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# Normalizer property of finite groups with almost simple subgroups

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**Abstract:** In this paper, we prove that all Coleman automorphisms of extension of an almost simple group by an abelian group or a simple group are inner. Using our methods we also show that the Coleman automorphisms of 2-power order of an odd order group by an almost simple group are inner. In particular, these groups have the normalizer property.

**Keywords:** almost simple group; simple group; Coleman automorphism; $q$-central automorphism; normalizer property

## 1. Introduction

Let $F$ be a finite group and $\mathrm{Aut}(F)$ be its automorphism group. We use $\mathbb{Z}F$ to denote the integral group ring of $F$. The normalizer problem (see [1], problem 43) asks whether $\mathrm{N}_{\mathrm{U}(\mathbb{Z}F)}(F) = F\zeta(\mathrm{U}(\mathbb{Z}F))$, where $\mathrm{N}_{\mathrm{U}(\mathbb{Z}F)}(F)$ is normalizer of $F$ in the unit group $\mathrm{U}(\mathbb{Z}F)$, $\zeta(\mathrm{U}(\mathbb{Z}F))$ is the center of $\mathrm{U}(\mathbb{Z}F)$. Write $\mathrm{Aut}_{\mathbb{Z}}(F) = \{\sigma_v \in \mathrm{Aut}(F) \mid x^{\sigma_v} = v^{-1}xv, v \in \mathrm{N}_{\mathrm{U}(\mathbb{Z}F)}(F), x \in F\}$, then $\mathrm{Aut}_{\mathbb{Z}}(F)$ is a subgroup of $\mathrm{Aut}(F)$. Set $\mathrm{Out}_{\mathbb{Z}}(F) = \mathrm{Aut}_{\mathbb{Z}}(F)/\mathrm{Inn}(F)$. Jackowski and Marciniak [2] proved that $\mathrm{N}_{\mathrm{U}(\mathbb{Z}F)}(F) = F\zeta(\mathrm{U}(\mathbb{Z}F))$ is equivalent to $\mathrm{Aut}_{\mathbb{Z}}(F) = \mathrm{Inn}(F)$. Thus, we call that $F$ has the normalizer property provided that $\mathrm{Out}_{\mathbb{Z}}(F) = 1$.

Hertweck and Kimmerle [3] introduced the Coleman automorphism, i.e., a $\varphi \in \mathrm{Aut}(F)$ is said to be Coleman automorphism if for any $q \in \pi(F)$ and any $Q \in \mathrm{Syl}_q(F)$, there exists a $h \in F$ with $\varphi|_Q = \mathrm{conj}(h)|_Q$. Denoted by $\mathrm{Aut}_{\mathrm{Col}}(F)$ the Coleman automorphism group of $F$. Write $\mathrm{Out}_{\mathrm{Col}}(F) = \mathrm{Aut}_{\mathrm{Col}}(F)/\mathrm{Inn}(F)$. In [4], Gross introduced the $q$-central automorphism, i.e., a $\theta \in \mathrm{Aut}(F)$ is called a $q$-central if there exists a $q \in \pi(F)$ and some $Q \in \mathrm{Syl}_q(F)$ such that $\theta|_Q = id|_Q$. Obviously, modifying the Coleman automorphism with an inner automorphism, then the Coleman automorphism of $F$ is $q$-central for any $q \in \pi(F)$.

Coleman automorphisms come up in the study of the normalizer problem. By Coleman's lemma [5] and Krempa's result [1], we only show that $\mathrm{Out}_{\mathrm{Col}}(F) = 1$ or $\mathrm{Out}_{\mathrm{Col}}(F)$ is a $2'$-group, then $\mathrm{Aut}_{\mathbb{Z}}(F) =$

Inn($F$). For example, let $F$ be a simple group or a nilpotent group. Then $\mathrm{Out}_{\mathrm{Col}}(F) = 1$(see [3]). Related results on this subject can be found in [6–10].

The purpose of this paper is to investigate normalizer property of finite groups with almost simple subgroups. Recall that a finite group $A$ is called to be an almost simple group provided that there exists some non-abelian simple group $S$ with $S \leq A \leq \mathrm{Aut}(S)$. Note that Van Antwerpen [10] gave a group $C_{15} \rtimes C_2$ for which $\mathrm{Out}_{\mathrm{Col}}(C_{15} \rtimes C_2) \cong C_2$. In this paper, we consider that $F$ is an almost simple group by a simple group or an odd order group by an almost simple group. We shall show that $\mathrm{Out}_{\mathrm{Col}}(F) = 1$ or $\mathrm{Out}_{\mathrm{Col}}(F)$ is of odd order. In particular, these groups have the normalizer property. Our notation is standard, refer to [1, 3, 11].

## 2. Preliminaries

**Lemma 2.1.** [3] Assume that $S$ is a simple group. Then there is $q \in \pi(S)$ such that every $q$-central automorphisms of $S$ is inner.

**Lemma 2.2.** Let $S$ be a non-abelian simple group. Then $C_{\mathrm{Aut}(S)}(\mathrm{Inn}(S)) = 1$.

*Proof.* By hypothesis, then $\zeta(S) = 1$ and $S \simeq \mathrm{Inn}(S)$. Set $\sigma : S \to \mathrm{Inn}(S)$ is an isomorphism. Thus, for any $\theta \in C_{\mathrm{Aut}(S)}(\mathrm{Inn}(S)), g \in G$, we obtain $\theta^{-1}\sigma(g)\theta = \sigma(g)$, that is, $\sigma(g^{\theta}) = \sigma(g)$. It follows that $g^{\theta} = g$, which implies that $\theta = 1$. Hence $C_{\mathrm{Aut}(S)}(\mathrm{Inn}(S)) = 1$. □

**Lemma 2.3.** Let $J \leq F$. Then $C_F(J) = 1$ if and only if $\zeta(H) = 1$ for every $H$ such that $J \leq H \leq F$.

*Proof.* The assertion is obvious. □

**Lemma 2.4.** Assume that $A$ is an almost simple group. Then $\zeta(A) = 1$.

*Proof.* By Lemma 2.2 and Lemma 2.3, the conclusion holds. □

**Lemma 2.5.** [6] Let $\rho \in \mathrm{Aut}(F)$ be of $p$-power order and $E \leq F$, where $p \in \pi(F)$. If $\rho|_E = \mathrm{conj}(h)|_E$ for some $h \in F$, then there exists a $\delta \in \mathrm{Inn}(F)$ such that $\rho\delta|_E = id|_E$ and $o(\rho\delta) = p^i$, where $i$ is positive integer.

**Lemma 2.6.** [3] Let $\rho \in \mathrm{Aut}_{\mathrm{Col}}(F)$ and $M \trianglelefteq F$. Then
(1) $\rho|_M \in \mathrm{Aut}(M)$,
(2) $\rho|_{F/M} \in \mathrm{Aut}_{\mathrm{Col}}(F/M)$.

**Lemma 2.7.** Assume that $A$ is an almost simple group. Then $\mathrm{Aut}_{\mathrm{Col}}(A) = \mathrm{Inn}(A)$.

*Proof.* Let $\rho \in \mathrm{Aut}_{\mathrm{Col}}(A)$. We shall prove that $\rho \in \mathrm{Inn}(A)$. By hypothesis $S \leq A \leq \mathrm{Aut}(S)$. It follows from Lemma 2.1 that there exists some $q \in \pi(S)$ such that every $q$-central automorphism of $S$ is inner. Let $Q \in \mathrm{Syl_q}(A)$. Since $\rho \in \mathrm{Aut}_{\mathrm{Col}}(A)$, then $\rho|_Q = \mathrm{conj}(a)|_Q$ for some $a \in A$. In general, we may suppose that $\rho|_Q = id|_Q$ by Lemma 2.5. Write $R = Q \cap S$, hence $R \in \mathrm{Syl_q}(S)$ and $\rho|_R = id|_R$. Note that $S \trianglelefteq A$, by Lemma 2.6(1), we obtain that $\rho|_S$ is $q$-central. Hence, $\rho|_S \in \mathrm{Inn}(S)$, i.e., there exists a $g \in S$ with $\rho|_S = \mathrm{conj}(g)|_S$. Write $\eta = \rho\mathrm{conj}(g^{-1})$, then $\eta|_S = id|_S$. By Lemma 2.2 and $S$ identifies with $\mathrm{Inn}(S)$, we obtain that $C_A(S) = C_{\mathrm{Aut}(S)}(S) \cap A = 1$. Thus, for any $y \in A$ and $x \in S$, we have $(y^{-1}xy)^{\eta} = (y^{-1})^{\eta}xy^{\eta} = y^{-1}xy$, which implies that $y^{\eta}y^{-1} \in C_A(S) = 1$. Hence, $\eta = id$, i.e., $\rho \in \mathrm{Inn}(A)$. □

**Lemma 2.8.** [12] Let $\rho \in \mathrm{Aut}(F)$ be of $p$-power order. If there exists some $H \lhd F$ such that $\rho|_H = id|_H$ and $\rho|_{F/H} = id|_{F/H}$, then $\rho|_{F/O_p(\zeta(H))} = id|_{F/O_p(\zeta(H))}$. Assume further that there exists a $T \in \mathrm{Syl_p}(F)$ such that $\rho|_T = id|_T$. Then $\rho \in \mathrm{Inn}(F)$.

**Lemma 2.9.** [6] Let $\rho \in \mathrm{Aut}(F)$ be of $p$-power order, and let $H \lhd F$ with $H^\rho = H$. Assume further that $\rho|_{F/H} \in \mathrm{Inn}(F/H)$. Then there is $\tau \in \mathrm{Inn}(F)$ such that $\rho\tau|_{F/H} = id|_{F/H}$ and $o(\rho\tau) = p^j$, where $j$ is positive integer.

**Lemma 2.10.** [3] Let $\rho \in \mathrm{Aut}(F)$ and $H \lhd F$ with $H^\rho = H$, and assume that $Q \in \mathrm{Syl}(H)$. If $\rho|_Q = \mathrm{conj}(g)|_Q$ for some $g \in F$, then $K = HC_F(Q) \trianglelefteq F$ and $K^\rho = K$. Moreover, $\rho|_{F/K} = \mathrm{conj}(g)|_{F/K}$.

**Lemma 2.11.** [3] Let $M$ be a $2'$-group. Then $\mathrm{Out_{Col}}(M)$ is also a $2'$-group.

## 3. Proof of The Theorems

**Theorem 3.1.** Let $A$ be an almost simple normal subgroup of $F$. If $F/A$ is an abelian group, then $\mathrm{Out_{Col}}(F) = 1$. In particular, $\mathrm{Aut}_{\mathbb{Z}}(F) = \mathrm{Inn}(F)$.

*Proof.* Let $\varphi \in \mathrm{Aut_{Col}}(F)$ and let $\varphi$ be of $p$-power order, where $p \in \pi(F)$. We shall prove that $\varphi \in \mathrm{Inn}(F)$. By hypothesis $A$ is almost simple, then $S \leq A \leq \mathrm{Aut}(S)$. Now, we show that $\varphi|_S$ is a $q$-central, where $q \in \pi(S)$ and $S$ is non-abelian simple. Let $Q \in \mathrm{Syl_q}(A)$, then there exists some $T \in \mathrm{Syl_q}(F)$ such that $Q \leq T$. Note that $\varphi \in \mathrm{Aut_{Col}}(F)$, thus $\varphi|_T = \mathrm{conj}(g)|_T$ for some $g \in F$. In general, we suppose that $\varphi|_T = id|_T$ by Lemma 2.5. Write $R = Q \cap S$, hence $R \in \mathrm{Syl_q}(S)$ and $\varphi|_R = id|_R$. By Lemma 2.6(1), we obtain that $\varphi|_A$ is an automorphism of $A$. Denote by $R^S$ the normal closure of $R$ in $S$. Since $S$ is non-abelian simple, then $R^S = S$. Note that $R^S =< s^{-1}rs : s \in S, r \in R >$ and $S \trianglelefteq A$. Hence, for any $s \in S, r \in R$, $(s^{-1}rs)^\varphi = (s^\varphi)^{-1}rs^\varphi \in S$, which implies that $\varphi|_S \in \mathrm{Aut}(S)$. Hence, $\varphi|_S$ is $q$-central. By Lemma 2.1, we have $\varphi|_S \in \mathrm{Inn}(S)$, that is, there exists a $h \in S$ with $\varphi|_S = \mathrm{conj}(h)|_S$. Again by Lemma 2.5, we may suppose that $\varphi|_S = id|_S$. By Lemma 2.2 and $S$ identifies with $\mathrm{Inn}(S)$, we obtain that $C_A(S) = C_{\mathrm{Aut}(S)}(S) \cap A = 1$. Thus, for any $y \in A$ and $x \in S$, we have $(y^{-1}xy)^\rho = (y^{-1})^\varphi xy^\varphi = y^{-1}xy$, which implies that $y^\varphi y^{-1} \in C_A(S) = 1$. Hence,

$$\varphi|_A = id|_A. \tag{3.1}$$

By Lemma 2.6(2), $\varphi|_{F/A} \in \mathrm{Aut_{Col}}(F/A)$. Note that $F/A$ is abelian, which implies that

$$\varphi|_{F/A} = id_{F/A}. \tag{3.2}$$

Now, by Lemma 2.8, we obtain that

$$\varphi|_{F/O_p(\zeta(A))} = id_{F/O_p(\zeta(A))}. \tag{3.3}$$

By Lemma 2.4, we have $O_p(\zeta(A)) = 1$. Hence, by (3.3), $\varphi = id$. □

**Corollary 3.2.** Let $S$ be a simple normal subgroup of $F$. If $F/S$ is an abelian group, then $\mathrm{Out_{Col}}(F) = 1$. In particular, $\mathrm{Aut}_{\mathbb{Z}}(F) = \mathrm{Inn}(F)$.

*Proof.* If $S$ is abelian simple, this is a consequence of Proposition 3.1 in [6]. Next, we suppose that $S$ is non-abelian simple. Hence the assertion holds by Theorem 3.1. □

**Theorem 3.3.** Let $A$ be an almost simple normal subgroup of $F$. If $F/A$ is a simple group, then $\mathrm{Out}_{\mathrm{Col}}(F) = 1$. In particular, $\mathrm{Aut}_{\mathbb{Z}}(F) = \mathrm{Inn}(F)$.

*Proof.* Let $\rho \in \mathrm{Aut}_{\mathrm{Col}}(F)$ and let $\rho$ be of $p$-power order, where $p \in \pi(F)$. We shall prove that $\rho \in \mathrm{Inn}(F)$. If $F/A$ is abelian simple, then the conclusion holds by Theorem 3.1. Next, we suppose that $F/A$ is non-abelian simple. It follows from Lemma 2.6(2) and Lemma 2.1 that $\rho|_{F/A} \in \mathrm{Inn}(F/A)$, that is, $\rho|_{F/A} = \mathrm{conj}(x)|_{F/A}$ for some $x \in F$. In general, by Lemma 2.9, we may suppose that

$$\rho|_{F/A} = id|_{F/A}. \tag{3.4}$$

First, we show that $\rho|_A \in \mathrm{Aut}_{\mathrm{Col}}(A)$. Since $\rho \in \mathrm{Aut}_{\mathrm{Col}}(F)$, then there is a $k \in F$ such that

$$\rho|_Q = \mathrm{conj}(k)|_Q, \tag{3.5}$$

where $Q \in \mathrm{Syl}(A)$. Set $H = AC_F(Q)$. By Lemma 2.10,

$$\rho|_{F/H} = \mathrm{conj}(k)|_{F/H}. \tag{3.6}$$

Note that $H \geq A$. By (3.4), we deduce that

$$\rho|_{F/H} = id|_{F/H}. \tag{3.7}$$

Consequently, by (3.6) and (3.7), we obtain that $\mathrm{conj}(k)|_{F/H} = id|_{F/H}$, this implies that $kH \in \zeta(F/H)$. Note that $H/A \trianglelefteq F/A$ and $F/A$ is non-abelian simple, then $H/A = 1$ or $H/A = F/A$. From this, we deduce that $\zeta(F/H) = 1$. Hence, $k \in H$. Note further that $H = AC_F(Q) = C_F(Q)A$, we may suppose that $k = ra$, where $r \in C_F(Q)$, $a \in A$. By (3.5),

$$\rho|_Q = \mathrm{conj}(k)|_Q = \mathrm{conj}(ra)|_Q = \mathrm{conj}(a)|_Q. \tag{3.8}$$

By (3.8), we have $\rho|_A \in \mathrm{Aut}_{\mathrm{Col}}(A)$. Since $A$ is almost simple, then $\rho|_A \in \mathrm{Inn}(A)$ by Lemma 2.7, i.e., there is a $b \in A$ with $\rho|_A = \mathrm{conj}(b)|_A$. Set $\varphi = \rho\mathrm{conj}(b^{-1})$. In general, we suppose that $\varphi$ is of $p$-power order, and

$$\varphi|_A = id|_A. \tag{3.9}$$

By (3.4), we also have

$$\varphi|_{F/A} = id|_{F/A}. \tag{3.10}$$

Hence, by Lemma 2.8,

$$\varphi|_{F/O_p(\zeta(A))} = id|_{F/O_p(\zeta(A))}. \tag{3.11}$$

By Lemma 2.4, $O_p(\zeta(A)) = 1$. Thus, by (3.11), we have that $\varphi = id$, i.e., $\rho \in \mathrm{Inn}(F)$. □

**Corollary 3.4.** Let $S$ be a simple normal subgroup of $F$. If $F/S$ is a simple group, then $\mathrm{Out}_{\mathrm{Col}}(F) = 1$. In particular, $\mathrm{Aut}_{\mathbb{Z}}(F) = \mathrm{Inn}(F)$.

*Proof.* If $S$ is abelian simple, this is a consequence of Theorem 1.2 in [6]. Next, we suppose that $S$ is non-abelian simple. Consequently, the assertion holds by Theorem 3.3. □

**Theorem 3.5.** Let $M$ be a normal subgroup of odd order of $F$. If $F/M$ is an almost simple group, then $\mathrm{Out}_{\mathrm{Col}}(F)$ is of odd order. In particular, $\mathrm{Aut}_{\mathbb{Z}}(F) = \mathrm{Inn}(F)$.

*Proof.* Let $\rho \in \mathrm{Aut_{Col}}(F)$ and let $\rho$ be of 2-power order. We shall prove that $\rho \in \mathrm{Inn}(F)$. By Lemma 2.6(2), $\rho|_{F/M} \in \mathrm{Aut_{Col}}(F/M)$. Since $F/M$ is almost simple, then, by Lemma 2.7, $\rho|_{F/M} \in \mathrm{Inn}(F/M)$, i.e., $\rho|_{F/M} = \mathrm{conj}(x)|_{F/M}$ for some $x \in F$. In general, we may suppose that

$$\rho|_{F/M} = id|_{F/M}. \tag{3.12}$$

First, we show that $\rho|_M \in \mathrm{Aut_{Col}}(M)$. Since $\rho \in \mathrm{Aut_{Col}}(F)$, then

$$\rho|_P = \mathrm{conj}(t)|_P, \tag{3.13}$$

where $t \in F, P \in \mathrm{Syl}(M)$. Set $H = M\mathrm{C}_F(P)$, by Lemma 2.10, $H \trianglelefteq F$ and $H^\rho = H$. Moreover,

$$\rho|_{F/H} = \mathrm{conj}(t)|_{F/H}. \tag{3.14}$$

Note that $H \geq M$. By (3.12), we have

$$\rho|_{F/H} = id|_{F/H}. \tag{3.15}$$

By (3.14) and (3.15), $\mathrm{conj}(t)|_{G/H} = id|_{F/H}$, which implies that $tH \in \zeta(F/H)$. Since $F/M$ is almost simple, then we may suppose that $S/M \leq F/M \leq \mathrm{Aut}(S/M)$. Note that $H/M \lhd F/M$ and $S/M \lhd F/M$, so either $H/M \cap S/M = 1$ or $S/M \leq H/M$. If $H/M \cap S/M = 1$, then $[H/M, S/M] = 1$. It follows from Lemma 2.2 that $H = M$. If $S/M \leq H/M$, then $\zeta(H/M) = 1, \zeta(F/M) = 1$ by Lemma 2.3. From this, we deduce that $\zeta(F/H) = 1$, that is, $t \in H$. Note further that $H = M\mathrm{C}_F(P) = \mathrm{C}_F(P)M$, we may suppose that $t = cm$, where $c \in \mathrm{C}_F(P), m \in M$. By (3.13), we have

$$\rho|_P = \mathrm{conj}(t)|_P = \mathrm{conj}(cm)|_P = \mathrm{conj}(m)|_P. \tag{3.16}$$

Thus (3.16) implies that $\rho|_M \in \mathrm{Aut_{Col}}(M)$. Next, by Lemma 2.11,

$$\rho|_M = id|_M. \tag{3.17}$$

Hence, by Lemma 2.8,

$$\rho|_{F/\mathrm{O}_2(\zeta(M))} = id|_{F/\mathrm{O}_2(\zeta(M))}. \tag{3.18}$$

But note that $\mathrm{O}_2(\zeta(M)) = 1$, so (3.18) implies that $\rho = id$. □

**Corollary 3.6.** Let $M$ be a normal subgroup of odd order of $F$. If $F/M$ is a non-abelian simple group, then $\mathrm{Out_{Col}}(F)$ is of odd order. In particular, $\mathrm{Aut}_{\mathbb{Z}}(F) = \mathrm{Inn}(F)$.

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## Conflict of interest

The authors declare there is no conflicts of interest.

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