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LOCAL WELL-POSEDNESS OF SOLUTIONS TO THE BOUNDARY LAYER EQUATIONS FOR COMPRESSIBLE TWO-FLUID FLOW

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Abstract. In this paper, we consider the two-dimensional (2D) two-fluid boundary layer system, which is a hyperbolic-degenerate parabolic-elliptic coupling system derived from the compressible isentropic two-fluid flow equations with nonslip boundary condition for the velocity. The local existence and uniqueness is established in weighted Sobolev spaces under the monotonicity assumption on tangential velocity along normal direction based on a nonlinear energy method by employing a nonlinear cancelation technic introduced in [R. Alexandre, Y.-G. Wang, C.-J. Xu and T. Yang, J. Amer. Math. Soc., 28 (2015), 745-784; N. Masmoudi and T.K. Wong, Comm. Pure Appl. Math., 68(2015), 1683-1741] and developed in [C.-J. Liu, F. Xie and T. Yang, Comm. Pure Appl. Math., 72(2019), 63-121].

1. Introduction.

1.1. Problem and main result. The boundary-layer theory has been developed by Ludwig Prandtl in 1904 (see [31]). Although this theory is now more than 110 years old, it is nowadays still being applied in industry and research, because many important fields of fluid mechanics (i.e. aeronautics, ship hydrodynamics, automobile aerodynamics) refer to flows at high Reynolds numbers. Mathematical

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analysis on the boundary layer theory has been extensively studied in different contexts, cf. [1, 4, 3, 5, 6, 7, 8, 9, 10, 11, 16, 17, 21, 22, 23, 27, 28, 29, 30, 31, 34, 35, 36, 37, 40, 42] and the references therein for incompressible or compressible Navier-Stokes boundary layer. Boundary layer problem on some more complex fluids such as magnetohydrodynamic was also made great progresses, cf. [12, 18, 19, 20, 24, 25, 26, 32, 33, 38, 39, 41]. The main object in this paper is to establish the local well-posedness of the two-phase boundary layer equations, which are derived from 2D two-phase flow with non-slip boundary condition on the velocity.

Two-phase or multi-phase flows are concerned with flows with two or more components and have a wide range of applications in nature, engineering, and biomedicine. Examples include sediment transport, geysers, volcanic eruptions, clouds, rain in natural and climate systems, mixture of oil and natural gas in extraction tubes of oil exploitation, oil transportation, steam generators, cooling systems, mixture of hot water and vapor of water in cooling tubes of nuclear power stations in energy production, bubble columns, aeration systems, tumor biology, anticancer therapies, developmental biology, plant physiology in chemical engineering, medical and genetic engineering, bioengineering, and so on. Multi-phase flow is much more complicated than single-phase flow due to the existence of a moving and deformable interface and its interactions with multi-phases [13, 14, 15].

We are concerned with this kind of compressible isentropic two-fluid flow, which can describe the mixture of two fluids of different densities or the mixture of fluid and particles, and is derived from physical considerations in [13] (see also [2]). Viscous compressible isentropic two-fluid flow equations read:

$$\begin{cases} \partial_t m + \nabla\cdot(m\mathbf{u}) = 0, \\ \partial_t n + \nabla\cdot(n\mathbf{u}) = 0, \\ \partial_t((m+n)\mathbf{u}) + \nabla\cdot((m+n)\mathbf{u}\otimes\mathbf{u}) + \nabla p(m,n) \\ = \mu\Delta\mathbf{u} + (\mu+\lambda)\nabla(\nabla\cdot\mathbf{u}). \end{cases} \tag{1}$$

Here the unknowns m, n represent the densities of two-fluid flows respectively, and $\mathbf{u}$ and $p(m,n)$ represent the velocity field and pressure. The pressure $p(m,n)$ function is expressed as

$$p = p(m,n) = m^\gamma + n^\beta \tag{2}$$

with $\beta, \gamma \geq 1$. The two-fluid flows have been investigated extensively. However the two-fluid boundary layer theory is comparatively mathematically underdeveloped and there are few results.

We assume the viscosity and shear coefficients have the same order of a small parameter ε. On the boundary, the non-slip boundary condition $\mathbf{u}|_{y=0}$ is imposed on velocity field. As in [29], the scaling is used as follows:

$$t = t, \quad x = x, \quad \tilde{y} = \varepsilon^{-\frac{1}{2}} y.$$

Then, set $\mathbf{u} = (u_1, u_2)$ and

$$u(t,x,\tilde{y}) = u_1(t,x,y), \quad v(t,x,\tilde{y}) = \varepsilon^{-\frac{1}{2}} u_2(t,x,y).$$

When ε tends to 0, by asymptotic expansion and taking out leading order terms, one obtains the following 2D two-fluid flow boundary layer equations (still denoting

$\tilde{y}$ by y for simplicity of notation):

$$\begin{cases} \partial_t m + \partial_x (mu) + \partial_y (mv) = 0, \\ \partial_t n + \partial_x (nu) + \partial_y (nv) = 0, \\ (m+n) \{\partial_t u + u\partial_x u + v\partial_y u\} + \partial_x p(t,x) = \partial_y^2 u, \\ \left(m^\gamma + n^\beta\right)(t,x,y) = p(t,x). \end{cases} \tag{3}$$

From $(3)_1 \times \gamma m^{\gamma-1} + (3)_2 \times \beta n^{\beta-1}$ together with $(3)_4$, one can obtain the following hyperbolic-degenerate parabolic-elliptic coupling system with special case $\gamma = \beta$ by setting $\rho = \frac{1}{m+n}$

$$\begin{cases} \partial_t \rho + u\partial_x \rho + v\partial_y \rho = -\dfrac{\rho}{\gamma p}(\partial_t p + u\partial_x p), \\ \partial_t u + u\partial_x u + v\partial_y u = \rho\partial_y^2 u - \rho\partial_x p, \\ \partial_x u + \partial_y v = -\dfrac{1}{\gamma p}(\partial_t p + u\partial_x p), \end{cases} \tag{4}$$

where the unknowns $u := u(t,x,y)$, $v := v(t,x,y)$ and $\rho := \rho(t,x,y)$ denote the tangential and normal components of the velocity field and the "total density" in the boundary layer respectively. The given function $p(t,x)$ is the pressure for the outer flow.

Throughout the paper, we focus on the initial-boundary value problem for the above two-fluid boundary layer equations (4) in the periodic domain $\Omega := \mathbb{T}\times\mathbb{R}^+ := \{(x,y) : x \in \mathbb{R}/\mathbb{Z}, 0 \le y < +\infty\}$ with the initial data

$$u|_{t=0} = u_0(x,y), \quad \rho|_{t=0} = \rho_0(x,y) \tag{5}$$

boundary and far-field conditions

$$u(t,x,y)|_{y=0} = v(t,x,y)|_{y=0} = 0, \quad \lim_{y\to\infty} u = U(t,x), \quad \lim_{y\to\infty} \rho = \rho^\infty(t,x), \tag{6}$$

Both U and ρ^∞ are the trace of tangential velocity and density satisfying the following "matching" conditions:

$$\partial_t U + U\partial_x U = -\rho^\infty \partial_x p, \tag{7}$$

$$\partial_t \rho^\infty + U\partial_x \rho^\infty = -\rho^\infty \left(\partial_t \ln p^{\frac{1}{\gamma}} + U\partial_x \ln p^{\frac{1}{\gamma}}\right). \tag{8}$$

In addition, the initial data $u_0 := u_0(x,y), \rho_0 := \rho_0(x,y)$ and the outer flow U satisfy the compatibility conditions:

$$u_0|_{y=0} = 0, \quad \lim_{y\to+\infty} u_0 = U|_{t=0}, \quad \lim_{y\to+\infty} \rho_0 = \rho^\infty|_{t=0}. \tag{9}$$

To state the main result, we first introduce the function space in which the initial-boundary value problem (4)-(6) will be solved under the strictly monotonic assumption on the tangential velocity in the normal variable (i.e., vorticity):

$$\omega := \partial_y u > 0. \tag{10}$$

The space $H_\delta^{s,\lambda}$ for $\omega:\Omega\to\mathbb{R}$ and $\tilde\rho:=\rho-\rho^\infty:\Omega\to\mathbb{R}$ is defined by

$$H_\delta^{s,\lambda}:=\Bigg\{(\omega,\tilde\rho):\|\omega\|_{H^{s,\lambda}}<+\infty,\|\tilde\rho\|_{H^{s,\lambda-1}}<+\infty,\sum_{|\alpha|\le 2}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|_{L^\infty}\le\frac{1}{\delta},$$
$$\sum_{|\alpha|\le 2}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|_{L^\infty}\le\frac{1}{\delta},\langle y\rangle^\lambda\omega\ge\delta\ \text{ and }\ \rho\ge\delta\Bigg\},$$

where $s\ge 6,\lambda\ge 1,\delta\in(0,1),D^\alpha=\partial_x^{\alpha_1}\partial_y^{\alpha_2}$ with the index $\alpha=(\alpha_1,\alpha_2)$, and the weighted H^s norm $\|\cdot\|_{H^{s,\lambda}(\Omega)}$ for a function f is defined by

$$\|f\|^2_{H^{s,\lambda}(\Omega)}:=\sum_{|\alpha|\le s}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha f\|^2_{L^2(\Omega)},\tag{11}$$

$\langle y\rangle=1+y$.

Another weighted norms $\|\cdot\|_{H_g^{s,\lambda}(\Omega)}$ for the vorticity ω and $\|\cdot\|_{H_h^{s,\lambda-1}(\Omega)}$ for $\tilde\rho$ are respectively given by

$$\|\omega\|^2_{H_g^{s,\lambda}(\Omega)}:=\left\|\langle y\rangle^\lambda g_s\right\|^2_{L^2(\Omega)}+\sum_{\substack{|\alpha|\le s\\ \alpha_1\le s-1}}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|^2_{L^2(\Omega)}\tag{12}$$

and

$$\|\tilde\rho\|^2_{H_h^{s,\lambda-1}(\Omega)}:=\left\|\langle y\rangle^{\lambda-1}h_s\right\|^2_{L^2(\Omega)}+\sum_{\substack{|\alpha|\le s\\ \alpha_1\le s-1}}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|^2_{L^2(\Omega)},\tag{13}$$

where

$$g_s:=\partial_x^s\omega-a\partial_x^s(u-U)\quad\text{and}\quad h_s:=\partial_x^s\tilde\rho-b\partial_x^s(u-U)$$

with $a:=\frac{\partial_y\omega}{\omega}$ and $b:=\frac{\partial_y\tilde\rho}{\omega}$, provided that $w>0$.

Similar to those in [27], the difference between norms $\|\cdot\|_{H^{s,\lambda}(\Omega)}$ (resp. $\|\cdot\|_{H^{s,\lambda-1}(\Omega)}$) and $\|\cdot\|_{H_g^{s,\lambda}(\Omega)}$ (resp. $\|\cdot\|_{H_h^{s,\lambda-1}(\Omega)}$) is that the weighted L^2-norm of $\partial_x^s\omega$ (resp. $\partial_x^s\tilde\rho$) is replaced by that of g_s (resp. h_s) which avoids the loss of x-derivative by the nonlinear cancellation. On the other hand, these two weighted norms are the almost equivalent. That is, there exist positive constants C_1 and C_2 so that the following inequality holds for any $(\omega,\tilde\rho)\in H_\delta^{s,\lambda}(\Omega)$ and an integer $s(\ge 6)$, $\lambda\ge 1$ and $\delta\in(0,1)$:

$$C_1^{-1}\|\omega\|_{H_g^{s,\lambda}}\le\|\omega\|_{H^{s,\lambda}}+\|u-U\|_{H^{s,\lambda-1}}\le C_1\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2(\mathbb{T})}\right).\tag{14}$$

Remark 1. Unlike the norm $\|\omega\|_{H_g^{s,\lambda}}$, the weighted norm $\|\tilde\rho\|_{H_h^{s,\lambda}}$ does not share the above almost equivalent relationship with the norm $\|\tilde\rho\|_{H^{s,\lambda}}$. However the following inequality holds

$$\|\tilde\rho\|_{H^{s,\lambda}}\le\|\tilde\rho\|_{H_h^{s,\lambda}}+C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right),\tag{15}$$

which is enough to solve the problem.

The main theorem can be stated as follows.

Theorem 1.1. *Given any even integer $s\ge 6$ and real numbers λ,δ satisfying $\lambda\ge 1$ and $\delta\in(0,1)$, assume the following conditions on the initial data and the outer flow (ρ^∞,U):*

(i) Suppose that the initial data $u_0 - U(0,x) \in H^{s,\lambda-1}, \partial_y u_0 \in H^{s,\lambda}_{2\delta}$ *and* $\rho_0 - \rho^\infty(0,x) \in H^{s,\lambda-1}_{2\delta}$ *satisfy the compatibility conditions* (9).

(ii) The outer flow (ρ^∞, U) *is supposed to satisfy*

$$\sup_t \sum_{l=0}^{\frac{s}{2}+1} \left\|\partial_t^l U\right\|_{H^{s-2l+2}(\mathbb{T})} + \sup_t \sum_{l=0}^{\frac{s}{2}+1} \left\|\partial_t^l \rho^\infty\right\|_{H^{s-2l+2}(\mathbb{T})} < +\infty. \tag{16}$$

Then there exist a time $T := T(s, \lambda, \delta, \|w_0\|_{H_g^{s,\lambda}}, \|\rho_0 - \rho^\infty\|_{H_h^{s,\lambda-1}}, \rho^\infty, U)$ *such that the initial-boundary value problem* (4)-(6) *has a unique classical solution* (ρ, u, v) *satisfying*

$$u - U \in L^\infty\left([0,T]; H^{s,\lambda-1}\right) \cap C([0,T]; H^s - w),$$
$$\partial_y u \in L^\infty\left([0,T]; H^{s,\lambda}_\delta\right) \cap C([0,T]; H^s - w),$$

and

$$\rho - \rho^\infty \in L^\infty\left([0,T]; H^{s,\lambda-1}\right) \cap C([0,T]; H^s - \omega),$$

where $H^s - w$ *is the space* H^s *endowed with its weak topology.*

1.2. **Regularized system and preliminaries.** In order to prove the local-in-time existence of the initial-boundary value problem (4)-(6) we consider the regularized equations for any $\epsilon > 0$,

$$\begin{cases} \partial_t \rho^\epsilon + u^\epsilon \partial_x \rho^\epsilon + v^\epsilon \partial_y \rho^\epsilon = -\dfrac{\rho^\epsilon}{\gamma p}\left(\partial_t p + u^\epsilon \partial_x p\right) + \epsilon^2 \partial_x^2 \rho^\epsilon, \\ \partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon + v^\epsilon \partial_y u^\epsilon = \rho^\epsilon \partial_y^2 u^\epsilon - \rho^\epsilon \partial_x p + \epsilon^2 \partial_x^2 u^\epsilon, \\ \partial_x u^\epsilon + \partial_y v^\epsilon = -\dfrac{1}{\gamma p}\left(\partial_t p + u^\epsilon \partial_x p\right) \end{cases} \tag{17}$$

with the initial data

$$u^\epsilon|_{t=0} = u_0(x,y), \quad \rho^\epsilon|_{t=0} = \rho_0(x,y), \tag{18}$$

boundary conditions

$$u^\epsilon|_{y=0} = v^\epsilon|_{y=0} = 0, \quad \lim_{y\to\infty} u^\epsilon = U(t,x), \quad \lim_{y\to\infty} \rho^\epsilon = \rho^\infty(t,x) \tag{19}$$

and the regularized "matching" conditions

$$\partial_t U + U \partial_x U = -\rho^\infty \partial_x p + \epsilon^2 \partial_x^2 U, \tag{20}$$

$$\partial_t \rho^\infty + U \partial_x \rho^\infty = -\rho^\infty \left(\partial_t \ln p^{\frac{1}{\gamma}} + U \partial_x \ln p^{\frac{1}{\gamma}}\right) + \epsilon^2 \partial_x^2 \rho^\infty. \tag{21}$$

Introducing the unknows $\tilde{u}^\epsilon = u^\epsilon - U$, $\tilde{\rho}^\epsilon = \rho^\epsilon - \rho^\infty$ and $\omega^\epsilon = \partial_y u^\epsilon$, then we have the regularized equation for $\tilde{u}^\epsilon$

$$\partial_t \tilde{u}^\epsilon + u^\epsilon \partial_x \tilde{u}^\epsilon + v^\epsilon \partial_y \tilde{u}^\epsilon = \rho^\epsilon \partial_y^2 \tilde{u}^\epsilon - \tilde{\rho}^\epsilon \partial_x p - \tilde{u}^\epsilon \partial_x U + \epsilon^2 \partial_x^2 \tilde{u}^\epsilon \tag{22}$$

and the regularized equations for $(\tilde{\rho}^\epsilon, \omega^\epsilon)$

$$\begin{cases} \partial_t \tilde{\rho}^\epsilon + u^\epsilon \partial_x \tilde{\rho}^\epsilon + v^\epsilon \partial_y \tilde{\rho}^\epsilon \\ = -\tilde{u}^\epsilon \partial_x \rho^\infty - \tilde{\rho}^\epsilon \partial_t \ln p^{\frac{1}{\gamma}} - \rho^\infty \tilde{u}^\epsilon \partial_x \ln p^{\frac{1}{\gamma}} - \tilde{\rho}^\epsilon u^\epsilon \partial_x \ln p^{\frac{1}{\gamma}} + \epsilon^2 \partial_x^2 \tilde{\rho}^\epsilon, \\ \partial_t \omega^\epsilon + u^\epsilon \partial_x \omega^\epsilon + v^\epsilon \partial_y \omega^\epsilon \\ = \dfrac{\omega^\epsilon}{\gamma p}\left(\partial_t p + u^\epsilon \partial_x p\right) + \partial_y \rho^\epsilon \left(\partial_y \omega^\epsilon - \partial_x p\right) + \rho^\epsilon \partial_y^2 \omega^\epsilon + \epsilon^2 \partial_x^2 \omega^\epsilon \end{cases} \tag{23}$$

with the initial data

$$\tilde{\rho}^\epsilon|_{t=0} = \rho_0(x,y) - \rho^\infty(0,x), \quad \omega^\epsilon|_{t=0} = \omega_0 := \partial_y u_0(x,y) \tag{24}$$

and boundary condition

$$\partial_y \omega^\epsilon|_{y=0} = 0. \tag{25}$$

The velocity field (u^ϵ, v^ϵ) is also given

$$u^\epsilon(t,x,y) = U - \int_y^{+\infty} \omega^\epsilon(t,x,\tilde{y})d\tilde{y}, \tag{26}$$

$$v^\epsilon(t,x,y) = -\int_0^y \partial_x u^\epsilon(t,x,\tilde{y})d\tilde{y} - \frac{1}{\gamma p}\int_0^y (\partial_t p + u^\epsilon \partial_x p)(t,x,\tilde{y})d\tilde{y} \tag{27}$$

by the equation

$$\partial_x u^\epsilon + \partial_y v^\epsilon = -\frac{1}{\gamma p}(\partial_t p + u^\epsilon \partial_x p). \tag{28}$$

As in [27], the local existence of the regularized initial-boundary value problem (23)-(25) can be solved in $[0,T]$, where

$$T := T(s,\gamma,\delta,\epsilon,\rho_0,\omega_0,\rho^\infty,U).$$

The rest need to derive uniform-in-ϵ estimates on $(\tilde{\rho}^\epsilon, \omega^\epsilon)$, which are respectively obtained in Sections 2 and 3. From now on we drop the superscript ϵ for simplicity of notations.

In what follows, we will collect some inequalities without proof, which play important roles in establishing uniform-in-ϵ estimates on the regularized system in next two sections.

First, we present the following Hardy type inequality and Sobolev type inequality (see Lemmas B.1 and B.2 in [27]) for the proof of the following Lemma 1.4 and some other parts in the paper.

Lemma 1.2 (Hardy type inequality; see Lemma B.1 in [27]). *Let $f : \Omega \to \mathbb{R}$. Then (i) if $\lambda > -\frac{1}{2}$ and $\lim_{y\to+\infty} f(x,y) = 0$, then*

$$\left\|\langle y\rangle^\lambda f\right\|_{L^2(\Omega)} \le \frac{2}{2\lambda+1}\left\|\langle y\rangle^{\lambda+1}\partial_y f\right\|_{L^2(\Omega)}; \tag{29}$$

(ii) if $\lambda < -\frac{1}{2}$, then

$$\left\|\langle y\rangle^\lambda f\right\|_{L^2(\Omega)} \le \sqrt{-\frac{1}{2\lambda+1}}\left\|f|_{y=0}\right\|_{L^2(\mathbb{T})} - \frac{2}{2\lambda+1}\left\|\langle y\rangle^{\lambda+1}\partial_y f\right\|_{L^2(\Omega)}. \tag{30}$$

Lemma 1.3 (Sobolev type inequality; see Lemma B.2 in [27]). *Let $f : \Omega \to \mathbb{R}$. Then there exists universal constant C such that*

$$\|f\|_{L^\infty(\Omega)} \le C\left(\|f\|_{L^2(\Omega)} + \|\partial_x f\|_{L^2(\Omega)} + \left\|\partial_y^2 f\right\|_{L^2(\Omega)}\right). \tag{31}$$

In the next lemma, we will use the weighted norms $\|\omega\|_{H_g^{s,\lambda}}$ and $\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}$ to control certain L^2 and L^∞ norms of u, v, ω, g_k, h_k and their derivatives.

Lemma 1.4 (Weighted L^2 and L^∞ estimates). *Let the vector field (u,v) defined on Ω satisfy the condition $\partial_x u + \partial_y v = -\frac{1}{\gamma p}(\partial_t p + u\partial_x p)$, the Dirichlet boundary condition $u|_{y=0} = v|_{y=0} = 0$ and $\lim_{y\to+\infty} u = U$, then we have the following estimates:*

Weighted L^2 ***estimates.***

(i) For all $k = 0, 1, \cdots, s$,

$$\left\| \langle y \rangle^{\lambda-1} \partial_x^k (u - U) \right\|_{L^2} \le C \left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right). \tag{32}$$

(ii) For all $k = 0, 1, \cdots, s-1$,

$$\left\| \langle y \rangle^{-1} \left(\partial_x^k v + y \tilde{U} \right) \right\|_{L^2} \le C \left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right), \tag{33}$$

where $\tilde{U}_k := \tilde{U}_k(t,x) = \partial_t \partial_x^k \ln p^{1/\gamma} + \partial_x^{k+1} U + \sum_{i=0}^k \binom{k}{i} \left(\partial_x^{i+1} \ln p^{1/\gamma} \right) \partial_x^{k-i} U$.

(iii) For all $|\alpha| \le s$,

$$\left\| \langle y \rangle^{\lambda+\alpha_2} D^\alpha \omega \right\|_{L^2} \le \begin{cases} C \left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right) & \text{if } \alpha = (s,0), \\ \|\omega\|_{H_g^{s,\lambda}} \quad \text{if } \alpha \ne (s,0). \end{cases} \tag{34}$$

(iv) For all $k = 1, 2, \cdots, s$,

$$\left\| \langle y \rangle^{\lambda} g_k \right\|_{L^2} \le \begin{cases} C \left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right) & \text{if } k = 1, 2, \cdots, s-1, \\ \|\omega\|_{H_g^{s,\lambda}} \quad \text{if } k = s, \end{cases} \tag{35}$$

where the quantity $g_k := \partial_x^k w - \frac{\partial_y w}{w} \partial_x^k (u - U)$.

(v) For all $k = 1, 2, \cdots, s$,

$$\left\| \langle y \rangle^{\lambda-1} h_k \right\|_{L^2} \le \begin{cases} \|\tilde{\rho}\|_{H_h^{s,\lambda-1}} + C \left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right) & \text{if } k = 1, 2, \cdots, s-1, \\ \|\tilde{\rho}\|_{H_h^{s,\lambda-1}} \quad \text{if } k = s, \end{cases} \tag{36}$$

where the quantity $h_k := \partial_x^k \tilde{\rho} - \frac{\partial_y \tilde{\rho}}{w} \partial_x^k (u - U)$.

Weighted L^∞ ***estimates.***

(vi) For all $k = 0, 1, \cdots, s-1$,

$$\left\| \partial_x^k u \right\|_{L^\infty} \le C \left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} \right). \tag{37}$$

(vii) For all $k = 0, 1, \cdots, s-2$,

$$\left\| \langle y \rangle^{-1} \partial_x^k v \right\|_{L^\infty} \le C \left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} + 1 \right). \tag{38}$$

(viii) For all $|\alpha| \le s-2$,

$$\left\| \langle y \rangle^{\lambda+\alpha_2} D^\alpha \omega \right\|_{L^\infty} \le C \|\omega\|_{H_g^{s,\lambda}}. \tag{39}$$

(ix) For all $|\alpha| \le s-2$,

$$\| \langle y \rangle^{\lambda+\alpha_2-1} D^\alpha \tilde{\rho} \|_{L^\infty} \le C \|\tilde{\rho}\|_{H_h^{s,\lambda-1}}. \tag{40}$$

The inequalities (i)-(iv) and (vi)-(viii) can be found in [27] under the incompressible condition $\partial_x u + \partial_y v = 0$. These properties still hold in the case that incompressible condition is replaced by the compressible condition and the proofs are given in [6]. Here we give the proof of property (v).

Proof. Noticing $h_k = \partial_x^k \tilde\rho - \frac{\tilde\rho}{\omega}\partial_x^k \tilde u$, when $k = 1, \cdots, s-1$, we have

$$\begin{aligned}\left\|\langle y\rangle^{\lambda-1}h_k\right\|_{L^2} \le& \left\|\langle y\rangle^{\lambda-1}\partial_x^k\tilde\rho\right\|_{L^2} + \left\|\langle y\rangle^{\lambda-1}\frac{\tilde\rho}{\omega}\partial_x^k\tilde u\right\|_{L^2}\\ \le& \|\langle y\rangle^{\lambda-1}\partial_x^k\tilde\rho\|_{L^2} + \frac{1}{\delta^2}\left\|\langle y\rangle^{\lambda-1}\partial_x^k\tilde u\right\|_{L^2}\\ \le& \|\tilde\rho\|_{H_h^{s,\lambda-1}} + C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2}\right).\end{aligned}$$

Here we have used $\|\langle y\rangle^\lambda \partial_y\tilde\rho\|_{L^\infty} \le \delta$, $\langle y\rangle^\lambda\omega \ge \frac{1}{\delta}$ and (i).

When $k = s$, it directly follows from the definition of the norm $\|\tilde\rho\|_{H_h^{s,\lambda-1}}$

$$\|\langle y\rangle^{\lambda-1}h_s\|_{L^2} \le \|\tilde\rho\|_{H_h^{s,\lambda-1}}.$$

□

Finally, two interpolation inequalities will be introduced to estimate nonlinear terms as follows.

Lemma 1.5 (Interpolation inequalities)**.** *For $\lambda \in \mathbb{R}$ and integer $s \ge 6$, any $\theta = (\theta_1, \theta_2)$, $\tilde\theta = (\tilde\theta_1, \tilde\theta_2)$ with $|\theta| + |\tilde\theta| \le s+2$, $\max(|\theta|, |\tilde\theta|) \le s$,*

$$\left\|\langle y\rangle^{2\lambda+\theta_1+\tilde\theta_2} D^\theta\omega D^{\tilde\theta}\omega\right\|_{L^2} \le C\|\omega\|^2_{H_g^{s,\lambda}} \tag{41}$$

and

$$\left\|\langle y\rangle^{2\lambda+\theta_1+\tilde\theta_2-1} D^\theta\tilde\rho D^{\tilde\theta}\omega\right\|_{L^2} \le C\|\omega\|_{H_g^{s,\lambda}}\|\tilde\rho\|_{H_h^{s,\lambda-1}}. \tag{42}$$

The rest of the paper is organized as follows. In section 2, the uniform weighted H^s estimates of ω are given in two steps. Subsection 2.1 and 2.2. In section 3, the uniform weighted H^s estimates of $\tilde\rho = \rho - \rho^\infty$ are given in Subsection 3.1 and 3.2. In section 5.1 and 5.2, the local-in-time existence and uniqueness of the solution to the initial boundary problem (4)-(6) will be proved respectively based on the uniform weighted estimates derived in Section 2 and Section 3.

2. Uniform weighted H^s estimates on the regularized ω.

In this section, we will derive uniform-in-ϵ estimates on ω in two steps. Weighted L^2 estimates on $D^\alpha\omega$ with $|\alpha| \le s, \alpha_1 \le s-1$ and weighted L^2 estimates on g_s are respectibely obtained in Subsections 2.1 and 2.2.

2.1. The case of $|\alpha| \le s, \alpha_1 \le s-1$.

Using the standard energy method, we will derive weighted L^2 estimates on $D^\alpha\omega$ for $|\alpha| \le s$ and $\alpha_1 \le s-1$.

Lemma 2.1. *Let $s \ge 6$ be an even integer, let the hypotheses for U, ρ^∞, P given in Theorem 1.1 hold, then we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le s,\alpha_1\le s-1}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|^2_{L^2}\\ &+\frac{1}{2}\sum_{|\alpha|\le s,\alpha_1\le s-1}\left\|\langle y\rangle^{\lambda+\alpha_2}\sqrt{\rho}\partial_y D^\alpha\omega\right\|^2_{L^2} + \sum_{|\alpha|\le s,\alpha_1\le s-1}\left\|\langle y\rangle^{\lambda+\alpha_2}\partial_x D^\alpha\omega\right\|^2_{L^2}\\ \le& C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^\infty}+1\right)^{s+2}\left(\|\tilde\rho\|_{H_h^{s,\lambda-1}} + \|\partial_x^s\rho^\infty\|_{L^\infty}+1\right)^s.\end{aligned} \tag{43}$$

Proof. Differentiating the equation (23)$_1$ α_1 times with respect to x and α_2 times with respect to y, one has

$$\begin{aligned}
&\partial_t D^\alpha\omega + u\partial_x D^\alpha\omega + v\partial_y D^\alpha\omega \\
=&-\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}D^\beta u\partial_x D^{\alpha-\beta}\omega - \sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}D^\beta v\partial_y D^{\alpha-\beta}\omega \\
&+\sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^{\alpha-\beta}\omega D^\beta\left(\partial_t \ln p^{1/\gamma} + u\partial_x \ln p^{1/\gamma}\right) \\
&+\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial_y D^\beta\rho D^{\alpha-\beta}\left(\partial_y\omega - \partial_x p\right) \\
&+\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}D^\beta\rho\partial_y^2 D^{\alpha-\beta}\omega + \rho\partial_y^2 D^\alpha\omega + \epsilon^2\partial_x^2 D^\alpha\omega.
\end{aligned}\tag{44}$$

Multiplying the equation (44) by $\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega$ and integrating the resulting equation over Ω, then we estimate term by term as follows.

(1) It is easy to get

$$\int_\Omega \partial_t D^\alpha\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega = \frac{1}{2}\frac{d}{dt}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|_{L^2}^2. \tag{45}$$

$$\int_\Omega \epsilon^2\partial_x^2 D^\alpha\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega = -\epsilon^2\left\|\langle y\rangle^{\lambda+\alpha_2}\partial_x D^\alpha\omega\right\|_{L^2}^2. \tag{46}$$

(2) Integrating by parts and using the equation (28) and Lemma 1.4, one has

$$\begin{aligned}
&\int_\Omega (u\partial_x D^\alpha\omega + v\partial_y D^\alpha\omega)\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega \\
=&-\frac{1}{2}\int_\Omega\left(\partial_x u + \partial_y v\right)\langle y\rangle^{2\lambda+2\alpha_2}(D^\alpha\omega)^2 \\
&-(\lambda+\alpha_2)\int_\Omega \langle y\rangle^{2\lambda+2\alpha_2-1}v(D^\alpha\omega)^2 \\
&+\frac{1}{2}\int_{\mathbb{T}} v\langle y\rangle^{2\lambda+2\alpha_2}(D^\alpha\omega)^2\Big|_{y=0}^{y=+\infty} \\
=&\frac{1}{2}\int_\Omega\left(\partial_t\ln p^{1/\gamma} + u\partial_x \ln p^{1/\gamma}\right)\langle y\rangle^{2\lambda+2\alpha_2}(D^\alpha\omega)^2 \\
&-(\lambda+\alpha_2)\int_\Omega \frac{v}{1+y}\langle y\rangle^{2\lambda+2\alpha_2}(D^\alpha\omega)^2 \\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2(\mathbb{T})} + 1\right)\|\omega\|_{H_g^{s,\lambda}}^2.
\end{aligned}\tag{47}$$

Here notice that the boundary term $\frac{1}{2}\int_{\mathbb{T}} v\langle y\rangle^{2\lambda+2\alpha_2}(D^\alpha\omega)^2\Big|_{y=0}^{y=+\infty}$ vanishes and $|\partial_t \ln p^{1/\gamma}| + |\partial_x \ln p^{1/\gamma}| \le C$.

(3) The term $\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega D^\beta u\partial_x D^{\alpha-\beta}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega$ is estimated in two cases of $\beta_2 = 0$ and $\beta_2 > 0$.

When $\beta_2 = 0$, one has

$$\begin{aligned}
&\int_\Omega \partial_x^{\beta_1}u\partial_x^{\alpha_1-\beta_1+1}\partial_y^{\alpha_2}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega \\
\le&\left\|\partial_x^{\beta_1}u\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2}\partial_x^{\alpha_1-\beta_1+1}\partial_y^{\alpha_2}\omega\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|_{L^2} \\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2(\mathbb{T})}\right)\|\omega\|_{H_g^{s,\lambda}}^2
\end{aligned}\tag{48}$$

provided $\beta_1 \leq s-1$.

When $\beta_2 \geq 1$, one has with $e_1=(1,0), e_2=(0,1)$

$$\begin{aligned}&\int_\Omega D^\beta u \partial_x D^{\alpha-\beta}\omega \langle y\rangle^{2\lambda+2\alpha_2} D^\alpha \omega\\ &\leq \left\|\langle y\rangle^{\lambda+\alpha_2} D^{\beta-e_2}\omega D^{\alpha-\beta+e_1}\omega\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2} D^\alpha\omega\right\|_{L^2}\\ &\leq C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2(\mathbb{T})}\right)\|\omega\|^2_{H_g^{s,\lambda}},\end{aligned}\tag{49}$$

which together with (48) yields

$$\begin{aligned}&\sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\int_\Omega D^\beta u \partial_x D^{\alpha-\beta}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ \leq& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2(\mathbb{T})}\right)\|\omega\|^2_{H_g^{s,\lambda}}.\end{aligned}\tag{50}$$

(4) The term $\sum_{0<\beta\leq\alpha}\binom{\alpha}{\beta}\int_\Omega D^\beta v\partial_y D^{\alpha-\beta}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega$ is estimated in five cases.

When $\beta_2=0, \beta_1\leq s-2$, one has

$$\begin{aligned}&\int_\Omega D^\beta v\partial_y D^{\alpha-\beta}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ =&\int_\Omega \partial_x^{\beta_1}v\partial_x^{\alpha_1-\beta_1}\partial_y^{\alpha_2+1}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ \leq&\left\|\frac{\partial_x^{\beta_1}v}{1+y}\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2+1}\partial_x^{\alpha_1-\beta_1}\partial_y^{\alpha_2+1}\omega\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|_{L^2}\\ \leq& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2(\mathbb{T})}+1\right)\|\omega\|^2_{H_g^{s,\lambda}}.\end{aligned}\tag{51}$$

When $\beta_2=0, \beta_1=s-1$, which means $\alpha_1=s-1$ and $\alpha_2\leq 1$, we have

$$\begin{aligned}&\int_\Omega D^\beta v\partial_y D^{\alpha-\beta}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ =&\int_\Omega \partial_x^{\beta_1}v\partial_y^{\alpha_2+1}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ \leq& C\left\|\frac{\partial_x^{s-1}v+y\tilde{U}_{s-1}}{1+y}\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2+1}\partial_y^{\alpha_2+1}\omega\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|_{L^2}\\ &+C\left\|\tilde{U}_{s-1}\right\|_{L^\infty(\mathbb{T})}\left\|\langle y\rangle^{\lambda+\alpha_2+1}\partial_y^{\alpha_2+1}\omega\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|_{L^2}\\ \leq& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^\infty(\mathbb{T})}+1\right)\|\omega\|^2_{H_g^{s,\lambda}}.\end{aligned}\tag{52}$$

When $\beta_2=1$, $\beta_1=s-1$, which means $\alpha_1=s-1$ and $\alpha_2\leq 1$, we have

$$\begin{aligned}&\int_\Omega D^\beta v\partial_y D^{\alpha-\beta}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ =&\int_\Omega \partial_x^{\beta_1}\left(\partial_x u+\partial_t\ln p^{1/\gamma}+u\partial_x\ln p^{1/\gamma}\right)\partial_y^{\alpha_2}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ \leq& C\big(\|\langle y\rangle^{\lambda-1}\partial_x^s\tilde{u}\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2}\partial_y^{\alpha_2}\omega\|_{L^\infty}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\\ &+\|\partial_x^s U\|_{L^\infty(\mathbb{T})}\|\langle y\rangle^{\lambda+\alpha_2}\partial_y^{\alpha_2}\omega\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\end{aligned}\tag{53}$$

$$
\begin{aligned}
&+\|\partial_t\partial_x^{s-1}\ln p^{1/\gamma}\|_{L^\infty(\mathbb{T})}\|\langle y\rangle^{\lambda+\alpha_2}\partial_y^{\alpha_2}\omega\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\\
&+\|\partial_x^{s-1}(u\partial_x\ln p^{1/\gamma})\|_{L^\infty}\|\langle y\rangle^{\lambda+\alpha_2}\partial_y^{\alpha_2}\omega\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\Big)\\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^\infty}+1\right)\|\omega\|_{H_g^{s,\lambda}}^2.
\end{aligned}
$$

When $\beta_2=1$, $\beta_1\le s-2$, we have

$$
\begin{aligned}
&\int_\Omega D^\beta v\partial_y D^{\alpha-\beta}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\
=&\int_\Omega \partial_x^{\beta_1}\left(\partial_x u+\partial_t\ln p^{1/\gamma}+u\partial_x\ln p^{1/\gamma}\right)\partial_x^{\alpha_1-\beta_1}\partial_y^{\alpha_2}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\
\le& C\big(\|\partial_x^{\beta_1+1}u\|_{L^\infty}\|\langle y\rangle^{\lambda+\alpha_2}\partial_x^{\alpha_1-\beta_1}\partial_y^{\alpha_2}\omega\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\\
&+\|\partial_t\partial_x^{\beta_1}\ln p^{1/\gamma}\|_{L^\infty(\mathbb{T})}\|\langle y\rangle^{\lambda+\alpha_2}\partial_x^{\alpha_1-\beta_1}\partial_y^{\alpha_2}\omega\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\\
&+\|\partial_x^{\beta_1}(u\partial_x\ln p^{1/\gamma})\|_{L^\infty}\|\langle y\rangle^{\lambda+\alpha_2}\partial_x^{\alpha_1-\beta_1}\partial_y^{\alpha_2}\omega\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\Big)\\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}+1\right)\|\omega\|_{H_g^{s,\lambda}}^2.
\end{aligned}
\tag{54}
$$

When $\beta_2\ge 2$, we have

$$
\begin{aligned}
&\int_\Omega D^\beta v\partial_y D^{\alpha-\beta}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\
=&\int_\Omega D^{\beta-2e_2}\left(\partial_x\omega+\omega\partial_x\ln p^{1/\gamma}\right)D^{\alpha-\beta+e_2}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\
\le& C\big(\|\langle y\rangle^{\lambda+\beta_2-2}D^{\beta-2e_2+e_1}\omega\langle y\rangle^{\lambda+\alpha_2-\beta_2+1}D^{\alpha-\beta+e_2}\omega\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\\
&+\|\langle y\rangle^{\lambda+\beta_2-2}D^{\beta-2e_2}(\omega\partial_x\ln p^{1/\gamma})\langle y\rangle^{\lambda+\alpha_2-\beta_2+1}D^{\alpha-\beta+e_2}\omega\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\Big)\\
\le& C\|\omega\|_{H_g^{s,\lambda}}^3
\end{aligned}
\tag{55}
$$

provided that Lemma 1.5 and $\lambda\ge 1$.

Combing estimates (51)-(55), we get

$$
\begin{aligned}
&\sum_{0<\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega D^\beta u\partial_x D^{\alpha-\beta}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^\infty}+1\right)\|\omega\|_{H_g^{s,\lambda}}^2.
\end{aligned}
\tag{56}
$$

(5) The term $\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega D^{\alpha-\beta}\omega D^\beta(\partial_t\ln p^{1/\gamma}+u\partial_x\ln p^{1/\gamma})\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega$ is estimated in two cases.

When $\beta_2=0$, we have

$$
\begin{aligned}
&\int_\Omega D^{\alpha-\beta}\omega D^\beta\left(\partial_t\ln p^{1/\gamma}+u\partial_x\ln p^{1/\gamma}\right)\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\
=&\int_\Omega D^{\alpha-\beta}\omega\partial_x^{\beta_1}\left(\partial_t\ln p^{1/\gamma}+u\partial_x\ln p^{1/\gamma}\right)\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\
\le& C\Big(\|\langle y\rangle^{\lambda+\alpha_2}D^{\alpha-\beta}\omega\|_{L^2}\|\partial_t\partial_x^{\beta_1}\ln p^{1/\gamma}\|_{L^\infty(\mathbb{T})}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\\
&+\|\langle y\rangle^{\lambda+\alpha_2}D^{\alpha-\beta}\omega\|_{L^2}\|\partial_x^{\beta_1}(u\partial_x\ln p^{1/\gamma})\|_{L^\infty}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\Big)\\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^\infty}+1\right)\|\omega\|_{H_g^{s,\lambda}}^2.
\end{aligned}
\tag{57}
$$

Here we use $\beta_1 \le s-1$.

When $\beta_2 \ge 1$, then we have

$$\begin{aligned}
&\int_\Omega D^{\alpha-\beta}\omega D^\beta\left(\partial_t \ln p^{1/\gamma} + u\partial_x \ln p^{1/\gamma}\right)\langle y\rangle^{2\lambda+2\alpha_2} D^\alpha\omega \\
=&\int_\Omega D^{\alpha-\beta}\omega D^{\beta-e_2}\left(\omega\partial_x \ln p^{1/\gamma}\right)\langle y\rangle^{2\lambda+2\alpha_2} D^\alpha\omega \\
\le&\left\|\langle y\rangle^{\lambda+\alpha_2-\beta_2}D^{\alpha-\beta}\omega\langle y\rangle^{\lambda+\beta_2-1}D^{\beta-e_2}(\omega\partial_x \ln p^{1/\gamma})\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|_{L^2} \\
\le& C\|\omega\|^3_{H_g^{s,\lambda}},
\end{aligned} \tag{58}$$

provided Lemma 1.5 and $\lambda \ge 1$.

Combing estimates (57) and (58), we obtain

$$\begin{aligned}
&\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega D^{\alpha-\beta}\omega D^\beta\left(\partial_t \ln p^{1/\gamma} + u\partial_x \ln p^{1/\gamma}\right)\langle y\rangle^{2\lambda+2\alpha_2} D^\alpha\omega \\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^\infty} + 1\right)\|\omega\|^2_{H_g^{s,\lambda}}.
\end{aligned} \tag{59}$$

(6) Now we estimate the term

$$\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega \partial_y D^\beta\rho D^{\alpha-\beta}(\partial_y\omega - \partial_x p)\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega. \tag{60}$$

When $|\beta| = |\alpha| = s$, one has for special pressure p satisfying $\partial_x p = 0$

$$\begin{aligned}
&\int_\Omega \partial_y D^\alpha\tilde\rho\left(\partial_y\omega - \partial_x p\right)\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega \\
=&-\int_\Omega D^\alpha\tilde\rho\partial_y^2\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega - \int_\Omega 2(\lambda+\alpha_2)D^\alpha\tilde\rho\partial_y\omega\langle y\rangle^{2\lambda+2\alpha_2-1}D^\alpha\omega \\
&-\int_\Omega D^\alpha\tilde\rho\partial_y\omega\langle y\rangle^{2\lambda+2\alpha_2}\partial_y D^\alpha\omega + \int_{\mathbb{T}} D^\alpha\tilde\rho\partial_y\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega|_{y=0}^{y=\infty} \\
:=&\sum_{i=1}^4 I_i.
\end{aligned} \tag{61}$$

First, It is obvious that $I_4 = 0$. The terms I_1-I_3 are estimated step by step.

$$\begin{aligned}
I_1 \le&\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|_{L^2}\left\|\langle y\rangle^{\lambda+2}\partial_y^2\omega\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|_{L^2} \\
\le& C\|\tilde\rho\|_{H_h^{s,\lambda-1}}\|\omega\|^2_{H_g^{s,\lambda}}.
\end{aligned} \tag{62}$$

Here $\lambda \ge 1$ is required.

$$\begin{aligned}
I_2 \le& C\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|_{L^2}\|\partial_y\omega\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|_{L^2} \\
\le& C\|\tilde\rho\|_{H_h^{s,\lambda-1}}\|\omega\|^2_{H_g^{s,\lambda}}
\end{aligned} \tag{63}$$

and

$$\begin{aligned}
I_3 \le& C\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|_{L^2}\|\langle y\rangle\partial_y\omega\|_{L^\infty}\|\langle y\rangle^{\lambda+\alpha_2}\partial_y D^\alpha\omega\|_{L^2} \\
\le&\frac{1}{8}\left\|\langle y\rangle^{\lambda+\alpha_2}\sqrt{\rho}\partial_y D^\alpha\omega\right\|^2_{L^2} + C\|\tilde\rho\|^2_{H_h^{s,\lambda-1}}\|\omega\|^2_{H_g^{s,\lambda}}.
\end{aligned} \tag{64}$$

When $|\beta|=0$ and $|\alpha|=s$, then $\alpha_2\geq 1$, we have

$$\begin{aligned}&\int_\Omega \partial_y\tilde\rho D^\alpha(\partial_y\omega-\partial_x p)\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ =&\int_\Omega \partial_y\tilde\rho\partial_y D^\alpha\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ \leq&\frac{1}{8}\left\|\langle y\rangle^{\lambda+\alpha_2}\sqrt{\rho}\partial_y D^\alpha\omega\right\|_{L^2}^2+C\delta\|\omega\|^2_{H^{s,\lambda}_g}.\end{aligned}\tag{65}$$

When $0<|\beta|<s$, we get by using Lemma 1.5

$$\begin{aligned}&\int_\Omega \partial_y D^\beta\tilde\rho D^{\alpha-\beta}\partial_y\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ =&\int_\Omega D^{\beta+e_2}\tilde\rho D^{\alpha-\beta+e_2}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ \leq&\left\|\langle y\rangle^{\lambda+\beta_2}D^{\beta+e_2}\tilde\rho\langle y\rangle^{\lambda+\alpha_2-\beta_2+1}D^{\alpha-\beta+e_2}\omega\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|_{L^2}\\ \leq&C\left\|\tilde\rho\right\|_{H^{s,\lambda-1}_h}\|\omega\|^2_{H^{s,\lambda}_g}.\end{aligned}\tag{66}$$

Collecting (62)-(66), we obtain

$$\begin{aligned}&\sum_{0\leq\beta\leq\alpha}\int_\Omega\binom{\alpha}{\beta}\partial_y D^\beta\rho\cdot D^{\alpha-\beta}(\partial_y\omega-\partial_x p)\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ \leq&\frac{1}{4}\left\|\langle y\rangle^{\lambda+\alpha_2}\sqrt{\rho}\partial_y D^\alpha\omega\right\|^2_{L^2}+C\left(\|\tilde\rho\|_{H^{s,\lambda-1}_h}+\|\tilde\rho\|^2_{H^{s,\lambda-1}_h}+\delta\right)\|\omega\|^2_{H^{s,\lambda}_g}.\end{aligned}\tag{67}$$

(7) Next we estimate the term $\sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta}\int_\Omega D^\beta\rho\partial_y^2D^{\alpha-\beta}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega$. When $\alpha=(s-1,1)$ and $|\beta|=1$, one has

$$\begin{aligned}&\int D^\beta\rho\partial_y^2D^{\alpha-\beta}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ \leq&C\left\|\langle y\rangle^{\lambda+\beta_2-1}D^\beta\rho\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2-\beta_2+1}\partial_y D^{\alpha-\beta+e_2}\omega\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|_{L^2}\\ \leq&\frac{1}{12}\left\|\sqrt{\rho}\langle y\rangle^{\lambda+\alpha_2-\beta_2+1}\partial_y D^{\alpha-\beta+e_2}\omega\right\|^2_{L^2}+C(\|\tilde\rho\|_{H^{s,\lambda-1}_h}+\|\partial_x^s\rho^\infty\|_{L^\infty})^2\|\omega\|^2_{H^{s,\lambda}_g}.\end{aligned}\tag{68}$$

When $2\leq|\beta|\leq s-2$, which means $|\alpha-\beta+2e_2|\leq s$, we have

$$\begin{aligned}&\int D^\beta\rho\partial_y^2D^{\alpha-\beta}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ \leq&C\|\langle y\rangle^{\lambda+\beta_2-1}D^\beta\rho\|_{L^\infty}\|\langle y\rangle^{\lambda+\alpha_2-\beta_2+2}D^{\alpha-\beta+2e_2}\omega\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\\ \leq&C(\|\tilde\rho\|_{H^{s,\lambda-1}_h}+\|\partial_x^s\rho^\infty\|_{L^\infty})\|\omega\|^2_{H^{s,\lambda}_g}.\end{aligned}\tag{69}$$

When $|\beta|\geq s-2$, we get

$$\begin{aligned}&\int D^\beta\rho\partial_y^2D^{\alpha-\beta}\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega\\ \leq&C\|\langle y\rangle^{\lambda+\beta_2-1}D^\beta\tilde\rho\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2-\beta_2+2}D^{\alpha-\beta+2e_2}\omega\|_{L^\infty}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\\ &+C\|D^\beta\rho^\infty\|_{L^\infty}\|\langle y\rangle^{\lambda+\alpha_2+2}D^{\alpha-\beta+2e_2}\omega\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^2}\\ \leq&C\left(\|\tilde\rho\|_{H^{s,\lambda-1}_h}+\|\partial_x^s\rho^\infty\|_{L^\infty(\mathbb{T})}\right)\|\omega\|^2_{H^{s,\lambda}_g}.\end{aligned}\tag{70}$$

(8) Finally we deal with the term $\int_\Omega\rho\partial_y^2D^\alpha\omega\langle y\rangle^{2\lambda+2\alpha_2}D^\alpha\omega$.

Integrating by parts, one has

$$\begin{aligned}
&\int_{\Omega} \rho \partial_y^2 D^\alpha \omega \langle y\rangle^{2\lambda+2\alpha_2} D^\alpha \omega \\
=& -\left\|\sqrt{\rho}\langle y\rangle^{\lambda+\alpha_2}(\partial_y D^\alpha \omega)\right\|_{L^2}^2 - \int_{\Omega} \partial_y \tilde{\rho} \partial_y D^\alpha \omega \langle y\rangle^{2\lambda+2\alpha_2} D^\alpha \omega \\
&- 2(\lambda+\alpha_2)\int_{\Omega} \rho \partial_y D^\alpha \omega \langle y\rangle^{2\lambda+2\alpha_2-1} D^\alpha \omega \\
&+ \int_{\mathbb{T}} \rho \partial_y D^\alpha \omega \langle y\rangle^{2\lambda+2\alpha_2} D^\alpha \omega|_{y=0}^{y=\infty} \\
:=& -\left\|\sqrt{\rho}\langle y\rangle^{\lambda+\alpha_2}(\partial_y D^\alpha \omega)\right\|_{L^2}^2 + \sum_{i=5}^{7} I_i.
\end{aligned} \tag{71}$$

Noticing that $\rho \geq \delta$, thus we have

$$\begin{aligned}
I_5 \leq& \|\partial_y \tilde{\rho}\|_{L^\infty} \|\langle y\rangle^{\lambda+\alpha_2} \partial_y D^\alpha \omega\|_{L^2} \|\langle y\rangle^{\lambda+\alpha_2} D^\alpha \omega\|_{L^2} \\
\leq& \frac{1}{12}\left\|\langle y\rangle^{\lambda+\alpha_2}\sqrt{\rho}\partial_y D^\alpha \omega\right\|_{L^2}^2 + C\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}^2 \|\omega\|_{H_g^{s,\lambda}}^2
\end{aligned} \tag{72}$$

and

$$\begin{aligned}
I_6 \leq& C\|\rho\|_{L^\infty} \|\langle y\rangle^{\lambda+\alpha_2} \partial_y D^\alpha \omega\|_{L^2} \|\langle y\rangle^{\lambda+\alpha_2} D^\alpha \omega\|_{L^2} \\
\leq& \frac{1}{8}\left\|\langle y\rangle^{\lambda+\alpha_2}\sqrt{\rho}\partial_y D^\alpha \omega\right\|_{L^2}^2 + C\|\rho\|_{L^\infty}^2 \|\omega\|_{H_g^{s,\lambda}}^2.
\end{aligned} \tag{73}$$

By using the boundary reduction and tedious calculations, the term I_7 is estimated in the following lemma.

Lemma 2.2 (The boundary reduction)**.** *Under the hypotheses of Lemma 2.1, for any $1 \leq k \leq \frac{s}{2}$, there are some constant C which does not depend on ϵ, such that,*

$$\begin{aligned}
&\rho^k \partial_y^{2k+1} \omega|_{y=0} \\
=& \sum_{\mathcal{A}_{m,n,p}} C\left(D^{\alpha^1}\omega \cdots D^{\alpha^m}\omega \cdot D^{\beta^1}\rho \cdots D^{\beta^n}\rho \partial_t^{l_1} \ln p^{1/\gamma} \cdots \partial_t^{l_p} \ln p^{1/\gamma}\right)|_{y=0},
\end{aligned} \tag{74}$$

where $\mathcal{A}_{m,n,p}$ is denoted by

$$\Big\{ \sum_{i=1}^{n} |\alpha^i| + \sum_{j=1}^{m} |\beta^i| \leq 2k+1, \max(|\alpha^i|) \leq 2k, \max(|\beta^j|) \leq 2k-1, \max(l_q) \leq k-1$$
$$\text{for } i = 1, \cdots, m,\ j = 1, \cdots, n \text{ and } q = 1, \cdots, p (m \leq k+1, n \leq k, p \leq k-1)\Big\}.$$

Proof. When $k = 1$, we have

$$\rho \partial_y^3 \omega|_{y=0} = \omega \partial_x \omega|_{y=0} - 2\partial_y \rho \partial_y^2 \omega|_{y=0}.$$

It is obvious that (74) holds.

Assume (74) hold for k, then taking $2k+1$ times derivatives on the equation $(23)_2$, one has

$$\begin{aligned}
&\rho^{k+1}\partial_y^{2k+3}\omega|_{y=0}\\
=&\big\{(\partial_t-\epsilon^2\partial_x^2)(\rho^k\partial_y^{2k+1}\omega)-(\partial_t-\epsilon^2\partial_x^2)\rho^k\partial_y^{2k+1}\omega+2\epsilon^2\partial_x\rho^k\partial_x\partial_y^{2k+1}\omega\\
&+\rho^k\sum_{i=1}^{2k+1}C_{2k+1}^i\partial_y^{i-1}\omega\partial_x\partial_y^{2k+1-i}\omega-\rho^k\sum_{i=2}^{2k+1}C_{2k+1}^i\partial_y^{i-2}\partial_x\omega\partial_y^{2k+2-i}\omega\\
&-(2k+2)\rho^k\partial_y^{2k+1}\omega\partial_t\ln p^{1/\gamma}\rho^k+\sum_{i=0}^{2k}C_{2k+1}^i\partial_y^{i+1}\rho\partial_y^{2k+2-i}\omega\\
&+\rho^k\sum_{i=1}^{2k+1}C_{2k+1}^i\partial_y^i\rho\partial_y^{2k+3-i}\omega\big\}|_{y=0}.
\end{aligned}\tag{75}$$

Here we used the boundary conditions $u|_{y=0}=v|_{y=0}=0$ and $\partial_y\omega|_{y=0}=0$.

By checking the index, it is enough to deal with the first and second terms on the right-hand side of (75).

Firstly, we have

$$\begin{aligned}
&\left(\partial_t-\epsilon^2\partial_x^2\right)\rho^k\partial_y^{2k+1}\omega\\
=&k\rho^{k-1}(\partial_t-\epsilon^2\partial_x^2)\rho\partial_y^{2k+1}\omega-\epsilon^2k(k-1)\rho^{k-2}(\partial_x\rho)^2\partial_y^{2k+1}\omega\\
=&k\rho^k\partial_t\ln p^{1/\gamma}\partial_y^{2k+1}\omega-\epsilon^2k(k-1)\rho^{k-2}(\partial_x\rho)^2\partial_y^{2k+1}\omega,
\end{aligned}\tag{76}$$

which satisfies Lemma 2.2. Here we use $(4)_1$ and $u|_{y=0}=v|_{y=0}=0$.

Now we move to deal with the second term on the right-hand side of (75).

$$\begin{aligned}
&(\partial_t-\epsilon^2\partial_x^2)(\rho^k\partial_y^{2k+1}\omega)\\
=&(\partial_t-\epsilon^2\partial_x^2)\sum_{\mathcal{A}_{m,n,p}}C\left(D^{\alpha^1}\omega\cdots D^{\alpha^m}\omega D^{\beta^1}\rho\cdots D^{\beta^n}\rho\partial_t^l\ln p^{1/\gamma}\right).
\end{aligned}\tag{77}$$

Three cases should be considered:

(1) The derivative operator $\partial_t-\epsilon^2\partial_x^2$ on $D^\alpha\omega$;

(2) The derivative operator $\partial_t-\epsilon^2\partial_x^2$ on $D^\alpha\rho$;

(3) ∂_x^2 separate to $D^\alpha\omega$ and $D^\alpha\rho$.

Case 1. From $(23)_2$ and boundary conditions, we have

$$\begin{aligned}
\left(\partial_t-\epsilon^2\partial_x^2\right)(D^\theta\omega)=&-\sum_{0<\beta\le\theta}\tbinom{\theta}{\beta}D^\beta u\partial_xD^{\theta-\beta}\omega-\sum_{0<\beta\le\theta}\tbinom{\theta}{\beta}D^\beta v\partial_yD^{\theta-\beta}\omega\\
&+D^\theta\omega\partial_t\ln p^{1/\gamma}+\sum_{0<\beta\le\theta}\tbinom{\theta}{\beta}\partial_yD^{\theta-\beta}\rho\partial_yD^\beta\omega\\
&+\sum_{0\le\beta\le\theta}\tbinom{\theta}{\beta}D^\beta\rho\partial_y^2D^{\theta-\beta}\omega,
\end{aligned}\tag{78}$$

where $|\theta|\le 2k$. We can check each term, which all satisfies Lemma 2.2.

Case 2. From $(4)_1$ and boundary conditions, we have

$$\begin{aligned}
\left(\partial_t-\epsilon^2\partial_x^2\right)(D^\kappa\rho)=&-\sum_{0<\beta\le\kappa}\tbinom{\kappa}{\beta}D^\beta u\partial_xD^{\kappa-\beta}\rho-\sum_{0<\beta\le\kappa}\tbinom{\kappa}{\beta}D^\beta v\partial_yD^{\kappa-\beta}\rho\\
&-\sum_{0\le\beta\le\kappa}\tbinom{\kappa}{\beta}D^\beta\rho D^{\kappa-\beta}\partial_t\ln p^{1/\gamma},
\end{aligned}\tag{79}$$

where we notice $|\kappa| \le 2k-1$. Similar to case 1, we also can check each term, which all satisfies Lemma 2.2.

Case 3. This situation is much easier than cases 1 and 2. We only need to check the terms like $-2\epsilon^2\partial_x(D^{\alpha^i}\omega)\partial_x(D^{\beta^j}\rho)$, $i \le m, j \le n$, it is obvious that all these terms satisfy Lemma 2.2. This completes the proof of Lemma 2.2. □

Now we continue to estimate I_7 in two cases.

When $\alpha_2 = 2k$, we have from Lemma 2.2,

$$\begin{aligned} I_7 =& \int_{\mathbb{T}} \rho\partial_y D^\alpha \omega D^\alpha \omega dx|_{y=0} \\ =& C\sum_{\mathcal{A}_{m,n,p}} \int_{\mathbb{T}} \rho^{1-k} D^\alpha\omega \partial_x^{\alpha_1}\left(D^{\theta^1}\omega\cdots D^{\theta^m}\omega D^{\kappa^1}\rho\cdots D^{\kappa^n}\rho\right) dx|_{y=0} \\ &- C\sum_{\mathcal{A}_{m,n,p}} \int_{\mathbb{T}} \rho^{1-k} D^\alpha\omega\left(\sum_{i=1}^{\alpha_1} C_{\alpha_1}^i \partial_x^i(\rho^k)\partial_x^{\alpha_1-i}\partial_y^{2k+1}\omega\right) dx|_{y=0}. \end{aligned}$$

It is observed that the highest order of the term $\partial_x^{\alpha_1} D^{\theta_i}\omega$ is s and the highest order of the term $\partial_x^{\alpha_1} D^{\kappa_i}\tilde{\rho}$ is $s-1$. At most one of all terms like $\partial_x^{\alpha_1} D^{\theta_i}\omega$ can attain the order s, the order of other terms is smaller than $s-2$ due to $s \ge 6$. In addition, the maximum of the index α_1 is $s-2$ since s is even and $\alpha_1 \le s-1$, which implies $\alpha_1 = s-2k$ is also even. As in [27], employing Cauchy inequality and trace estimate, one has

$$\begin{aligned} I_7 \le& \frac{1}{12}\sum_{|\alpha|\le s, \alpha_1 \le s-1} \left\|\langle y\rangle^{\lambda+\alpha_2}\sqrt{\rho}\partial_y D^\alpha\omega\right\|_{L^2}^2 \\ &+ C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^\infty} + 1\right)^{s+2}\left(\|\tilde{\rho}\|_{H_h^{s,\lambda-1}} + \|\partial_x^s\rho^\infty\|_{L^\infty} + 1\right)^s. \end{aligned} \tag{80}$$

When $\alpha_2 = 2k+1$, it is easy to follow $\partial_x^{\alpha_1}\partial_y\omega|_{y=0} = 0$ due to $\partial_y\omega|_{y=0} = 0$. Thus we only need to consider the case of $k \ge 1$. Meanwhile, $\alpha_1 \ge 1$ since s is even. Then by employing integration by parts in the variable x, we have

$$\begin{aligned} I_7 =& \int_{\mathbb{T}} \rho\partial_y D^\alpha\omega D^\alpha\omega dx|_{y=0} \\ =& -\int_{\mathbb{T}} \partial_x\rho\partial_x^{\alpha_1-1}\partial_y^{\alpha_2+1}\omega\partial_x^{\alpha_1}\partial_y^{\alpha_2}\omega dx|_{y=0} - \int_{\mathbb{T}} \rho\partial_x^{\alpha_1-1}\partial_y^{\alpha_2+1}\omega\partial_x^{\alpha_1+1}\partial_y^{\alpha_2}\omega dx|_{y=0}. \end{aligned}$$

Similar to the case $\alpha_2 = 2k$, the same inequality (80) is obtained and the details are omitted here.

Finally combining all the above estimates, we obtain the estimate (43). This completes the proof of Lemma 2.1. □

2.2. **The case of $\alpha_1 = s$.** Recalling notations $g_s = \partial_x^s\omega - \frac{\partial_y\omega}{\omega}\partial_x^s\tilde{u}$, $a = \frac{\partial_y\omega}{\omega}$ and the equations satisfied by $(\omega, \tilde{u})$ for special pressure $p = p(t)$

$$\partial_t\omega + u\partial_x\omega + v\partial_y\omega = \frac{\omega}{\gamma}\partial_t \ln p + \partial_y\rho\partial_y\omega + \epsilon^2\partial_x^2\omega + \rho\partial_y^2\omega \tag{81}$$

and

$$\partial_t\tilde{u} + u\partial_x\tilde{u} + v\partial_y\tilde{u} = \rho\partial_y^2\tilde{u} + \epsilon^2\partial_x^2\tilde{u} - \tilde{u}\partial_x U, \tag{82}$$

differentiating the equations (81) and (82) s times with respect to x respectively, one has

$$\begin{aligned}&(\partial_t + u\partial_x + v\partial_y)\,\partial_x^s\omega + \sum_{i=1}^{s}\binom{s}{i}\partial_x^i u\partial_x^{s-i+1}\omega + \sum_{i=1}^{s-1}\binom{s}{i}\partial_x^i v\partial_x^{s-i}\partial_y\omega + \partial_x^s v\partial_y\omega\\&= \partial_x^s\left(\frac{\omega}{\gamma}\partial_t \ln p\right) + \partial_x^s(\partial_y\rho\partial_y\omega) + \partial_x^s(\rho\partial_y^2\omega) + \epsilon^2\partial_x^{s+2}\omega\end{aligned} \tag{83}$$

and

$$\begin{aligned}&(\partial_t + u\partial_x + v\partial_y)\,\partial_x^s\tilde{u} + \sum_{i=1}^{s}\binom{s}{i}\partial_x^i u\partial_x^{s-i+1}\tilde{u} + \sum_{i=1}^{s-1}\binom{s}{i}\partial_x^i v\partial_x^{s-i}\partial_y\tilde{u} + \partial_x^s v\omega\\&= \partial_x^s\left(\rho\partial_y^2\tilde{u}\right) + \epsilon^2\partial_x^{s+2}\tilde{u} - \partial_x^s\left(\tilde{u}\partial_x U\right).\end{aligned} \tag{84}$$

Subtracting (84)$\times\frac{\partial_y\omega}{\omega}$ from (83), we get the equation satisfied by g_s by tedious calculations

$$\begin{aligned}&\left(\partial_t + u\partial_x + v\partial_y - \epsilon^2\partial_x^2 - \rho\partial_y^2\right) g_s\\=&2\epsilon^2\left(\partial_x^{s+1}\tilde{u} - \frac{\partial_x\omega}{\omega}\partial_x\tilde{u}\right)\partial_x a + 2\rho\partial_y a g_s - g_1\partial_x^s U\\&-\left(\partial_y^2\rho a + 2\partial_y\rho\frac{\partial_y^2\omega}{\omega} - \partial_y\rho a^2 + \frac{a}{\gamma}\partial_t\ln p\right)\partial_x^s\tilde{u}\\&-\sum_{j=1}^{s-1}\binom{s}{j}\partial_x^{s-j}u g_{j+1} - \sum_{j=1}^{s-1}\binom{s}{j}\partial_x^{s-j}v\left(\partial_x^j\partial_y\omega - a\partial_x^j\partial_y\tilde{u}\right)\\&+a\sum_{j=0}^{s-1}\binom{s}{j}\partial_x^j\tilde{u}\partial_x^{s-j+1}U + \sum_{i=1}^{s}\binom{s}{i}\partial_x^i\rho\partial_y^2\partial_x^{s-i}\omega - a\sum_{i=1}^{s}\binom{s}{i}\partial_x^i\rho\partial_y^2\partial_x^{s-i}\tilde{u}\\&+\partial_x^s\left(\partial_y\rho\partial_y\omega\right) + \partial_x^s\left(\frac{\omega}{\gamma}\partial_t\ln p\right).\end{aligned} \tag{85}$$

Lemma 2.3. *Under the assumptions of Theorem 1.1, we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\langle y\rangle^\lambda g_s\right\|_{L^2}^2 + \frac{\epsilon^2}{2}\left\|\langle y\rangle^\lambda\partial_x g_s\right\|_{L^2}^2 + \frac{3}{4}\left\|\sqrt{\rho}\langle y\rangle^\lambda\partial_y g_s\right\|_{L^2}^2\\\le&\frac{1}{4}\left\|\langle y\rangle^{\lambda+1}\sqrt{\rho}\partial_y(\partial_y\partial_x^{s-1}\omega)\right\|_{L^2}^2\\&+C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^{s+1}U\|_{L^2(\mathbb{T})} + 1\right)^3\left(\|\tilde{\rho}\|_{H_h^{s,\lambda-1}} + \|\partial_x^s\rho^\infty\|_{L^\infty(\mathbb{T})} + 1\right)^4.\end{aligned} \tag{86}$$

Proof. Multiplying the equation (85) by $\langle y\rangle^{2\lambda}g_s$ and integrating the resulting equation over Ω, then estimate term by term as follows.

(1) It is obvious to get

$$\int_\Omega \partial_t g_s\langle y\rangle^{2\lambda}g_s = \frac{1}{2}\frac{d}{dt}\left\|\langle y\rangle^\lambda g_s\right\|_{L^2}^2 \tag{87}$$

and

$$\int_\Omega \epsilon^2\partial_x^2 g_s\langle y\rangle^{2\lambda}g_s = -\epsilon^2\|\langle y\rangle^\lambda\partial_x g_s\|_{L^2}^2. \tag{88}$$

(2) Integrating by parts and using Lemma 1.4, one has

$$\int_\Omega (u\partial_x g_s + v\partial_y g_s)\langle y\rangle^{2\lambda}g_s \tag{89}$$

$$
\begin{aligned}
&=-\frac{1}{2}\int_\Omega(\partial_x u+\partial_y v)\langle y\rangle^{2\lambda}g_s^2-\lambda\int_\Omega v\langle y\rangle^{2\lambda-1}g_s^2+\int_{\mathbb{T}} v\langle y\rangle^{2\lambda}g_s^2\big|_{y=0}^{y=+\infty}\\
&=\frac{1}{2}\int_\Omega \partial_t\ln p^{1/\gamma}\langle y\rangle^{2\lambda}g_s^2-\lambda\int_\Omega\frac{v}{1+y}\langle y\rangle^{2\lambda}g_s^2\\
&\le\frac{1}{2}\|\partial_t\ln p^{1/\gamma}\|_{L^\infty}\|\langle y\rangle^\lambda g_s\|_{L^2}^2+\lambda\left\|\frac{v}{1+y}\right\|_{L^\infty}\left\|\langle y\rangle^\lambda g_s\right\|_{L^2}^2\\
&\le C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}+1\right)\|\omega\|_{H_g^{s,\lambda}}^2.
\end{aligned}
$$

(3) Integrating by part in variable y, we have

$$
\begin{aligned}
\int_\Omega\rho\partial_y^2g_s\langle y\rangle^{2\lambda}g_s=&-\left\|\sqrt{\rho}\langle y\rangle^\lambda\partial_yg_s\right\|_{L^2}^2-\int_\Omega\partial_y\rho\langle y\rangle^{2\lambda}\partial_yg_sg_s\\
&-2\lambda\int_\Omega\rho\langle y\rangle^{2\lambda-1}\partial_yg_s\cdot g_s+\int_{\mathbb{T}}\rho\partial_yg_s\cdot g_s|_{y=0}\\
:=&-\left\|\sqrt{\rho}\langle y\rangle^\lambda\partial_yg_s\right\|_{L^2}^2+\sum_{i=8}^{10}I_i.
\end{aligned}
\tag{90}
$$

Firstly, the terms I_8 and I_9 are estimated as follows.

$$
\begin{aligned}
I_8\le&\frac{1}{24}\|\langle y\rangle^\lambda\sqrt{\rho}(\partial_yg_s)\|_{L^2}^2+C\|\langle y\rangle^\lambda g_s\|_{L^2}^2\|\partial_y\rho\|_{L^\infty}^2\\
\le&\frac{1}{24}\|\langle y\rangle^\lambda\sqrt{\rho}(\partial_yg_s)\|_{L^2}^2+C\|\tilde\rho\|_{H_h^{s,\lambda-1}}^2\|\omega\|_{H_g^{s,\lambda}}^2
\end{aligned}
\tag{91}
$$

and

$$
\begin{aligned}
I_9\le&\frac{1}{24}\left\|\langle y\rangle^\lambda\sqrt{\rho}\partial_yg_s\right\|_{L^2}^2+C\|\rho\|_{L^\infty}^2\left\|\langle y\rangle^\lambda g_s\right\|_{L^2}^2\\
\le&\frac{1}{24}\left\|\langle y\rangle^\lambda\sqrt{\rho}(\partial_yg_s)\right\|_{L^2}^2+C\left(\|\tilde\rho\|_{H_h^{s,\lambda-1}}+\|\rho^\infty\|_{L^\infty(\mathbb{T})}\right)^2\|\omega\|_{H_g^{s,\lambda}}^2.
\end{aligned}
\tag{92}
$$

Next we deal with the boundary term I_{10}. Using the boundary conditions $\partial_y\omega|_{y=0}=0$ and $u|_{y=0}=0$, one has

$$
\partial_yg_s|_{y=0}=\left.\frac{\partial_y^2\omega}{\omega}\partial_x^sU\right|_{y=0}.
\tag{93}
$$

Thus, we have by employing Cauchy inequality, $\left|\frac{\partial_y^2\omega}{\omega}|_{y=0}\right|\le\frac{1}{\delta^2}$ and trace estimate

$$
\begin{aligned}
I_{10}=&\int_{\mathbb{T}}\rho\left.\frac{\partial_y^2\omega}{\omega}\partial_x^sUg_s\right|_{y=0}dx\\
\le&\frac{1}{2\delta^2}\int_{\mathbb{T}}(\partial_x^sU)^2dx+\frac{1}{2\delta^2}\int_{\mathbb{T}}\rho^2g_s^2|_{y=0}dx\\
\le&\frac{1}{12}\left\|\langle y\rangle^\lambda\sqrt{\rho}\partial_yg_s\right\|_{L^2}^2\\
&+C\left(\|\tilde\rho\|_{H_h^{s,\lambda-1}}+\|\rho^\infty\|_{L^\infty(\mathbb{T})}+1\right)^4\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2(\mathbb{T})}\right)^2.
\end{aligned}
\tag{94}
$$

(4) The term $\int_\Omega 2\epsilon^2\left(\partial_x^{s+1}\tilde u-\frac{\partial_x\omega}{\omega}\partial_x\tilde u\right)\partial_xa\langle y\rangle^{2\lambda}g_s$ is estimated as follows.

Due to $\delta\le\langle y\rangle^\lambda\omega,\ \ \sum_{|\alpha|\le2}|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega|^2\le\delta^{-2}$, which means $|\langle y\rangle\partial_xa|\le\delta^{-2}$, we have

$$
2\epsilon^2\int_\Omega\partial_x^{s+1}\tilde u\partial_xa\langle y\rangle^{2\lambda}g_s\le2\delta^{-2}\epsilon^2\|\langle y\rangle^{\lambda-1}\partial_x^{s+1}\tilde u\|_{L^2}\|\langle y\rangle^\lambda g_s\|_{L^2}.
\tag{95}
$$

Noted that $\omega\partial_y\left(\frac{\partial_x^{s+1}\tilde{u}}{\omega}\right)=g_{s+1}=\partial_x g_s+\partial_x a\partial_x^s\tilde{u}$, we have

$$\begin{aligned}\|\langle y\rangle^{\lambda-1}\partial_x^{s+1}\tilde{u}\|_{L^2}=&\left\|\langle y\rangle^{\lambda-1}\omega\frac{\partial_x^{s+1}\tilde{u}}{\omega}\right\|_{L^2}\\ \leq&\frac{1}{\delta}\left\|\langle y\rangle^{-1}\frac{\partial_x^{s+1}\tilde{u}}{\omega}\right\|_{L^2}\\ \leq&C\|\partial_x^{s+1}U\|_{L^2}+C\left\|\langle y\rangle^{\lambda}\omega\partial_y\left(\frac{\partial_x^{s+1}\tilde{u}}{\omega}\right)\right\|_{L^2}\\ \leq&C\left(\|\partial_x^{s+1}U\|_{L^2}+\|\langle y\rangle^{\lambda}\partial_x g_s\|_{L^2}+\|\langle y\rangle^{\lambda}\partial_x a\partial_x^s\tilde{u}\|_{L^2}\right).\end{aligned}\tag{96}$$

Here we use the fact $\delta\leq\langle y\rangle^{\lambda}\omega\leq\frac{1}{\delta}$ and Lemma 1.2.

Thus, we get

$$\begin{aligned}&2\epsilon^2\int_\Omega\partial_x^{s+1}\tilde{u}\partial_x a\langle y\rangle^{2\lambda}g_s\\ \leq&C\epsilon^2\|\langle y\rangle^{\lambda-1}\partial_x^{s+1}\tilde{u}\|_{L^2}\|\langle y\rangle^{\lambda}g_s\|_{L^2}\\ \leq&C\epsilon^2\left(\|\partial_x^{s+1}U\|_{L^2}+\|\langle y\rangle^{\lambda}\partial_x g_s\|_{L^2}+\|\langle y\rangle^{\lambda}\partial_x a\partial_x^s\tilde{u}\|_{L^2}\right)\|\langle y\rangle^{\lambda}g_s\|_{L^2}.\end{aligned}\tag{97}$$

Similarly, we obtain

$$\begin{aligned}&\int_\Omega 2\epsilon^2\frac{\partial_x\omega}{\omega}\partial_x\tilde{u}\partial_x a\langle y\rangle^{2\lambda}g_s\\ \leq&2\epsilon^2\left\|\frac{\partial_x\omega}{\omega}\right\|_{L^\infty}\|\langle y\rangle\partial_x a\|_{L^\infty}\|\langle y\rangle^{\lambda-1}\partial_x\tilde{u}\|_{L^2}\|\langle y\rangle^{\lambda}g_s\|_{L^2}\\ \leq&2\epsilon^2\delta^{-4}\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2}\right)\|\omega\|_{H_g^{s,\lambda}}.\end{aligned}\tag{98}$$

Thus we have

$$\begin{aligned}&\int_\Omega 2\epsilon^2\left(\partial_x^{s+1}\tilde{u}-\frac{\partial_x\omega}{\omega}\partial_x\tilde{u}\right)\partial_x a\cdot\langle y\rangle^{2\lambda}g_s\\ \leq&\frac{\epsilon^2}{2}\left\|\langle y\rangle^{\lambda}\partial_x g_s\right\|_{L^2}^2+C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^{s+1}U\|_{L^2}\right)\|\omega\|_{H_g^{s,\lambda}}.\end{aligned}\tag{99}$$

(5) The terms $\int_\Omega 2\rho\partial_y a g_s\langle y\rangle^{2\lambda}g_s$ and $-\int_\Omega g_1\partial_x^s U\langle y\rangle^{2\lambda}g_s$ are controlled due to $|\partial_y a|\leq C$.

$$\int_\Omega 2\rho\partial_y a g_s\langle y\rangle^{2\lambda}g_s\leq C\left(\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}+\|\rho^\infty\|_{L^\infty}\right)\|\omega\|_{H_g^{s,\lambda}}^2\tag{100}$$

and

$$-\int_\Omega g_1\partial_x^s U\langle y\rangle^{2\lambda}g_s\leq\|\partial_x^s U\|_{L^\infty(\mathbb{T})}\|\omega\|_{H_g^{s,\lambda}}\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2(\mathbb{T})}\right).\tag{101}$$

(6) Next we estimate the term

$$-\int_\Omega\left(\partial_y^2\rho a+2\partial_y\rho\frac{\partial_y^2\omega}{\omega}-\partial_y\rho a^2+\frac{a}{\gamma}\partial_t\ln p\right)\partial_x^s\tilde{u}\langle y\rangle^{2\lambda}g_s,\tag{102}$$

which is divided into four terms to estimate.

$$\begin{aligned} &-\int_\Omega 2\partial_y^2\rho a\partial_x^s\tilde{u}\langle y\rangle^{2\lambda}g_s \\ \le& \left\|2\partial_y^2\rho\langle y\rangle a\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda-1}(\partial_x^s\tilde{u})\right\|_{L^2}\left\|\langle y\rangle^\lambda g_s\right\|_{L^2} \\ \le& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\|\omega\|_{H_g^{s,\lambda}}. \end{aligned} \tag{103}$$

Here we used $\|\langle y\rangle a\|_{L^\infty}\le C$.

$$\begin{aligned} &-\int_\Omega 2\partial_y\rho\frac{\partial_y^2\omega}{\omega}\partial_x^s\tilde{u}\langle y\rangle^{2\lambda}g_s \\ \le& \left\|2\langle y\rangle\partial_y\rho\frac{\partial_y^2\omega}{\omega}\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda-1}(\partial_x^s\tilde{u})\right\|_{L^2}\left\|\langle y\rangle^\lambda g_s\right\|_{L^2} \\ \le& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\|\omega\|_{H_g^{s,\lambda}}. \end{aligned} \tag{104}$$

Here we use $\left\|\langle y\rangle\frac{\partial_y^2\omega}{\omega}\right\|_{L^\infty}\le C$ and Lemma 1.4.

$$\begin{aligned} &-\int_\Omega 2\partial_y\rho a^2\partial_x^s\tilde{u}\langle y\rangle^{2\lambda}g_s \\ \le& \left\|2\langle y\rangle\partial_y\rho a^2\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda-1}\partial_x^s\tilde{u}\right\|_{L^2}\left\|\langle y\rangle^\lambda g_s\right\|_{L^2} \\ \le& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\|\omega\|_{H_g^{s,\lambda}} \end{aligned} \tag{105}$$

and

$$\begin{aligned} &\int_\Omega \frac{a}{\gamma}\partial_t\ln p\partial_x^s\tilde{u}\langle y\rangle^{2\lambda}g_s \\ \le& \left\|\langle y\rangle\frac{a}{\gamma}\partial_t\ln p\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda-1}\partial_x^s\tilde{u}\right\|_{L^2}\left\|\langle y\rangle^\lambda g_s\right\|_{L^2} \\ \le& C\|\omega\|_{H_g^{s,\lambda}}\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right). \end{aligned} \tag{106}$$

Combining the estimates (103)-(106), we obtain

$$\begin{aligned} &-\int_\Omega\left(\partial_y^2\rho a+2\partial_y\rho\frac{\partial_y^2\omega}{\omega}-\partial_y\rho a^2+\frac{a}{\lambda}\partial_t\ln p\right)\partial_x^s\tilde{u}\langle y\rangle^{2\lambda}g_s \\ \le& C\|\omega\|_{H_g^{s,\lambda}}\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\left(\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}+1\right). \end{aligned} \tag{107}$$

(7) The estimate on the term $-\sum_{j=1}^{s-1}\binom{s}{j}\int_\Omega\partial_x^{s-j}ug_{j+1}\langle y\rangle^{2\lambda}g_s$.
In fact, one has for all $j=1\cdots s-1$

$$\begin{aligned} &\int_\Omega\partial_x^{s-j}ug_{j+1}\langle y\rangle^{2\lambda}g_s \\ \le& \left\|\partial_x^{s-j}u\right\|_{L^\infty}\|\langle y\rangle^\lambda g_{j+1}\|_{L^2}\|\langle y\rangle^\lambda g_s\|_{L^2} \\ \le& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\|\omega\|^2_{H_g^{s,\lambda}}. \end{aligned} \tag{108}$$

(8) The estimate on the term $-\sum_{j=1}^{s-1}\binom{s}{j}\int_\Omega\partial_x^{s-j}v(\partial_x^j\partial_y\omega-a\partial_x^j\partial_y\tilde{u})\langle y\rangle^{2\lambda}g_s$ is divided into two cases.

Case 1. When $j=1$, we have

$$\begin{aligned}
&\int_\Omega \partial_x^{s-1}v(\partial_x\partial_y\omega - a\partial_x\partial_y\tilde{u})\langle y\rangle^{2\lambda}g_s\\
\le&\int_\Omega \frac{\partial_x^{s-1}v+y\partial_x^sU}{1+y}\langle y\rangle(\partial_x\partial_y\omega - a\partial_x\partial_y\tilde{u})\langle y\rangle^{2\lambda}g_s\\
&+\int_\Omega \partial_x^sU\langle y\rangle(\partial_x\partial_y\omega - a\partial_x\partial_y\tilde{u})\langle y\rangle^{2\lambda}g_s\\
\le&\left\|\frac{\partial_x^{s-1}v+y\partial_x^sU}{1+y}\right\|_{L^2}\|\langle y\rangle^{\lambda+1}(\partial_x\partial_y\omega - a\partial_x\partial_y\tilde{u})\|_{L^\infty}\|\langle y\rangle^\lambda g_s\|_{L^2}\\
&+\|\partial_x^sU\|_{L^\infty}\|\langle y\rangle^{\lambda+1}(\partial_x\partial_y\omega - a\partial_x\partial_y\tilde{u})\|_{L^2}\|\langle y\rangle^\lambda g_s\|_{L^2}\\
\le&C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^\infty}\right)\|\omega\|^2_{H_g^{s,\lambda}}.
\end{aligned}\tag{109}$$

Case 2. When $j=2,\cdots,s-1$, we have

$$\begin{aligned}
&\int_\Omega \partial_x^{s-j}v(\partial_x^j\partial_y\omega - a\partial_x^j\partial_y\tilde{u})\langle y\rangle^{2\lambda}g_s\\
=&\int_\Omega \frac{\partial_x^{s-j}v}{1+y}\langle y\rangle(\partial_x^j\partial_y\omega - a\partial_x^j\partial_y\tilde{u})\langle y\rangle^{2\lambda}g_s\\
\le&\left\|\frac{\partial_x^{s-j}v}{1+y}\right\|_{L^\infty}\|\langle y\rangle^{\lambda+1}(\partial_x^j\partial_y\omega - a\partial_x^j\partial_y\tilde{u})\|_{L^2}\|\langle y\rangle^\lambda g_s\|_{L^2}\\
\le&C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}+1\right)\|\omega\|^2_{H_g^{s,\lambda}}.
\end{aligned}\tag{110}$$

Combining the estimates (109)-(110), we get

$$\begin{aligned}
&-\sum_{j=1}^{s-1}\binom{s}{j}\int_\Omega \partial_x^{s-j}v(\partial_x^j\partial_y\omega - a\partial_x^j\partial_y\tilde{u})\langle y\rangle^{2\lambda}g_s\\
\le&C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^\infty}+1\right)\|\omega\|^2_{H_g^{s,\lambda}}.
\end{aligned}\tag{111}$$

(9) The estimate on the term $-\sum_{j=0}^{s-1}\binom{s}{j}\int_\Omega a\partial_x^j\tilde{u}\partial_x^{s-j+1}U\langle y\rangle^{2\lambda}g_s$.
In fact, one has for all $j=0,\cdots,s-1$

$$\begin{aligned}
&-\sum_{j=1}^{s-1}\int_\Omega a\partial_x^j\tilde{u}\partial_x^{s-j+1}U\langle y\rangle^{2\lambda}g_s\\
\le&C\left\|\langle y\rangle a\cdot\partial_x^{s+1}U\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda-1}\partial_x^j\tilde{u}\right\|_{L^2}\|\langle y\rangle^\lambda g_s\|_{L^2}\\
\le&C\left\|\partial_x^{s+1}U\right\|_{L^\infty}\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\|\omega\|_{H_g^{s,\lambda}}.
\end{aligned}\tag{112}$$

Here we used $\|\langle y\rangle a\|_{L^\infty}\le C$.

(10) The estimate on the term $\sum_{i=1}^{s}\binom{s}{i}\int_\Omega \partial_x^i\rho\partial_y^2\partial_x^{s-i}\omega\langle y\rangle^{2\lambda}g_s$ is divided into four cases.

Case 1. When $i=1$, we have

$$\begin{aligned}
&\int_\Omega \partial_x\rho\partial_y^2\partial_x^{s-1}\omega\langle y\rangle^{2\lambda}g_s\\
\le&C\|\partial_x\rho\|_{L^\infty}\|\langle y\rangle^{\lambda+1}\sqrt{\rho}\partial_y(\partial_y\partial_x^{s-1}\omega)\|_{L^2}\|\langle y\rangle^\lambda g_s\|_{L^2}
\end{aligned}\tag{113}$$

$$\leq\frac{1}{4}\|\langle y\rangle^{\lambda+1}\sqrt{\rho}\partial_y(\partial_y\partial_x^{s-1}\omega)\|_{L^2}^2+C\|\partial_x\rho\|_{L^\infty}^2\|\langle y\rangle^{\lambda}g_s\|_{L^2}^2$$
$$\leq\frac{1}{4}\left\|\langle y\rangle^{\lambda+1}\sqrt{\rho}\partial_y(\partial_y\partial_x^{s-1}\omega)\right\|_{L^2}^2+C\left(\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}+\|\partial_x\rho^\infty\|_{L^\infty}\right)^2\|\omega\|_{H_g^{s,\lambda}}^2.$$

Case 2. When $1<i<s-1$, we have

$$\int_\Omega\partial_x^i\rho\partial_y^2\partial_x^{s-i}\omega\langle y\rangle^{2\lambda}g_s\leq C\left(\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}+\|\partial_x^s\rho^\infty\|_{L^\infty}\right)\|\omega\|_{H_g^{s,\lambda}}^2. \tag{114}$$

Case 3. When $i=s-1$, we have

$$\begin{aligned}
&\int_\Omega\partial_x^{s-1}\rho\partial_y^2\partial_x\omega\langle y\rangle^{2\lambda}g_s\\
\leq&\int_\Omega\partial_x^{s-1}\tilde{\rho}\partial_y^2\partial_x\omega\langle y\rangle^{2\lambda}g_s+\int_\Omega\partial_x^{s-1}\rho^\infty\partial_y^2\partial_x\omega\langle y\rangle^{2\lambda}g_s\\
\leq&\|\langle y\rangle^{\lambda-1}\partial_x^{s-1}\tilde{\rho}\|_{L^2}\|\langle y\rangle\partial_y^2\partial_x\omega\|_{L^\infty}\|\langle y\rangle^{\lambda}g_s\|_{L^2}+\|\partial_x^{s-1}\rho^\infty\|_{L^\infty}\|\omega\|_{H_g^{s,\lambda}}^2\\
\leq&\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\|\omega\|_{H_g^{s,\lambda}}^2+\|\partial_x^{s-1}\rho^\infty\|_{L^\infty}\|\omega\|_{H_g^{s,\lambda}}^2.
\end{aligned}\tag{115}$$

Case 4. When $i=s$, we have

$$\begin{aligned}
&\int_\Omega\partial_x^s\rho\partial_y^2\omega\langle y\rangle^{2\lambda}g_s\\
=&\int_\Omega h_s\partial_y^2\omega\langle y\rangle^{2\lambda}g_s+\int_\Omega b\partial_x^s\tilde{u}\partial_y^2\omega\langle y\rangle^{2\lambda}g_s+\int_\Omega\partial_x^s\rho^\infty\partial_y^2\omega\langle y\rangle^{2\lambda}g_s\\
\leq&\|\langle y\rangle^{\lambda-1}h_s\|_{L^2}\|\langle y\rangle\partial_y^2\omega\|_{L^\infty}\|\langle y\rangle^{\lambda}g_s\|_{L^2}\\
&+\|b\|_{L^\infty}\|\langle y\rangle^{\lambda-1}\partial_x^s\tilde{u}\|_{L^2}\|\langle y\rangle\partial_y^2\omega\|_{L^\infty}\|\langle y\rangle^{\lambda}g_s\|_{L^2}\\
&+\|\partial_x^s\rho^\infty\|_{L^\infty}\|\langle y\rangle^{\lambda}\partial_y^2\omega\|_{L^2}\|\langle y\rangle^{\lambda}g_s\|_{L^2}\\
\leq&C\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\|\omega\|_{H_g^{s,\lambda}}^2+C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\|\omega\|_{H_g^{s,\lambda}}^2\\
&+C\|\partial_x^s\rho^\infty\|_{L^\infty}\|\omega\|_{H_g^{s,\lambda}}^2.
\end{aligned}\tag{116}$$

Collecting the estimates (113)-(116), we obtain

$$\begin{aligned}
&\sum_{i=1}^s\binom{s}{i}\int_\Omega\partial_x^i\rho\partial_y^2\partial_x^{s-i}\omega\langle y\rangle^{2\lambda}g_s\\
\leq&\frac{1}{4}\left\|\langle y\rangle^{\lambda+1}\sqrt{\rho}\partial_y(\partial_y\partial_x^{s-1}\omega)\right\|_{L^2}^2+C\left(\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}+\|\partial_x^s\rho^\infty\|_{L^\infty}+1\right)^2\|\omega\|_{H_g^{s,\lambda}}^2\\
&+C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\|\omega\|_{H_g^{s,\lambda}}^2.
\end{aligned}\tag{117}$$

(11) The estimate on the term $\sum_{i=1}^s\binom{s}{i}\int_\Omega a\partial_x^i\rho\partial_y^2\partial_x^{s-i}\tilde{u}\langle y\rangle^{2\lambda}g_s$ is divided into three cases.

Case 1. When $i=1,\cdots,s-2$, we have

$$\begin{aligned}
&\int_\Omega a\partial_x^i\rho\partial_y\partial_x^{s-i}\omega\langle y\rangle^{2\lambda}g_s\\
\leq&\|a\|_{L^\infty}\left\|\langle y\rangle^{\lambda}\partial_x^i\rho\partial_y\partial_x^{s-i}\omega\right\|_{L^2}\|\langle y\rangle^{\lambda}g_s\|_{L^2}\\
\leq&C\left(\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}+\|\partial_x^{s-2}\rho^\infty\|_{L^\infty}\right)\|\omega\|_{H_g^{s,\lambda}}^2.
\end{aligned}\tag{118}$$

Here we used Lemma 1.5.

Case 2. When $i = s-1$, we have

$$\begin{aligned}&\int_\Omega a\partial_x^{s-1}\rho\partial_y\partial_x\omega\langle y\rangle^{2\lambda}g_s\\ \le&\|a\|_{L^\infty}\|\langle y\rangle^\lambda\partial_x^{s-1}\rho\partial_y\partial_x\omega\|_{L^2}\|\langle y\rangle^\lambda g_s\|_{L^2}\\ \le&C\left(\|\tilde\rho\|_{H_h^{s,\lambda-1}}+\|\partial_x^{s-1}\rho^\infty\|_{L^\infty}\right)\|\omega\|^2_{H_g^{s,\lambda}}.\end{aligned}\tag{119}$$

Case 3. When $i = s$, we have

$$\begin{aligned}&\int_\Omega a\partial_x^s\rho\partial_y\omega\langle y\rangle^{2\lambda}g_s\\ =&\int_\Omega ah_s\partial_y\omega\langle y\rangle^{2\lambda}g_s+\int_\Omega ab\partial_x^s\tilde u\partial_y\omega\langle y\rangle^{2\lambda}g_s+\int_\Omega a\partial_x^s\rho^\infty\partial_y\omega\langle y\rangle^{2\lambda}g_s\\ \le&C\left(\|\tilde\rho\|_{H_h^{s,\lambda-1}}+\|\partial_x^s\rho^\infty\|_{L^\infty}\right)\|\omega\|^2_{H_g^{s,\lambda}}+C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^\infty}\right)\|\omega\|^2_{H_g^{s,\lambda}}.\end{aligned}\tag{120}$$

Combining the estimates (118)-(120), we obtain

$$\begin{aligned}&\sum_{i=1}^s\binom{s}{i}\int_\Omega a\partial_x^i\rho\partial_y^2\partial_x^{s-i}\tilde u\langle y\rangle^{2\lambda}g_s\\ \le&C\left(\|\tilde\rho\|_{H_h^{s,\lambda-1}}+\|\partial_x^s\rho^\infty\|_{L^\infty}\right)\|\omega\|^2_{H_g^{s,\lambda}}+C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^\infty}\right)\|\omega\|^2_{H_g^{s,\lambda}}.\end{aligned}\tag{121}$$

(12) The estimate on the term $\int_\Omega\partial_x^s\left(\partial_y\rho\partial_y\omega\right)\langle y\rangle^{2\lambda}g_s=\sum_{i=0}^s\binom{s}{i}\int_\Omega\partial_x^i\partial_y\rho\partial_x^{s-i}\partial_y\omega$ is divided into three cases.

Case 1. When $i = s$, one has by integration by parts

$$\begin{aligned}&\int_\Omega\partial_x^s\partial_y\tilde\rho\partial_y\omega\langle y\rangle^{2\lambda}g_s\\ =&-\int_\Omega\partial_x^s\tilde\rho\partial_y^2\omega\langle y\rangle^{2\lambda}g_s-\int_\Omega2\lambda\partial_x^s\tilde\rho\partial_y\omega\langle y\rangle^{2\lambda-1}g_s\\ &-\int_\Omega\partial_x^s\tilde\rho\partial_y\omega\langle y\rangle^{2\lambda}\partial_yg_s+\int_{\mathbb T}\partial_x^s\tilde\rho\partial_y\omega\langle y\rangle^{2\lambda}g_s|_{y=0}^{y=+\infty}\\ :=&\sum_{i=11}^{14}I_i.\end{aligned}\tag{122}$$

$$\begin{aligned}I_{11}=&-\int_\Omega h_s\partial_y^2\omega\langle y\rangle^{2\lambda}g_s-\int_\Omega b\partial_x^s\tilde u\partial_y^2\omega\langle y\rangle^{2\lambda}g_s\\ \le&\|h_s\|_{L^2}\left\|\langle y\rangle^\lambda\partial_y^2\omega\right\|_{L^\infty}\left\|\langle y\rangle^\lambda g_s\right\|_{L^2}\\ &+\|b\|_{L^\infty}\|\partial_x^s\tilde u\|_{L^2}\left\|\langle y\rangle^\lambda\partial_y^2\omega\right\|_{L^\infty}\left\|\langle y\rangle^\lambda g_s\right\|_{L^2}\\ \le&C\|\tilde\rho\|_{H_h^{s,\lambda-1}}\|\omega\|^2_{H_g^{s,\lambda}}+C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^\infty}\right)\|\omega\|^2_{H_g^{s,\lambda}}.\end{aligned}\tag{123}$$

$$\begin{aligned}I_{12}=&-2\lambda\int_\Omega h_s\partial_y\omega\langle y\rangle^{2\lambda-1}g_s-2\lambda\int_\Omega b\partial_x^s\tilde u\partial_y\omega\langle y\rangle^{2\lambda-1}g_s\\ \le&2\lambda\|h_s\|_{L^2}\|\langle y\rangle^{\lambda-1}\partial_y\omega\|_{L^\infty}\|\langle y\rangle^\lambda g_s\|_{L^2}\\ &+2\lambda\|b\|_{L^\infty}\|\partial_x^s\tilde u\|_{L^2}\|\langle y\rangle^{\lambda-1}\partial_y\omega\|_{L^\infty}\|\langle y\rangle^\lambda g_s\|_{L^2}\\ \le&C\|\tilde\rho\|_{H_h^{s,\lambda-1}}\|\omega\|^2_{H_g^{s,\lambda}}+C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^\infty}\right)\|\omega\|^2_{H_g^{s,\lambda}}.\end{aligned}\tag{124}$$

$$
\begin{aligned}
I_{13} =& -\int_\Omega h_s\partial_y\omega\langle y\rangle^{2\lambda}\partial_y g_s - \int_\Omega b\partial_x^s\tilde{u}\partial_y\omega\langle y\rangle^{2\lambda}\partial_y g_s\\
\le& C\|h_s\|_{L^2}\|\langle y\rangle^\lambda\partial_y\omega\|_{L^\infty}\|\langle y\rangle^\lambda\sqrt{\rho}\partial_y g_s\|_{L^2}\\
&+C\|b\|_{L^\infty}\|\partial_x^s\tilde{u}\|_{L^2}\|\langle y\rangle^\lambda\partial_y\omega\|_{L^\infty}\|\langle y\rangle^\lambda\sqrt{\rho}\partial_y g_s\|_{L^2}\\
\le&\frac{1}{12}\|\langle y\rangle^\lambda\sqrt{\rho}\partial_y g_s\|_{L^2}^2 + C\|\tilde{\rho}\|^2_{H_h^{s,\lambda-1}}\|\omega\|^2_{H_g^{s,\lambda}}\\
&+C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^\infty}\right)^2\|\omega\|^2_{H_g^{s,\lambda}}.
\end{aligned}
\tag{125}
$$

The boundary integral I_{14} vanishes due to boundary and far field conditions.

Collecting all the estimates I_{11}-I_{14}, we have

$$
\begin{aligned}
&\int_\Omega \partial_x^s\partial_y\tilde{\rho}\partial_y\omega\langle y\rangle^{2\lambda}g_s\\
\le&\frac{1}{12}\|\langle y\rangle^\lambda\sqrt{\rho}\partial_y g_s\|_{L^2}^2 + C\|\tilde{\rho}\|^2_{H_h^{s,\lambda-1}}\|\omega\|^2_{H_g^{s,\lambda}}\\
&+C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^\infty}\right)^2\|\omega\|^2_{H_g^{s,\lambda}}\\
&+C\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\|\omega\|^2_{H_g^{s,\lambda}} + C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^\infty}\right)\|\omega\|^2_{H_g^{s,\lambda}}.
\end{aligned}
\tag{126}
$$

Case 2. When $i=0$, we have

$$
\begin{aligned}
&\int_\Omega \partial_y\tilde{\rho}\partial_x^s\partial_y\omega\langle y\rangle^{2\lambda}g_s\\
=&\int_\Omega \partial_y\tilde{\rho}\left(\partial_y g_s+\partial_y a\partial_x^s\tilde{u}+ag_s+a^2\partial_x^s\tilde{u}\right)\langle y\rangle^{2\lambda}g_s\\
\le&\frac{1}{12}\|\sqrt{\rho}\langle y\rangle^\lambda\partial_y g_s\|_{L^2}^2 + C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2}\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\|\omega\|_{H_g^{s,\lambda}}\\
&+C\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\|\omega\|_{H_g^{s,\lambda}} + C\|\tilde{\rho}\|^2_{H_h^{s,\lambda-1}}\|\omega\|^2_{H_g^{s,\lambda}}.
\end{aligned}
\tag{127}
$$

Case 3. For $i=1,\cdots,s-1$, we have

$$
\int_\Omega \partial_x^i\partial_y\rho\partial_x^{s-i}\partial_y\omega\langle y\rangle^{2\lambda}g_s \le C\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\|\omega\|^2_{H_g^{s,\lambda}}.
\tag{128}
$$

In summary, we get

$$
\begin{aligned}
&\int_\Omega \partial_x^s\left(\partial_y\rho\partial_y\omega\right)\langle y\rangle^{2\lambda}g_s\\
\le&\frac{1}{6}\left\|\sqrt{\rho}\langle y\rangle^\lambda\partial_y g_s\right\|_{L^2}^2 + C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^\infty}\right)^2\|\omega\|^2_{H_g^{s,\lambda}}\\
&+C\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\|\omega\|^2_{H_g^{s,\lambda}} + C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^\infty}\right)\|\omega\|^2_{H_g^{s,\lambda}}\\
&+C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2}\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\|\omega\|_{H_g^{s,\lambda}} + C\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\|\omega\|_{H_g^{s,\lambda}}\\
&+C\|\tilde{\rho}\|^2_{H_h^{s,\lambda-1}}\|\omega\|^2_{H_g^{s,\lambda}}.
\end{aligned}
\tag{129}
$$

(13) Finally we estimate the term $\int_\Omega \partial_x^s\left(\frac{\omega}{\gamma}\partial_t\ln p\right)\langle y\rangle^{2\lambda}g_s$.

Noticing that $\partial_x p=0$, thus we have

$$
\int_\Omega \partial_t \ln p^{1/\gamma}\partial_x^s\omega\langle y\rangle^{2\lambda}g_s
\tag{130}
$$

$$
\begin{aligned}
&= \int_\Omega \partial_t \ln p^{1/\gamma} g_s \langle y\rangle^{2\lambda} g_s + \int_\Omega \partial_t \ln p^{1/\gamma} \cdot a\partial_x^s \tilde{u} \cdot \langle y\rangle^{2\lambda} g_s \\
&\le \|\partial_t \ln p^{1/\gamma}\|_{L^\infty} \|\langle y\rangle^\lambda g_s\|_{L^2}^2 \\
&\quad + \|\partial_t \ln p^{1/\gamma}\|_{L^\infty} \|\langle y\rangle a\|_{L^\infty} \|\langle y\rangle^{\lambda-1}\partial_x^s \tilde{u}\|_{L^2} \|\langle y\rangle^\lambda g_s\|_{L^2} \\
&\le C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2}\right)\|\omega\|_{H_g^{s,\lambda}}.
\end{aligned}
$$

Combining all the above estimates, we complete the proof of Lemma 2.3. □

3. Uniform weighted H^s estimates on the regularized $\tilde{\rho}$.

3.1. The case of $|\alpha| \le s, \alpha_1 \le s-1$.

Lemma 3.1. *Under the assumption of Theorem 1.1, we have*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\le s,\alpha_1\le s-1} \left\|\langle y\rangle^{\lambda+\alpha_2-1} D^\alpha \tilde{\rho}\right\|_{L^2}^2 + \epsilon^2 \sum_{|\alpha|\le s,\alpha_1\le s-1} \left\|\langle y\rangle^{\lambda+\alpha_2-1} \partial_x D^\alpha \tilde{\rho}\right\|_{L^2}^2 \\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2} + 1\right)^2 \left(\|\tilde{\rho}\|_{H_h^{s,\lambda-1}} + \left\|\partial_x^{s+1}\rho^\infty\right\|_{L^\infty} + 1\right)^2.
\end{aligned} \tag{131}
$$

Proof. Differentiating the equation $(23)_1$ α_1 times with respect to x and α_2 times with respect to y, we have

$$
\begin{aligned}
&\partial_t D^\alpha \tilde{\rho} + u\partial_x D^\alpha \tilde{\rho} + v\partial_y D^\alpha \tilde{\rho} \\
=& -\sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} D^\beta u \cdot \partial_x D^{\alpha-\beta}\tilde{\rho} - \sum_{0<\beta\le\alpha} \binom{\alpha}{\beta} D^\beta v \cdot \partial_y D^{\alpha-\beta}\tilde{\rho} \\
&- \sum_{0\le\beta\le\alpha} \binom{\alpha}{\beta} D^\beta \tilde{u} \cdot D^{\alpha-\beta+e_1}\rho^\infty - \sum_{0\le\beta\le\alpha} \binom{\alpha}{\beta} D^\beta \tilde{\rho} \cdot D^{\alpha-\beta}\partial_t \ln p \\
&+ \epsilon^2 \partial_x^2 D^\alpha \rho.
\end{aligned} \tag{132}
$$

Multiplying the equation (132) by $\langle y\rangle^{2\lambda+2\alpha_2-2} D^\alpha \tilde{\rho}$ and integrating the resulting equation over Ω, we estimate each term step by step as follows.

(1) It is obvious to get

$$
\int_\Omega \partial_t D^\alpha \tilde{\rho} \cdot \langle y\rangle^{2\lambda+2\alpha_2-2} D^\alpha \tilde{\rho} = \frac{1}{2}\frac{d}{dt}\|\langle y\rangle^{\lambda+\alpha_2-1} D^\alpha \tilde{\rho}\|_{L^2}^2. \tag{133}
$$

(2) Integrating by parts and using the equation (28) and Lemma 1.4, one has

$$
\begin{aligned}
&\int_\Omega \left(u\partial_x D^\alpha \tilde{\rho} + v\partial_y D^\alpha \tilde{\rho}\right) \langle y\rangle^{2\lambda+2\alpha_2-2} D^\alpha \tilde{\rho} \\
=& -\frac{1}{2}\int_\Omega \left(\partial_x u + \partial_y v\right) \langle y\rangle^{2\lambda+2\alpha_2-2} (D^\alpha \tilde{\rho})^2 \\
&- (\lambda+\alpha_2-1)\int_\Omega v\langle y\rangle^{2\lambda+2\alpha_2-3} (D^\alpha \tilde{\rho})^2 \\
&+ \int_{\mathbb{T}} v\langle y\rangle^{2\lambda+2\alpha_2-2} (D^\alpha \tilde{\rho})^2\Big|_{y=0}^{y=+\infty} \\
=& \frac{1}{2}\int_\Omega \partial_t \ln p^{1/\gamma} \langle y\rangle^{2\lambda+2\alpha_2-2} (D^\alpha \tilde{\rho})^2 \\
&- (\lambda+\alpha_2-1)\int_\Omega \frac{v}{1+y}\langle y\rangle^{2\lambda+2\alpha_2-2} (D^\alpha \tilde{\rho})^2 \\
:=& K_1 + K_2,
\end{aligned} \tag{134}
$$

where

$$K_1 = \frac{1}{2}\int_\Omega \partial_t \ln p^{1/\gamma}\langle y\rangle^{2\lambda+2\alpha_2-2}(D^\alpha\tilde{\rho})^2 \le C\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde{\rho}\right\|_{L^2}^2 \tag{135}$$

and

$$\begin{aligned}K_2 =&-(\lambda+\alpha_2-1)\int_\Omega \frac{v}{1+y}\langle y\rangle^{2\lambda+2\alpha_2-2}(D^\alpha\tilde{\rho})^2\\ \le& C\left\|\frac{v}{1+y}\right\|_{L^\infty}\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde{\rho}\|_{L^2}^2\\ \le& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2}\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}^2.\end{aligned} \tag{136}$$

Substituting (135) and (136) into (134), we have

$$\begin{aligned}&\int_\Omega (u\partial_x D^\alpha\tilde{\rho}+v\partial_y D^\alpha\tilde{\rho})\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde{\rho}\\ \le& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2}+1\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}^2.\end{aligned} \tag{137}$$

(3) The estimate on the term $-\sum_{0<\beta\le\alpha}\int_\Omega\binom{\alpha}{\beta}D^\beta u\cdot\partial_x D^{\alpha-\beta}\tilde{\rho}\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde{\rho}$ is divided into three cases.

Case 1. When $\beta=(\beta_1,0)$, $1\le\beta_1\le s-1$, we have from Lemma 1.4

$$\begin{aligned}&\int_\Omega \partial_x^{\beta_1}u\cdot\partial_x D^{\alpha-\beta}\tilde{\rho}\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde{\rho}\\ \le&\|\partial_x^{\beta_1}u\|_{L^\infty}\|\langle y\rangle^{\lambda+\alpha_2-1}\partial_x D^{\alpha-\beta}\tilde{\rho}\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde{\rho}\|_{L^2}\\ \le& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2}\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}^2.\end{aligned} \tag{138}$$

Case 2. When $\beta_2=1$, $0\le\beta_1\le s-1$, we further discuss for three different cases. Firstly, one has for $1\le\beta_1\le s-1$

$$\int_\Omega \partial_x^{\beta_1}\omega\cdot\partial_x^{\alpha_1-\beta_1+1}\partial_y^{\alpha_2-1}\tilde{\rho}\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde{\rho}\le C\|\omega\|_{H_g^{s,\lambda}}\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}^2. \tag{139}$$

Secondly, we consider $\beta_1=0$, which means $\alpha_2\ge1$, when $\alpha_1=s-1$, then we have

$$\begin{aligned}&\int_\Omega \omega\cdot\partial_x^s\tilde{\rho}\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde{\rho}\\ =&\int_\Omega \omega\cdot h_s\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde{\rho}+\int_\Omega \omega\cdot b\partial_x^s\tilde{u}\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde{\rho}\\ \le& C\|\omega\|_{H_g^{s,\lambda}}\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}^2+C\|\omega\|_{H_g^{s,\lambda}}(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2})\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}.\end{aligned} \tag{140}$$

Finally, when $\alpha_1\le s-2$, which means $\beta_1\le\alpha_1\le s-2$, then we have

$$\int_\Omega \partial_x^{\beta_1}\omega\cdot\partial_x^{\alpha_1+1}\partial_y^{\alpha_2-1}\tilde{\rho}\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde{\rho}\le C\|\omega\|_{H_g^{s,\lambda}}\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}^2. \tag{141}$$

Case 3. When $\beta_2\ge2$, we have

$$\int_\Omega D^{\beta-e_2}\omega\cdot D^{\alpha-\beta+e_1}\tilde{\rho}\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde{\rho}\le C\|\omega\|_{H_g^{s,\lambda}}\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}^2. \tag{142}$$

Combining the estimates (138)-(142), we get

$$\begin{aligned}&-\sum_{0<\beta\leq\alpha}\int_\Omega \binom{\alpha}{\beta} D^\beta u\cdot\partial_x D^{\alpha-\beta}\tilde\rho\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho\\ \leq& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\left(\|\omega\|_{H_g^{s,\lambda}}+\|\tilde\rho\|_{H_h^{s,\lambda-1}}\right)\|\tilde\rho\|_{H_h^{s,\lambda-1}}.\end{aligned}\tag{143}$$

(4) The estimate on the term $-\sum_{0<\beta\leq\alpha}\int_\Omega\binom{\alpha}{\beta}D^\beta v\cdot\partial_y D^{\alpha-\beta}\tilde\rho\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho$ is divided into three cases.

Case 1. $\beta_2=0$, one has for $1\leq\beta_1\leq s-2$

$$\begin{aligned}&\int_\Omega\partial_x^{\beta_1}v\cdot D^{\alpha-\beta+e_2}\tilde\rho\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho\\ \leq&\left\|\frac{\partial_x^{\beta_1}v}{1+y}\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2}D^{\alpha-\beta+e_2}\tilde\rho\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|_{L^2}\\ \leq& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}+1\right)\|\tilde\rho\|^2_{H_h^{s,\lambda-1}}\end{aligned}\tag{144}$$

and for $\beta_1=s-1$

$$\begin{aligned}&\int_\Omega\partial_x^{\beta_1}v\cdot D^{\alpha-\beta+e_2}\tilde\rho\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho\\ =&\int_\Omega\frac{\partial_x^{\beta_1}v+y\tilde U_{\beta_1}}{1+y}\cdot D^{\alpha-\beta+e_2}\tilde\rho\cdot\langle y\rangle^{2\lambda+2\alpha_2-1}D^\alpha\tilde\rho\\ &-\int_\Omega y\tilde U_{\beta_1}\cdot D^{\alpha-\beta+e_2}\tilde\rho\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho\\ \leq&\left\|\frac{\partial_x^{\beta_1}v+y\tilde U_{\beta_1}}{1+y}\right\|_{L^2}\|\langle y\rangle^{\lambda+\alpha_2}D^{\alpha-\beta+e_2}\tilde\rho\|_{L^\infty}\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\|_{L^2}\\ &+\left\|\tilde U_{\beta_1}\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2}D^{\alpha-\beta+e_2}\tilde\rho\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|_{L^2}\\ \leq& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^\infty}+1\right)\|\tilde\rho\|^2_{H_h^{s,\lambda-1}}.\end{aligned}\tag{145}$$

Case 2. $\beta_2=1$, one has for $\beta_1=0$

$$\begin{aligned}&-\int_\Omega\partial_x^{\beta_1}(\partial_xu+\partial_t\ln p^{1/\gamma})\cdot D^{\alpha-\beta+e_2}\tilde\rho\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho\\ \leq&\|\partial_xu\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^{\alpha-\beta+e_2}\tilde\rho\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|_{L^2}\\ &+\left\|\partial_t\ln p^{1/\gamma}\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^{\alpha-\beta+e_2}\tilde\rho\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|_{L^2}\\ \leq& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}+\|\partial_t\ln p^{1/\gamma}\|_{L^\infty}\right)\|\tilde\rho\|^2_{H_h^{s,\lambda-1}}\end{aligned}\tag{146}$$

for $1\leq\beta_1\leq s-2$

$$\begin{aligned}&-\int_\Omega\partial_x^{\beta_1}(\partial_xu+\partial_t\ln p^{1/\gamma})\cdot D^{\alpha-\beta+e_2}\tilde\rho\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho\\ \leq&\left\|\partial_x^{\beta_1+1}u\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^{\alpha-\beta+e_2}\tilde\rho\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|_{L^2}\\ \leq& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\|\tilde\rho\|^2_{H_h^{s,\lambda-1}}\end{aligned}\tag{147}$$

and for $\beta_1 = s-1$

$$
\begin{aligned}
&-\int_\Omega \partial_x^{\beta_1}\left(\partial_x u + \partial_t \ln p^{1/\gamma}\right)\cdot D^{\alpha-\beta+e_2}\tilde\rho \cdot \langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho \\
\le& \|\partial_x^s \tilde u\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^{\alpha-\beta+e_2}\tilde\rho\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|_{L^2} \\
&+\|\partial_x^s U\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^{\alpha-\beta+e_2}\tilde\rho\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|_{L^2} \\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^\infty}\right)\|\tilde\rho\|^2_{H_h^{s,\lambda-1}}.
\end{aligned}
\tag{148}
$$

Summarizing the estimates (146)-(148), we have for $\beta_2 = 1$

$$
\begin{aligned}
&-\int_\Omega \partial_x^{\beta_1}\left(\partial_x u + \partial_t \ln p^{1/\gamma}\right)\cdot D^{\alpha-\beta+e_2}\tilde\rho \cdot \langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho \\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^\infty} + \|\partial_t \ln p^{1/\gamma}\|_{L^\infty}\right)\|\tilde\rho\|^2_{H_h^{s,\lambda-1}}.
\end{aligned}
\tag{149}
$$

Case 3. When $\beta_2 > 1$, we have for $\lambda \ge 1$

$$
\begin{aligned}
&-\int_\Omega \partial_x^{\beta_1+1}\partial_y^{\beta_2-2}\omega\cdot D^{\alpha-\beta+e_2}\tilde\rho\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho \\
\le& \left\|\langle y\rangle^{\lambda+\beta_2-2}\partial_x^{\beta_1+1}\partial_y^{\beta_2-2}\omega\right\|_{L^2}\left\|\langle y\rangle^{\lambda+\alpha_2-\beta_2}D^{\alpha-\beta+e_2}\tilde\rho\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|_{L^\infty} \\
\le& C\|\omega\|_{H_g^{s,\lambda}}\|\tilde\rho\|^2_{H_h^{s,\lambda-1}}.
\end{aligned}
\tag{150}
$$

Combining (144), (145), (149) and (150), we obtain

$$
\begin{aligned}
&-\sum_{0<\beta\le\alpha}\int_\Omega \binom{\alpha}{\beta}D^\beta v\cdot\partial_y D^{\alpha-\beta}\tilde\rho\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho \\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^\infty} + 1\right)\|\tilde\rho\|^2_{H_h^{s,\lambda-1}}.
\end{aligned}
\tag{151}
$$

(5) Now we estimate the term

$$
\begin{aligned}
&\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega D^\beta\tilde u\cdot D^{\alpha-\beta+e_1}\rho^\infty\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho \\
=&\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega \partial_x^{\beta_1}\partial_y^{\alpha_2}\tilde u\cdot\partial_x^{\alpha_1-\beta_1+1}\rho^\infty\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho.
\end{aligned}
\tag{152}
$$

Here we notice that $\alpha_2 = \beta_2$ since ρ^∞ is independent of y.

Case 1. When $\beta_2 = 0$, which implies $\alpha_2 = 0$ and $\beta_1 \le s-1$, we have

$$
\begin{aligned}
&\int_\Omega \partial_x^{\beta_1}\tilde u\cdot\partial_x^{\alpha_1-\beta_1+1}\rho^\infty\cdot\langle y\rangle^{2\lambda-2}D^\alpha\tilde\rho \\
\le& \left\|\langle y\rangle^{\lambda-1}\partial_x^{\beta_1}\tilde u\right\|_{L^2}\left\|\partial_x^{\alpha_1-\beta_1+1}\rho^\infty\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda-1}D^\alpha\tilde\rho\right\|_{L^2} \\
\le& \left\|\partial_x^{s+1}\rho^\infty\right\|_{L^\infty}\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2}\right)\|\tilde\rho\|_{H_h^{s,\lambda-1}}.
\end{aligned}
\tag{153}
$$

Case 2. When $\beta_2 \ge 1$, which implies $\alpha_2 \ge 1$, we get

$$
\begin{aligned}
&\int_\Omega \partial_x^{\beta_1}\partial_y^{\beta_2-1}\omega\cdot\partial_x^{\alpha_1-\beta_1+1}\rho^\infty\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde\rho \\
\le& \left\|\langle y\rangle^{\lambda+\alpha_2-1}\partial_x^{\beta_1}\partial_y^{\beta_2-1}\omega\right\|_{L^2}\left\|\partial_x^{\alpha_1-\beta_1+1}\rho^\infty\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho\right\|_{L^2} \\
\le& \|\partial_x^s\rho^\infty\|_{L^\infty}\|\omega\|_{H_g^{s,\lambda}}\|\tilde\rho\|_{H_h^{s,\lambda-1}}.
\end{aligned}
\tag{154}
$$

Combining (153) and (154), we obtain

$$\begin{aligned}&\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega D^\beta\tilde{u}\cdot D^{\alpha-\beta+e_1}\rho^\infty\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde{\rho}\\ \le&\left\|\partial_x^{s+1}\rho^\infty\right\|_{L^\infty}\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}.\end{aligned}\tag{155}$$

(6) Finally we estimate

$$\begin{aligned}&\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\int_\Omega D^\beta\tilde{\rho}\cdot D^{\alpha-\beta}\partial_t\ln p\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde{\rho}\\ =&\int_\Omega D^\alpha\tilde{\rho}\cdot\partial_t\ln p\cdot\langle y\rangle^{2\lambda+2\alpha_2-2}D^\alpha\tilde{\rho}\\ \le& C\|\partial_t\ln p\|_{L^\infty}\|\tilde{\rho}\|^2_{H_h^{s,\lambda-1}}.\end{aligned}\tag{156}$$

Collecting the above estimates (133), (137), (151),(155) and (156), we obtain the estimate (131) and this completes the proof of Lemma 3.1. □

3.2. **The case of $\alpha_1 = s$.** Recalling notations $h_s=\partial_x^s\tilde{\rho}-b\partial_x^s\tilde{u}$, $b=\frac{\partial_y\rho}{\omega}$ and the equations satisfied by $(\tilde{\rho},\tilde{u})$ for special pressure $p=p(t)$

$$\partial_t\tilde{\rho}+u\partial_x\tilde{\rho}+v\partial_y\tilde{\rho}=-\tilde{u}\partial_x\rho^\infty-\tilde{\rho}\partial_t\ln p^{1/\gamma}+\epsilon^2\partial_x^2\tilde{\rho}\tag{157}$$

and

$$\partial_t\tilde{u}+u\partial_x\tilde{u}+v\partial_y\tilde{u}=-\tilde{u}\partial_xU+\rho\partial_y^2\tilde{u}+\epsilon^2\partial_x^2\tilde{u}.\tag{158}$$

We get the equation for h_s from (157) and (158) by tedious calculations

$$\begin{aligned}&\left(\partial_t+u\partial_x+v\partial_y-\epsilon^2\partial_x^2\right)h_s\\ =&-h_1\cdot\partial_x^sU-\epsilon^2\left(\frac{\partial_x\partial_y\rho\cdot\partial_x\partial_yu}{\omega^2}+\partial_xb\cdot\frac{\partial_x\partial_yu}{\omega}-b\left(\frac{\partial_x\partial_yu}{\omega}\right)^2\right)\cdot\partial_x^s\tilde{u}\\ &+\left(b\partial_t\ln p^{1/\gamma}+b^2\cdot\partial_y^2\tilde{u}+b\rho\frac{\partial_y^2\omega}{\omega}\right)\partial_x^s\tilde{u}\\ &-\sum_{j=1}^{s-1}\binom{j}{s}\partial_x^{s-j}u\cdot h_{j+1}-\sum_{j=1}^{s-1}\binom{j}{s}\partial_x^{s-j}v\left(\partial_x^j\partial_y\tilde{\rho}-b\partial_x^j\partial_y\tilde{u}\right)\\ &-\sum_{i=0}^{s-1}\binom{s}{i}\partial_x^i\tilde{u}\cdot\partial_x^{s-i+1}\rho^\infty-\partial_x^s\left(\tilde{\rho}\cdot\partial_t\ln p^{1/\gamma}\right)+\sum_{i=1}^{s}\binom{s}{i}b\partial_x^{i+1}U\partial_x^{s-i}\tilde{u}\\ &-\partial_x^s\left(\rho\cdot\partial_y^2\tilde{u}\right)\cdot b+2\epsilon^2\partial_xb\cdot\partial_x^{s+1}\tilde{u}.\end{aligned}\tag{159}$$

We will give the weighted L^2 estimate on h_s in the following lemma.

Lemma 3.2. *Under the assumptions of Theorem 1.1, we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\langle y\rangle^{\lambda-1}h_s\right\|^2_{L^2}+\frac{\epsilon^2}{2}\left\|\langle y\rangle^{\lambda-1}\partial_xh_s\right\|^2_{L^2}\\ \le&\frac{\epsilon^2}{4}\left\|\langle y\rangle^\lambda\partial_xg_s\right\|^2_{L^2}+\frac{1}{4}\left\|\langle y\rangle^\lambda\sqrt{\rho}\partial_yg_s\right\|^2_{L^2}\\ &+C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^{s+1}\tilde{U}\|_{L^\infty}+1\right)^2\left(\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}+\|\partial_x^{s+1}\rho^\infty\|_{L^\infty}+1\right)^2.\end{aligned}\tag{160}$$

Proof. Multiplying the equation (159) by $\langle y\rangle^{2\lambda-2}h_s$ and integrating the resulting equation over Ω, then we estimate each term step by step as follows.

(1) It is obvious to get

$$\int_\Omega \partial_t h_s\cdot\langle y\rangle^{2\lambda-2}h_s=\frac{1}{2}\frac{d}{dt}\left\|\langle y\rangle^{\lambda-1}h_s\right\|_{L^2}^2. \tag{161}$$

(2) Integrating by parts and using the equation (28), one has

$$\begin{aligned}
&\int_\Omega\left(u\partial_x h_s+v\partial_y h_s\right)\langle y\rangle^{2\lambda-2}h_s\\
=&-\frac{1}{2}\int_\Omega(\partial_x u+\partial_y v)\langle y\rangle^{2\lambda-2}h_s^2-(\lambda-1)\int_\Omega v\langle y\rangle^{2\lambda-3}h_s^2\\
&+\int_{\mathbb{T}} v\langle y\rangle^{2\lambda-2}h_s^2\big|_{y=0}^{y=+\infty}\\
=&\frac{1}{2}\int_\Omega\partial_t\ln p^{1/\gamma}\langle y\rangle^{2\lambda-2}h_s^2-(\lambda-1)\int_\Omega\frac{v}{1+y}\langle y\rangle^{2\lambda-2}h_s^2\\
:=&K_3+K_4,
\end{aligned} \tag{162}$$

where

$$K_3=\frac{1}{2}\int_\Omega\partial_t\ln p^{1/\gamma}\langle y\rangle^{2\lambda-2}h_s^2\le C\left\|\langle y\rangle^{\lambda-1}h_s\right\|_{L^2}^2 \tag{163}$$

and

$$\begin{aligned}
K_4=&-(\lambda-1)\int_\Omega\frac{v}{1+y}\langle y\rangle^{2\lambda-2}h_s^2\\
\le& C\left\|\frac{v}{1+y}\right\|_{L^\infty}\|\langle y\rangle^{\lambda-1}h_s\|_{L^2}^2\\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2}\right)\|\tilde\rho\|_{H_h^{s,\lambda-1}}^2.
\end{aligned} \tag{164}$$

Thus, we have

$$\int_\Omega\left(u\partial_x h_s+v\partial_y h_s\right)\langle y\rangle^{2\lambda-2}h_s\le C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2}\right)\|\tilde\rho\|_{H_h^{s,\lambda-1}}^2. \tag{165}$$

(3) The estimate on the term $-\int_\Omega h_1\cdot\partial_x^s U\cdot\langle y\rangle^{2\lambda-2}h_s$.

$$\begin{aligned}
&-\int_\Omega h_1\cdot\partial_x^s U\cdot\langle y\rangle^{2\lambda-2}h_s\\
\le&\|\partial_x^s U\|_{L^\infty}\left\|\langle y\rangle^{\lambda-1}h_1\right\|_{L^2}\left\|\langle y\rangle^{\lambda-1}h_s\right\|_{L^2}\\
\le& C\|\partial_x^s U\|_{L^\infty}\left(\|\tilde\rho\|_{H_h^{s,\lambda-1}}+\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2}\right)\|\tilde\rho\|_{H_h^{s,\lambda-1}}.
\end{aligned} \tag{166}$$

(4) The estimate on the term $\int_\Omega \epsilon^2\left\{\frac{\partial_x\partial_y\rho\cdot\partial_x\partial_y u}{\omega^2}+\partial_x b\cdot\frac{\partial_x\partial_y u}{\omega}-b(\frac{\partial_x\partial_y u}{\omega})^2\right\}\cdot\partial_x^s\tilde u\cdot\langle y\rangle^{2\lambda-2}h_s$.

$$\int_\Omega\epsilon^2\left\{\frac{\partial_x\partial_y\rho\cdot\partial_x\partial_y u}{\omega^2}+\partial_x b\cdot\frac{\partial_x\partial_y u}{\omega}-b\left(\frac{\partial_x\partial_y u}{\omega}\right)^2\right\}\cdot\partial_x^s\tilde u\cdot\langle y\rangle^{2\lambda-2}h_s \tag{167}$$

$$\begin{aligned}
\le&\left\|\frac{\partial_x\partial_y\rho\partial_x\partial_y u}{\omega^2}+\partial_x b\frac{\partial_x\partial_y u}{\omega}-b\left(\frac{\partial_x\partial_y u}{\omega}\right)^2\right\|_{L^\infty}\|\langle y\rangle^{\lambda-1}\partial_x^s\tilde u\|_{L^2}\|\langle y\rangle^{\lambda-1}h_s\|_{L^2}\\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^s U\|_{L^2}\right)\|\tilde\rho\|_{H_h^{s,\lambda-1}}
\end{aligned}$$

provided $\langle y\rangle^\lambda\omega \ge \delta$ and $\sum_{|\alpha|\le 2}|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega|^2 + \sum_{|\alpha|\le 2}|\langle y\rangle^{\lambda+\alpha_2-1}D^\alpha\tilde\rho|^2 \le \delta^{-2}$.

(5) The estimate on the term $\int_\Omega\left(-b\partial_t\ln p^{1/\gamma} - b^2\cdot\partial_y^2\tilde u - b\rho\frac{\partial_y^2\omega}{\omega}\right)\partial_x^s\tilde u\cdot\langle y\rangle^{2\lambda-2}h_s$.

$$\begin{aligned}
&\int_\Omega\left(-b\partial_t\ln p^{1/\gamma} - b^2\cdot\partial_y^2\tilde u - b\rho\frac{\partial_y^2\omega}{\omega}\right)\partial_x^s\tilde u\cdot\langle y\rangle^{2\lambda-2}h_s\\
\le&\left\|-b\partial_t\ln p^{1/\gamma} - b^2\cdot\partial_y^2\tilde u - b\rho\frac{\partial_y^2\omega}{\omega}\right\|_{L^\infty}\int_\Omega\left|\partial_x^s\tilde u\cdot\langle y\rangle^{2\lambda-2}h_s\right|\\
\le&C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\tilde\rho\|_{H_h^{s,\lambda-1}}+\|\rho^\infty\|_{L^\infty}+1\right)\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\|\tilde\rho\|_{H_h^{s,\lambda-1}}
\end{aligned}\tag{168}$$

since

$$\begin{aligned}
&\left\|-b\partial_t\ln p^{1/\gamma} - b^2\cdot\partial_y^2\tilde u - b\rho\frac{\partial_y^2\omega}{\omega}\right\|_{L^\infty}\\
\le&C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\tilde\rho\|_{H_h^{s,\lambda-1}}+\|\rho^\infty\|_{L^\infty}+1\right).
\end{aligned}\tag{169}$$

(6) The estimate on the term $-\sum_{j=1}^{s-1}\binom{j}{s}\int_\Omega\partial_x^{s-j}u\cdot h_{j+1}\cdot\langle y\rangle^{2\lambda-2}h_s$.

$$\begin{aligned}
&\sum_{j=1}^{s-1}\binom{j}{s}\int_\Omega\partial_x^{s-j}u\cdot h_{j+1}\cdot\langle y\rangle^{2\lambda-2}h_s\\
\le&\sum_{j=1}^{s-1}\binom{j}{s}\left\|\partial_x^{s-j}u\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda-1}h_{j+1}\right\|_{L^2}\left\|\langle y\rangle^{\lambda-1}h_s\right\|_{L^2}\\
\le&C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\left(\|\tilde\rho\|_{H_h^{s,\lambda-1}}+\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\|\tilde\rho\|_{H_h^{s,\lambda-1}}.
\end{aligned}\tag{170}$$

(7) The estimate on the term $\sum_{j=1}^{s-1}\binom{j}{s}\int_\Omega\partial_x^{s-j}v(\partial_x^j\partial_y\tilde\rho - b\partial_x^j\partial_y\tilde u)\cdot\langle y\rangle^{2\lambda-2}h_s$ is divided into two cases.

Case 1. When $j=1$, we have

$$\begin{aligned}
&\left|\int_\Omega\partial_x^{s-1}v(\partial_x\partial_y\tilde\rho - b\partial_x\partial_y\tilde u)\cdot\langle y\rangle^{2\lambda-2}h_s\right|\\
\le&\left|\int_\Omega\frac{\partial_x^{s-1}v+y\partial_x^sU}{1+y}(\partial_x\partial_y\tilde\rho - b\partial_x\partial_y\tilde u)\cdot\langle y\rangle^{2\lambda-1}h_s\right|\\
&+\left|\int_\Omega\partial_x^sU(\partial_x\partial_y\tilde\rho - b\partial_x\partial_y\tilde u)\cdot\langle y\rangle^{2\lambda-1}h_s\right|\\
\le&\left\|\frac{\partial_x^{s-1}v+y\partial_x^sU}{1+y}\right\|_{L^2}\|\langle y\rangle^\lambda(\partial_x\partial_y\tilde\rho - b\partial_x\partial_y\tilde u)\|_{L^\infty}\|\langle y\rangle^{\lambda-1}h_s\|_{L^2}\\
&+\|\partial_x^sU\|_{L^\infty}\|\langle y\rangle^\lambda(\partial_x\partial_y\tilde\rho - b\partial_x\partial_y\tilde u)\|_{L^2}\|\langle y\rangle^{\lambda-1}h_s\|_{L^2}\\
\le&C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^\infty}\right)\left(\|\omega\|_{H_g^{s,\lambda}}+\|\tilde\rho\|_{H_h^{s,\lambda-1}}\right)\|\tilde\rho\|_{H_h^{s,\lambda-1}}.
\end{aligned}\tag{171}$$

Case 2. When $j=2,\cdots,s-1$, we have

$$\int_\Omega\partial_x^{s-j}v(\partial_x^j\partial_y\tilde\rho - b\partial_x^j\partial_y\tilde u)\cdot\langle y\rangle^{2\lambda-2}h_s\tag{172}$$

$$
\begin{aligned}
&\leq \left\|\frac{\partial_x^{s-j} v}{1+y}\right\|_{L^\infty} \left\|\langle y\rangle^{\lambda}(\partial_x^j \partial_y \tilde{\rho} - b\partial_x^j \partial_y \tilde{u})\right\|_{L^2} \left\|\langle y\rangle^{\lambda-1} h_s\right\|_{L^2} \\
&\leq C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^\infty}\right)\left(\|\omega\|_{H_g^{s,\lambda}} + \|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}.
\end{aligned}
$$

Combining the above estimates (172)-(172), we get

$$
\begin{aligned}
&\sum_{j=1}^{s-1} \binom{j}{s} \int_\Omega \partial_x^{s-j} v(\partial_x^j \partial_y \tilde{\rho} - b\partial_x^j \partial_y \tilde{u}) \cdot \langle y\rangle^{2\lambda-2} h_s \\
&\leq C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^\infty}\right)\left(\|\omega\|_{H_g^{s,\lambda}} + \|\tilde{\rho}\|_{H_h^{s,\lambda-1}}\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}.
\end{aligned}
\tag{173}
$$

(8) The estimate on the term $\sum_{i=0}^{s-1}\binom{s}{i}\int_\Omega \partial_x^i \tilde{u}\cdot\partial_x^{s-i+1}\rho^\infty\langle y\rangle^{2\lambda-2}h_s$.

$$
\begin{aligned}
&\sum_{i=0}^{s-1} \int_\Omega \binom{s}{i} \partial_x^i \tilde{u} \cdot \partial_x^{s-i+1} \rho^\infty \langle y\rangle^{2\lambda-2} h_s \\
&\leq C \sum_{i=0}^{s-1} \left\|\partial_x^{s-i+1}\rho^\infty\right\|_{L^\infty} \left\|\langle y\rangle^{\lambda-1}\partial_x^i \tilde{u}\right\|_{L^2} \left\|\langle y\rangle^{\lambda-1} h_s\right\|_{L^2} \\
&\leq C\left\|\partial_x^{s+1}\rho^\infty\right\|_{L^\infty}\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2}\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}.
\end{aligned}
\tag{174}
$$

(9) The estimate on the term $\sum_{i=0}^{s}\binom{s}{i}\int_\Omega \partial_x^i\rho\cdot\partial_x^{s-i}\partial_y^2\tilde{u}\cdot b\cdot\langle y\rangle^{2\lambda-2}h_s$ is divided into three cases.

Case 1. When $i=0$, we have

$$
\begin{aligned}
&\int_\Omega \rho \cdot \partial_x^s \partial_y^2 \tilde{u} \cdot b \cdot \langle y\rangle^{2\lambda-2} h_s \\
=&\int_\Omega \rho \cdot \partial_y g_s \cdot b \cdot \langle y\rangle^{2\lambda-2} h_s + \int_\Omega \rho \cdot \partial_y a \cdot \partial_x^s \tilde{u} \cdot b \cdot \langle y\rangle^{2\lambda-2} h_s \\
&+ \int_\Omega \rho \cdot a \cdot g_s \cdot b \cdot \langle y\rangle^{2\lambda-2} h_s + \int_\Omega \rho \cdot a^2 \partial_x^s \tilde{u} \cdot b \cdot \langle y\rangle^{2\lambda-2} h_s \\
\leq&\|\sqrt{\rho}\langle y\rangle^{\lambda}\partial_y g_s\|_{L^2}\|\langle y\rangle^{\lambda-1}h_s\|_{L^2}\|\sqrt{\rho}b\|_{L^\infty} \\
&+ \|\rho\partial_y ab|_{L^\infty}\|\langle y\rangle^{\lambda-1}\partial_x^s\tilde{u}\|_{L^2}\|\langle y\rangle^{\lambda-1}h_s\|_{L^2} \\
&+ \|\rho ab\|_{L^\infty}\|\langle y\rangle^{\lambda}g_s\|_{L^2}\|\langle y\rangle^{\lambda-1}h_s\|_{L^2} + \|\rho a^2 b|_{L^\infty}\|\langle y\rangle^{\lambda-1}\partial_x^s\tilde{u}\|_{L^2}\|\langle y\rangle^{\lambda-1}h_s\|_{L^2} \\
\leq&\frac{1}{8}\|\langle y\rangle^{\lambda}\sqrt{\rho}\partial_y g_s\|_{L^2}^2 \\
&+ C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2}\right)\left(\|\tilde{\rho}\|_{H_h^{s,\lambda-1}} + \|\rho^\infty\|_{L^\infty}\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}.
\end{aligned}
\tag{175}
$$

Case 2. When $i=1,\cdots,s-1$, we have

$$
\begin{aligned}
&\int_\Omega \partial_x^i \rho \cdot \partial_x^{s-i} \partial_y^2 \tilde{u} \cdot b \cdot \langle y\rangle^{2\lambda-2} h_s \\
=&\int_\Omega \partial_x^i \tilde{\rho} \cdot \partial_x^{s-i} \partial_y \omega \cdot b \cdot \langle y\rangle^{2\lambda-2} h_s + \int_\Omega \partial_x^i \rho^\infty \cdot \partial_x^{s-i} \partial_y \omega \cdot b \cdot \langle y\rangle^{2\lambda-2} h_s \\
\leq&C\left(\|\tilde{\rho}\|_{H_h^{s,\lambda-1}} + \|\partial_x^{s-1}\rho^\infty\|_{L^\infty}\right)\|\omega\|_{H_g^{s,\lambda}}\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}.
\end{aligned}
\tag{176}
$$

Case 3. When $i = s$, we have

$$\begin{aligned}
&\int_\Omega \partial_x^s \rho \cdot \partial_y^2 \tilde{u} \cdot b \cdot \langle y\rangle^{2\lambda-2} h_s \\
=&\int_\Omega h_s \cdot \partial_y^2 \tilde{u} \cdot b \cdot \langle y\rangle^{2\lambda-2} h_s + \int_\Omega \partial_x^s \rho^\infty \cdot \partial_y^2 \tilde{u} \cdot b \cdot \langle y\rangle^{2\lambda-2} h_s \\
&+ \int_\Omega b^2 \partial_x^s \tilde{u} \cdot \partial_y^2 \tilde{u} \langle y\rangle^{2\lambda-2} h_s \\
\le& \|b\partial_y^2 \tilde{u}\|_{L^\infty} \|\langle y\rangle^{\lambda-1} h_s\|_{L^2}^2 + \|b\partial_x^s \rho^\infty\|_{L^\infty} \|\langle y\rangle^{\lambda} \partial_y^2 \tilde{u}\|_{L^2} \|\langle y\rangle^{\lambda-1} h_s\|_{L^2} \\
&+ \|b\partial_y^2 \tilde{u}\|_{L^\infty} \|\langle y\rangle^{\lambda-1} \partial_x^s \tilde{u}\|_{L^2} \|\langle y\rangle^{\lambda-1} h_s\|_{L^2} \\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2} + \|\tilde{\rho}\|_{H_h^{s,\lambda-1}} + \|\partial_x^s \rho^\infty\|_{L^\infty}\right) \\
&\cdot \left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^\infty}\right) \|\tilde{\rho}\|_{H_h^{s,\lambda-1}}.
\end{aligned} \tag{177}$$

Here we use $|b| \le \delta^{-2}$.

Combining all the estimates (175)-(177), we obtain

$$\begin{aligned}
&\sum_{i=0}^s \int_\Omega \binom{s}{i} \partial_x^i \rho \cdot \partial_x^{s-i} \partial_y^2 \tilde{u} \cdot b \cdot \langle y\rangle^{2\lambda-2} h_s \\
\le& \frac{1}{8} \|\langle y\rangle^\lambda \sqrt{\rho} \partial_y g_s\|_{L^2}^2 + C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2} + \|\tilde{\rho}\|_{H_h^{s,\lambda-1}} + \|\partial_x^s \rho^\infty\|_{L^\infty}\right) \\
&\cdot \left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^\infty}\right) \|\tilde{\rho}\|_{H_h^{s,\lambda-1}}.
\end{aligned} \tag{178}$$

(10) The estimate on the term $\int_\Omega \partial_x^s(\tilde{\rho} \cdot \partial_t \ln p) \cdot \langle y\rangle^{2\lambda-2} h_s$.

$$\begin{aligned}
&\int_\Omega \partial_x^s(\tilde{\rho} \cdot \partial_t \ln p^{1/\gamma}) \cdot \langle y\rangle^{2\lambda-2} h_s \\
=& \int_\Omega h_s \cdot \partial_t \ln p^{1/\gamma} \cdot \langle y\rangle^{2\lambda-2} h_s + \int_\Omega b\partial_x^s \tilde{u} \cdot \partial_t \ln p^{1/\gamma} \cdot \langle y\rangle^{2\lambda-2} h_s \\
\le& \|\partial_t \ln p^{1/\gamma}\|_{L^\infty} \|\langle y\rangle^{\lambda-1} h_s\|_{L^2}^2 + \|b\partial_t \ln p^{1/\gamma}\|_{L^\infty} \|\langle y\rangle^{\lambda-1} \partial_x^s \tilde{u}\|_{L^2} \|\langle y\rangle^{\lambda-1} h_s\|_{L^2} \\
\le& C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2}\right) \|\tilde{\rho}\|_{H_h^{s,\lambda-1}} + C\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}^2.
\end{aligned} \tag{179}$$

(11) The estimate the term $\sum_{i=1}^s \binom{s}{i} \int_\Omega b\partial_x^{i+1} U \partial_x^{s-i} \tilde{u} \cdot \langle y\rangle^{2\lambda-2} h_s$.

$$\begin{aligned}
&\sum_{i=1}^s \binom{s}{i} \int_\Omega b\partial_x^{i+1} U \partial_x^{s-i} \tilde{u} \cdot \langle y\rangle^{2\lambda-2} h_s \\
\le& C\sum_{i=1}^s \|\partial_x^{i+1} U\|_{L^\infty} \|\langle y\rangle^{\lambda-1} \partial_x^{s-i} \tilde{u}\|_{L^2} \|\langle y\rangle^{\lambda-1} h_s\|_{L^2} \\
\le& C\|\partial_x^{s+1} U\|_{L^\infty} \left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2}\right) \|\tilde{\rho}\|_{H_h^{s,\lambda-1}}.
\end{aligned} \tag{180}$$

(12) The estimate on the term $\int_\Omega 2\epsilon^2 \partial_x b \cdot \partial_x^{s+1} \tilde{u} \cdot \langle y\rangle^{2\lambda-2} h_s$.
Recalling (96)

$$\|\langle y\rangle^{\lambda-1} \partial_x^{s+1} \tilde{u}\|_{L^2} \le C\left(\|\partial_x^{s+1} U\|_{L^2} + \|\langle y\rangle^\lambda \partial_x g_s\|_{L^2} + \|\langle y\rangle^{\lambda-1} \partial_x^s \tilde{u}\|_{L^2}\right), \tag{181}$$

and we have

$$\begin{aligned}
&\int_\Omega 2\epsilon^2\partial_x b\cdot\partial_x^{s+1}\tilde{u}\cdot\langle y\rangle^{2\lambda-2}h_s\\
\leq&2\epsilon^2\left\|\partial_x b\right\|_{L^\infty}\left\|\langle y\rangle^{\lambda-1}\partial_x^{s+1}\tilde{u}\right\|_{L^2}\left\|\langle y\rangle^{\lambda-1}h_s\right\|_{L^2}\\
\leq&\frac{\epsilon^2}{4}\left\|\langle y\rangle^{\lambda}\partial_x g_s\right\|_{L^2}^2+C\epsilon^2\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^sU\|_{L^2}\right)\|\tilde{\rho}\|_{H_h^{s,\lambda-1}}+C\epsilon^2\|\tilde{\rho}\|^2_{H_h^{s,\lambda-1}}.
\end{aligned}\tag{182}$$

Combining the above all estimates, we obtain the estimate (160). This completes the proof of Lemma 3.2. □

4. **Closeness of priori estimate.** In this section, we will close the *priori* estimate. Firstly, combining Lemmas 2.1, 2.3, 3.1 and 3.2, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\omega\|^2_{H_g^{s,\lambda}}+\|\tilde{\rho}\|^2_{H_h^{s,\lambda-1}}\right)+\frac{1}{4}\sum_{|\alpha|\leq s,\alpha_1\leq s-1}\|\langle y\rangle^{\lambda+\alpha_2}\partial_y D^\alpha\omega\|^2_{L^2}\\
&+\epsilon^2\sum_{|\alpha|\leq s,\alpha_1\leq s-1}\|\langle y\rangle^{\lambda+\alpha_2}\partial_x D^\alpha\omega\|^2_{L^2}+\epsilon^2\sum_{|\alpha|\leq s,\alpha_1\leq s-1}\|\langle y\rangle^{\lambda+\alpha_2-1}\partial_x D^\alpha\tilde{\rho}\|^2_{L^2}\\
&+\frac{\epsilon^2}{4}\|\langle y\rangle^{\lambda}\partial_x g_s\|^2_{L^2}+\frac{\epsilon^2}{2}\|\langle y\rangle^{\lambda-1}\partial_x h_s\|^2_{L^2}\\
\leq&C\left(\|\omega\|_{H_g^{s,\lambda}}+\|\partial_x^{s+1}U\|_{L^\infty}+1\right)^{s+2}\left(\|\tilde{\rho}\|_{H_h^{s,\lambda}}+\|\partial_x^{s+1}\rho^\infty\|_{L^\infty}+1\right)^{s}\\
\leq&C\left(\|\omega\|^2_{H_g^{s,\lambda}}+\|\tilde{\rho}\|^2_{H_h^{s,\lambda-1}}\right)^{s+2}+F(t),
\end{aligned}\tag{183}$$

where

$$F(t)=C\left(\|\partial_x^{s+1}U\|^2_{L^\infty}+\|\partial_x^{s+1}\rho^\infty\|^2_{L^\infty}+1\right)^{s+2}.\tag{184}$$

Thus, it follows from the comparison principal of ordinary differential equations that

$$\begin{aligned}
&\|\omega\|^2_{H_g^{s,\lambda}}+\|\tilde{\rho}\|^2_{H_h^{s,\lambda-1}}\\
\leq&\left\{\|\omega_0\|^2_{H_g^{s,\lambda}}+\|\tilde{\rho}_0\|^2_{H_h^{s,\lambda-1}}+\int_0^t F(\tau)d\tau\right\}\\
&\cdot\left\{1-(s+1)C\left(\|\omega_0\|^2_{H_g^{s,\lambda}}+\|\tilde{\rho}_0\|^2_{H_h^{s,\lambda-1}}+\int_0^t F(\tau)d\tau\right)^{s+1}t\right\}^{-\frac{1}{s+1}}\\
=:&\mathcal{H}
\end{aligned}\tag{185}$$

as long as the quantity within the second set of braces on the right-hand side of the inequality (185) is positive.

Lemma 4.1. *Let $s\geq 6$ be an even integer, $\lambda\geq 1$ be a real number and hypotheses for $(U,\rho^\infty,p)(t,x)$ given in Theorem 1.1 hold. Assume that $(\omega,\tilde{\rho})$ is a classical solution to initial-boundary value problem (23)-(25) in $[0,T]$ satisfying that*

$$\omega\in L^\infty(0,T;H^{s,\lambda}),\quad \tilde{\rho}\in L^\infty(0,T;H^{s,\lambda-1}).$$

Then, the inequality (185) holds for small time, Taking derivative $D^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$ on the equation $(23)_2$ and multiplying $\langle y\rangle^{\lambda+\alpha_2}$, one has

$$
\begin{aligned}
&\partial_t\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega + \sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\langle y\rangle^{\lambda+\alpha_2}D^\beta u\cdot\partial_x D^{\alpha-\beta}\omega\\
&+\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\langle y\rangle^{\lambda+\alpha_2}D^\beta v\cdot\partial_y D^{\alpha-\beta}\omega\\
=&\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\cdot\partial_t\ln p^{1/\gamma} + \sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\langle y\rangle^{\lambda+\alpha_2}D^\beta\rho\cdot\partial_y^2 D^{\alpha-\beta}\omega\\
&+\sum_{0\le\beta\le\alpha}\binom{\alpha}{\beta}\langle y\rangle^{\lambda+\alpha_2}\partial_y D^\beta\rho\cdot\partial_y D^{\alpha-\beta}\omega + \epsilon^2\langle y\rangle^{\lambda+\alpha_2}\partial_x^2 D^\alpha\omega.
\end{aligned}
\tag{186}
$$

Claim 1. *From Lemma 4.4, when $|\beta|\le 2$, we have the following results*

$$
\left\|\langle y\rangle^{\beta_2}D^\beta u\right\|_{L^\infty} \le \begin{cases} C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2}\right), & \beta_2 = 0,\\ C\|\omega\|_{H_g^{s,\lambda}}, & \beta_2\ge 1.\end{cases}
\tag{187}
$$

$$
\left\|\langle y\rangle^{\beta_2-1}D^\beta v\right\|_{L^\infty} \le C\left(\|\omega\|_{H_g^{s,\lambda}} + \|\partial_x^s U\|_{L^2} + 1\right).
\tag{188}
$$

$$
\left\|\langle y\rangle^{\beta_2-2}D^\beta \rho\right\|_{L^\infty} \le C\left(\|\tilde\rho\|_{H_h^{s,\lambda-1}} + \|\partial_x^2\rho^\infty\|_{L^\infty}\right).
\tag{189}
$$

$$
\left\|\langle y\rangle^{\beta_2-1}\partial_y D^\beta \rho\right\|_{L^\infty} \le C\,\|\tilde\rho\|_{H_h^{s,\lambda-1}}.
\tag{190}
$$

Then by direct calculations, we have

$$
\left\|\partial_t\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|_{L^\infty} \le C\mathcal{H} + C\left(\|\partial_x^s U\|_{L^2} + \|\partial_x^2\rho^\infty\|_{L^\infty} + 1\right)^2
\tag{191}
$$

and

$$
\begin{aligned}
&\sum_{|\alpha|\le 2}\left\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\right\|_{L^\infty}\\
\le&\sum_{|\alpha|\le 2}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega_0\|_{L^\infty} + \sum_{|\alpha|\le 2}\int_0^t\|\partial_t\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^\infty}d\tau\\
\le&\sum_{|\alpha|\le 2}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega_0\|_{L^\infty} + \sum_{|\alpha|\le 2}\sup_{0\le\tau\le t}\|\partial_t\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega\|_{L^\infty}t\\
\le&\sum_{|\alpha|\le 2}\|\langle y\rangle^{\lambda+\alpha_2}D^\alpha\omega_0\|_{L^\infty}\\
&+C\mathcal{H}t + C\sup_{0\le\tau\le t}\left(\|\partial_x^s U\|_{L^2} + \|\partial_x^2\rho^\infty\|_{L^\infty} + 1\right)^2 t.
\end{aligned}
\tag{192}
$$

By using the case $\alpha = (0,0)$ in (186), we also have

$$
\begin{aligned}
\langle y\rangle^\lambda\omega \ge&\langle y\rangle^\lambda\omega_0 - \int_0^t\|\partial_t\langle y\rangle^\lambda\omega\|_{L^\infty}d\tau\\
\ge&\langle y\rangle^\lambda\omega_0 - \sup_{0\le\tau\le t}\|\partial_t\langle y\rangle^\lambda\omega\|_{L^\infty}t\\
\ge&\langle y\rangle^\lambda\omega_0 - C\mathcal{H}t - C\sup_{0\le\tau\le t}\left(\|\partial_x^s U\|_{L^2} + \|\partial_x^2\rho^\infty\|_{L^\infty} + 1\right)^2 t.
\end{aligned}
\tag{193}
$$

Taking derivative $D^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$ on the equation $(23)_1$ and multiplying $\langle y\rangle^{\lambda+\alpha_2-1}$, one has

$$\begin{aligned}
&\partial_t \langle y\rangle^{\lambda+\alpha_2-1} D^\alpha \tilde{\rho} + \sum_{0\le\beta\le\alpha} \tbinom{\alpha}{\beta} \langle y\rangle^{\lambda+\alpha_2-1} D^\beta u \cdot \partial_x D^{\alpha-\beta}\tilde{\rho} \\
&\quad + \sum_{0\le\beta\le\alpha} \tbinom{\alpha}{\beta} \langle y\rangle^{\lambda+\alpha_2-1} D^\beta v \cdot \partial_y D^{\alpha-\beta}\tilde{\rho} \\
&= \sum_{0\le\beta\le\alpha} \tbinom{\alpha}{\beta} \langle y\rangle^{\lambda+\alpha_2-1} D^\beta \tilde{u} \cdot \partial_x D^{\alpha-\beta}\rho^\infty - \langle y\rangle^{\lambda+\alpha_2-1} D^\alpha\tilde{\rho}\cdot\partial_t \ln p^{1/\gamma} \\
&\quad + \epsilon^2 \langle y\rangle^{\lambda+\alpha_2-1}\partial_x^2 D^\alpha \tilde{\rho}.
\end{aligned} \tag{194}$$

Using Claim 1, we obtain

$$\begin{aligned}
&\sum_{|\alpha|\le 2} \|\langle y\rangle^{\lambda+\alpha_2-1} D^\alpha\tilde{\rho}\|_{L^\infty} \\
&\le \sum_{|\alpha|\le 2} \|\langle y\rangle^{\lambda+\alpha_2-1} D^\alpha\tilde{\rho}_0\|_{L^\infty} + \sum_{|\alpha|\le 2}\int_0^t \|\partial_t\langle y\rangle^{\lambda+\alpha_2-1} D^\alpha\tilde{\rho}\|_{L^\infty} d\tau \\
&\le \sum_{|\alpha|\le 2} \|\langle y\rangle^{\lambda+\alpha_2-1} D^\alpha\tilde{\rho}_0\|_{L^\infty} + \sum_{|\alpha|\le 2}\sup_{0\le\tau\le t} \|\partial_t\langle y\rangle^{\lambda+\alpha_2-1} D^\alpha\tilde{\rho}\|_{L^\infty} t \\
&\le \sum_{|\alpha|\le 2} \|\langle y\rangle^{\lambda+\alpha_2-1} D^\alpha\omega_0\|_{L^\infty} + C\mathcal{H}t \\
&\quad + C\sup_{0\le\tau\le t}\left(\|\partial_x^s U\|_{L^2} + \|\partial_x^3\rho^\infty\|_{L^\infty}+1\right)^2 t.
\end{aligned} \tag{195}$$

Similar to (193), we also have

$$\begin{aligned}
\langle y\rangle^{\lambda-1}\tilde{\rho} \ge& \langle y\rangle^{\lambda-1}\tilde{\rho}_0 - \int_0^t \|\partial_t\langle y\rangle^{\lambda-1}\tilde{\rho}\|_{L^\infty} d\tau \\
\ge& \langle y\rangle^{\lambda-1}\tilde{\rho}_0 - \sup_{0\le\tau\le t}\|\partial_t\langle y\rangle^{\lambda-1}\tilde{\rho}\|_{L^\infty} t \\
\ge& \langle y\rangle^{\lambda-1}\tilde{\rho}_0 - C\mathcal{H}t \\
&- C\sup_{0\le\tau\le t}\left(\|\partial_x^s U\|_{L^2} + \|\partial_x^3\rho^\infty\|_{L^\infty}+1\right)^2 t,
\end{aligned} \tag{196}$$

which implies

$$\rho \ge \rho_0 - C\mathcal{H}t - C\sup_{0\le\tau\le t}\left(\|\partial_x^s U\|_{L^2} + \|\partial_x^3\rho^\infty\|_{L^\infty}+1\right)^2 t. \tag{197}$$

Lemma 4.2. *Let $s\ge 6$ be an even integer, $\lambda\ge 1$, $\delta\in(0,1)$ and $\epsilon\in(0,1)$. If $\omega_0\in H_{2\delta}^{s,\lambda}$ and $\tilde{\rho}_0\in H_{2\delta}^{s,\lambda-1}$. $F(t)$ defined by (184) satisfies $|F(t)|\le M$. Then there exists a uniform life span $T>0$, which is independent of ϵ, such that the regularized system has solutions $\omega\in C([0,T];H_\delta^{s,\lambda})\cap C^1([0,T];H^{s-2,\lambda})$ and $\tilde{\rho}\in C([0,T];H_\delta^{s,\lambda-1})\cap C^1([0,T];H^{s-2,\lambda-1})$ with the following uniform in ϵ estimates:*
(i)(Uniform weighted H^s estimates) For any $\epsilon\in[0,1]$ and any $t\in[0,T]$,

$$\|\omega\|^2_{H_g^{s,\lambda}} + \|\tilde{\rho}\|^2_{H_h^{s,\lambda-1}} \le 4\left(\|\omega_0\|^2_{H_g^{s,\lambda}} + \|\tilde{\rho}_0\|^2_{H_h^{s,\lambda-1}}\right). \tag{198}$$

(ii) (Uniform weighted L^∞ upper bound) For any $\epsilon \in [0,1]$ and any $t \in [0,T]$,

$$\sum_{|\alpha|\le 2} \left\| \langle y\rangle^{\lambda+\alpha_2} D^\alpha \omega \right\|_{L^\infty} \le \delta^{-1} \tag{199}$$

and

$$\sum_{|\alpha|\le 2} \left\| \langle y\rangle^{\lambda+\alpha_2-1} D^\alpha \tilde\rho \right\|_{L^\infty} \le \delta^{-1}. \tag{200}$$

(iii)(Uniform weighted L^∞ lower bound) For any $\epsilon \in [0,1]$ and any $t \in [0,T]$,

$$\langle y\rangle^\lambda \omega \ge \delta \quad \text{and} \quad \rho \ge \delta. \tag{201}$$

Proof. Now we assume $F(t) = C(\|\partial_x^{s+1} U\|^2_{L^\infty} + \|\partial_x^{s+1}\rho^\infty\|^2_{L^\infty} + 1)^{s+2} \le M$. If T_1 is chosen by

$$\min\left\{ \frac{\|\omega_0\|^2_{H_g^{s,\lambda}} + \|\tilde\rho_0\|^2_{H_h^{s,\lambda-1}}}{M}, \frac{1-2^{-s-1}}{C(s+1)2^{s+1}(\|\omega_0\|^2_{H_g^{s,\lambda}} + \|\tilde\rho_0\|^2_{H_h^{s,\lambda-1}})^{s+1}} \right\}, \tag{202}$$

then by (185), we have for all $t \in [0,T_1]$

$$\|\omega\|^2_{H_g^{s,\lambda}} + \|\tilde\rho\|^2_{H_h^{s,\lambda-1}} \le 4(\|\omega_0\|^2_{H_g^{s,\lambda}} + \|\tilde\rho_0\|^2_{H_h^{s,\lambda-1}}). \tag{203}$$

In fact, we also have for $t \in [0,T_1]$

$$\mathcal{H} \le 4\left(\|\omega_0\|^2_{H_g^{s,\lambda}} + \|\tilde\rho_0\|^2_{H_h^{s,\lambda-1}} \right). \tag{204}$$

Secondly, due to $\sum_{|\alpha|\le 2} \|\langle y\rangle^{\lambda+\alpha_2} D^\alpha \omega_0\|_{L^\infty} \le \frac{1}{2\delta}$, if T_2 is chosen as

$$T_2 = \min\left\{ T_1, \frac{\delta^{-1}}{16C(\|\omega_0\|^2_{H_g^{s,\lambda}} + \|\tilde\rho_0\|^2_{H_h^{s,\lambda-1}})}, \frac{\delta^{-1}}{4CM^{\frac{1}{s+2}}} \right\}, \tag{205}$$

then from (192), we have for all $t \in [0,T_2]$

$$\sum_{|\alpha|\le 2} \left\| \langle y\rangle^{\lambda+\alpha_2} D^\alpha \omega \right\|_{L^\infty} \le \delta^{-1}. \tag{206}$$

Here we used the fact

$$\sup_{0\le\tau\le t} \left(\|\partial_x^s U\|_{L^2} + \|\partial_x^2 \rho^\infty\|_{L^\infty} + 1 \right) \le C \sup_{0\le\tau\le t} F^{\frac{1}{2(s+2)}} \le CM^{\frac{1}{2(s+2)}}.$$

Thirdly, due to $\langle y\rangle^\lambda \omega_0 \ge 2\delta$, if T_3 is chosen as

$$T_3 = \min\left\{ T_1, \frac{\delta}{8C(\|\omega_0\|^2_{H_g^{s,\lambda}} + \|\tilde\rho_0\|^2_{H_h^{s,\lambda-1}})}, \frac{\delta}{2CM^{\frac{1}{s+2}}} \right\}, \tag{207}$$

then from (193), we have for all $t \in [0,T_3]$

$$\langle y\rangle^\lambda \omega \ge \delta. \tag{208}$$

In addition, due to $\sum_{|\alpha|\le 2} \|\langle y\rangle^{\lambda+\alpha_2-1} D^\alpha \tilde\rho_0\|_{L^\infty} \le \frac{1}{2\delta}$, if T_4 is chosen as

$$T_4 = \min\left\{ T_1, \frac{\delta^{-1}}{16C(\|\omega_0\|^2_{H_g^{s,\lambda}} + \|\tilde\rho_0\|^2_{H_h^{s,\lambda-1}})}, \frac{\delta^{-1}}{4CM^{\frac{1}{s+2}}} \right\}, \tag{209}$$

then from (195), we get for all $t \in [0, T_4]$

$$\sum_{|\alpha|\le 2} \left\| \langle y\rangle^{\lambda+\alpha_2-1} D^\alpha \tilde\rho \right\|_{L^\infty} \le \frac{1}{\delta}. \tag{210}$$

Finally, due to $\rho_0 \ge 2\delta$, if T_5 is chosen as

$$T_5 = \min\left\{ T_1, \frac{\delta}{8C(\|\omega_0\|^2_{H^{s,\lambda}_g} + \|\tilde\rho_0\|^2_{H^{s,\lambda-1}_h})}, \frac{\delta}{2CM^{\frac{1}{s+2}}} \right\}, \tag{211}$$

then from (197), we obtain for all $t \in [0, T_5]$

$$\rho \ge \delta. \tag{212}$$

Then we choose $T := \min\{T_1, T_2, T_3, T_4, T_5\}$ and this completes the proof of Lemma 4.2. □

5. **Local-in-time existence and uniqueness.**

5.1. **Local-in-time existence.**

Using almost equivalence relation (14) and uniform weighted H^s estimate (199), we have

$$\begin{aligned}
&\sup_{0\le t\le T} \left(\|\omega^\epsilon\|^2_{H^{s,\lambda}} + \|\tilde u^\epsilon\|^2_{H^{s,\lambda-1}} + \|\tilde\rho^\epsilon\|^2_{H^{s,\lambda-1}} \right)\\
\le& C \sup_{0\le t\le T} \left(\|\omega^\epsilon\|^2_{H^{s,\lambda}_g} + \|\tilde\rho^\epsilon\|^2_{H^{s,\lambda-1}_h} + \|\partial_x^s U\|^2_{L^2} \right)\\
\le& 4C \left(\|\omega^\epsilon_0\|^2_{H^{s,\lambda}_g} + \|\tilde\rho^\epsilon_0\|^2_{H^{s,\lambda-1}_h} + \|\partial_x^s U\|^2_{L^2} \right).
\end{aligned} \tag{213}$$

Furthermore, we also know that $\partial_t \omega^\epsilon, \partial_t \tilde u$ and $\partial_t \tilde\rho$ are uniformly bounded in $L^\infty(0,T; H^{s-2,\lambda})$ and $L^\infty(0,T; H^{s-2,\lambda-1})$, respectively. By the Lions-Aubin lemma and compact embedding of $H^{s,\lambda}$ in $H^{s'}_{loc}$, we have, after taking a subsequence, as $\epsilon_k \to 0^+$

$$\left\{\begin{array}{lll}
\omega^{\epsilon_k} \overset{*}{\rightharpoonup} \omega & in & L^\infty(0,T; H^{s,\lambda}),\\
\omega^{\epsilon_k} \to \omega & in & C(0,T; H^{s'}_{loc}),\\
\tilde u^{\epsilon_k} \overset{*}{\rightharpoonup} \tilde u & in & L^\infty(0,T; H^{s,\lambda-1}),\\
\tilde u^{\epsilon_k} \to \tilde u & in & C(0,T; H^{s'}_{loc}),\\
\tilde\rho^{\epsilon_k} \overset{*}{\rightharpoonup} \tilde\rho & in & L^\infty(0,T; H^{s,\lambda-1}),\\
\tilde\rho^{\epsilon_k} \to \tilde\rho & in & C(0,T; H^{s'}_{loc})
\end{array}\right. \tag{214}$$

for all $s' < s$, where $\omega = \partial_y u \in L^\infty(0,T; H^{s,\lambda}) \cap \bigcap_{s'<s} C(0,T; H^{s'}_{loc})$, $\tilde u \in L^\infty(0,T; H^{s,\lambda-1}) \cap \bigcap_{s'<s} C(0,T; H^{s'}_{loc})$ and $\tilde\rho \in L^\infty(0,T; H^{s,\lambda-1}) \cap \bigcap_{s'<s} C(0,T; H^{s'}_{loc})$. Using the local uniform convergence of $\partial_x u^{\epsilon_k}$, we also have the pointwise convergence of v^{ϵ_k}, as $\epsilon_k \to 0^+$

$$v^{\epsilon_k} = -\int_0^y \partial_x u^{\epsilon_k} dy - y\partial_t \ln p^{1/\gamma} \to -\int_0^y \partial_x u dy - y\partial_t \ln p^{1/\gamma} =: v. \tag{215}$$

Combining (214) and (215), one may justify the pointwise convergence of all terms in the regularized equation (23)-(25). Thus, passing to the limit $\epsilon_k \to 0^+$ in (23)-(25) and regularized Bernoulli's law (7) and (8), we know that the limit (u, v, ρ) solves the problem (4)-(6) with the Bernoulli's law (20) and (21) in the classical sense.

5.2. **Uniqueness.** Assume (u_1, v_1, ρ_1) and (u_2, v_2, ρ_2) are all the solutions to the initial-boundary value problem (4)-(6) and $\omega_i = \partial_y u_i$ $(i = 1, 2)$, then set $\bar{u} = u_1 - u_2, \bar{v} = v_1 - v_2, \bar{\rho} = \rho_1 - \rho_2, \bar{\omega} = \omega_1 - \omega_2$, we obtain the following equations

$$\begin{cases} \left(\partial_t + u_1\partial_x + v_1\partial_y - \rho_1\partial_y^2\right)\bar{\omega} + \bar{u}\partial_x\omega_2 + \bar{v}\partial_y\omega_2 \\ = \bar{\omega}\partial_t \ln p^{1/\gamma} + \partial_y\rho_1\partial_y\bar{\omega} + \partial_y\bar{\rho}\partial_y\omega_2 + \bar{\rho}\partial_y^2\omega_2, \\ \left(\partial_t + u_1\partial_x + v_1\partial_y\right)\bar{\rho} + \bar{u}\partial_x\rho_2 + \bar{v}\partial_y\rho_2 = -\bar{\rho}\partial_t \ln p^{1/\gamma}, \\ \left(\partial_t + u_1\partial_x + v_1\partial_y - \rho_1\partial_y^2\right)\bar{u} + \bar{u}\partial_x u_2 + \bar{v}\partial_y u_2 = \bar{\rho}\partial_y^2 u_2, \\ \partial_x\bar{u} + \partial_y\bar{v} = 0, \\ \bar{\omega}|_{t=0} = \omega_{10} - \omega_{20}, \\ \bar{\rho}|_{t=0} = \rho_{10} - \rho_{20}, \\ \bar{u}|_{t=0} = u_{10} - u_{20}, \\ (\bar{u}, \bar{v})|_{y=0} = 0, \quad \partial_y\bar{\omega}|_{y=0} = 0. \end{cases} \tag{216}$$

Further set $\hat{\omega} = \bar{\omega} - a_2\bar{u}$ and $\hat{\rho} = \bar{\rho} - b_2\bar{u}$ with $a_2 = \frac{\partial_y\omega_2}{\omega_2}$ and $b_2 = \frac{\partial_y\rho_2}{\omega_2}$. By direct calculations, we get

$$\begin{cases} \left(\partial_t + u_1\partial_x + v_1\partial_y - \rho_1\partial_y^2\right)\hat{\omega} \\ = A\hat{\omega} + \partial_y\rho_1\partial_y\hat{\omega} + \hat{\rho}\left(\partial_y^2\omega_2 - a_2\partial_y^2 u_2\right) + \partial_y\omega_2\partial_y\hat{\rho} + B\bar{u}, \\ \left(\partial_t + u_1\partial_x + v_1\partial_y\right)\hat{\rho} = -\hat{\rho}\left(\partial_t \ln p^{1/\gamma} + \partial_y^2 u_2\right) - \rho_1 b_2\partial_y\hat{\omega} - \rho_1 a_2 b_2\hat{\omega} + C\bar{u}, \\ \hat{\omega}|_{t=0} = (\omega_{10} - \omega_{20}) - a_{20}(u_{10} - u_{20}), \\ \hat{\rho}|_{t=0} = (\rho_{10} - \rho_{20}) - b_{20}(u_{10} - u_{20}), \\ \hat{\omega}|_{y=0} = (\omega_1 - \omega_2)|_{y=0}, \quad \partial_y\hat{\omega}|_{y=0} = 0, \end{cases} \tag{217}$$

where

$$\begin{aligned} A =& \partial_t \ln p^{1/\gamma} + a_2\partial_y\rho_1 + b_2\partial_y\omega_2 + 2\rho_1\partial_y a_2, \\ B =& -\left(\partial_t + u_1\partial_x + v_1\partial_y - \rho_1\partial_y^2\right)a_2 - \partial_x\omega_2 + a_2\partial_x u_2 + a_2\partial_t \ln p^{1/\gamma} + \partial_y\rho_1\partial_y a_2 \\ &+ \partial_y\rho_1 a_2^2 + \partial_y\omega_2\partial_y b_2 + \partial_y\omega_2 a_2 b_2 + 2\rho_1 a_2\partial_y a_2 + \left(\partial_y^2\omega_2 - a_2\partial_y^2 u_2\right)b_2, \\ C =& -\left(\partial_t + u_1\partial_x + v_1\partial_y\right)b_2 - \partial_x\rho_2 + b_2\partial_x u_2 + b_2\partial_t \ln p^{1/\gamma} + \rho_1 b_2\partial_y a_2 \\ &+ \rho_1 b_2 a_2^2 + \partial_y^2 u_2 b_2. \end{aligned}$$

It follows from direct calculations

$$\|A\|_{L^\infty}, \|\langle y\rangle B\|_{L^\infty}, \|C\|_{L^\infty} \leq C,$$

where

$$C = C(T, U, \rho^\infty, p, \|\tilde{\rho}\|_{H^3_{\lambda-1}}, \|\tilde{\omega}\|_{H^4_\lambda}) > 0.$$

Multiplying the equations $(217)_1$ and $(217)_2$ by $\langle y\rangle^{2\lambda}\hat{\omega}$ and $\langle y\rangle^{2\lambda-2}\hat{\rho}$ respectively, then integrating the resulting equations over $\mathbb{T}\times[0,+\infty)$, we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|\langle y\rangle^{\lambda}\hat{\omega}\|_{L^2}^2+\|\langle y\rangle^{\lambda-1}\hat{\rho}\|_{L^2}^2\right)\\ \le& C\left(\|\langle y\rangle^{\lambda}\hat{\omega}\|_{L^2}^2+\|\langle y\rangle^{\lambda-1}\hat{\rho}\|_{L^2}^2+\|\langle y\rangle^{\lambda-1}\bar{u}\|_{L^2}^2\right).\end{aligned}\tag{218}$$

$\bar{u}$ can be expressed by $\bar{u}=\omega_2\int_0^y\frac{\hat{\omega}}{\omega_2}dy$ because of $\hat{\omega}=\omega_2\partial_y\left(\frac{\bar{u}}{\omega_2}\right)$ and $\bar{u}|_{y=0}=0$. Thus one has

$$\begin{aligned}\|\langle y\rangle^{\lambda-1}\bar{u}\|_{L^2}\le&\left\|\langle y\rangle^{\lambda-1}\omega_2\int_0^y\frac{\hat{\omega}}{\omega_2}dy\right\|_{L^2}\\ \le&\|\langle y\rangle^{\lambda}\omega_2\|_{L^\infty}\left\|\langle y\rangle^{-1}\int_0^y\frac{\hat{\omega}}{\omega_2}dy\right\|_{L^2}\\ \le& C\|\langle y\rangle^{\lambda}\hat{\omega}\|_{L^2}.\end{aligned}\tag{219}$$

Substituting (219) into (218), we have

$$\frac{d}{dt}\left(\|\langle y\rangle^{\lambda}\hat{\omega}\|_{L^2}^2+\|\langle y\rangle^{\lambda-1}\hat{\rho}\|_{L^2}^2\right)\le C\left(\|\langle y\rangle^{\lambda}\hat{\omega}\|_{L^2}^2+\|\langle y\rangle^{\lambda-1}\hat{\rho}\|_{L^2}^2\right).\tag{220}$$

Applying Gronwall's inequality, we further obtain due to $\hat{\omega}|_{t=0}=0$ and $\hat{\rho}|_{t=0}=0$

$$\|\langle y\rangle^{\lambda}\hat{\omega}\|_{L^2}^2+\|\langle y\rangle^{\lambda-1}\hat{\rho}\|_{L^2}^2\equiv 0,\tag{221}$$

which means $\hat{\omega}\equiv 0$ and $\hat{\rho}\equiv 0$. Consequently, it follows that $\bar{u}\equiv 0$ from $\bar{u}=\omega_2\int_0^y\frac{\hat{\omega}}{\omega_2}dy$. We also further obtain $\rho_1=\rho_2$ due to $\bar{\rho}=\hat{\rho}+b_2\bar{u}=0$. Finally we get $v_1=v_2$ from $(4)_3$. This completes the proof of uniqueness.

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