}x.$$

Similar to the derivation process of (2.18), one obtains

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} \le (m+1)\, y_1^{\frac{mp}{m+1}} \left(\lambda R^{s-\alpha} y_1^{\frac{1-m(p-1)}{m+1}} - \kappa_2^{-1}\kappa_3^{-\frac{mp}{m+1}}\right). \tag{2.23}$$

Taking $u_0(x)$ so small that

$$\kappa_{10} = \lambda R^{s-\alpha}\left[y_1(0)\right]^{\frac{1-m(p-1)}{m+1}} - \kappa_2^{-1}\kappa_3^{-\frac{mp}{m+1}} < 0,$$

then by (2.23), it holds that

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} \le \kappa_{10}(m+1)\, y_1^{\frac{mp}{mp+1}}.$$

Integrating both sides of the above inequality with respect to the time variable on $(0,t)$, one can claim that

$$y_1 \le \left\{\left[y_1(0)\right]^{\frac{1-m(p-1)}{m+1}} + [1-m(p-1)]\kappa_{10} t\right\}^{\frac{m+1}{1-m(p-1)}},$$

which means that there exists a finite time

$$T_2 = [m(p-1)-1]^{-1}\,\kappa_{10}^{-1}\left[y_1(0)\right]^{\frac{1-m(p-1)}{m+1}}$$

such that

$$\lim_{t\to T_2^-} y_1(t) = \lim_{t\to T_2^-}\int_\Omega |x|^{-s} u^{m+1}\mathrm{d}x = 0,$$

that is, $u(x,t)$ will vanish in finite time T_2.

If $p \in \left(1, \frac{N+Nm}{Nm+m+1}\right]$, then by (2.19), it holds that

$$\begin{aligned}\lambda R^{s-\alpha}\int_\Omega |x|^{-s} u^{mp+\ell+1}\mathrm{d}x \ge \lambda \int_\Omega |x|^{-\alpha} u^{mp+\ell+1}\mathrm{d}x \\ &= \frac{1}{mp+\ell+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^{-s} u^{mp+\ell+1}\mathrm{d}x \\ &\quad + \left(\ell m^{p-1} + p m^p\right)\left(\frac{2mp-m+\ell}{p}\right)^{-p}\int_\Omega \left|\nabla u^{\frac{2mp-m+\ell}{p}}\right|^p \mathrm{d}x.\end{aligned}$$

Similar to the derivation process of (2.22), one obtains

$$\frac{\mathrm{d}y_2}{\mathrm{d}t} \le y_2^{\frac{2mp+\ell-m}{mp+\ell+1}} \left(\lambda R^{s-\alpha} y_2^{\frac{1-m(p-1)}{mp+\ell+1}} - \kappa_8 \right). \tag{2.24}$$

Selecting $u_0(x)$ so small that

$$\kappa_{11} = \lambda R^{s-\alpha} \left[y_2(0) \right]^{\frac{1-m(p-1)}{mp+\ell+1}} - \kappa_8 < 0,$$

then by (2.24), it holds that

$$\frac{\mathrm{d}y_2}{\mathrm{d}t} \le \kappa_{11} y_2^{\frac{2mp+\ell-m}{mp+\ell+1}}.$$

Integrating both sides of the above inequality with respect to the time variable on $(0, t)$, one can conclude that

$$y_2 \le \left\{ \left[y_2(0) \right]^{\frac{1-m(p-1)}{mp+\ell+1}} + [1 - m(p-1)]\kappa_{11} t \right\}^{\frac{mp+\ell+1}{1-m(p-1)}},$$

which tells us that there is a finite time

$$T_3 = [m(p-1) - 1]^{-1} \kappa_{11}^{-1} \left[y(0) \right]^{\frac{1-m(p-1)}{mp+\ell+1}}$$

such that

$$\lim_{t \to T_3^-} y_2(t) = \lim_{t \to T_3^-} \int_\Omega |x|^{-s} u^{mp+\ell+1} \mathrm{d}x = 0,$$

namely, $u(x, t)$ will vanish in finite time T_3. The proof of Theorem 1.2 is complete. □

Now it remains to prove the non-extinction result.

Proof of Theorem 1.3. Let λ_1 be the first eigenvalue of the following eigenvalue problem and ψ be the corresponding eigenfunction

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda u|u|^{p-2}, & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega. \end{cases} \tag{2.25}$$

Assume that $\psi > 0$ and $\max\limits_{x\in\Omega} \psi(x) = 1$. Define a function $f(t)$ for $t \ge 0$ by

$$f(t) = d^{\frac{1}{m(p-1)-q}} (1 - e^{-ct})^{\frac{1}{1-q}},$$

where $d \in (0, 1)$, and $0 < c < [m(p-1) - q] d^{\frac{1-q}{q-m(p-1)}}$. It is easily seen that $f(0) = 0$, $f(t) \in (0, 1)$ for $t > 0$, and

$$f' + d^{-1} f^{m(p-1)} - f^q < 0. \tag{2.26}$$

Let

$$v_1(x, t) = f(t)\psi(x).$$

Our next objective is to prove that $v_1(x,t)$ is a non-extinction weak sub-solution of problem (1.1). Denote $\Omega_t = \Omega \times (0,t)$ for any $t > 0$. With the help of (2.26) and the definition of $\psi(x)$, by a series of simple calculations, one can obtain

$$\begin{aligned}
\kappa_{12} :&= \iint\limits_{\Omega_t} |x|^{-s} v_{1\tau}(x,\tau)\phi \mathrm{d}x\mathrm{d}\tau + \iint\limits_{\Omega_t} |\nabla v_1^m|^{p-2}\nabla v_1^m \cdot \nabla\phi \mathrm{d}x\mathrm{d}\tau - \lambda \iint\limits_{\Omega_t} |x|^{-\alpha} v_1^q \phi \mathrm{d}x\mathrm{d}\tau \\
&= \iint\limits_{\Omega_t} |x|^{-s} f_\tau(\tau)\psi(x)\phi \mathrm{d}x\mathrm{d}\tau - \lambda \iint\limits_{\Omega_t} |x|^{-\alpha} f^q(t)\psi^q(x)\phi \mathrm{d}x\mathrm{d}\tau \\
&\quad + \iint\limits_{\Omega_t} f^{m(p-1)}(\tau)|\nabla\psi^m|^{p-2}\nabla\psi^m \cdot \nabla\phi \mathrm{d}x\mathrm{d}\tau \\
&< \iint\limits_{\Omega_t} |x|^{-s}\left(f^q - d^{-1} f^{m(p-1)}\right)\psi(x)\phi \mathrm{d}x\mathrm{d}\tau - \lambda \iint\limits_{\Omega_t} |x|^{-\alpha} f^q(\tau)\psi^q(x)\phi \mathrm{d}x\mathrm{d}\tau \\
&\quad + \lambda_1 \iint\limits_{\Omega_t} f^{m(p-1)}(\tau)\psi^{m(p-1)}(x)\phi \mathrm{d}x\mathrm{d}\tau \\
&< \underbrace{\iint\limits_{\Omega_t} \left(|x|^{\alpha-s} + \lambda_1 R^\alpha - \lambda\psi^q\right)\phi |x|^{-\alpha} f^q(\tau)\, \mathrm{d}x\mathrm{d}\tau}_{\kappa_{13}}.
\end{aligned}$$

Since $O \notin \Omega$, one can see that there is a point $(x^\star, \tau^\star) \in \Omega_t$ such that

$$\kappa_{13} = \left(|x^\star|^{\alpha-s} + \lambda_1 R^\alpha - \lambda\psi^q(x^\star)\right) \iint\limits_{\Omega_t} \phi |x|^{-\alpha} f^q(\tau)\, \mathrm{d}x\mathrm{d}\tau.$$

If λ is so large that $|x^\star|^{\alpha-s} + \lambda_1 R^\alpha - \lambda\psi^q(x^\star) < 0$, then one has $\kappa_{12} < \kappa_{13} < 0$, which tells us that $v_1(x,t)$ is a non-extinction weak sub-solution of problem (1.1).

Let $v_2(x,t)$ be a weak solution of the problem as follows

$$\begin{cases} |x|^{-s}u_t - \mathrm{div}(|\nabla u^m|^{p-2}\nabla u^m) = \lambda|x|^{-\alpha}(u_+ + 1)^q, & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.27}$$

Then $v_2(x,t)$ is a weak super-solution of problem (1.1). Now, by slightly modifying the proof of Theorem 4.3 in [15], we are about to prove that $v_1(x,t) \le v_2(x,t)$. Select the test function $\phi_\epsilon(x,t) = H_\epsilon\left(v_1^m(x,t) - v_2^m(x,t)\right)$, where $H_\epsilon(r)$ is a monotone increasing smooth approximation of

$$H(r) = \begin{cases} 1, & r > 0, \\ 0, & \text{otherwise}. \end{cases}$$

Moreover, one can verify that $H_\epsilon(r)$ satisfies $\lim\limits_{\epsilon\to 0} H_\epsilon'(r) = \delta(r)$. By virtue of the definitions of $v_1(x,t)$

and $v_2(x,t)$, one has

$$\iint\limits_{\Omega_t} |x|^{-s}(v_1-v_2)_\tau H_\epsilon(v_1^m-v_2^m)\,\mathrm{d}x\mathrm{d}\tau + \iint\limits_{\Omega_t} H_\epsilon'(v_1^m-v_2^m)\left(|\nabla v_1^m|^{p-2}\nabla v_1^m-|\nabla v_2^m|^{p-2}\nabla v_2^m\right)\cdot\nabla(v_1^m-v_2^m)\,\mathrm{d}x\mathrm{d}\tau \le \lambda\iint\limits_{\Omega_t}|x|^{-\alpha}\left(v_1^q-(v_{2+}+1)^q\right)H_\epsilon(v_1^m-v_2^m)\,\mathrm{d}x\mathrm{d}\tau. \tag{2.28}$$

Letting $\epsilon\to 0$, (2.28) leads to

$$\int\limits_\Omega |x|^{-s}(v_1-v_2)_+\,\mathrm{d}x \le \lambda q\iint\limits_{\Omega_t}|x|^{-\alpha}(v_1-(v_{2+}+1))_+\,\mathrm{d}x\mathrm{d}\tau \le \lambda q R^{s-\alpha}\iint\limits_{\Omega_t}|x|^{-s}(v_1-v_2)_+\,\mathrm{d}x\mathrm{d}\tau.$$

Gronwall's inequality tells us that

$$\int\limits_\Omega |x|^{-s}(v_1-v_2)_+\,\mathrm{d}x = 0$$

holds for all $t>0$, which means that $v_1(x,t)\le v_2(x,t)$ a.e. in $\Omega\times(0,+\infty)$. Then by a standard iterated process, one sees that problem (1.1) admits a non-extinction weak solution $u(x,t)$ satisfying $v_1(x,t)\le u(x,t)\le v_2(x,t)$.

On the other hand, one can also show that

$$v_3(x,t) = \left[t-m(p-1)t\right]^{\frac{1}{1-m(p-1)}}\psi(x)$$

is a non-extinction weak sub-solution of problem (1.1) with $q=m(p-1)$ provided that λ is suitably large. Let $v_4(x,t)$ be a weak solution of problem (2.27) with $q=m(p-1)$. Repeating the arguments in the case $q<m(p-1)$, one knows that problem (1.1) admits at least a non-extinction solution $u(x,t)$ satisfying $v_3(x,t)\le u(x,t)\le v_4(x,t)$. The proof of Theorem 1.3 is complete. □

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Conflict of interest

The authors declare there is no conflict of interest.

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