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Research article

Brake orbits with minimal period estimates of first-order variant subquadratic Hamiltonian systems

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Abstract: Under a generalized subquadratic growth condition, brake orbits are guaranteed via the homological link theorem. Moreover, the minimal period estimate is given by Morse index estimate and L_0-index estimate.

Keywords: Hamiltonian system; Sobolev spaces; weakly continuous functionals; L_0-index; brake orbit; anisotropic growth

1. Introduction

This paper concerns the existence of τ-periodic brake orbits ($\tau > 0$) of the autonomous first-order Hamiltonian system

$$\begin{cases} J\dot{z}(t) = -\nabla H(z(t)), & \\ z(-t) = Nz(t), & t \in \mathbb{R}, \\ z(t+\tau) = z(t), & \end{cases} \tag{1.1}$$

where $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ with $H(Nz) = H(z)$, $z \in \mathbb{R}^{2n}$, $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ and $N = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}$ with I_n the $n \times n$ identity matrix.

As shown in [1, 2], for $\vec{x} = (x_1, \cdots, x_n)$ and $\vec{y} = (y_1, \cdots, y_n)$, we set

$$V(\vec{x}, \vec{y}) = \mathrm{diag}\{x_1, \cdots, x_n, y_1, \cdots, y_n\} \in \mathbb{R}^{2n\times 2n}.$$

For $z = (p_1, \cdots, p_n, q_1, \cdots, q_n)$, we have

$$V(\vec{x}, \vec{y})(z) = (x_1 p_1, \cdots, x_n p_n, y_1 q_1, \cdots, y_n q_n).$$

Below are the conditions cited from [3] with minor modifications.

(H1) $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, $H(Nz) = H(z),\ z \in \mathbb{R}^{2n}$.

(H2) There exist $\gamma_i > 0$ $(i = 1, \cdots, n)$ such that

$$\lim_{|z|\to+\infty} \frac{H(z)}{\omega(z)} = 0,$$

where $\omega(z) = \sum_{i=1}^{n} \left(|p_i|^{1+\gamma_i} + |q_i|^{1+\frac{1}{\gamma_i}} \right)$.

(H3) There exist $\beta > 1$ and $c_1,\ c_2,\ \alpha_i,\ \beta_i > 0$ with $\alpha_i + \beta_i = 1$ $(1 \le i \le n)$ such that

$$\min\{H(z), H(z) - \nabla H(z) \cdot V(z)\} \ge c_1|z|^\beta - c_2, \quad z \in \mathbb{R}^{2n},$$

where $V(z) = V(\vec{\alpha}, \vec{\beta})(z)$ with $\vec{\alpha} = (\alpha_1, \cdots, \alpha_n), \vec{\beta} = (\beta_1, \cdots, \beta_n)$.

(H4) There exists $\lambda \in [1, \frac{\beta^2}{\beta+1})$ such that

$$|H''_{zz}(z)| \le c_2(|z|^{\lambda-1} + 1), \quad z \in \mathbb{R}^{2n},$$

where H''_{zz} means the Hessian matrix of H.

(H5) $H(0) = 0$ and $H(z) > 0,\ |\nabla H(z)| > 0$ for $z \ne 0$.

Note that (H2) is a variant subquadratic growth condition which has superquadratic growth behaviors in some components and has subquadratic growth behaviors in other components, while [4] provided one other kind of variant subquadratic growth condition, we also call such conditions anisotropic growth conditions.

In the last decades, brake orbit problems have been investigated deeply, see [5–13] and references therein. In [14], the existence of brake orbits and symmetric brake orbits were proved under the classical superquadratic growth conditions. Meanwhile, the minimal period estimates were given by comparing the L_0-index iterations. Later, in [15], the authors obtained the same minimal period estimates under a weak growth condition which has super-quadratic growth only on some J-invariant plane. In [4, 16], the authors considered first-order anisotropic convex Hamiltonian systems and reduced the existence problem of brake orbits to the dual variation problem, moreover, in [4], the minmality of period for brake orbits was obtained. In [1], the authors removed the convex assumption in [16] and obtained brake orbits with minimal period estimates under more general anisotropic growth conditions which are variant superquadratic growth conditions.

The following is the main result of this paper.

Theorem 1.1. *If H is a Hamiltonian function satisfying (H1)–(H5), then there exists $\tilde{\tau} > 0$ such that when $\tau \ge \tilde{\tau}$, the system (1.1) has a nontrivial brake orbit z with the L_0-index estimate*

$$i_{L_0}(z, \frac{\tau}{2}) \le 0. \tag{1.2}$$

Futhermore, if the above brake orbit z also satisfies

(H6) $H''_{zz}(z(t)) \ge 0$*, $t \in \mathbb{R}$ and $\int_0^{\frac{\tau}{2}} H''_{qq}(z(t))\mathrm{d}t > 0$, where $H''_{qq}(z)$ means the Hessian matrix w.r.t. q for $z = (p, q),\ p,\ q \in \mathbb{R}^n$.*

Then the brake orbit z has minimal period τ or $\frac{\tau}{2}$.

We remind the readers that the minimal period $\frac{\tau}{2}$ may not be eliminated generally. See Remark 4.2 in [14], for example, the minimal period is $\frac{\tau}{2}$ under the condition (H6). In [2], we also consider the symmetric brake orbit case under the above conditions with small changes using different index iteration inequalities.

If $\tilde{z}$ is a brake orbit for the system (1.1), then $z(t) = \tilde{z}(\frac{\tau}{2}t)$ satisfies

$$\begin{cases} J\dot{z}(t) = -\frac{\tau}{2}\nabla H(z(t)), \\ z(-t) = Nz(t), \\ z(t+2) = z(t). \end{cases} \tag{1.3}$$

The converse is also true. So finding brake orbits for the system (1.1) is equivalent to finding 2-periodic brake orbits for the system (1.3).

In Section 2, we recall the L_0-index theory and the related Sobolev space. In Section 3, we prove the existence of a nontrivial brake orbit with minimal period 2 or 1.

2. Preliminaries

The Maslov-type index theory is higly-developed and widly-used to study the existence, minimality of period, multiplicity and stability of periodic solutions of Hamiltonian systems, see [17]. And to estimate the minimal period for brake orbits, Liu and his cooperators introduced the L_0-index theory —a topologically variant Maslov-type index theory, see the monograph [18] and the recent survey paper [19].

We denote by $\mathcal{L}(\mathbb{R}^{2n})$ the set of all $2n \times 2n$ real matrices, and denote by $\mathcal{L}_s(\mathbb{R}^{2n})$ its subset of symmetric ones. The symplectic group $\mathrm{Sp}(2n)$ for $n \in \mathbb{N}$ and the symplectic path $\mathcal{P}_\tau(2n)$ in $\mathrm{Sp}(2n)$ starting from the identity I_{2n} on $[0, \tau]$ are denoted respectively by

$$\begin{aligned} \mathrm{Sp}(2n) &= \{M \in \mathcal{L}(\mathbb{R}^{2n}) \mid M^T JM = J\}, \\ \mathcal{P}_\tau(2n) &= \{\gamma \in C([0,\tau], \mathrm{Sp}(2n)) \mid \gamma(0) = I_{2n}\}. \end{aligned}$$

As showed in [18], for the Lagrangian subspaces $L_0 = \{0\}\times\mathbb{R}^n$ and $L_1 = \mathbb{R}^n\times\{0\}$, there are two pairs of integers $(i_{L_k}(\gamma, \tau), \nu_{L_k}(\gamma, \tau)) \in \mathbb{Z}\times\{0,\ 1,\ \cdots,\ n\}$ $(k = 0, 1)$ associated with $\gamma \in \mathcal{P}_\tau(2n)$ on the interval $[0, \tau]$, called the Maslov-type index associated with L_k for $k = 0, 1$ or the L_k-index of γ in short. When $\tau = 1$, we simply write $(i_{L_k}(\gamma), \nu_{L_k}(\gamma))$.

The L_0-iteration paths $\gamma^j : [0, j] \to \mathrm{Sp}(2n)$ of $\gamma \in \mathcal{P}_1(2n)$ (see [18]) are defined by

$$\gamma^1(t) = \gamma(t), \quad t \in [0, 1],$$

$$\gamma^2(t) = \begin{cases} \gamma(t), & t \in [0, 1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1), & t \in [1, 2] \end{cases}$$

and more generally, for $j \in \mathbb{N}$,

$$\gamma^{2j}(t) = \begin{cases} \gamma^{2j-1}(t), & t \in [0, 2j-1], \\ N\gamma(2j-t)N[\gamma^2(2)]^j, & t \in [2j-1, 2j], \end{cases}$$

$$\gamma^{2j+1}(t) = \begin{cases} \gamma^{2j}(t), & t \in [0, 2j], \\ \gamma(t-2j)[\gamma^2(2)]^j, & t \in [2j, 2j+1]. \end{cases}$$

Then we denote by $(i_{L_0}(\gamma^j), \nu_{L_0}(\gamma^j))$ the L_0-index of γ^j on the interval $[0, j]$.

Assume $B(t) \in C([0,\tau], \mathcal{L}_s(\mathbb{R}^{2n}))$ satisfies $B(t+\tau) = B(t)$ and $B(\frac{\tau}{2}+t)N = NB(\frac{\tau}{2}-t)$, consider the fundamental solution γ_B of the following linear Hamiltonian system

$$\begin{cases} J\dot{z}(t) = -B(t)z(t), & t \in [0,\tau], \\ z(0) = I_{2n}. \end{cases}$$

Then $\gamma_B \in \mathcal{P}_\tau(2n)$. Note that γ_B^k satisfies

$$\begin{cases} J\dot{z}(t) = -B(t)z(t), & t \in [0,k\tau], \\ z(0) = I_{2n}. \end{cases}$$

The L_0-index of γ_B is denoted by $(i_{L_0}(B), \nu_{L_0}(B))$, called the L_0-index pair with respect to B.

Moreover, if z is a brake orbit of the system (1.1), set $B(t) = H''(z(t))$, denote by $(i_{L_0}(z), \nu_{L_0}(z))$ the L_0-index of γ_B, called the L_0-index pair with respect to z.

See [17] for the Maslov-type index $(i_1(\gamma), \nu_1(\gamma))$ of $\gamma \in \mathcal{P}(2n)$. And we refer to [18] for the indices $(i^{L_0}_{\sqrt{-1}}(\gamma), \nu^{L_0}_{\sqrt{-1}}(\gamma))$ and $(i^{L_0}_{\sqrt{-1}}(B), \nu^{L_0}_{\sqrt{-1}}(B))$ for $\tau = 1$.

Below are some basic results needed in this paper.

Lemma 2.1. *([11]) For $\gamma \in \mathcal{P}(2n)$, there hold*

$$i_1(\gamma^2) = i_{L_0}(\gamma) + i_{L_1}(\gamma) + n \quad \text{and} \quad \nu_1(\gamma^2) = \nu_{L_0}(\gamma) + \nu_{L_1}(\gamma).$$

Lemma 2.2. *([14]) Suppose $B(t) \in C([0,2], \mathcal{L}_s(\mathbb{R}^{2n}))$ with $B(t+2) = B(t)$ and $B(1+t)N = NB(1-t)$. If $B(t) \geq 0$ for all $t \in [0,2]$, then*

$$i_{L_0}(B) + \nu_{L_0}(B) \geq 0 \quad \text{and} \quad i^{L_0}_{\sqrt{-1}}(B) \geq 0.$$

Lemma 2.3. *([14]) Suppose $B(t) \in C([0,2], \mathcal{L}_s(\mathbb{R}^{2n}))$ with $B(t+2) = B(t)$ and $B(1+t)N = NB(1-t)$. If $B(t) = \begin{pmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{pmatrix} \geq 0$ and $\int_0^1 S_{22}(t)\,\mathrm{d}t > 0$, then $i_{L_0}(B) \geq 0$.*

Lemma 2.4. *([18]) The Maslov-type index iteration inequalities are presented below.*

1° *For $\gamma \in \mathcal{P}(2n)$ and $k \in 2\mathbb{N}-1$, there holds*

$$i_{L_0}(\gamma^k) \geq i_{L_0}(\gamma^1) + \frac{k-1}{2}(i_1(\gamma^2) + \nu_1(\gamma^2) - n).$$

2° *For $\gamma \in \mathcal{P}(2n)$ and $k \in 2\mathbb{N}$, there holds*

$$i_{L_0}(\gamma^k) \geq i_{L_0}(\gamma^1) + i^{L_0}_{\sqrt{-1}}(\gamma^1) + \left(\frac{k}{2} - 1\right)(i_1(\gamma^2) + \nu_1(\gamma^2) - n).$$

Now we introduce the Sobolev space $E = W_{L_0}$ and its subspaces as in [10, 14].

$$\begin{aligned} E = W_{L_0} &= \left\{ z \in W^{\frac{1}{2},2}(\mathbb{R}/2\mathbb{Z}, \mathbb{R}^{2n}) \mid z(-t) = Nz(t) \text{ for a.e. } t \in \mathbb{R} \right\} \\ &= \left\{ z \in W^{\frac{1}{2},2}(\mathbb{R}/2\mathbb{Z}, \mathbb{R}^{2n}) \,\middle|\, z(t) = \sum_{k\in\mathbb{Z}} \exp(k\pi tJ)h_k,\ h_k \in L_0 \right\}. \end{aligned}$$

For $m \in \mathbb{N}$, define

$$\begin{aligned} E^{\pm} &= \left\{ z \in W_{L_0} \mid z(t) = \sum_{\pm k \in \mathbb{Z}} \exp(k\pi t J) h_k,\ h_k \in L_0 \right\}, \\ E^0 &= L_0, \\ E_m &= \left\{ z \in W_{L_0} \mid z(t) = \sum_{k=-m}^{m} \exp(k\pi t J) h_k,\ h_k \in L_0 \right\}, \end{aligned}$$

and set $E_m^+ := E_m \cap E^+$, $E_m^- := E_m \cap E^-$. Then $E = E^0 \oplus E^- \oplus E^+$ and $E_m = E^0 \oplus E_m^- \oplus E_m^+$. Moreover $\{E_m, P_m\}$ forms a Galerkin approximation scheme of the unbounded self-adjoint operator $-J\frac{\mathrm{d}}{\mathrm{d}t}$ defined on $L^2([0,2]; L_0)$, where $P_m : E \to E_m$ denotes the orthogonal projection. Furthermore, define the following bounded self-adjoint operator $\mathcal{A}$ on E

$$\langle \mathcal{A}z, \zeta \rangle = \int_0^2 -J\dot{z} \cdot \zeta \, \mathrm{d}t, \quad z,\ \zeta \in W^{1,2}([0,2]; L_0) \subseteq E,$$

and, obviously, $\langle \mathcal{A}z, z \rangle = 2(\|z^+\|^2 - \|z^-\|^2)$, $\mathcal{A}z = \mathcal{A}z^+ - \mathcal{A}z^-$, $z \in E$.

Remark 2.1. *([1]) For $z \in E$, there holds $V(\vec{x}, \vec{y})z \in E$. And for $z \in E_m$, we have $V(\vec{x}, \vec{y})z \in E_m$. As for the Fourier expression for $V(\vec{x}, \vec{y})z$, see [1] for details. Note that for V defined in (H2) and $z \in E$, we have $V(z) \in E$. Moreover, a simple computation shows that*

$$\langle \mathcal{A}z, V(z) \rangle = \frac{1}{2} \langle \mathcal{A}z, z \rangle, \quad z \in E.$$

In our case, assume $B(t) \in C([0,2], \mathcal{L}_s(\mathbb{R}^{2n}))$ satisfies $B(t+2) = B(t)$ and $B(1+t)N = NB(1-t)$, define the following bounded self-adjoint compact operator $\mathcal{B}$

$$\langle \mathcal{B}z, \zeta \rangle = \int_0^2 B(t)z \cdot \zeta \mathrm{d}t, \quad z,\ \zeta \in E. \tag{2.1}$$

For any $d > 0$, denote by $M_d^-(\cdot)$, $M_d^0(\cdot)$, $M_d^+(\cdot)$ the eigenspaces corresponding to the eigenvalues λ belonging to $(-\infty, -d]$, $(-d, d)$, $[d, +\infty)$ respectively. Set $(\mathcal{A} - \mathcal{B})^\sharp = (\mathcal{A} - \mathcal{B}|_{\mathrm{Im}(\mathcal{A}-\mathcal{B})})^{-1}$. The following result is crucial to esmiate the L_0-index.

Lemma 2.5. *([20,21]) For $B(t) \in C([0,2], \mathcal{L}_s(\mathbb{R}^{2n}))$ satisfying $B(t+2) = B(t)$, $B(1+t)N = NB(1-t)$ and $0 < d \leq \frac{1}{4} \left\| (\mathcal{A} - \mathcal{B})^\sharp \right\|^{-1}$, there exists $m_0 > 0$ such that for $m \geq m_0$, we have*

$$\begin{aligned} &\dim M_d^+(P_m(\mathcal{A} - \mathcal{B})P_m) = mn - i_{L_0}(B) - \nu_{L_0}(B). \\ &\dim M_d^-(P_m(\mathcal{A} - \mathcal{B})P_m) = mn + n + i_{L_0}(B). \\ &\dim M_d^0(P_m(\mathcal{A} - \mathcal{B})P_m) = \nu_{L_0}(B). \end{aligned}$$

3. Main results

As shown in [10,14], searching for brake orbits for the system (1.3) can be transformed into finding critical points of the following functional

$$g(z) = \frac{\tau}{2} \int_0^2 H(z) \, \mathrm{d}t - \frac{1}{2} \langle \mathcal{A}z, z \rangle, \quad z \in E.$$

By (H4), we have $g \in C^2(E, \mathbb{R})$, then, let us now set $g_m = g|_{E_m}$, $m \in \mathbb{N}$. To find the critical points of g_m , we shall prove that g_m satisfies the hypotheses of the homological link Theorem 4.1.7 in [22]. The following several lemmas are essential.

Lemma 3.1. *If H(z) satifies (H1),(H3) and (H4), then the above functional g satisfies (PS)* condition with respect to* $\{E_m\}_{m\in\mathbb{N}}$, *i.e., any sequence* $\{z_m\} \subset E$ *satisfying* $z_m \in E_m$, $g_m(z_m)$ *is bounded and* $\nabla g_m(z_m) \to 0$ *as* $m \to +\infty$ *possesses a convergent subsequence in E.*

Proof. We follow the ideas in [3].

Let $\{z_m\}$ be a sequence such that $|g(z_m)| \le c_3$ and $\nabla g_m(z_m) \to 0$ as $m \to \infty$, where $c_3 > 0$. To prove the lemma, it is enough to show that $\{z_m\}$ is bounded.

For m large enough, by Remark 2.1 and (H3), we have

$$\begin{aligned} c_3 + \|z_m\| &\ge g(z_m) - \langle \nabla g_m(z_m), V(z_m)\rangle \\ &= \frac{\tau}{2}\int_0^2 (H(z_m) - \nabla H(z_m)\cdot V(z_m))\ \mathrm{d}t \\ &\ge \frac{\tau}{2}\int_0^2 (c_1|z_m|^\beta - c_2)\ \mathrm{d}t, \end{aligned}$$

then there exists $c_4 > 0$ such that

$$\|z_m\|_{L^\beta} \le c_4(1 + \|z_m\|^{\frac{1}{\beta}}). \tag{3.1}$$

For large m, we have

$$\|z_m^\pm\| \ge \|\langle \nabla g_m(z_m), z_m^\pm\rangle\| = \left|\frac{\tau}{2}\int_0^2 \nabla H(z_m)\cdot z_m^\pm\ \mathrm{d}t - \langle \mathcal{A}z_m, z_m^\pm\rangle\right|. \tag{3.2}$$

By (3.2), (H4), Hölder's inequality and the embedding theorem, we obtain

$$\begin{aligned} \|z_m^\pm\|^2 &= \pm\frac{1}{2}\langle \mathcal{A}z_m, z_m^\pm\rangle \\ &\le \frac{\tau}{4}\Big|\int_0^2 \nabla H(z_m)\cdot z_m^\pm\ \mathrm{d}t\Big| + \frac{1}{2}\|z_m^\pm\| \\ &\le c_5\int_0^2 (|z_m|^\lambda + 1)|z_m^\pm|\ \mathrm{d}t + \frac{1}{2}\|z_m^\pm\| \\ &\le c_5\left(\int_0^2 \left(|z_m|^\lambda\right)^{\frac{\beta}{\lambda}}\ \mathrm{d}t\right)^{\frac{\lambda}{\beta}}\left(\int_0^2 |z_m^\pm|^{\frac{\beta}{\beta-\lambda}}\ \mathrm{d}t\right)^{\frac{\beta-\lambda}{\beta}} + c_5\|z_m^\pm\|_{L^1} + \frac{1}{2}\|z_m^\pm\| \\ &= c_5\left(\int_0^2 |z_m|^\beta\ \mathrm{d}t\right)^{\frac{\lambda}{\beta}}\left(\int_0^2 |z_m^\pm|^{\frac{\beta}{\beta-\lambda}}\ \mathrm{d}t\right)^{\frac{\beta-\lambda}{\beta}} + c_5\|z_m^\pm\|_{L^1} + \frac{1}{2}\|z_m^\pm\| \\ &\le c_6(1 + \|z_m\|_{L^\beta}^\lambda)\|z_m^\pm\|, \end{aligned} \tag{3.3}$$

where $\beta > \lambda \ge 1$ for (H3), (H4) and $c_5,\ c_6 > 0$ are suitable constants.

Combining (3.1) and (3.3), for m large enough, there exists $c_7 > 0$ such that

$$\|z_m^\pm\| \le c_7(1 + \|z_m\|^{\frac{\lambda}{\beta}}). \tag{3.4}$$

Set $\widehat{z_m} = z_m - z_m^0 = z_m^+ + z_m^-$. By (H4), (3.4) and the embedding theorem, we obtain

$$\begin{aligned}\left|\int_0^2 [H(z_m) - H(z_m^0)]\,\mathrm{d}t\right| &= \left|\int_0^2 \int_0^1 \nabla H_z(z_m^0 + s\widehat{z_m})\cdot\widehat{z_m}\,\mathrm{d}s\mathrm{d}t\right| \\ &\le \int_0^2 2^\lambda c_8\left(|z_m^0|^\lambda + |\widehat{z_m}|^\lambda + 1\right)|\widehat{z_m}|\,\mathrm{d}t \\ &\le c_9\left(1 + \|z_m\|^{\lambda+\frac{\lambda}{\beta}}\right),\end{aligned} \tag{3.5}$$

where $c_8, c_9 > 0$ are suitable constants. From (3.4) and (3.5), we see

$$\begin{aligned}\frac{\tau}{2}\int_0^2 H(z_m^0)\,\mathrm{d}t &= g(z_m) + \frac{1}{2}\langle\mathcal{A}z_m, z_m\rangle - \frac{\tau}{2}\int_0^2 [H(z_m) - H(z_m^0)]\,\mathrm{d}t \\ &\le c_{10}\left(1 + \|z_m\|^{\lambda+\frac{\lambda}{\beta}}\right),\end{aligned} \tag{3.6}$$

where $c_{10} > 0$. From (H3), it follows that

$$\int_0^2 H(z_m^0)\,\mathrm{d}t \ge \int_0^2 \left(c_1|z_m^0|^\beta - c_2\right)\,\mathrm{d}t. \tag{3.7}$$

From (3.6) and (3.7), we see that

$$|z_m^0| \le c_{11}\left(1 + \|z_m\|^{\frac{\lambda+\lambda\beta}{\beta^2}}\right), \tag{3.8}$$

where $c_{11} > 0$. From (3.4), (3.8) and $\frac{\lambda+\lambda\beta}{\beta^2} < 1$, we see $\{z_m\}$ is bounded.

For $u_0 \in E_1^+$ with $\|u_0\| = 1$, define $S = \left(E^- \oplus E^0\right) + u_0$.

Lemma 3.2. *If $H(z)$ satifies ($H1$),($H4$) and ($H5$), then there exists $\tilde{\tau} > 0$ such that for $\tau \ge \tilde{\tau}$, there holds $\inf_S g > 0$.*

Proof. The ideas come from [23].

For $z \in S$, we have

$$g(z) = \frac{\tau}{2}\int_0^2 H(z)\,\mathrm{d}t + \|z^-\|^2 - 1. \tag{3.9}$$

There exist two cases to be considered.

Case (i) If $\|z^-\| > 1$, then by (H5), we have

$$g(z) = \frac{\tau}{2}\int_0^2 H(z)\,\mathrm{d}t + \|z^-\|^2 - 1 > 0.$$

Case (ii) If $\|z^-\| \le 1$, set $\Omega = \{z \in S \mid \|z^-\| \le 1\}$, then Ω is weakly compact and convex.

Since the functional $z \mapsto \int_0^2 H(z)\,\mathrm{d}t$ is weakly continuous, then the functional achieves its minimum on Ω, assume the minimum is σ achieved at $u^- + u_0 \in S$. Since $u_0 \neq 0$, we have $u^- + u_0 \neq 0$, then $\sigma > 0$ by (H5).

Set $\tilde{\tau} = \frac{2}{\sigma}$, for $\tau > \tilde{\tau}$, by (3.9), we have

$$g(z) \ge \frac{\tau\sigma}{2} - 1 > 0.$$

Therefore, the lemma holds.

Choose $\mu > 0$ large enough such that $\sigma_i = \frac{\mu}{1+\gamma_i} > 1$ and $\tau_i = \frac{\mu}{1+\frac{1}{\gamma_i}} > 1$. For $\rho > 0$, we set

$$L_\rho(z) = (\rho^{\sigma_1-1}p_1, \cdots, \rho^{\sigma_n-1}p_n, \rho^{\tau_1-1}q_1 \cdots, \rho^{\tau_n-1}q_n),$$

where $z = (p_1, \cdots, p_n, q_1, \cdots, q_n) \in E$. Note that L_ρ is well-defined on E by Remark 2.1. The operator L_ρ is linear bounded and invertible and $\|L_\rho\| \leq 1$, if $\rho \leq 1$.

For any $z = z^0 + z^- + z^+ \in E$, we have

$$\langle \mathcal{A}L_\rho z, L_\rho z\rangle = \rho^{\mu-2}\langle \mathcal{A}z, z\rangle = 2\rho^{\mu-2}(\|z^+\|^2 - \|z^-\|^2). \tag{3.10}$$

Lemma 3.3. *If H satisfies (H2), then there exists $\rho > 1$ large enough such that $\sup\limits_{L_\rho(\partial Q)} g < 0$, where $Q = \{z \in E^+ \mid \|z\| \leq \rho\}$.*

Proof. For any $\epsilon > 0$, by $(H2)$, there exists M_ϵ such that

$$H(z) \leq \epsilon \sum_{i=1}^{n} \left(|p_i|^{1+\gamma_i} + |q_i|^{1+\frac{1}{\gamma_i}}\right) + M_\epsilon, \quad z \in \mathbb{R}^{2n}. \tag{3.11}$$

For $z \in \partial Q$, from (3.10) and (3.11), we have

$$\begin{aligned} g(L_\rho z) &= \frac{\tau}{2}\int_0^2 H(L_\rho z)\,\mathrm{d}t - \frac{1}{2}\langle \mathcal{A}L_\rho z, L_\rho z\rangle \\ &\leq \frac{\tau\varepsilon}{2}\sum_{i=1}^{n}\int_0^2 \left(\rho^{(\sigma_i-1)(1+\gamma_i)}|p_i|^{1+\gamma_i} + \rho^{(\tau_i-1)(1+\frac{1}{\gamma_i})}|q_i|^{1+\frac{1}{\gamma_i}}\right)\mathrm{d}t \\ &\quad + M_\epsilon\tau - \rho^\mu \\ &\leq (n\tau\epsilon c_{12} - 1)\rho^\mu + M_\epsilon\tau, \end{aligned} \tag{3.12}$$

where $c_{12} > 0$ is the embedding constant.

Choose $\epsilon > 0$ such that $n\tau\epsilon c_{12} < 1$, then for $\rho > 1$ large enough, we have $\sup\limits_{L_\rho(\partial Q)} g < 0$.

Lemma 3.4. *Set $S_m = S \cap E_m$ and $Q_m = Q \cap E_m$. For $\rho > 1$ defined as above, we have $L_\rho(\partial Q_m)$ and S_m homologically link.*

Proof. Since $\rho > 1$, $\rho > \|L_\rho^{-1}\| = \|L_{\frac{1}{\rho}}\|$. By direct computation, we can check that $PL_\rho : E^+ \to E^+$ is liner, bounded and invertible (see [24]). Let $\widetilde{P}_m : E_m \to E_m^+$ be the orthogonal projection. Note that $L_\rho(E_m) \subset E_m$ by Remark 2.1, then $\left(\widetilde{P}_m L_\rho\right)|_{E_m} : E_m^+ \to E_m^+$ is also linear, bounded and invertible.

Then the assertion follows from Lemma 2.8 in [3].

Theorem 3.1. *Assume H satisfies (H1)–(H5), then there exists $\tilde{\tau} > 0$ such that for $\tau \geq \tilde{\tau}$, the system (1.3) possesses a nontrivial 2-periodic brake orbit z satisfying*

$$i_{L_0}(z, 1) \leq 0. \tag{3.13}$$

Proof. The proof is standard, we proceed as that in [10, 14].

For any $m \in \mathbb{N}$, Lemmas 3.1–3.4 show that $g_m = g|_{E_m}$ satisfies the hypotheses of the homological link Theorem 4.1.7 in [22], so g_m possesses a critical point z_m satisfying

$$0 < \inf_S g \le g(z_m) \le \sup_{L_\rho(Q)} g. \tag{3.14}$$

By Lemma 3.1, when $\tau \ge \tau_0$, we may suppose $z_m \to z \in E$ as $m \to \infty$, then $g(z) > 0$ and $\nabla g(z) = 0$. By (H5), we see the critical point z of g is a classical nontrivial 2-periodic brake orbit of the system (1.3).

Now we show (3.13) holds. Let $\mathcal{B}$ be the operator for $B(t) = \frac{\tau}{2}H''_{zz}(z(t))$ defined by (2.1), then

$$\|g''(x) - (\mathcal{B} - \mathcal{A})\| \to 0 \quad \text{as} \quad \|x - z\| \to 0, \;\; x \in E. \tag{3.15}$$

By (3.15), there exists $r_0 > 0$ such that

$$\|g''(x) - (\mathcal{B} - \mathcal{A})\| < d, \quad x \in B_{r_0} = \{x \in E \mid \|x - z\| \le r_0\},$$

where $d = \frac{1}{4}\|(B - A)^\sharp\|^{-1}$.

Hence, for m large enough, there holds

$$\|g''_m(x) - P_m(\mathcal{B} - \mathcal{A})P_m\| < \frac{d}{2}, \quad x \in B_{r_0} \cap E_m. \tag{3.16}$$

For $x \in B_{r_0} \cap E_m$ and $w \in M_d^+(P_m(\mathcal{B} - \mathcal{A})P_m) \setminus \{0\}$, (3.16) implies that

$$\begin{aligned} \langle g''_m(x)w, w\rangle &\ge \langle P_m(\mathcal{B} - \mathcal{A})P_m w, w\rangle - \|g''_m(x) - P_m(\mathcal{B} - \mathcal{A})P_m\| \cdot \|w\|^2 \\ &\ge \frac{d}{2}\|w\|^2 > 0. \end{aligned}$$

Then

$$\dim M^+(g''_m(x)) \ge \dim M_d^+(P_m(\mathcal{B} - \mathcal{A})P_m), \quad x \in B_{r_0} \cap E_m. \tag{3.17}$$

Note that

$$\begin{aligned} \dim M_d^-(P_m(\mathcal{B} - \mathcal{A})P_m) &= \dim M_d^+(P_m(\mathcal{A} - \mathcal{B})P_m), \\ \dim M_d^0(P_m(\mathcal{B} - \mathcal{A})P_m) &= \dim M_d^0(P_m(\mathcal{A} - \mathcal{B})P_m). \end{aligned} \tag{3.18}$$

By (3.17), (3.18) and the link theorem 4.1.7 in [22], for large m, we have

$$\begin{aligned} mn = \dim Q_m &\le m(z_m) + m^0(z_m) \\ &\le \dim M_d^-(P_m(\mathcal{B} - \mathcal{A})P_m) + \dim M_d^0(P_m(\mathcal{B} - \mathcal{A})P_m) \\ &= mn - i_{L_0}(z, 1). \end{aligned}$$

Hence, we obtain $i_{L_0}(z, 1) \le 0$.

Theorem 3.2. *Assume H satisfies (H1)–(H6), then there exists $\tilde{\tau}$ such that for $\tau \ge \tilde{\tau}$, the system (1.3) possesses a nontrivial brake orbit z with minimal period* 2 *or* 1.

Proof. The idea stems from [14], we proceed roughly.

For the nontrivial symmetric 2-periodic brake orbit z obtained in Theorem 3.1, assume its minimal period $\frac{2}{k}$ for some nonnegative integer k. Denote by $\gamma_{z,\frac{1}{k}}$ and γ_z the corresponding symplectic path on the interval $[0, \frac{1}{k}]$ and $[0, 1]$ respectively, then $\gamma_z = \gamma^k_{z,\frac{1}{k}}$.

As shown in [14], we have the L_1-index estimate

$$i_{L_1}(\gamma_{z,\frac{1}{k}}) + \nu_{L_1}(\gamma_{z,\frac{1}{k}}) \geq 1. \tag{3.19}$$

By (H6), we see $B(t) = H''(z(t))$ is semipositive, Lemmas 2.1 and 2.2 and Eq (3.19) imply that

$$i_1(\gamma^2_{z,\frac{1}{k}}) + \nu_1(\gamma^2_{z,\frac{1}{k}}) - n = i_{L_0}(\gamma_{z,\frac{1}{k}}) + \nu_{L_0}(\gamma_{z,\frac{1}{k}}) + i_{L_1}(\gamma_{z,\frac{1}{k}}) + \nu_{L_1}(\gamma_{z,\frac{1}{k}}) \geq 1. \tag{3.20}$$

By Lemmas 2.2 and 2.3, we see

$$i_{L_0}(\gamma_{z,\frac{1}{k}}) \geq 0 \quad \text{and} \quad i^{L_0}_{\sqrt{-1}}(\gamma_{z,\frac{1}{k}}) \geq 0. \tag{3.21}$$

If k is odd, by Lemma 2.4, we see

$$i_{L_0}(\gamma_z) \geq i_{L_0}(\gamma_{z,\frac{1}{k}}) + \frac{k-1}{2}\left[i_1(\gamma^2_{z,\frac{1}{k}}) + \nu_1(\gamma^2_{z,\frac{1}{k}}) - n\right] \tag{3.22}$$

From (3.13), (3.20)–(3.22), we see $k = 1$.

If k is even, If k is even, by Lemma 2.4, we see

$$i_{L_0}(\gamma_z) \geq i_{L_0}(\gamma_{z,\frac{1}{k}}) + i^{L_0}_{\sqrt{-1}}(\gamma_{z,\frac{1}{k}}) + (\frac{k}{2} - 1)\left[i_1(\gamma^2_{z,\frac{1}{k}}) + \nu_1(\gamma^2_{z,\frac{1}{k}}) - n\right]. \tag{3.23}$$

From (3.13), (3.20), (3.21) and (3.23), we have $k = 2$.

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Conflict of interest

The authors declare there is no conflicts of interest.

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