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Research article

Optimal investment-reinsurance strategy with derivatives trading under the joint interests of an insurer and a reinsurer

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Abstract: Considering the common interests of an insurer and a reinsurer, the optimal investment-reinsurance problem with derivatives trading is studied. Suppose that both parties would invest a stock and a risk-free asset for capital appreciation, the insurer could purchase reinsurance and trade derivatives, the optimization problem is formulated by maximizing the expected exponential utility of two parties' wealth processes. The corresponding HJB equations are built for optimal strategy through the dynamic programming principle. In addition, derivatives trading is evaluated based on the certainty-equivalence principle. A numerical study directly illustrates how model parameters influence optimal strategies.

Keywords: derivatives trading; investment-reinsurance; HJB equations; certainty-equivalence; utility function

1. Introduction

Insurance transfers risk and provides commercial security, as well as increasing the deposit base, national welfare, and macroeconomic stability [1]. It is well known that investment is the primary way for insurers to maintain and appreciate their funds, while reinsurance plays an important role as an effectual way for insurers to transfer risk. Moreover, optimization plays a significant role in decision-making as it helps to select the best and make the right decision [2]. Hence, the exploration of optimal investment-reinsurance strategy becomes an appealing topic risk management and actuarial science. The most common criteria to deal with optimal investment-reinsurance problem includes mean-variance criterion [3–5], minimizing ruin probability [6–8] and maximizing expected utility [9–13], etc. Among them, the expected utility theory, as the basis of the uncertainty decision-making problem in economics, has received a lot of attention in insurance studies.

There are two parties involved in reinsurance contracting. The insurer purchases reinsurance to hedge risks while the reinsurer sells reinsurance contracts for a profit with the consideration of risk control. Therefore, the investment-reinsurance optimization problem, which takes the common interests of an insurer and a reinsurer into account has recently attracted significant attention and has been developed in different ways. Firstly, the total wealth process is established through weightedly summing the wealth from an insurer and a reinsurer. For example, Zhou et al. [14] and Zhao et al. [15] discussed investment-reinsurance optimal strategies through the expected utility criterion and mean-variance criterion respectively. Secondly, the model is constructed by maximizing the product of the exponential utility of an insurer and a reinsurer to find the optimal strategies, for example, Li et al. [16] and Huang et al. [17]. Thirdly, the weighted sum of mean-variance criteria for two parties is optimized to investigate the optimal excess-of-loss and proportional reinsurance strategies respectively, see [18] and [19]. Finally, the utilities of the insurer's and the reinsurer's wealth processes are maximized respectively to find optimal strategies [20].

Among the references mentioned above, there is no research in which risk hedge of financial derivatives is involved in the research of finding optimal the reinsurance-investment strategy. However, derivatives are attracting attention increasingly in portfolio selection and financial risk management for the purpose of risk hedging and speculative role [21]. Liu and Pan [22] introduced derivatives investment into an incomplete financial market and found that the performance of portfolio was improved significantly. The optimal asset allocation with derivatives was derived under the recursive utility function and the expected utility function in Husuku [23] and Fu et al. [24] respectively. And the performance of portfolios with derivatives trading was verified to be better improved [25–27]. Baranoff et al. [28] mentioned that derivatives hedging was a financial tool for life insurers to mitigate the risk of their activities. Wang and Gu [29] and Xue et al. [30] also studied the optimal investment-reinsurance strategy with derivatives under the Black-Scholes model or Heston's volatility model respectively, but the interests of the reinsurer are not considered.

To sum up, the investment-reinsurance issues have been studied extensively. Some consider the risk hedge of derivatives [29,30], while some take account the common interests of an insurer and a reinsurer [14–20]. However, it is ready to be investigated that the problem with the consideration of derivatives and two parties' interests together. This paper will explore this problem. Since neither the stock nor risk-free asset can bring differential exposure for the imperfect instantaneous correlation, and hence the market is incomplete. Following references [22] and [30], derivatives could be introduced to complete the market. The total wealth process for the insurer and the reinsurer will be described by the weighted sum of their surpluses. The investment-reinsurance model will be built by the expected utility criterion and the optimal strategy will be explored from the point view of theoretical induction and simulation analysis.

The paper is organized as follows. The fundamental models and formulation of the wealth processes are introduced in Section 2. Section 3 constructed the Hamilton-Jacobi-Bellman (HJB) equation and derived the optimal strategy. Section 4 quantifies the value of derivatives trading. Numerical simulation is displayed in Section 5. Section 6 concludes.

2. Models

Throughout this paper, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered complete probability space with the right-continuous filtration $\mathcal{F} := \{\mathcal{F}_t\}_{t\in[0,T]}$, which contains all random variables and stochastic processes in our discussion, where $0 < T < \infty$ represents the time horizon.

2.1. The surplus processes of the reinsurance

We use a Brownian motion with drift to describe the claim process as follows,

$$dC(t) = adt - \sigma_0 dW_0(t), \tag{2.1}$$

where $a > 0, \sigma_0 > 0$ and $\{W_0(t)\}$ is a standard Brownian motion. The premium rate for the insurer is $c_0 = a(1+p)$ and $p > 0$ is a loading factor. The rate of proportional reinsurance premium is $c_1 = a(1+p')q(t)$, where $p' > 0$ is the reinsurer's relative safety loading and $0 \le q(t) \le 1$.

Therefore, under the proportional reinsurance treaty $q(t)$, the surplus process of the insurer is given by

$$dR_1(t) = c_0 dt - (1-q(t))dC(t) - c_1 dt = a(p - p'q(t))dt + (1-q(t))\sigma_0 dW_0(t). \tag{2.2}$$

The surplus process of the reinsurer is

$$dR_2(t) = ap'q(t)dt + q(t)\sigma_0 dW_0(t). \tag{2.3}$$

2.2. Heston's stochastic volatility model

Both the insurer and the reinsurer will invest in a risk-free asset and a risky asset (e.g., a stock). And the former earns constant interest rate *r*. Assume that the price process $S(t)$ of risky asset satisfies Heston's stochastic volatility model,

$$\begin{cases} dS(t) = S(t)[(r + vL(t))dt + \sqrt{L(t)}dW_1(t)], \\ dL(t) = \alpha(\delta - L(t))dt + \sigma[\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t)]\sqrt{L(t)}. \end{cases} \tag{2.4}$$

Here, both $W_1(t)$ and $W_2(t)$ are standard Brownian motions and independent. $L(t)$ is the an instantaneous variance process and long-run mean $\delta > 0$, mean reversion rate $\alpha > 0$ and volatility coefficient $\sigma > 0$. The price and volatility of risk assets are correlated by coefficient $\rho \in (-1,1)$. In addition, $2\delta\alpha \ge \sigma^2$ is required to ensure that the process $L(t)$ is strictly positive.

2.3. Dynamic model of derivatives price

Besides risky and free-risk assets, the insurer is also accessible to choose derivatives transaction to hedge risk. Following Liu and Pan [22], let the derivatives' price be $O(t) = g(S(t), L(t))$, where $S(t)$ and $L(t)$ are the price and volatility of the derivatives at time *t* respectively. These kinds of derivatives might include most tradable derivatives. Suppose that the insurer trades a European call option with maturity τ and strike price K, and its time-*t* price is

$$g(S(t), L(t)) = c(S(t), L(t); K, \tau).$$

Similarly, a particular pricing kernel is introduced into the financial market to price all the risk factors, which can also price any derivatives. Namely, we get

$$d\Pi(t) = -[rdt + v\sqrt{L(t)}dW_1(t) + \xi\sqrt{L(t)}dW_2(t)]\Pi(t), \Pi(0) = 1,$$

where v and ξ represent respectively the premiums for diffusion risk W_1 and additional risk W_2. Therefore, the price dynamics of the derivatives meets (see Liu and Pan [22])

$$\begin{aligned} dO(t) &= rO(t)dt + (g_s S(t) + \sigma\rho g_l)[vL(t)dt + \sqrt{L(t)}dW_1(t)] \\ &\quad + \sqrt{1-\rho^2}\sigma g_l[\xi L(t)dt + \sqrt{L(t)}dW_2(t)], \end{aligned} \tag{2.5}$$

where g_s and g_l are the partial derivatives of $O(t) = g(S(t), L(t))$.

2.4. Wealth process

Assuming that $\varphi_1(t)$ and $\psi(t)$ represent the dollar amounts that the insurer invested in the stock and the European option at time *t*, respectively, $\varphi_2(t)$ denotes the dollar amount of stock invested by the reinsurer at time t. Then the time-*t* wealth of the insurer is described by as follows,

$$\begin{aligned} dX_1(t) &= \psi(t)\frac{dO(t)}{O(t)} + \varphi_1(t)\frac{dS(t)}{S(t)} + r[X_1(t) - \psi(t) - \varphi_1(t)]dt + dR_1(t) \\ &= \{rX_1(t) + \varphi_1(t)vL(t) + [(g_s S(t) + \sigma\rho g_l)v + \sigma\sqrt{1-\rho^2}g_l\xi]\frac{\psi(t)L(t)}{O(t)} \\ &\quad + (ap - p'q(t))\}dt + [\varphi_1(t) + (g_s S(t) + \sigma\rho g_l)\frac{\psi(t)}{O(t)}]\sqrt{L(t)}dW_1(t) \\ &\quad + \sqrt{1-\rho^2}\frac{\sigma g_l\psi(t)}{O(t)}\sqrt{L(t)}dW_2(t) + (1-q(t))\sigma_0 dW_0(t), X_1(0) = x_{10} > 0. \end{aligned} \tag{2.6}$$

Here, the four parts in the first equation describe the insurer's time-*t* wealth from stock investment, derivative investment, risk-free assets and reinsurance, respectively.

Similarly, the reinsurer's wealth process $X_2(t)$ is composed of three parts including stock investment, risk-free assets and reinsurance.

$$\begin{aligned} dX_2(t) &= \varphi_2(t)\frac{dS(t)}{S(t)} + r[X_2(t) - \varphi_2(t)]dt + dR_2(t) \\ &= [rX_2(t) + \varphi_2(t)vL(t) + ap'q(t)]dt + \varphi_2(t)\sqrt{L(t)}dW_1(t) \\ &\quad + q(t)\sigma_0 dW_0(t), X_2(0) = x_{20} > 0. \end{aligned} \tag{2.7}$$

Based on references [14,15], the following process is established in consideration of two parties' common interests.

$$Z(t) = (1-\omega)X_1(t) + \omega X_2(t),$$

where $\omega \in [0,1]$ is a decision weight. A larger ω shows more concern on the willingness of the reinsurer, vice versa. So,

$$\begin{aligned} dZ(t) &= \{rZ(t)+[\theta^{W_1}(t)+\omega\varphi_2(t)]vL(t)+(1-\omega)\theta^{W_2}(t)\xi L(t) \\ &\quad +a[(1-\omega)p-(1-2\omega)p'q(t)]\}dt+[\theta^{W_1}(t)+\omega\varphi_2(t)]\sqrt{L(t)}dW_1(t) \\ &\quad +(1-\omega)\theta^{W_2}(t)\sqrt{L(t)}dW_2(t)+[(1-\omega)-(1-2\omega)q(t)]\sigma_0 dW_0(t), \end{aligned} \tag{2.8}$$

where

$$\theta^{W_1}(t)=(1-\omega)\varphi_1(t)+(1-\omega)(g_s S(t)+\sigma\rho g_l)\frac{\psi(t)}{O(t)} \tag{2.9}$$

and

$$\theta^{W_2}(t)=\sigma\sqrt{1-\rho^2}\,\frac{\psi(t)g_l}{O(t)}, \tag{2.10}$$

θ^{W_1} and θ^{W_2} measure the risk exposure of the insurer under the price shock W_1 and the volatility shock W_2, respectively.

Definition 2.1. If the following three conditions are satisfied for any $0\le t\le T$, then a risk-exposure-reinsurance strategy $\pi(\cdot):=\left(\varphi_2(\cdot),\theta^{W_1}(\cdot),\theta^{W_2}(\cdot),q(\cdot)\right)$ is admissible.

(i) $\pi(\cdot)$ is $\mathcal{F}-$predictable process which satisfies that $\mathbb{P}(\int_0^T|\theta^{W_i}(t)\sqrt{L(t)}|^2dt<\infty=1), i=1,2$ and $\mathbb{P}(\int_0^T|\varphi_2(t)\sqrt{L(t)}|^2dt<\infty=1)$;

(ii) $q(t)\in[0,1]$;

(iii) stochastic process in Eq (2.8) has a pathwise unique solution.

3. Solution for optimal strategy

The objective in this paper is to optimize the expected utility for the joint interests of the insurer and the reinsurer, as follows,

$$\sup_{\pi\in\mathcal{A}} E[u(Z(T))],$$

where $u(\cdot)$ is a utility function. The following indirect utility function might be defined based on dynamic programming principle.

$$H(t,z,l)=\sup_{\pi\in\mathcal{A}} E[u\left(Z(T)\right)|\ Z(t)=z, L(t)=l], \tag{3.1}$$

whose corresponding HJB equation is

$$\begin{cases} H_t(t,z,l)+_{\pi}^{\mathcal{A}}H(t,z,l)=0, \\ H(T,z,l)=u(z), \end{cases} \tag{3.2}$$

where

$$
\begin{aligned}
{}_{\pi}^{\mathcal{A}}H(t,z,l) = H_t &+ \{rz + [\theta^{W_1}(t)+\omega\varphi_2(t)]vl + (1-\omega)\xi l\theta^{W_2}(t) + a[(1-\omega)p-(1-2\omega)p'q(t)]\}H_z \\
&+\frac{1}{2}\{l[\theta^{W_1}(t)+\omega\varphi_2(t)]^2 + l(1-\omega)^2(\theta^{W_2}(t))^2 + [(1-\omega)-(1-2\omega)q(t)]^2\sigma_0{}^2\}H_{zz} \qquad (3.3)\\
&+\alpha(\delta-l)H_l + \frac{1}{2}\sigma^2 lH_{ll} + \{\rho[\theta^{W_1}(t)+\omega\varphi_2(t)] + (1-\omega)\sqrt{1-\rho^2}\theta^{W_2}(t)\}\sigma lH_{zl}.
\end{aligned}
$$

Assuming that

$$u(z) = -\frac{1}{\gamma}e^{-\gamma z}$$

and $\gamma > 0$ is the risk aversion coefficient. The exponential function is the unique utility function, under which a fair premium induced from the "zero utility" principle is independent of reserves level ([31]). And hence it plays an important role in actuarial sciences. Then we have

$$H(t,z,l) = \sup_{\pi\in\mathcal{A}} E[-\frac{1}{\gamma}e^{-\gamma Z(T)} |\ Z(t)=z, L(t)=l]. \qquad (3.4)$$

Theorem 3.1. Suppose that there are non-redundant derivatives available for trading at any $t\in[0,T]$, then the following conclusions are obtained.

1) For given wealth process $Z(t)$ and variance process $L(t)$, the solution of the Eq (3.2) is shown by

$$H(t,z,l) = -\frac{1}{\gamma}e^{a(t)z+b(t)l+c(t)}, \qquad (3.5)$$

where

$$a(t) = -\gamma e^{r(T-t)}, \qquad (3.6)$$

$$b(t) = \begin{cases} \dfrac{(v^2+\xi^2)[e^{(\xi\sigma\sqrt{1-\rho^2}+v\sigma\rho+\alpha)(t-T)}-1]}{2(\xi\sigma\sqrt{1-\rho^2}+v\sigma\rho+\alpha)}, & \text{if } \xi\sigma\sqrt{1-\rho^2}+v\sigma\rho+\alpha\neq 0,\\ \dfrac{v^2+\xi^2}{2}(t-T), & \text{if } \xi\sigma\sqrt{1-\rho^2}+v\sigma\rho+\alpha = 0, \end{cases} \qquad (3.7)$$

and

$$c(t) = \int_t^T [(1-\omega)a(p'-p)a(s) + \frac{a^2p'^2}{2\sigma_0^2} - \alpha\delta b(s)]ds. \qquad (3.8)$$

2) The optimal weighted amount of stocks invested by an insurer and a reinsurer is

$$\hat{\phi}(t) = (1-\omega)\hat{\varphi}_1(t) + \omega\hat{\varphi}_2(t) = -\frac{g_s S(t)[\xi+\sigma\sqrt{1-\rho^2}b(t)] + \sigma g_l(\rho\xi - v\sqrt{1-\rho^2})}{\sigma g_l\gamma\sqrt{1-\rho^2}e^{r(T-t)}}, \qquad (3.9)$$

and the optimal amount of derivative by the insurer is

$$\hat{\psi}(t)=\frac{[\xi+\sigma\sqrt{1-\rho^2}b(t)]O(t)}{(1-\omega)\sigma g_l\gamma\sqrt{1-\rho^2}e^{r(T-t)}}. \tag{3.10}$$

3) There exists $\tau_1,\tau_2\in[0,T]$ such that the optimal reinsurance strategy is

$$\hat{q}(t)=\begin{cases}0, & t\in[0,\tau_1),\\ \dfrac{ap'+(1-\omega)a(t)\sigma_0^2}{(1-2\omega)a(t)\sigma_0^2}, & t\in[\tau_1,\tau_2],\\ 1, & t\in(\tau_2,T].\end{cases} \tag{3.11}$$

Proof. First, a solution to solve the HJB Eq (3.2) is constructed by formula

$$H(t,z,l)=-\frac{1}{\gamma}e^{a(t)z+b(t)l+c(t)}, \tag{3.12}$$

where $a(T)=-\gamma$ and $b(T)=c(T)=0$. Then

$$H_t=[a'(t)z+b'(t)l+c'(t)]H\,,\ \ H_z=a(t)H\,,\ \ H_{zz}=a^2(t)H\,,$$

$$H_l=b(t)H\,,\ \ H_{ll}=b^2(t)H\ \ \text{and}\ \ H_{zl}=a(t)b(t)H\,.$$

Substituting these partial derivatives into Eq (3.3), we obtain

$$\begin{aligned}&a'(t)z+b'(t)l+c'(t)+a(t)\{rz+[\theta^{W_1}(t)+\omega\varphi_2(t)]vl+(1-\omega)\theta^{W_2}(t)\xi l+a[(1-\omega)p\\ &-(1-2\omega)p'q(t)]\}+\frac{1}{2}a^2(t)\{[\theta^{W_1}(t)+\omega\varphi_2(t)]^2l+(1-\omega)^2(\theta^{W_2}(t))^2l\\ &+\sigma_0^2[(1-\omega)-(1-2\omega)q(t)]^2\}+b(t)\alpha(\delta-l)+\frac{1}{2}b^2(t)\sigma^2l\\ &+\sigma la(t)b(t)\{\rho[\theta^{W_1}(t)+\omega\varphi_2(t)]+(1-\omega)\sqrt{1-\rho^2}\theta^{W_2}(t)\}=0.\end{aligned} \tag{3.13}$$

The first-order condition implies that

$$v+a(t)[\hat{\theta}^{W_1}(t)+\omega\hat{\varphi}_2(t)]+b(t)\sigma\rho=0,$$

$$\xi+a(t)\hat{\theta}^{W_2}(t)+b(t)\sigma\sqrt{1-\rho^2}=0,$$

and

$$ap'+\sigma_0^2[(1-\omega)-(1-2\omega)q*(t)]a(t)=0.$$

Simplification reduces to

$$\hat{\theta}^{W_1}(t)+\omega\hat{\varphi}_2(t)=-\frac{v+\sigma\rho b(t)}{a(t)},$$

$$\hat{\theta}^{W_2}(t)=-\frac{\xi+\sigma\sqrt{1-\rho^2}b(t)}{(1-\omega)a(t)},$$

and

$$q^*(t)=\frac{ap'+(1-\omega)a(t)\sigma_0^2}{(1-2\omega)a(t)\sigma_0{}^2}.$$

Second, substituting $\hat{\phi}(t)$ and $\hat{\psi}(t)$ solved from Eqs (2.9) and (2.10) into Eq (3.13) yield

$$a'(t)\ =-ra(t),$$

$$b'(t)\ =\frac{1}{2}(v^2+\xi^2)+(\xi\sigma\sqrt{1-\rho^2}+\sigma\rho v+\alpha)b(t),$$

and

$$c'(t)=a(1-\omega)(p'-p)a(t)+\frac{a^2p'^2}{2\sigma_0^2}-\alpha\delta b(t).$$

Finally, $a(t),b(t)$ and $c(t)$ will be solved by the theorem of ordinary differential equations.

Remark 3.1. It can be easily found from Theorem 3.1 that the optimal reinsurance strategy is irrelevant to the stock price and the level of current wealth. Meanwhile, the level of current wealth and the choice of reinsurance strategy do not affect the optimal investment strategy in stock and derivative.

Remark 3.2. Although the derivatives complicate the solving process of the problem, we still obtain a closed-form solution for the optimal derivatives strategy. There is positive correlation between the optimal derivative strategy and the volatility risk premium ξ, but the optimal derivative strategy is irrespective of the risk aversion γ, which is consistent with Xue et al. [30].

Remark 3.3. Different with Xue et al. [30], the interest of reinsurer is considered in this paper. And it can be seen from (3.10) that the optimal amount $\hat{\psi}(t)$ of the insurer's investment in derivative is inversely proportional to the decision weight ω. That is, when the interest of the insurer is paid more attention, the insurer needs to reduce the amount in derivative so as to achieve the optimal utility for both parties.

Corollary 3.1. (*Without Derivative*) When $\psi(t)\equiv 0$ in the wealth process of the insurer (Eq (2.6)) which means no derivatives trading is involved, the optimal weighted amount of stocks is

$$\hat{\phi}^W(t)=-\frac{v+b^W(t)\sigma\rho}{a^W(t)}, \tag{3.14}$$

and the optimal reinsurance strategy is $\hat{q}^W(t)=\hat{q}(t)$.

Remark 3.4. We found from Corollary 3.1 that the optimal reinsurance strategy does not change whether the insurer trades derivatives or not. However, if there's no derivatives trading, the investment opportunities of the insurer will be restricted and the need for volatility exposures will be altered, which will result in different optimal stock investment.

4. Valuing derivatives trading

Derivatives serves as a tool to hedge risks. But the analytical solution in Section 3 cannot give the answer whether the insurer could profit from derivatives trading. Following Xue et al. [30], we will quantify the worth of derivatives trading through certainty-equivalence principle here.

Suppose that the volatility of initial market is L_0, and hence the optimal expected utility in Eq (3.1) at time *0* is rewritten as $H(0,Z_0,L_0)$. Suppose that χ^* is the certainty-equivalent wealth of derivatives trading. Therefore, the value of derivatives transaction could be measured through

$$H(0,Z_0,L_0)=H^W(0,Z_0+(1-\omega)\chi^*,L_0),$$

where

$$H^W(t,z,l)=\sup_{\pi^W\in\mathcal{A}^W}E[u(Z(T))|\ Z(t)=z,L(t)=l]=-\frac{1}{\gamma}e^{a^W(t)z+b^W(t)l+c^W(t)} \tag{4.1}$$

is the expected exponential utility of the weighted wealth when there is no derivatives trading.
Theorem 4.1. The certainty-equivalent wealth of derivatives trading is

$$\chi^*=\frac{\Delta b(0)L_0+\Delta c(0)}{(1-\omega)a(0)}>0, \tag{4.2}$$

where $\Delta b(0)=b(0)-b^W(0)$ and $\Delta c(0)=c(0)-c^W(0)$.
Proof. Letting $\psi(t)\equiv 0$, (3.2) can be reduced to the following equation

$$\begin{aligned} &a^W(t)'z+b^W(t)'l+c^W(t)'+a^W(t)\{rz+\phi^W(t)vl+a[(1-\omega)p-(1-2\omega)p'q^W(t)]\}\\ &+\frac{1}{2}a^W(t)^2l\{\phi^W(t)^2+[(1-\omega)-(1-2\omega)q^W(t)]^2\sigma_0^2\}+b^W(t)\alpha(\delta-l)\\ &+\frac{1}{2}b^W(t)^2\sigma^2l+a^W(t)b^W(t)\phi^W(t)\rho\sigma l=0, \end{aligned} \tag{4.3}$$

then we substitute Eq (3.14) into Eq (4.3) and get

$$a^W(t)'=-ra^W(t),$$
$$b^W(t)'=\frac{1}{2}v^2+(\sigma\rho v+\alpha)b^W(t)+\frac{1}{2}\sigma^2(\rho^2-1)b^W(t)^2,$$

and

$$c^W(t)'=a^W(t)a(1-\omega)(p'-p)+\frac{a^2p'^2}{2\sigma_0^2}-\alpha\delta b^W(t).$$

By solving the above differential equations, the following results yield

$$a^W(t)=-\gamma e^{r(T-t)}, \tag{4.4}$$

$$b^W(t)=\frac{-\frac{v^2}{2k}\sinh(k(T-t))}{\cosh(k(T-t))+\frac{\sigma\rho v+\alpha}{2k}\sinh(k(T-t))}, \tag{4.5}$$

and

$$c^W(t)=\int_t^T[a^W(s)a(1-\omega)(p'-p)+\frac{a^2p'^2}{2\sigma_0^2}-\alpha\delta b^W(s)]ds, \tag{4.6}$$

where $k=\frac{\sqrt{(\sigma\rho v+\alpha)^2+\left(1-\rho^2\right)\sigma^2 v^2}}{2}$, $\sinh(t)=\frac{e^t-e^{-t}}{2}$ and $\cosh(t)=\frac{e^t+e^{-t}}{2}$ are hyperbolic functions.

Besides, based on the definitions of H and H^W (Eqs (3.12) and (4.1)), we have

$$[b(0)-b^W(0)]L_0+c(0)-c^W(0)-(1-\omega)a(0)\chi^*=0.$$

Meanwhile, $[b(0)-b^W(0)]L_0+c(0)-c^W(0)<0$ is derived in Xue et al. [30] (Theorem 4.1). And it is easily seen that $1-\omega>0$ and $a^W(t)=a(t)=-\gamma e^{r(T-t)}<0$ for $\forall t\in[0,T]$. Therefore, we obtain

$$\chi^*=\frac{[b(0)-b^W(0)]L_0+c(0)-c^W(0)}{(1-\omega)a(0)}>0.$$

Remark 4.1. We find from Theorem 4.1that the insurer will always benefit from investing in derivatives; hence, derivatives investment is attractive to the insurer, which is consistent with the conclusion in Xue et al. [30]. However, the interests of the reinsurer were not considered in Xue et al. [30]. This paper shows a positive correlation between the value of derivatives trading and the decision weight, which tells us that more emphasis on the reinsurer gives the insurer more profits in derivatives investment.

5. Numerical simulation analysis

We numerically discuss the influence of model parameters on the optimal strategy and the value of derivatives trading based on Eqs (3.9)–(3.11), (3.14) and (4.2). Following Liu and Pan [22], we apply the calibrated parameters to the Heston's model, and set the other parameters according to [15,30].

The parameters in the models are assumed as follows: $T=5$, $\omega=0.3$, $\gamma=0.1$, $\sigma=0.25$, $\xi=-6$, $r=0.05$, $a=0.2$, $p'=0.3$, $p=0.2$, $\sigma_0=0.75$, $K=2$, $v=4$, $\alpha=5$, $\rho=-0.4$, $\tau=4$, $\sqrt{L_0}=0.15$, $\delta=(0.13)^2$, $S(0)=2$. The results are displayed in Figures 1–6 and Table 1. Here, ϕ denotes the optimal weighted dollar amount of stocks invested by the insurer and the reinsurer while ψ denotes the optimal dollar amount of option invested by the insurer.

5.1. Optimal investment strategy

From analytical solution of the optimal investment strategy of European option in Theorem 3.1, the insurer will choose a more prudent and conservative investment strategy to reduce the risk of investing in derivatives, which is consistent with Figure 1.

From Figures 2–Figure 4, we could give the following conclusions.

1) The investment in derivatives is in a short position while the investment in stocks is in a long position.
2) Larger γ means less investment cash in both stocks and derivatives. As the insurer and reinsurer become less aggressive, they will naturally reduce investment to avoid risks.
3) With the increase of σ, the amount invested in stocks and derivatives gradually decrease. That is, even increasing volatility brings higher returns, the corresponding risks also become the prime factor which the company considers in decision-making.
4) As the increase of ξ, the investment in stocks changes from long position to short position, and the investment in derivatives obeys in opposite direction. Especially when $\xi=0$, the optimal

investment amount of stocks and derivative is approximately zero, which tells us that zero risk premium makes investing risk assets be unattractive.

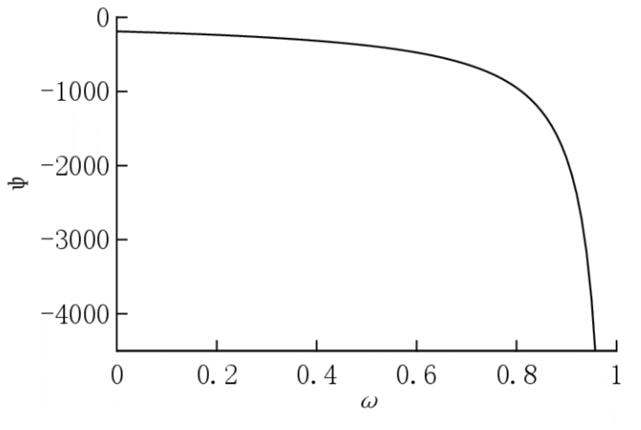

Figure 1. Effect of the decision weight ω on derivative strategy.

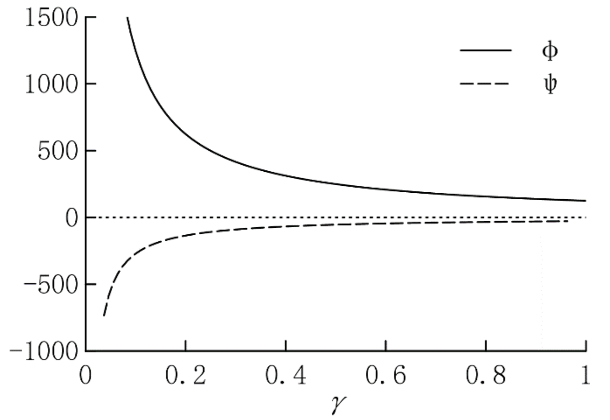

Figure 2. Effect of the risk aversion γ on investment strategy.

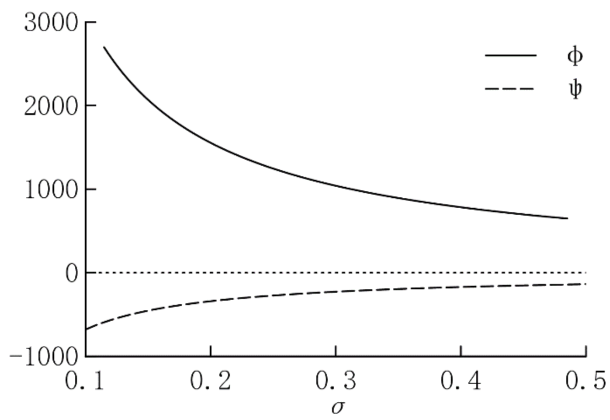

Figure 3. Effect of the volatility of the volatility σ on investment strategy.

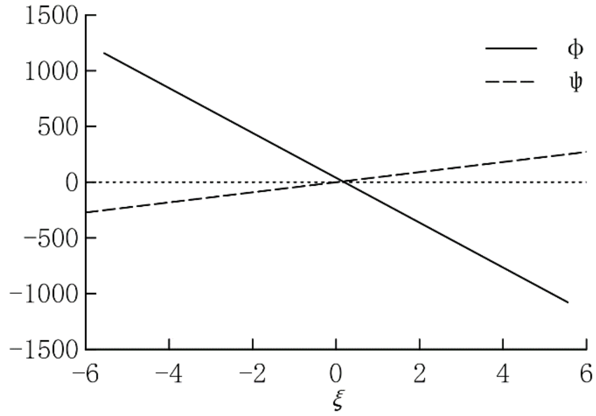

Figure 4. Effect of the volatility risk premium on investment strategy.

5.2. Valuing derivatives trading

Table 1 displays that $\phi(t), |\psi(t)|, \phi^W$ and χ^* decrease proportionally when risk aversion coefficient γ increases. That is, when both the insurer and reinsurer become more risk averse, and hence the relative value at risk of venture capital become greater, they will naturally reduce investment to avoid risks.

Table 1. Optimal investment strategy with different decision weight ω and risk aversion γ.

ω	$\gamma=0.01$				$\gamma=0.1$			
	ϕ	ψ	ϕ^W	χ^*	ϕ	ψ	ϕ^W	χ^*
0	12471.91	-1900.92	400.00	63.61	1247.19	-190.09	40.00	6.36
0.1	12471.91	-2112.13	400.00	70.68	1247.19	-211.21	40.00	7.07
0.3	12471.91	-2715.60	400.00	90.87	1247.19	-271.56	40.00	9.09
0.5	12471.91	-3801.83	400.00	127.22	1247.19	-380.18	40.00	12.72
0.7	12471.91	-6336.39	400.00	212.04	1247.19	-633.64	40.00	21.20
0.9	12471.91	-19009.17	400.00	636.11	1247.19	-1900.92	40.00	63.61
ω	$\gamma=0.5$				$\gamma=1$			
	ϕ	ψ	ϕ^W	χ^*	ϕ	ψ	ϕ^W	χ^*
0	249.44	-38.02	8.00	1.27	124.72	-19.01	4.00	0.64
0.1	249.44	-42.24	8.00	1.41	124.72	-21.12	4.00	0.71
0.3	249.44	-54.31	8.00	1.82	124.72	-27.15	4.00	0.91
0.5	249.44	-76.04	8.00	2.54	124.72	-38.02	4.00	1.27
0.7	249.44	-126.73	8.00	4.24	124.72	-63.36	4.00	2.12
0.9	249.44	-380.18	8.00	12.72	124.72	-190.09	4.00	6.36

In addition, the increase of decision weight will raise the value of derivatives. Especially, its value at $\omega=0.9$ is about ten times the one at $\omega=0$. Taking $\gamma=0.1, \omega=0.3$ as an example, the insurer needs short-selling options of about $271.56 and invest $1247.19 in stocks with the reinsurer to achieve optimal utility. In contrast, the insurer and the reinsurer only need to invest $40 in the stock if no derivatives trading is involved. So, the insurer without derivatives trading must be more cautious in investment since derivatives could hedge the risks of stock investment. Further, prioritizing the reinsurer's interests in determining reinsurance contracts could increase the gains of derivatives trading.

5.3. Optimal reinsurance strategy

We divide the influence of decision weight ω on proportional reinsurance into two parts, $\omega < 0.5$ and $\omega > 0.5$. The former means that the insurer's interest outweighs that of the reinsurer, vice versa.

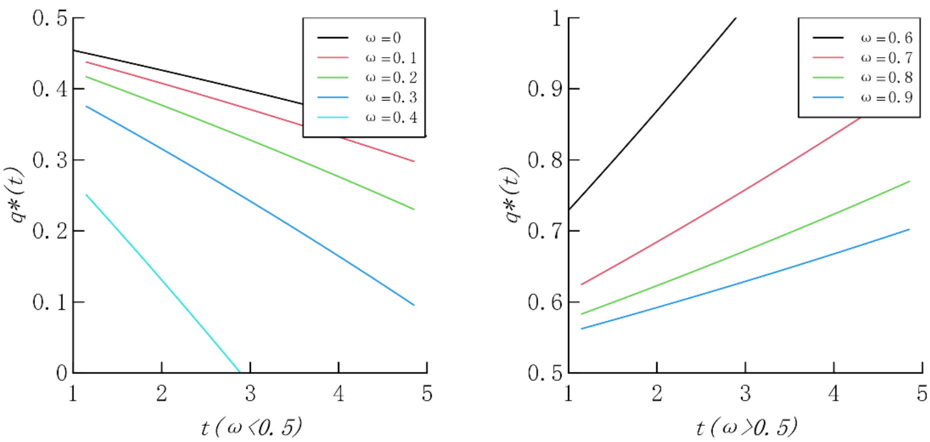

Figure 5. Effect of the decision weight ω on reinsurance strategy.

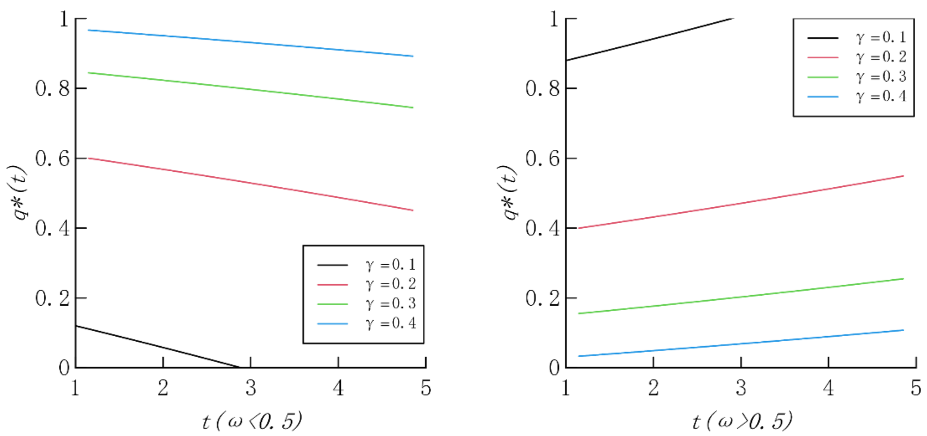

Figure 6. Effect of the risk aversion γ on reinsurance strategy.

Figure 5 tells us that the reinsurer stands in a more leading position as ω becomes larger, and hence the reinsurance proportion $q^*(t)$ gradually decreases since the reinsurer generally expects more risk liability for the insurer.

It can be found from Figure 6 that larger risk aversion γ results in more investment in reinsurance for the insurer since its interest is prioritized when $\omega < 0.5$. On the contrary, when $\omega > 0.5$, the reinsurer's decision is laid with more weight, the reinsurer is averse to risk with the increase of γ, so the proportion of optimal reinsurance decreases, which leads the insurer to be left to take on more risk by themselves.

From Figures 5 and 6, we can also conclude that the optimal reinsurance proportion decreases over time when $\omega < 0.5$, and increases when $\omega > 0.5$.

6. Conclusions

This article studies a derivative-based investment-reinsurance problem in consideration of the expected utility criterion. Different from Xue et al. [30], the common interests of an insurer and a reinsurer are considered here. Besides a risk-free asset and a stock, the insurer also invests a derivative. Based on maximizing the expected exponential utility of the weighted sum of both parties' wealth processes, the analytical solution of optimal investment-reinsurance strategy is obtained through solving the HJB equation. Numerical study gives the analysis of the impact of parameters in model on optimal strategies.

The main theoretical contribution of this paper is that the common interests of the insurer and the reinsurer are involved and derivatives trading is allowed to management risk for the insurer in the process of finding the optimal investment-reinsurance strategy. When decision weight ω becomes larger, the reinsurer stands in a more leading position, which results in a reduction in the optimal reinsurance proportion. Therefore, at this time the insurer requires to purchase more derivatives to enhance the value of derivatives so as to attain the optimal utility of both parties. This study could be expanded by other directions. For example, the CEV model could be used to describe the risky asset's price for the sake of volatility clustering, volatility smile and thick-tailed nature of return distributions. Meanwhile, the discussion about Pareto optimality is not necessary here since we assume that the amount of reinsurance that the insurer needs to purchase is equivalent to the amount sold by reinsurer. However, a cooperative game framework would be employed to find optimal investment-reinsurance strategy if the assumption becomes much more flexible, and hence Pareto optimality might be incorporated. We will explore them problems in the following study.

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Conflict of interest

The authors declare there is no conflict of interest.

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