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GLOBAL DYNAMICS OF SOME SYSTEM OF SECOND-ORDER DIFFERENCE EQUATIONS

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Abstract. In this paper, we study the boundedness and persistence of positive solution, existence of invariant rectangle, local and global behavior, and rate of convergence of positive solutions of the following systems of exponential difference equations

$$x_{n+1} = \frac{\alpha_1 + \beta_1 e^{-x_{n-1}}}{\gamma_1 + y_n}, \; y_{n+1} = \frac{\alpha_2 + \beta_2 e^{-y_{n-1}}}{\gamma_2 + x_n},$$
$$x_{n+1} = \frac{\alpha_1 + \beta_1 e^{-y_{n-1}}}{\gamma_1 + x_n}, \; y_{n+1} = \frac{\alpha_2 + \beta_2 e^{-x_{n-1}}}{\gamma_2 + y_n},$$

where the parameters α_i, β_i, γ_i for $i \in \{1, 2\}$ and the initial conditions x_{-1}, x_0, y_{-1}, y_0 are positive real numbers. Some numerical example are given to illustrate our theoretical results.

1. **Introduction.** Mathematical models of population dynamics are often described by difference equations and systems of difference equations. In particular, the population models involving exponential difference equations are quite popular, although their stability analysis can be complicated. In recent years, the global asymptotic behavior of the difference equations of exponential form has been one of the main topics in the theory of difference equations (see [2, 3, 4, 5, 7, 12, 13, 14, 15, 16, 17, 8, 20] and reference cited therein).

In [4], El-Metwally et al. investigated the following population model:

$$x_{n+1} = \alpha + \beta x_{n-1} e^{-x_n}, \tag{1}$$

where the parameters α and β are positive numbers and the initial conditions x_{-1} and x_0 are arbitrary non-negative numbers. Later in [5], Fotiades et al. studied the existence, uniqueness and attractivity of prime period two solution of this equation.

Papaschinopoulos et al. [15] and Papaschinopoulos and Schinas [17] investigated the dynamical properties of two-species model described by systems of difference equations, which is natural extension of single-species population model depicted in 1.

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Ozturk et al. [12] have investigated the following difference equation:

$$y_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, \tag{2}$$

where the parameters α, β, γ are positive numbers and the initial conditions are arbitrary non-negative numbers.

Papaschinopoulos et al. [16] have studied the following systems of two difference equations of exponential form:

$$x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, \ y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\eta + x_{n-1}},$$
$$x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + x_{n-1}}, \ y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\eta + y_{n-1}},$$
$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + y_{n-1}}, \ y_{n+1} = \frac{\delta + \epsilon e^{-y_n}}{\eta + x_{n-1}},$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \eta$ are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive numbers.

In 2016, Wang and Feng [20] have investigated the dynamics of positive solution of the following difference equation which is naturally a new form of single-species model described in 1:

$$x_{n+1} = \alpha + \beta x_n e^{-x_{n-1}},$$

where the parameters α and β are positive numbers and the initial conditions x_{-1} and x_0 are arbitrary non-negative numbers.

Motivated by the aforementioned study, our goal in this paper is to investigate the qualitative behavior of positive solutions of some systems of exponential difference equations

$$x_{n+1} = \frac{\alpha_1 + \beta_1 e^{-x_{n-1}}}{\gamma_1 + y_n}, \ y_{n+1} = \frac{\alpha_2 + \beta_2 e^{-y_{n-1}}}{\gamma_2 + x_n}, \tag{3}$$

$$x_{n+1} = \frac{\alpha_1 + \beta_1 e^{-y_{n-1}}}{\gamma_1 + x_n}, \ y_{n+1} = \frac{\alpha_2 + \beta_2 e^{-x_{n-1}}}{\gamma_2 + y_n}, \tag{4}$$

where the parameters $\alpha_i,\ \beta_i,\ \gamma_i$ for $i \in \{1, 2\}$ and the initial conditions x_{-1}, x_0, y_{-1}, y_0 are positive real numbers.

More precisely, we investigate the boundedness character, persistence, existence of invariant rectangle, local asymptotic stability and global behavior of unique positive equilibrium point, and rate of convergence of positive solutions of system 3 and 4 which converges to its unique positive equilibrium point. For applications and basic theory of difference equations we refer to [1, 6, 10, 11, 19].

2. **Preliminaries.**

In this section, we present some definitions and theorems which are used throughout this study.

Let us consider fourth-dimensional discrete dynamical system of the following form:

$$x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}), \quad y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}),\ n = 0, 1, \ldots. \tag{5}$$

where $f : I^2 \times J^2 \to I$ and $g : I^2 \times J^2 \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{x_n, y_n\}_{n=-1}^{\infty}$ of system 5 is uniquely determined by initial conditions $(x_i, y_i) \in I \times J$ for $i \in \{-1, 0\}$.

Along with system 5, we consider the corresponding vector map $F = (f, x_n, g, y_n)$. An equilibrium point of 5 is a point $(\overline{x}, \overline{y})$ that sitisfies

$$\overline{x} = f(\overline{x}, \overline{x}, \overline{y}, \overline{y}), \quad \overline{y} = g(\overline{x}, \overline{x}, \overline{y}, \overline{y}).$$

The point $(\overline{x}, \overline{x}, \overline{y}, \overline{y})$ is also called a fixed point of the vector map F.

Definition 2.1. Let $(\overline{x}, \overline{y})$ be an equilibrium point of system 5.

(i) An equilibrium point $(\overline{x}, \overline{y})$ is called stable if for any $\varepsilon > 0$ there is $\delta > 0$ such that for every initial conditions (x_{-1}, y_{-1}) and (x_0, y_0), if $||(x_{-1}, y_{-1}) - (\overline{x}, \overline{y})|| + ||(x_0, y_0) - (\overline{x}, \overline{y})|| < \delta$ implies that $||(x_n, y_n) - (\overline{x}, \overline{y})|| < \varepsilon$ for all $n > 0$, where $||.||$ is usual Euclidean norm in $\mathbb{R}^2$.

(ii) An equilibrium point $(\overline{x}, \overline{y})$ is called unstable if it is not stable.

(iii) An equilibrium point $(\overline{x}, \overline{y})$ is called locally asymptotically stable if it stable and if, in addition, there exists $r > 0$ such that $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$ for all (x_{-1}, y_{-1}) and (x_0, y_0) that satisfy $||(x_{-1}, y_{-1}) - (\overline{x}, \overline{y})|| + ||(x_0, y_0) - (\overline{x}, \overline{y})|| < r$.

(iv) An equilibrium point $(\overline{x}, \overline{y})$ is called global attractor if $(x_n, y_n) \to (\overline{x}, \overline{y})$ as $n \to \infty$.

(v) An equilibrium point $(\overline{x}, \overline{y})$ is called globally asymptotically stable if it stable and a global attractor.

Definition 2.2. Let $(\overline{x}, \overline{x}, \overline{y}, \overline{y})$ be a fixed point of a map $F = (f, x_n, g, y_n)$ where f and g are continuously differentiable functions at $(\overline{x}, \overline{y})$. The linearized system of 5 about the equilibrium point $(\overline{x}, \overline{y})$ is

$$X_{n+1} = J_F X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$ and J_F is the Jacobian matrix of system 5 about the equilibrium point $(\overline{x}, \overline{y})$.

Lemma 2.3. *(see [19]) Assume that $X_{n+1} = F(X_n), n = 0, 1, ...,$ is a system of difference equations such that $\overline{X}$ is a fixed point of F. If all eigenvalues of the Jacobian matrix J_F about $\overline{X}$ lie inside the open unit disk $|\lambda| < 1$, then $\overline{X}$ is locally asymptotically stable. If one of them has a modulus greater than one, then $\overline{X}$ is unstable.*

Definition 2.4. A positive solution $\{x_n, y_n\}_{n=-1}^{\infty}$ of system 5 is bounded and persists if there exist positive constants m, M and an interger $N \geq -1$ such that

$$m \leq x_n, y_n \leq M, \quad n \geq N.$$

In order to study the asymptotic behavior of positive equilibrium, we state the following lemma which is a slight modification of Theorem 1.16 of [6] and for readers convenience we state it without its proof.

Lemma 2.5. *Assume that $f : (0, \infty) \times (0, \infty) \to (0, \infty)$ and $g : (0, \infty) \times (0, \infty) \to (0, \infty)$ be continuous functions and a, b, c, d are positive real numbers with $a < b, c < d$. Moreover, suppose that $f : [a, b] \times [c, d] \to [a, b]$ and $g : [a, b] \times [c, d] \to [c, d]$ such that following conditions are satisfied:*

(i) $f(x, y), g(x, y)$ are decreasing with respect to x (resp. y) for all y (resp. x);

(ii) Let m_1, M_1, m_2, M_2 are real numbers such that

$$m_1 = f(M_1, M_2), M_1 = f(m_1, m_2), m_2 = g(M_1, M_2), M_2 = g(m_1, m_2) \qquad (6)$$

then $m_1 = M_1$ *and* $m_2 = M_2$.

Then the systems of difference equations

$$x_{n+1} = f(x_{n-1}, y_n),\ y_{n+1} = g(x_n, y_{n-1}), \tag{7}$$

$$x_{n+1} = f(x_n, y_{n-1}),\ y_{n+1} = g(x_{n-1}, y_n) \tag{8}$$

have a unique equilibrium point $(\overline{x}, \overline{y})$ *and every solution* (x_n, y_n) *of the system 7 (resp. 8) with* $x_{-1}, x_0 \in [a, b], y_{-1}, y_0 \in [c, d]$ *converges to the unique equilibrium* $(\bar{x}, \bar{y})$.

The following results give the rate of convergence of solutions of a system of difference equations

$$X_{n+1} = [A + B(n)]X_n \tag{9}$$

where X_n is a m-dimensional vector, $A \in \mathbf{C}^{m\times m}$ is a constant matrix, and $B : \mathbb{Z}^+ \longrightarrow \mathbf{C}^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \text{ when } n \to \infty, \tag{10}$$

where $||.||$ denotes any matrix norm which is associated with the vector norm

$$\|(x,\ y)\| = \sqrt{x^2 + y^2}.$$

Proposition 2.6 (Perron's theorem [18]). *Assume that condition 10 holds. If* X_n *is a solution of system 9, then either* $X_n = 0$ *for all large* n *or*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|X_n\|} \tag{11}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Proposition 2.7 (See [18]). *Assume that condition 10 holds. If* X_n *is a solution of system 9, then either* $X_n = 0$ *for all large* n *or*

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{12}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

3. Main results.

3.1. Global behavior of solutions of system 3.

3.1.1. *Boundedness and persistence.* In this section, we show the boundedness and persistence of the positive solutions of system 3.

Lemma 3.1. *Every positive solution* $\{(x_n, y_n)\}$ *of system 3 is bounded and persists.*

Proof. For any positive solution $\{(x_n, y_n)\}$ of system 3, one has

$$x_{n+1} \le \frac{\alpha_1 + \beta_1}{\gamma_1} = b_1, \quad y_{n+1} \le \frac{\alpha_2 + \beta_2}{\gamma_2} = d_1,\ n = 0, 1, 2, \dots \tag{13}$$

Furthermore, from system 3 and 13, we obtain that

$$x_{n+1} \ge \frac{\alpha_1 + \beta_1 e^{-b_1}}{\gamma_1 + d_1} = a_1, \quad y_{n+1} \ge \frac{\alpha_2 + \beta_2 e^{-d_1}}{\gamma_2 + b_1} = c_1,\ n = 2, 3, 4, \dots \tag{14}$$

From 13 and 14, it follows that

$$a_1 \le x_n \le b_1, \quad c_1 \le y_n \le d_1,\ n = 3, 4, 5, \dots$$

So the proof is complete. □

Lemma 3.2. *Let $\{(x_n, y_n)\}$ be a positive solution of system 3. Then $[a_1, b_1]\times[c_1, d_1]$ is an invariant set for system 3.*

Proof. The proof follows by induction. □

3.1.2. *Stability analysis.* In this section, we shall investigate the asymptotic behavior of system 3. Similar method can be found in [9].
Let $(\overline{x}, \overline{y})$ be the equilibrium point of system 3 then

$$\overline{x} = \frac{\alpha_1 + \beta_1 e^{-\overline{x}}}{\gamma_1 + \overline{y}}, \quad \overline{y} = \frac{\alpha_2 + \beta_2 e^{-\overline{y}}}{\gamma_2 + \overline{x}}.$$

The linearized form of system 3 about the equilibrium point $(\overline{x}, \overline{y})$ is given by

$$X_{n+1} = J_F(\overline{x}, \overline{y})X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$ and $J_F(\overline{x}, \overline{y}) = \begin{pmatrix} 0 & -\frac{\beta_1 e^{-\overline{x}}}{\gamma_1 + \overline{y}} & -\frac{\overline{x}}{\gamma_1 + \overline{y}} & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{\overline{y}}{\gamma_2 + \overline{x}} & 0 & 0 & -\frac{\beta_2 e^{-\overline{y}}}{\gamma_2 + \overline{x}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

In the following theorem, we show the asymptotic behavior of the positive solutions of system 3.

Theorem 3.3. *Suppose that the following relation holds true:*

$$\beta_1 < \gamma_1, \quad \beta_2 < \gamma_2. \tag{15}$$

Then system 3 has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every positive solution of system 3 tends to the unique positive equilibrium as $n \to \infty$.

Proof. Consider the following functions:

$$f(x, y) = \frac{\alpha_1 + \beta_1 e^{-x}}{\gamma_1 + y}, \quad g(x, y) = \frac{\alpha_2 + \beta_2 e^{-y}}{\gamma_2 + x},$$

where $x \in I_1 = [a_1, b_1], y \in I_2 = [c_1, d_1]$ which implies that $f(x, y) \in I_1, g(x, y) \in I_2$ and so that $f : I_1 \times I_2 \to I_1, g : I_1 \times I_2 \to I_2$. Then, it is easy to see that $f(x, y), g(x, y)$ are decreasing with respect to x (resp. y) for all y (resp. x). Let (m, M, r, R) be a solution of the system

$$m = f(M, R), \quad M = f(m, r),$$
$$r = g(M, R), \quad R = g(m, r).$$

Then, one has

$$m = \frac{\alpha_1 + \beta_1 e^{-M}}{\gamma_1 + R}, \; M = \frac{\alpha_1 + \beta_1 e^{-m}}{\gamma_1 + r}, \; r = \frac{\alpha_2 + \beta_2 e^{-R}}{\gamma_2 + M}, \; R = \frac{\alpha_2 + \beta_2 e^{-r}}{\gamma_2 + m}. \tag{16}$$

Moreover arguing as in the proof of Theorem 1.16 of [6], it suffices to assume that

$$m \le M, \quad r \le R. \tag{17}$$

From 16, we get

$$\beta_1 e^{-m} = M(\gamma_1 + r) - \alpha_1, \quad \beta_1 e^{-M} = m(\gamma_1 + R) - \alpha_1,$$
$$\beta_2 e^{-r} = R(\gamma_2 + m) - \alpha_2, \quad \beta_2 e^{-R} = r(\gamma_2 + M) - \alpha_2,$$

which imply that

$$\begin{aligned}\beta_1(e^{-m}-e^{-M}) &= \gamma_1(M-m)+Mr-mR,\\ \beta_2(e^{-r}-e^{-R}) &= \gamma_2(R-r)+mR-Mr.\end{aligned} \tag{18}$$

Then by adding the two relations 18, we obtain

$$\beta_1(e^{-m}-e^{-M})+\beta_2(e^{-r}-e^{-R})=\gamma_1(M-m)+\gamma_2(R-r).$$

Moreover, we get

$$\begin{aligned}e^R-e^r &= e^{\xi}(R-r),\ \min\{R,r\}\le\xi\le\max\{R,r\},\\ e^M-e^m &= e^{\theta}(M-m),\ \min\{M,m\}\le\theta\le\max\{M,m\}.\end{aligned} \tag{19}$$

Then from 19, imply that

$$\beta_1 e^{-m-M+\theta}(M-m)+\beta_2 e^{-r-R+\xi}(R-r)=\gamma_1(M-m)+\gamma_2(R-r). \tag{20}$$

Hence from 20, we have

$$\gamma_1(M-m)\left(1-\frac{\beta_1}{\gamma_1}e^{-m-M+\theta}\right)+\gamma_2(R-r)\left(1-\frac{\beta_2}{\gamma_2}e^{-r-R+\xi}\right)=0. \tag{21}$$

Finally, from 15, 17 and 21, it follows that $M=m$ and $R=r$. Therefore, from Lemma 2.5, it follows that system 3 has a unique positive equilibrium $(\overline{x},\overline{y})$ and every positive solution of system 3 tends to the unique positive equilibrium as $n\to\infty$. This completes the proof of the theorem. □

In the next theorem of this section, we will study the global asymptotic stability of the positive equilibrium of system 3.

Theorem 3.4. *Consider system 3 where 15 holds true. Also suppose that*

$$\frac{\beta_1 e^{-a_1}}{\gamma_1+c_1}+\frac{\beta_2 e^{-c_1}}{\gamma_2+a_1}+\frac{b_1d_1+\beta_1\beta_2 e^{-a_1-c_1}}{(\gamma_1+c_1)(\gamma_2+a_1)}<1. \tag{22}$$

Then the unique positive equilibrium point $(\overline{x},\overline{y})$ of system 3 is globally asymptotically stable.

Proof. First we will prove that $(\overline{x},\overline{y})$ is locally asymptotically stable. The charateristic equation of the Jacobian matrix $J_F(\overline{x},\overline{y})$ about $(\overline{x},\overline{y})$ is given by

$$\lambda^4+p_2\lambda^2+p_4=0, \tag{23}$$

where

$$p_2=\frac{\beta_1 e^{-\overline{x}}}{\gamma_1+\overline{y}}+\frac{\beta_2 e^{-\overline{y}}}{\gamma_2+\overline{x}}-\frac{\overline{x}.\overline{y}}{(\gamma_1+\overline{y})(\gamma_2+\overline{x})},$$
$$p_4=\frac{\beta_1 e^{-\overline{x}}}{\gamma_1+\overline{y}}.\frac{\beta_2 e^{-\overline{y}}}{\gamma_2+\overline{x}}.$$

From condition 22, we get

$$\begin{aligned}|p_2|+|p_4| &= \frac{\beta_1 e^{-\overline{x}}}{\gamma_1+\overline{y}}+\frac{\beta_2 e^{-\overline{y}}}{\gamma_2+\overline{x}}+\frac{\overline{x}.\overline{y}}{(\gamma_1+\overline{y})(\gamma_2+\overline{x})}+\frac{\beta_1 e^{-\overline{x}}}{\gamma_1+\overline{y}}.\frac{\beta_2 e^{-\overline{y}}}{\gamma_2+\overline{x}}\\ &\le \frac{\beta_1 e^{-a_1}}{\gamma_1+c_1}+\frac{\beta_2 e^{-c_1}}{\gamma_2+a_1}+\frac{b_1d_1+\beta_1\beta_2 e^{-a_1-c_1}}{(\gamma_1+c_1)(\gamma_2+a_1)}<1.\end{aligned}$$

Therefore, follows Remark 1.3.1 of reference [10], all the roots of equation 23 are of modulus less than 1, and it follows from Lemma 2.3 that the unique positive equilibrium point $(\overline{x},\overline{y})$ of system 3 is locally asymptotically stable. Using Theorem

3.3, we obtain that $(\overline{x}, \overline{y})$ is globally asymptotically stable. This completes the proof of the theorem. □

3.1.3. *Rate of convergence.* In this section, we give the rate of convergence of a solution that converges to the equilibrium of the systems 3. Similar method can be found in [8, 9].

Let $\{(x_n, y_n)\}$ be an arbitrary solution of system 3 such that $\lim_{n\to\infty} x_n = \overline{x}$, and $\lim_{n\to\infty} y_n = \overline{y}$, where $\overline{x} \in [a_1, b_1]$, and $\overline{y} \in [c_1, d_1]$. To find the error terms, one has from the system 3

$$\begin{aligned}
x_{n+1} - \overline{x} &= \frac{\alpha_1 + \beta_1 e^{-x_{n-1}}}{\gamma_1 + y_n} - \frac{\alpha_1 + \beta_1 e^{-\overline{x}}}{\gamma_1 + \overline{y}} \\
&= \frac{(\alpha_1 + \beta_1 e^{-x_{n-1}})(\gamma_1 + \overline{y}) - (\alpha_1 + \beta_1 e^{-\overline{x}})(\gamma_1 + y_n)}{(\gamma_1 + y_n)(\gamma_1 + \overline{y})} \\
&= \frac{-\alpha_1(y_n - \overline{y}) + \beta_1\gamma_1(e^{-x_{n-1}} - e^{-\overline{x}}) + \beta_1(e^{-x_{n-1}}\overline{y} - e^{-\overline{x}}y_n)}{(\gamma_1 + y_n)(\gamma_1 + \overline{y})} \\
&= \frac{-\alpha_1(y_n - \overline{y}) - \beta_1\gamma_1 e^{-x_{n-1}}(e^{x_{n-1}-\overline{x}} - 1)}{(\gamma_1 + y_n)(\gamma_1 + \overline{y})} \\
&+ \frac{\beta_1(e^{-x_{n-1}}\overline{y} - e^{-x_{n-1}}y_n + e^{-x_{n-1}}y_n - e^{-\overline{x}}y_n)}{(\gamma_1 + y_n)(\gamma_1 + \overline{y})} \\
&= -\frac{\beta_1 e^{-x_{n-1}}(e^{x_{n-1}-\overline{x}} - 1)}{(\gamma_1 + \overline{y})(x_{n-1} - \overline{x})}(x_{n-1} - \overline{x}) - \frac{\alpha_1 + \beta_1 e^{-x_{n-1}}}{(\gamma_1 + y_n)(\gamma_1 + \overline{y})}(y_n - \overline{y}),
\end{aligned}$$

and

$$\begin{aligned}
y_{n+1} - \overline{y} &= \frac{\alpha_2 + \beta_2 e^{-y_{n-1}}}{\gamma_2 + x_n} - \frac{\alpha_2 + \beta_2 e^{-\overline{y}}}{\gamma_2 + \overline{x}} \\
&= \frac{(\alpha_2 + \beta_2 e^{-y_{n-1}})(\gamma_2 + \overline{x}) - (\alpha_2 + \beta_2 e^{-\overline{y}})(\gamma_2 + x_n)}{(\gamma_2 + x_n)(\gamma_2 + \overline{x})} \\
&= \frac{-\alpha_2(x_n - \overline{x}) + \beta_2\gamma_2(e^{-y_{n-1}} - e^{-\overline{y}}) + \beta_2(e^{-y_{n-1}}\overline{x} - e^{-\overline{y}}x_n)}{(\gamma_2 + x_n)(\gamma_2 + \overline{x})} \\
&= \frac{-\alpha_2(x_n - \overline{x}) - \beta_2\gamma_2 e^{-y_{n-1}}(e^{y_{n-1}-\overline{y}} - 1)}{(\gamma_2 + x_n)(\gamma_2 + \overline{x})} \\
&+ \frac{\beta_2(e^{-y_{n-1}}\overline{x} - e^{-y_{n-1}}x_n + e^{-y_{n-1}}x_n - e^{-\overline{y}}x_n)}{(\gamma_2 + x_n)(\gamma_2 + \overline{x})} \\
&= -\frac{\alpha_2 + \beta_2 e^{-y_{n-1}}}{(\gamma_2 + x_n)(\gamma_2 + \overline{x})}(x_n - \overline{x}) - \frac{\beta_2 e^{-y_{n-1}}(e^{y_{n-1}-\overline{y}} - 1)}{(\gamma_2 + \overline{x})(y_{n-1} - \overline{y})}(y_{n-1} - \overline{y}).
\end{aligned}$$

Let $e_n^1 = x_n - \overline{x}$, and $e_n^2 = y_n - \overline{y}$, then one has

$$\begin{aligned}
e_{n+1}^1 &= a_n e_{n-1}^1 + b_n e_n^2, \\
e_{n+1}^2 &= c_n e_n^1 + d_n e_{n-1}^2,
\end{aligned}$$

where

$$\begin{aligned}
a_n &= -\frac{\beta_1 e^{-x_{n-1}}(e^{x_{n-1}-\overline{x}} - 1)}{(\gamma_1 + \overline{y})(x_{n-1} - \overline{x})}, \quad b_n = -\frac{\alpha_1 + \beta_1 e^{-x_{n-1}}}{(\gamma_1 + y_n)(\gamma_1 + \overline{y})}, \\
c_n &= -\frac{\alpha_2 + \beta_2 e^{-y_{n-1}}}{(\gamma_2 + x_n)(\gamma_2 + \overline{x})}, \quad d_n = -\frac{\beta_2 e^{-y_{n-1}}(e^{y_{n-1}-\overline{y}} - 1)}{(\gamma_2 + \overline{x})(y_{n-1} - \overline{y})}.
\end{aligned}$$

Moreover,

$$\lim_{n\to\infty} a_n = -\frac{\beta_1 e^{-\overline{x}}}{\gamma_1+\overline{y}}, \quad \lim_{n\to\infty} b_n = -\frac{\overline{x}}{\gamma_1+\overline{y}},$$
$$\lim_{n\to\infty} c_n = -\frac{\overline{y}}{\gamma_2+\overline{x}}, \quad \lim_{n\to\infty} d_n = -\frac{\beta_2 e^{-\overline{y}}}{\gamma_2+\overline{x}}.$$

So, the limiting system of the error terms can be written as

$$\begin{pmatrix} e^1_{n+1} \\ e^1_n \\ e^2_{n+1} \\ e^2_n \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\beta_1 e^{-\overline{x}}}{\gamma_1+\overline{y}} & -\frac{\overline{x}}{\gamma_1+\overline{y}} & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{\overline{y}}{\gamma_2+\overline{x}} & 0 & 0 & -\frac{\beta_2 e^{-\overline{y}}}{\gamma_2+\overline{x}} \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^1_n \\ e^1_{n-1} \\ e^2_n \\ e^2_{n-1} \end{pmatrix}$$

which similar to the linearized system of 3 about the equilibrium point $(\overline{x},\overline{y})$. Using Proposition 2.6 and 2.7, one has the following result.

Theorem 3.5. *Assume that $\{(x_n,y_n)\}$ be a positive solution of system 3 such that $\lim_{n\to\infty} x_n = \overline{x}$, and $\lim_{n\to\infty} y_n = \overline{y}$, where $\overline{x}\in[a_1,b_1]$ and $\overline{y}\in[c_1,d_1]$. Then the error vector $e_n = \begin{pmatrix} e^1_n \\ e^1_{n-1} \\ e^2_n \\ e^2_{n-1} \end{pmatrix}$ of every solution of (3) satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} (||e_n||)^{\frac{1}{n}} = |\lambda_i|, \quad \lim_{n\to\infty} \frac{||e_{n+1}||}{||e_n||} = |\lambda_i|, \quad i=1,2,3,4,$$

where λ_i is one of the charateristic roots of Jacobian matrix $J_F(\overline{x},\overline{y})$.

3.2. Global behavior of solutions of system 4.

3.2.1. *Boundedness and persistence.* In the following lemma, we study the boundedness and persistence of the positive solutions of system 4.

Lemma 3.6. *Every positive solution $\{(x_n,y_n)\}$ of system 4 is bounded and persists.*

Proof. Let $\{(x_n,y_n)\}$ be a positive solution of system 4. Similarly as Lemma 3.1, for $n=3,4,5,\ldots$ by induction, we get

$$x_n\in[a_2,b_2], \quad y_n\in[c_2,d_2],$$

where

$$a_2 = \frac{\alpha_1+\beta_1 e^{-\frac{\alpha_2+\beta_2}{\gamma_2}}}{\gamma_1+\frac{\alpha_1+\beta_1}{\gamma_1}}, \quad b_2 = \frac{\alpha_1+\beta_1}{\gamma_1},$$
$$c_2 = \frac{\alpha_2+\beta_2 e^{-\frac{\alpha_1+\beta_1}{\gamma_1}}}{\gamma_2+\frac{\alpha_2+\beta_2}{\gamma_2}}, \quad d_2 = \frac{\alpha_2+\beta_2}{\gamma_2}.$$

So the proof is complete. □

Corollary 3.7. *Let $\{(x_n,y_n)\}$ be a positive solution of system 4. Then $[a_2,b_2]\times[c_2,d_2]$ is an invariant set for system 4.*

3.2.2. *Stability analysis.* In this section, we shall investigate the asymptotic behavior of system 4. Let $(\overline{x}, \overline{y})$ be the equilibrium point of system 4 then

$$\overline{x} = \frac{\alpha_1 + \beta_1 e^{-\overline{y}}}{\gamma_1 + \overline{x}}, \quad \overline{y} = \frac{\alpha_2 + \beta_2 e^{-\overline{x}}}{\gamma_2 + \overline{y}}.$$

The linearized form of system 4 about the equilibrium point $(\overline{x}, \overline{y})$ is given by

$$X_{n+1} = J_F(\overline{x}, \overline{y}) X_n,$$

where $X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ y_n \\ y_{n-1} \end{pmatrix}$ and $J_F(\overline{x}, \overline{y}) = \begin{pmatrix} -\frac{\overline{x}}{\gamma_1 + \overline{x}} & 0 & 0 & -\frac{\beta_1 e^{-\overline{y}}}{\gamma_1 + \overline{x}} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{\beta_2 e^{-\overline{x}}}{\gamma_2 + \overline{y}} & -\frac{\overline{y}}{\gamma_2 + \overline{y}} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

In the following theorem, we show the asymptotic behavior of the positive solutions of system 4.

Theorem 3.8. *Suppose that the following relation holds true:*

$$\beta_1 \beta_2 < \gamma_1 \gamma_2. \tag{24}$$

Then system 4 has a unique positive equilibrium $(\overline{x}, \overline{y})$ and every positive solution of system 4 tends to the unique positive equilibrium as $n \to \infty$.

Proof. Consider the following functions:

$$f(x, y) = \frac{\alpha_1 + \beta_1 e^{-y}}{\gamma_1 + x}, \quad g(x, y) = \frac{\alpha_2 + \beta_2 e^{-x}}{\gamma_2 + y},$$

where $x \in I_3 = [a_2, b_2], y \in I_4 = [c_2, d_2]$ which implies that $f(x, y) \in I_3, g(x, y) \in I_4$ and so that $f : I_3 \times I_4 \to I_3, g : I_3 \times I_4 \to I_4$. Then, it is easy to see that $f(x, y), g(x, y)$ are decreasing with respect to x (resp. y) for all y (resp. x). Let (m, M, r, R) be a solution of the system

$$\begin{aligned} m = f(M, R), \quad M = f(m, r), \\ r = g(M, R), \quad R = g(m, r). \end{aligned}$$

Then, one has

$$m = \frac{\alpha_1 + \beta_1 e^{-R}}{\gamma_1 + M}, \; M = \frac{\alpha_1 + \beta_1 e^{-r}}{\gamma_1 + m}, \; r = \frac{\alpha_2 + \beta_2 e^{-M}}{\gamma_2 + R}, \; R = \frac{\alpha_2 + \beta_2 e^{-m}}{\gamma_2 + r}. \tag{25}$$

From 25, we get

$$\begin{aligned} \beta_1 e^{-r} = M(\gamma_1 + m) - \alpha_1, \quad \beta_1 e^{-R} = m(\gamma_1 + M) - \alpha_1, \\ \beta_2 e^{-m} = R(\gamma_2 + r) - \alpha_2, \quad \beta_2 e^{-M} = r(\gamma_2 + R) - \alpha_2, \end{aligned}$$

which imply that

$$\begin{aligned} \beta_1 (e^{-r} - e^{-R}) = \gamma_1 (M - m), \\ \beta_2 (e^{-m} - e^{-M}) = \gamma_2 (R - r). \end{aligned} \tag{26}$$

Moreover, we get

$$\begin{aligned} e^R - e^r = e^{\xi}(R - r), \; \min\{R, r\} \le \xi \le \max\{R, r\}, \\ e^M - e^m = e^{\theta}(M - m), \; \min\{M, m\} \le \theta \le \max\{M, m\}. \end{aligned} \tag{27}$$

Then from 26 and 27, we have

$$\begin{aligned} M-m &= \frac{\beta_1}{\gamma_1}(e^{-r}-e^{-R}) = \frac{\beta_1}{\gamma_1}e^{-r-R}(e^R-e^r) = \frac{\beta_1}{\gamma_1}e^{-r-R+\xi}(R-r), \\ R-r &= \frac{\beta_2}{\gamma_2}(e^{-m}-e^{-M}) = \frac{\beta_2}{\gamma_2}e^{-m-M}(e^M-e^m) = \frac{\beta_2}{\gamma_2}e^{-m-M+\theta}(M-m), \end{aligned} \tag{28}$$

and so

$$|M-m| \le \frac{\beta_1}{\gamma_1}|R-r|, \quad |R-r| \le \frac{\beta_2}{\gamma_2}|M-m|.$$

Therefore, we get

$$\left(1-\frac{\beta_1\beta_2}{\gamma_1\gamma_2}\right)|M-m| \le 0, \quad \left(1-\frac{\beta_1\beta_2}{\gamma_1\gamma_2}\right)|R-r| \le 0. \tag{29}$$

Finally, from 24 and 29, it follows that $M=m$ and $R=r$. Therefore, from Lemma 2.5, it follows that system 4 has a unique positive equilibrium $(\overline{x},\overline{y})$ and every positive solution of system 4 tends to the unique positive equilibrium as $n\to\infty$. This completes the proof of the theorem. □

In the next theorem of this section, we will study the global asymptotic stability of the positive equilibrium of system 4.

Theorem 3.9. *Consider system 4 where 24 holds true. Also suppose that*

$$\frac{b_2}{\gamma_1+a_2}+\frac{d_2}{\gamma_2+c_2}+\frac{b_2d_2+\beta_1\beta_2e^{-a_2-c_2}}{(\gamma_1+a_2)(\gamma_2+c_2)}<1. \tag{30}$$

Then the unique positive equilibrium point $(\overline{x},\overline{y})$ of system 4 is globally asymptotically stable.

Proof. First we will prove that $(\overline{x},\overline{y})$ is locally asymptotically stable. The charateristic equation of the Jacobian matrix $J_F(\overline{x},\overline{y})$ about $(\overline{x},\overline{y})$ is given by

$$\lambda^4+q_1\lambda^3+q_2\lambda^2+q_4=0, \tag{31}$$

where

$$\begin{aligned} q_1 &= \frac{\overline{x}}{\gamma_1+\overline{x}}+\frac{\overline{y}}{\gamma_2+\overline{y}}, \\ q_2 &= \frac{\overline{x}.\overline{y}}{(\gamma_1+\overline{x})(\gamma_2+\overline{y})}, \\ q_4 &= -\frac{\beta_1e^{-\overline{y}}}{\gamma_1+\overline{x}}.\frac{\beta_2e^{-\overline{x}}}{\gamma_2+\overline{y}}. \end{aligned}$$

From condition 30, we get

$$\begin{aligned} |q_1|+|q_2|+|q_4| &= \frac{\overline{x}}{\gamma_1+\overline{x}}+\frac{\overline{y}}{\gamma_2+\overline{y}}+\frac{\overline{x}.\overline{y}}{(\gamma_1+\overline{x})(\gamma_2+\overline{y})}+\frac{\beta_1e^{-\overline{y}}}{\gamma_1+\overline{x}}.\frac{\beta_2e^{-\overline{x}}}{\gamma_2+\overline{y}} \\ &\le \frac{b_2}{\gamma_1+a_2}+\frac{d_2}{\gamma_2+c_2}+\frac{b_2d_2+\beta_1\beta_2e^{-a_2-c_2}}{(\gamma_1+a_2)(\gamma_2+c_2)}<1. \end{aligned}$$

Therefore, follows Remark 1.3.1 of reference [10], all the roots of equation 31 are of modulus less than 1, and it follows from Lemma 2.3 that the unique positive equilibrium point $(\overline{x},\overline{y})$ of system 4 is locally asymptotically stable. Using Theorem 3.8, we obtain that $(\overline{x},\overline{y})$ is globally asymptotically stable. This completes the proof of the theorem. □

3.2.3. *Rate of convergence.* In this section, we give the rate of convergence of a solution that converges to the equilibrium of the systems 4.

Let $\{(x_n, y_n)\}$ be an arbitrary solution of system 4 such that $\lim_{n\to\infty} x_n = \overline{x}$, and $\lim_{n\to\infty} y_n = \overline{y}$, where $\overline{x} \in [a_2, b_2]$, and $\overline{y} \in [c_2, d_2]$. To find the error terms, one has from the system 4

$$\begin{aligned}
x_{n+1} - \overline{x} &= \frac{\alpha_1 + \beta_1 e^{-y_{n-1}}}{\gamma_1 + x_n} - \frac{\alpha_1 + \beta_1 e^{-\overline{y}}}{\gamma_1 + \overline{x}} \\
&= \frac{(\alpha_1 + \beta_1 e^{-y_{n-1}})(\gamma_1 + \overline{x}) - (\alpha_1 + \beta_1 e^{-\overline{y}})(\gamma_1 + x_n)}{(\gamma_1 + x_n)(\gamma_1 + \overline{x})} \\
&= \frac{-\alpha_1(x_n - \overline{x}) + \beta_1\gamma_1(e^{-y_{n-1}} - e^{-\overline{y}}) + \beta_1(e^{-y_{n-1}}\overline{x} - e^{-\overline{y}}x_n)}{(\gamma_1 + x_n)(\gamma_1 + \overline{x})} \\
&= \frac{-\alpha_1(x_n - \overline{x}) - \beta_1\gamma_1 e^{-y_{n-1}}(e^{y_{n-1}-\overline{y}} - 1)}{(\gamma_1 + x_n)(\gamma_1 + \overline{x})} \\
&+ \frac{\beta_1(e^{-y_{n-1}}\overline{x} - e^{-y_{n-1}}x_n + e^{-y_{n-1}}x_n - e^{-\overline{y}}x_n)}{(\gamma_1 + x_n)(\gamma_1 + \overline{x})} \\
&= -\frac{\alpha_1 + \beta_1 e^{-y_{n-1}}}{(\gamma_1 + x_n)(\gamma_1 + \overline{x})}(x_n - \overline{x}) - \frac{\beta_1 e^{-y_{n-1}}(e^{y_{n-1}-\overline{y}} - 1)}{(\gamma_1 + \overline{x})(y_{n-1} - \overline{y})}(y_{n-1} - \overline{y}),
\end{aligned}$$

and

$$\begin{aligned}
y_{n+1} - \overline{y} &= \frac{\alpha_2 + \beta_2 e^{-x_{n-1}}}{\gamma_2 + y_n} - \frac{\alpha_2 + \beta_2 e^{-\overline{x}}}{\gamma_2 + \overline{y}} \\
&= \frac{(\alpha_2 + \beta_2 e^{-x_{n-1}})(\gamma_2 + \overline{y}) - (\alpha_2 + \beta_2 e^{-\overline{x}})(\gamma_2 + y_n)}{(\gamma_2 + y_n)(\gamma_2 + \overline{y})} \\
&= \frac{-\alpha_2(y_n - \overline{y}) + \beta_2\gamma_2(e^{-x_{n-1}} - e^{-\overline{x}}) + \beta_2(e^{-x_{n-1}}\overline{y} - e^{-\overline{x}}y_n)}{(\gamma_2 + y_n)(\gamma_2 + \overline{y})} \\
&= \frac{-\alpha_2(y_n - \overline{y}) - \beta_2\gamma_2 e^{-x_{n-1}}(e^{x_{n-1}-\overline{x}} - 1)}{(\gamma_2 + y_n)(\gamma_2 + \overline{y})} \\
&+ \frac{\beta_2(e^{-x_{n-1}}\overline{y} - e^{-x_{n-1}}y_n + e^{-x_{n-1}}y_n - e^{-\overline{x}}y_n)}{(\gamma_2 + y_n)(\gamma_2 + \overline{y})} \\
&= -\frac{\beta_2 e^{-x_{n-1}}(e^{x_{n-1}-\overline{x}} - 1)}{(\gamma_2 + \overline{y})(x_{n-1} - \overline{x})}(x_{n-1} - \overline{x}) - \frac{\alpha_2 + \beta_2 e^{-x_{n-1}}}{(\gamma_2 + y_n)(\gamma_2 + \overline{y})}(y_n - \overline{y}).
\end{aligned}$$

Let $e_n^1 = x_n - \overline{x}$, and $e_n^2 = y_n - \overline{y}$, then one has

$$\begin{aligned}
e_{n+1}^1 &= f_n e_n^1 + g_n e_{n-1}^2, \\
e_{n+1}^2 &= h_n e_{n-1}^1 + k_n e_n^2,
\end{aligned}$$

where

$$\begin{aligned}
f_n &= -\frac{\alpha_1 + \beta_1 e^{-y_{n-1}}}{(\gamma_1 + x_n)(\gamma_1 + \overline{x})}, \quad g_n = -\frac{\beta_1 e^{-y_{n-1}}(e^{y_{n-1}-\overline{y}} - 1)}{(\gamma_1 + \overline{x})(y_{n-1} - \overline{y})}, \\
h_n &= -\frac{\beta_2 e^{-x_{n-1}}(e^{x_{n-1}-\overline{x}} - 1)}{(\gamma_2 + \overline{y})(x_{n-1} - \overline{x})}, \quad k_n = -\frac{\alpha_2 + \beta_2 e^{-x_{n-1}}}{(\gamma_2 + y_n)(\gamma_2 + \overline{y})}.
\end{aligned}$$

Moreover,

$$\lim_{n\to\infty} f_n = -\frac{\overline{x}}{\gamma_1 + \overline{x}}, \quad \lim_{n\to\infty} g_n = -\frac{\beta_1 e^{-\overline{y}}}{\gamma_1 + \overline{x}},$$

$$\lim_{n\to\infty} h_n = -\frac{\beta_2 e^{-\overline{x}}}{\gamma_2+\overline{y}}, \quad \lim_{n\to\infty} k_n = -\frac{\overline{y}}{\gamma_2+\overline{y}}.$$

So, the limiting system of the error terms can be written as

$$\begin{pmatrix} e^1_{n+1} \\ e^1_n \\ e^2_{n+1} \\ e^2_n \end{pmatrix} = \begin{pmatrix} -\frac{\overline{x}}{\gamma_1+\overline{x}} & 0 & 0 & -\frac{\beta_1 e^{-\overline{y}}}{\gamma_1+\overline{x}} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{\beta_2 e^{-\overline{x}}}{\gamma_2+\overline{y}} & -\frac{\overline{y}}{\gamma_2+\overline{y}} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e^1_n \\ e^1_{n-1} \\ e^2_n \\ e^2_{n-1} \end{pmatrix}$$

which similar to the linearized system of 4 about the equilibrium point $(\overline{x},\overline{y})$. Using Proposition 2.6 and 2.7, one has the following result.

Theorem 3.10. *Assume that* $\{(x_n,y_n)\}$ *be a positive solution of system 4 such that* $\lim_{n\to\infty} x_n = \overline{x}$, *and* $\lim_{n\to\infty} y_n = \overline{y}$, *where* $\overline{x}\in[a_2,b_2]$ *and* $\overline{y}\in[c_2,d_2]$. *Then the error vector* $e_n = \begin{pmatrix} e^1_n \\ e^1_{n-1} \\ e^2_n \\ e^2_{n-1} \end{pmatrix}$ *of every solution of 4 satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} (||e_n||)^{\frac{1}{n}} = |\lambda_i|, \quad \lim_{n\to\infty} \frac{||e_{n+1}||}{||e_n||} = |\lambda_i|, \quad i=1,2,3,4,$$

where λ_i *is one of the charateristic roots of Jacobian matrix* $J_F(\overline{x},\overline{y})$.

4. **Numerical simulations.** In an effort to affirm our theoretical dialogue, we consider several numerical examples. These examples represent different types of qualitative behavior of solutions of the systems 3 and 4. All plots in this section are drawn with MATLAB.

Example 4.1. Let $\alpha_1 = 30, \beta_1 = 1.4, \gamma_1 = 1.5, \alpha_2 = 45, \beta_2 = 2.5$, and $\gamma_2 = 2.8$. Then system 3 can be written as

$$x_{n+1} = \frac{30+1.4e^{-x_{n-1}}}{1.5+y_n}, \; y_{n+1} = \frac{45+2.5e^{-y_{n-1}}}{2.8+x_n}, \tag{32}$$

with initial conditions $x_{-1} = 0.59, x_0 = 0.61, y_{-1} = 0.96$, and $y_0 = 0.94$.

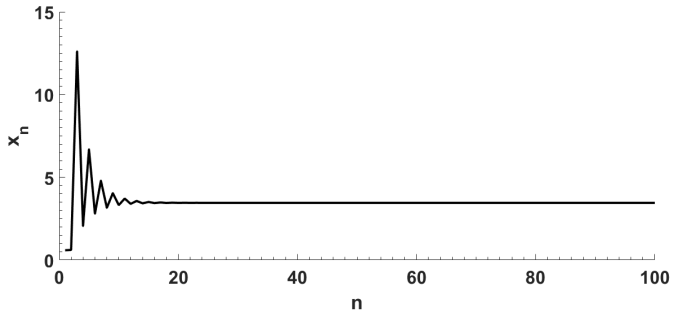

FIGURE 1. Plot of x_n for the system 32

In this case, the unique positive equilibrium point of the system 32 is given by $(\overline{x}, \overline{y}) = (3.455959, 7.193442)$. Moreover, the plot of x_n is shown in Figure 1, the plot of y_n is shown in Figure 2, and an attractor of the system 32 is shown in Figure 3.

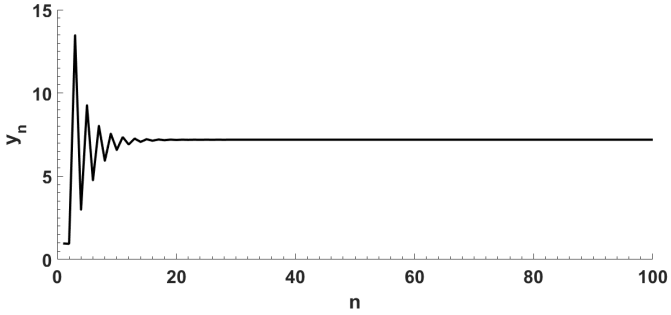

FIGURE 2. Plot of y_n for the system 32

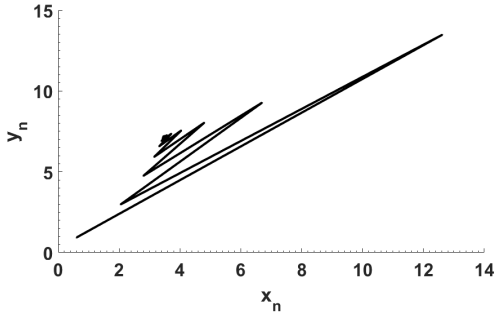

FIGURE 3. An attractor of the system 32

Example 4.2. Let $\alpha_1 = 0.2, \beta_1 = 19, \gamma_1 = 4, \alpha_2 = 0.3, \beta_2 = 20$, and $\gamma_2 = 2$. Then system 3 can be written as

$$x_{n+1} = \frac{0.2 + 19e^{-x_{n-1}}}{4 + y_n}, \ y_{n+1} = \frac{0.3 + 20e^{-y_{n-1}}}{2 + x_n}, \tag{33}$$

with initial conditions $x_{-1} = 1, x_0 = 1, y_{-1} = 3$, and $y_0 = 3$.

In this case, the unique positive equilibrium point of the system 33 is unstable. Moreover, the plot of x_n is shown in Figure 4, the plot of y_n is shown in Figure 5, and a phase portrait of system 33 is shown in Figure 6.

Example 4.3. Let $\alpha_1 = 401, \beta_1 = 1, \gamma_1 = 1.75, \alpha_2 = 395, \beta_2 = 1.5$, and $\gamma_2 = 1$. Then system 4 can be written as

$$x_{n+1} = \frac{401 + 1e^{-y_{n-1}}}{1.75 + x_n}, \ y_{n+1} = \frac{395 + 1.5e^{-x_{n-1}}}{1 + y_n}, \tag{34}$$

with initial conditions $x_{-1} = 15, x_0 = 24.5, y_{-1} = 15$, and $y_0 = 25$.

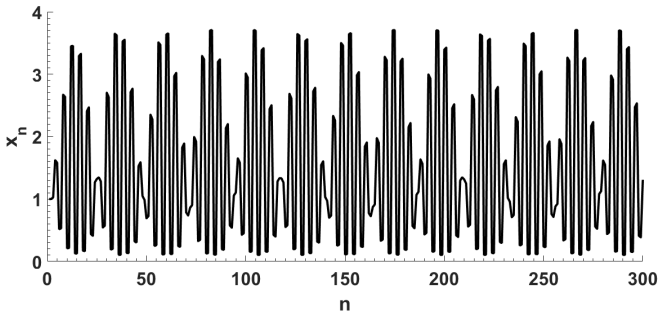

FIGURE 4. Plot of x_n for the system 33

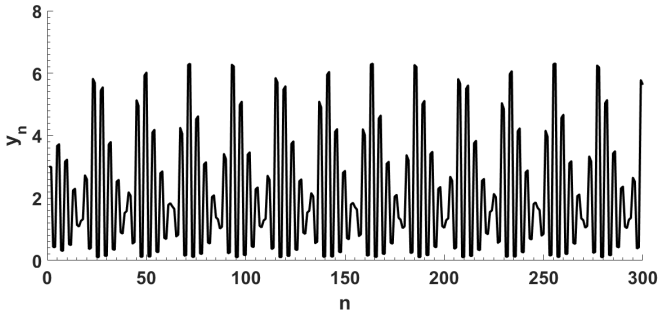

FIGURE 5. Plot of y_n for the system 33

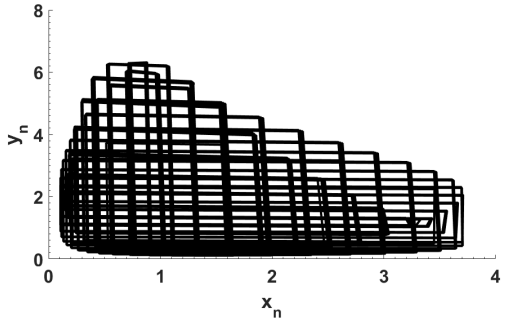

FIGURE 6. Phase portrait of system 33

In this case, the unique positive equilibrium point of the system 34 is given by $(\overline{x}, \overline{y}) = (19.169092, 19.380895)$. Moreover, the plot of x_n is shown in Figure 7, the plot of y_n is shown in Figure 8, and an attractor of the system 34 is shown in Figure 9.

Example 4.4. Let $\alpha_1 = 4, \beta_1 = 10, \gamma_1 = 1.6, \alpha_2 = 5.5, \beta_2 = 10.8$, and $\gamma_2 = 1$. Then system 4 can be written as

$$x_{n+1} = \frac{4 + 10e^{-y_{n-1}}}{1.6 + x_n}, \; y_{n+1} = \frac{5.5 + 10.8e^{-x_{n-1}}}{1 + y_n}, \tag{35}$$

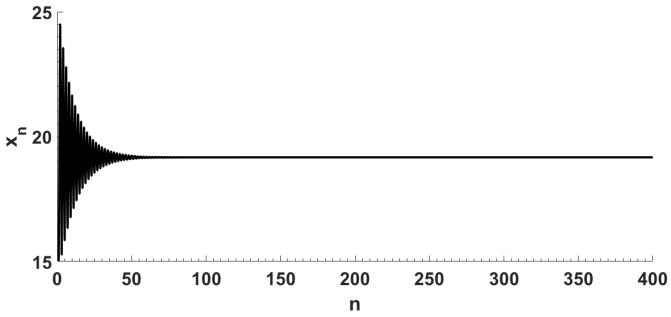

FIGURE 7. Plot of x_n for the system 34

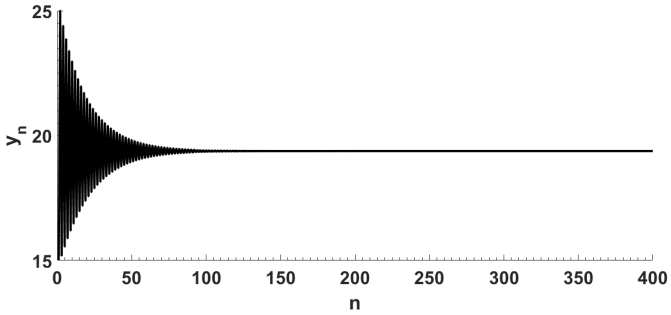

FIGURE 8. Plot of y_n for the system 34

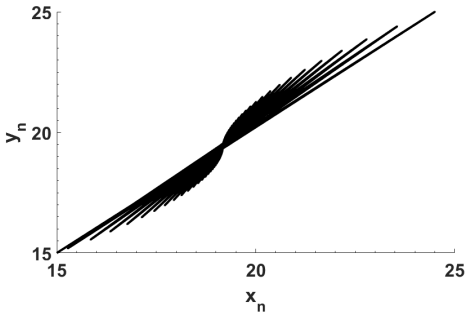

FIGURE 9. An attractor of the system 34

with initial conditions $x_{-1} = 1.3, x_0 = 1.5, y_{-1} = 2$, and $y_0 = 1.7$.

In this case, the unique positive equilibrium point of the system (35) is unstable. Moreover, the plot of x_n is shown in Figure 10, the plot of y_n is shown in Figure 11, and a phase portrait of system 35 is shown in Figure 12.

5. **Conclusion.** In this study, we investigate the qualitative behavior of some systems of exponential difference equations. We have proved the boundedness and

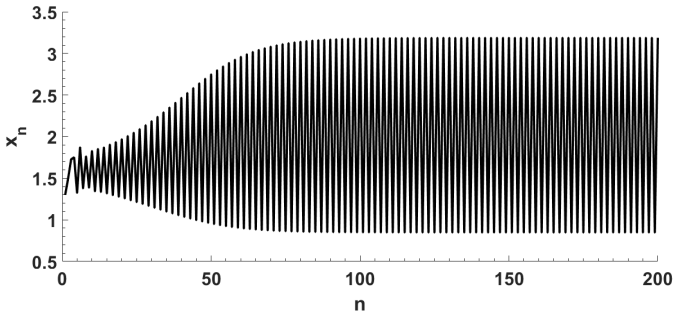

FIGURE 10. Plot of x_n for the system 35

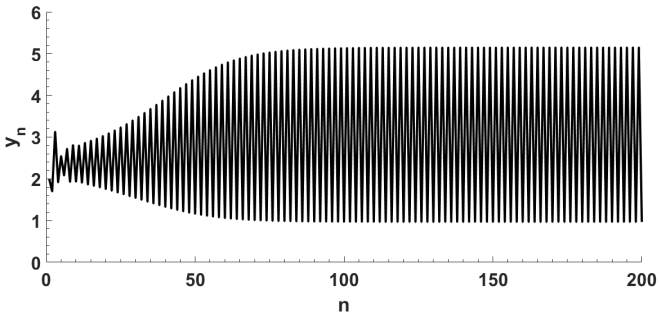

FIGURE 11. Plot of y_n for the system 35

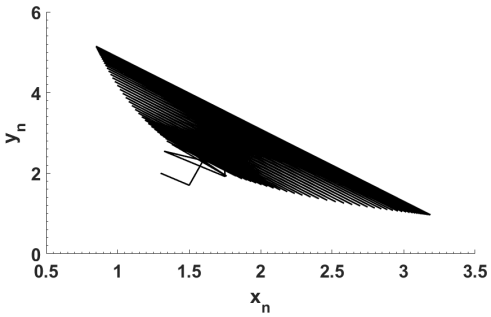

FIGURE 12. Phase portrait of system 35

persistence of positive solutions of system 3 and 4. Moreover, we have shown that unique positive equilibrium point of system 3 and 4 is locally as well as globally asymptotically stable under certain parametric conditions. Furthermore, the rate of convergence of positive solutions of 3 and 4 which converges to its unique positive equilibrium point is demonstracted. Finally, some illustrative numerical examples are provided.

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